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Lewis Carroll's "Pillow Problems": On the 1993 Centenary

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Abstract. We examine a few key problems and their solutions from the 13 probability problems. Some are badly posed with imaginative but incorrect solutions; others are difficult, interesting and with correct solutions. The work of Carroll (C. L. Dodgson) is used to illustrate the nature, standing and understanding of probability within the wider English mathematical community of his time. Additionally, a probabilistic controversy in which he was involved is discussed, and an Appendix presents a Markov chain formulation of published and unpublished problems and discussion of a further unpublished problem. One focus of the paper is the intuitive difficulty in distinguishing between events of probability zero and impossible events, and the teaching of such probability-based difficulties.

Key words and phrases: English probability, urn models, prior distribution, Bayes' theorem, sample space, uncountability, impossible event, Kolmogorov's axioms, Markov chains.

1. INTRODUCTION

Readers may know that Lewis Carroll, the author of *Alice in Wonderland* and *Through the Looking-Glass,* was an Oxford mathematician whose real name was Charles Lutwidge Dodgson (1832–1898). A great deal has been written about him as a result of these books on which his fame rests, but little has been said about the mathematical aspects of Dodgson's creativity, and even less about his interest in probability.

My own interest in Dodgson's probability arose in 1979 when I was giving talks to high school mathematics teachers on the importance of probability and statistics in the curriculum. When we got to Bayes' Theorem, one of them came to me with a problem on this topic from a book called *Pillow Problems and a Tangled Tale* by Lewis Carroll, published by Dover in 1958. The first part is a reprinting of the 4th edition of *Pillow Problems* which has a preface written by its author in March 1895. The introduction to the 1st edition of his *Curiosa Mathematica, Part II: Pillow Problems* is dated May 1893. It is the hundredth anniversary that the present article commemorates, from a probabilistic standpoint.

By 1983 I had acquired a modest library on Carroll, partly thanks to a nonmathematical friend who assumed all mathematicians are equal and interested in each other. It had become clear that there were many biographies; in these there was generally some attempt to say something about Dodgson's mathematics, citing some expert. The assessment was generally negative. In Lennon (1945), Eric Temple Bell comments (p. 271) in characteristic fashion: "His range was that of a freshman today in a good technical school, though the freshman would have clearer ideas about elementary things than CLD, and would not be so far behind his own times mathematically as CLD was behind his all his life." The main mathematical commentator until recent years has been Warren Weaver. The author of *Lady Luck*, on looking through Dodgson's unpublished manuscripts, assesses him thus (Weaver, 1954): "the more elementary aspects of calculus represented the upper limits of Dodgson's mathematical flights, and . . . even in calculus, he had such vague and inaccurate notions about infinitesimals that one must confess he lagged behind the best knowledge of his times." [Quoted by Clark (1979), p. 262.] The mathematical opinion in Hudson (1954), provided by C. A. Coulson, FRS, is no better. I was fortunate, however, to read as my first biography (a gift) that of Gattégno (1977) which notes the comment of Bourbaki (1974, p. 87) about "Kronecker's definitive form of theorems on linear equation systems . . . which are also elucidated, in an obscure treatise, with attention to detail characteristic of him, by the celebrated author of *Alice in Wonder-*

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land" and soon after encountered a comment by the authors of Chapter 2 of Kolmogorov and Yushkevich (1978, p. 69): "The concept of rank and the theorem of Kronecker–Capelli were discovered independently by several investigators. The first printed proof of this theorem is due to C. L. Dodgson (1832–1898), author of the splendid stories *Alice*. . . ." The assessments of Bell and Weaver indeed overlook Dodgson's quite startling contributions to linear algebra and to the theory of determinants, discussed recently by Seneta (1984, §§5–6) and Abeles (1986), even though "His suggestions had little effect on the mathematical world and the book (*An Elementary Treatise on Determinants with their Application to Simultaneous Linear Equations and Algebraical Geometry*, 1867) did not go into a second edition." (Beale, 1973, p. 30)

There was and is, to be sure, a continuing interest in Dodgson as a logician, especially in connection with certain paradoxes of his invention (Heath, 1967; Clark, 1979); in the construction of electoral systems (Black, 1958); as a constructor of games and puzzles (Gardner, 1960) and ciphers (recent work by Abeles, Hales and Lipson); and as a pioneer in photography. Numerous photographs of his are reproduced in the biographies mentioned and elsewhere. That photograph of the motivation for the *Alice* books, Alice Liddell (b. 4 May, 1852, d. 1934) as "the young beggar"–taken in 1859 and reproduced in Gattégno–is particularly striking in its undertones. (Among studies of Dodgson's association with Alice Liddell and female children in general, a recent film–shown in Australia as *Dreamchild*–presents it in rather sombre form.)

There was also, surprisingly, an Australian connection (Hudson, 1954): two cousins of Dodgson, Frank and Percy, emigrated to Australia, which he thought a "mad idea." (The Australian branch of the family persists, and two letters from Dodgson to Frank exist.) Some Australiana, possibly as a consequence, was to be found in Dodgson's library (Stern, 1981). In parallel to active Lewis Carroll Societies in England (2 branches) and North America, there existed until recently a Carroll Foundation in Australia, partly motivated by this connection.

At any rate, by 1983 Dodgson's probability seemed worth looking at, and I set out to do a thorough study of the probabilistic contents of the 72 problems of which *Pillow Problems* consist. All 72 are claimed to have been formulated and worked out at night while in bed, mentally, and the answer written down afterward. The questions are stated together in Chapter 1 with date of solution, and page references are given to the answer (answers are contained in Chapter 2), and to a solution (Chapter 3 gives the solution procedure by which the answers were mentally obtained). There are 13 probability problems altogether: 12 occur under the subheading "Chances" of the heading "Algebra." The 13th, No. 72, occurs under the mysterious heading "Transcendental Probabilities." All 13 problems are dated between March 1876 and August 1890.

My study of all 13 problems appeared as Seneta (1984), but it is little known. Even the Bayesian chronicle of Dale (1991), while it mentions Lewis Carroll, does not do so, as he deserves, in the context of English Bayesian probability. I shall therefore confine overlap in the present discussion to a few key problems. Only one of the 13 problems, No. 72 (described below), has attracted more than fleeting attention. On the basis of it, Weaver (1956) dismisses Dodgson as a probabilist, as he finally dismisses him as a mathematician.

It is a fact that some of Dodgson's probability problems, like No. 72, are badly posed, and his solution, while imaginative, is incorrect. On the other hand, for some difficult and interesting problems he obtains correct solutions, as we shall demonstrate. As a probabilist he is not important; but his work reflects the nature, standing and understanding of probability within the wider English mathematical community of the time. So, another motivation for this article is not only to entertain but to demonstrate how difficult some of the things in elementary probability, which we now take for granted, were to formulate and grasp only 100 years ago and thus how difficult our beginning students in statistics may find them.

To this end we shall use not only *Pillow Problems* but also a probabilistic controversy (also discussed by the author in Abeles, 1993) in which Dodgson was involved on the pages of a journal called the *Educational Times* which had a section called "Mathematical Questions and Solutions" and some unpublished material from the Weaver Collection at The Harry Ransom Humanities Research Center at the University of Texas at Austin. (The other main U.S. archival source of Carrolliana is The Parrish Collection at Princeton University Library.)

2. DODGSON AND ENGLISH PROBABILITY

In regard to probability, English mathematics by the 19th century (De Moivre's *Doctrine of Chances* had appeared in the 18th century) had taken directions predominantly different from the analytical approach of the Continent (Laplace, Poisson, Cournot, Chebyshev). The dominant influence in the 19th century seemed to have been the Cambridge group, headed in spirit (if not in location) by Augustus De Morgan (1806–1871), his influential works being De Morgan (1838) followed by his *Encyclopaedia Metropolitana* essay of 1845. Other representative works of the group were De Morgan (1847), Boole (1854), Todhunter (1865), Venn (1866) and Whitworth (1867). Within the group's work were essentially two directions, both represented in De Morgan.

Venn's and Boole's approach to probability is through logic. Dodgson was a strong logician, and it may be that he came to probability by contact with such writings. He certainly corresponded with Venn on improvement of "Venn diagrams." He had all the books mentioned above in his library, and, indeed, all of De Morgan's books (Stern, 1981).

On the other hand, De Morgan's (1838) *Essay on Probabilities* and Whitworth's (1867) *Choice and Chance* focused strongly on "inverse probabilities," and many illustrations are in terms of drawing counters from bags. Dodgson's probability problems tend to be urn models of this kind with several stages of drawing, and his approach to formultation of ignorance in terms of a prior distribution is similar to Whitworth's. These are thus the books he learned from.

From the writings of the French, he had in his library only Legendre's *Géométrie* and Serret's *Algèbre supérieure*, rather little, and with Laplace conspicuously absent.

The general question as to the prior distribution in the urn problems has been discussed extensively in Seneta (1984, §2), so we merely summarize the distinct approaches in a characteristic setting (taken from Whitworth–Question 136 in the 1901 edition):

> A purse contains ten coins, each of which is either a sovereign or a shilling: a coin is drawn and found to be a sovereign, what is the chance that this is the only sovereign?

If A_i, $i = 0, 1, \ldots, 10$, is the event that initially there are i sovereigns, De Morgan expresses initial ignorance in a similar setting through a uniform prior (the "principle of insufficient reason"): $P(A_i) = 1/11$, $i = 0, 1, 2, \ldots, 10$, as does Venn; whereas Whitworth's solution to the problem assumes that initially each coin in the bag has probability 1/2 of being a sovereign, so the prior distribution of the number of sovereigns is binomial: $P(A_i) = \binom{10}{i}(1/2)^i$, $i = 0, \ldots, 10$, presupposing a plausible mechanism for inputting coins.

3. DODGSON'S PROBABILITY PROBLEMS

We begin our discussion with Nos. 50, 72 and 45 from *Pillow Problems*, to give some insight into Dodgson's strengths and failings.

No. 50

> There are 2 bags, H and K, each containing 2 counters: and it is known that each counter is either black or white. A white counter is added to bag H, the bag is shaken up, and one counter is transferred (without looking at it) to bag K, where the process is repeated, a counter being transferred to bag H. What is now the chance of drawing a white counter from bag H?

The problem is to be understood as starting with 3 counters in bag H, the number of whites having possible values 0, 1, 2, 3 with prior distribution (0, 1/4, 1/2, 1/4); whereas the number of whites initially in bag K has possible values 0, 1, 2, 3 with prior (1/4, 1/2, 1/4, 0). Then a counter is transferred in turn between bags H and K, the counter being randomly chosen from its bag each time.

Each transfer modifies the probability distribution of the number of whites in each bag, the actual number of counters alternating between 3 and 2, both current distributions being needed for the modification. Mental solution would seem to require a phenomenal memory, even for just 2 transfers. This is perhaps the most complex problem of the set of the 13 probability problems; Dodgson devotes over a page of calculation in setting out the correct solution, 17/27. We see, from a modern perspective, that we are dealing with a simple version of an urn mixing model, along the lines of the Bernoulli–Laplace, or the Ehrenfest, Markov chain models for diffusion of heat. Dodgson's model too can be formulated as a Markov chain (see Appendix). We next pass to the famous No. 72.

No. 72

> A bag contains 2 counters, as to which nothing is known except that each is either black or white. Ascertain their colours without taking them out of the bag.

The prior distribution is again taken as binomial: (1/4, 1/2, 1/4) for BB, BW and WW: Dodgson's solution of this badly formulated problem proceeds by noting by total probability that if a black were added to the bag, the chance of then drawing a black would be 2/3. But the only composition of a bag with 3 counters which gives black a 2/3 chance of being drawn is two blacks and a white. Thus, he reasons, the initial composition (before the black was added) had to be one black and one white. Weaver (1956) describes the taking of the binomial prior as the first of the two "dreadful mistakes." But is even the second, obvious, mistake so dreadful? After all, Dodgson describes the problem as one in *Transcendental Probabilities* and his May 1893 Introduction in reference to it says: "To the casual reader it may seem abnormal, and even paradoxical; but I would have such a reader ask himself, candidly, the question 'Is Life not itself a Paradox?'" Is it then not impossible that No. 72 is just one of Lewis Carroll's jests? However, Dodgson had genuine difficulties when the state space in a random experiment was not finite. The next problem, again improperly formulated, leads us into this topic:

No. 45

> If an infinite number of rods be broken: find the chance that one at least is broken in the middle.

Dodgson's answer is "0.6321207 etc." His solution, not without ingenuity, begins by dividing each rod into $(n + 1)$ parts, where n is odd, and assuming the n points of division are the only points where a rod can break, each breakpoint being equally likely. The probability that a rod will not break in the middle is thus $1 - n^{-1}$. Now, taking the same number of rods as breakpoints (!) the probability of no rod breaking in the middle is $(1 - n^{-1})^n$, so the answer is $\lim(n \to \infty)$ $1 - (1 - n^{-1})^n = 1 - e^{-1} = 0.6321207$, etc. Dodgson thus proceeds by reducing the problem to a finite number of equally likely sample points.

We might nowadays ascribe some density over rod length to the breakpoint position, possibly the uniform; and come out with the answer zero. On the one hand, we might note that in Dodgson's time countability and uncountability were only just beginning to be handled by mathematicians such as Georg Cantor. On the other hand there were probabilists (such as Laplace) who had been treating "continuous probabilities" through probability densities for almost 100 years, even in England (such as Bayes). About such things a probabilistic dilletante such as Dodgson was not to know; one wonders if he was aware, even in his own time, of Crofton's (1885) encyclopaedia entry. Still, in his defence, No. 45 (dated May 1884) predates some of these developments, and a probabilistic episode which centred on this gap in Dodgson's understanding, and resulted in a substantial controversy, began in March 1885. This setting is particularly apropos, since it simultaneously illustrates his facility with *discrete* probability.

The following problem (Question 7695, posed by J. O'Regan, 1885) appeared in the *Educational Times* [see Abeles (1993) for general background]:

> Two persons play for a stake, each throwing two dice. They throw in turn, A commencing. A wins if he throws a 6, B if he throws a 7: the game ceasing as soon as either events happens. Show that A's chance is to B's as 30 to 31.

The "reasoning" of the published solution is as follows: out of the 36 possible outcomes of a toss of two dice, there are 5 yielding face value six, thus giving a probability 5/36. Likewise there are 6 outcomes giving face value seven. Thus the odds are (!) "no seven"/"no six" = (1 − 6/36)/(1 − 5/36) = 30/31. The answer is correct, but the reasoning is wrong: a situation tailor-made for Lewis Carroll. On March 14, 1885, Dodgson wrote to the editor about the erroneous solution method, giving a correct argument: if the probability of A's being successful on a single toss is a, and B's is b, the probability of A's winning is:

$$a + (1 - a)(1 - b)a + (1 - a)^2(1 - b)^2 a + \cdots$$
$$= a/\{1 - (1 - a)(1 - b)\}.$$

The odds of A's to B's winning are thus $a/\{(1 - a)b\}$ which with $a = 5/36$, $b = 6/36$ gives 30/31. So far so good: even with a countably infinite state space. What opens up a hornet's nest is his final statement:

> the ratio 30/31, is only *approximative*, the expectation of A and B being just *less* than the fractions 30/61, 31/61. If this were not so, the sum total of their expectations would equal 1; *i.e.* it would be absolutely certain one or other of them would win – whereas there is clearly a chance, though an indefinitely small one, that the game might go on forever without either winning.

How many of us have spent long periods discussing this kind of issue with intelligent and obstinate students? In modern probability, the conceivable sample point corresponding to the game going on forever has to be allocated probability zero, since the sum of probabilities of the countable set of sample points on which A or B wins (so the game finishes in a finite time) is unity. We do not, however, equate the statement that an event has probability zero with the statement that it cannot occur, as Dodgson is doing. Subsequently the Rev. T. C. Simmons, a far better mathematician than Dodgson, called Dodgson's final statement "extremely unmathematical" and had a view close to ours nowadays. Dodgson would not concede defeat, and they (and others), each trying to win the point, sailed off into the then uncharted waters of allocation of probabilities on the sample space [0, 1]. Specifically, in order to get away from the discrete setting and to show that *sensible* sets of probability (measure) zero can readily occur, Simmons puts in 1886 the Question 2000:

> A random point being taken on a given line, what is the chance of it coinciding with a previously assigned point?

We would allocate, in accordance with Simmons' formulation, Lebesgue measure to the Borel subsets of [0, 1] – the uniform distribution. Simmons argues correctly: if the point is k, probability of taking a point to its left is k and to its right is $1 - k$, so the probability must be zero. Dodgson says that if the probability of a specific point is zero, then the probability of any point is zero, yet some point is chosen, and concludes: "I re-affirm, as absolutely axiomatic, that when an event is *possible*, its chance of happening is *not* zero." Yet the whole thing was eating at him, and with Question 9588 (posed in 1889) he thought he finally had Simmons. We paraphrase this slightly:

> A random point being taken on a given line, find the chance of its dividing the line into two parts which are:
> (1) commensurable (2) incommensurable.

On the basis of Simmons' answer of zero to the previous problem, he says Simmons would argue that the probability of selecting a specific rational is zero, and that by adding the probabilities the probability of selecting a rational is zero [as the answer to (1)]. By the same argument however, he says, the probability of an irrational is zero: hence the probability of any number is zero: so there is, Dodgson thinks, a contradiction, since some point is selected. Dodgson's error, obvious to us now, is that in the case of irrationals he is using additivity over a noncountable set. We would deduce that the probability of selecting an irrational is one, using the countability of the rationals, if we accept the third of Kolmogorov's probability axioms. Countability is also at the root of Dodgson's difficulty with the "game going on forever." The difficulty reflects his obsession with "infinitesimals": quantities (e.g., probabilities) supposed, on account of some limiting process, say, to be less than any positive number, but not zero (so a corresponding event is not "impossible"). In the Kolmogorov axiomatization the continuity of countably additive probability measure, of course, ensures probability zero for a corresponding limiting event, which need not be null (that is, need not be "impossible"). But the Kolmogorov axiomatization came some 40 years after Dodgson's probability dablings, and it is easy to be wise with the benefit of hindsight.

APPENDIX: MARKOV CHAINS AND UNPUBLISHED PROBLEMS

Problem No. 50 may be formulated as a Markov chain on the states (i, j), i indicating the number of white counters in that bag which currently has a total of 3 counters and j the number of whites in that bag which has a total of 2 counters. Transitions occur with transfer of a counter. Relabel the following sequence of states as 1 to 11: (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), (1, 2), (2, 1), (3, 0), (2, 2), (3, 1), (3, 2). Then the chain consists of 5 closed irreducible aperiodic sets: {1, 2}, {3, 4, 5}, {6, 7, 8}, {9, 10}, {11}. The nonzero transition probabilities are given by: $p_{12} = 1$, $p_{21} = 2/3$, $p_{22} = 1/3$; $p_{35} = 1$, $p_{44} = 2/3$, $p_{45} = 1/3$, $p_{53} = 1/3$, $p_{54} = 2/3$; $p_{67} = 2/3$, $p_{68} = 1/3$, $p_{76} = 1/3$, $p_{77} = 2/3$, $p_{86} = 1$; $p_{99} = 1/3$, $p_{9,10} = 2/3$, $p_{10,9} = 1$; $p_{11,11} = 1$. The corresponding 5 stationary-limiting distribution vectors are: (2/5, 3/5), (1/10, 6/10, 3/10), (3/10, 6/10, 1/10), (3/5, 2/5), (1). Hence, irrespective of the initial distribution, there will be a limiting ("long-term") distribution vector $\{\pi_i\}$, $i = 1, \ldots, 11$, which will be a convex combination of the 5 stationary-limiting vectors, and it will depend on the initial distribution. Hence the marginal limiting distribution for the number of whites in the bag with 3 counters has sample space (0, 1, 2, 3) with probabilities $(\pi_1 + \pi_3, \pi_2 + \pi_4 + \pi_6, \pi_5 + \pi_7 + \pi_9, \pi_8 + \pi_{10} + \pi_{11})$; and for the bag with 2 counters: (0, 1, 2) with probabilities $(\pi_2 + \pi_5 + \pi_8, \pi_1 + \pi_4 + \pi_7 + \pi_{10}, \pi_3 + \pi_6 + \pi_9 + \pi_{11})$.

Note that if one adds an additional closed state (0, 0) (and relabels it 0) – although it can never occur with Dodgson's formulation – symmetry (which would have pleased Dodgson) of the resulting 12-state chain occurs about its centre. The Markov chain formulation and the decomposable structure make it easy to compute – by matrix powering – the probability structure of the number of whites of either bag at any time point, given an initial distribution. The computation of the probability of then choosing a white from the bag with 3 counters is straightforward. Dodgson's initial distribution with the 11-state relabelling is (0, 1/16, 0, 1/8, 1/8, 1/16, 1/4, 1/16, 1/8, 1/8, 1/16).

Of course the more usual mixing models (Laplace, Ehrenfest) with their structure of a single closed set have the advantage of a unique stationary distribution, although the Ehrenfest, with its periodicity, has no limiting distribution. One realization of Dodgson's process, which can be extended in structure, is confined to a single closed subset.

Curiously, an unpublished problem from the Weaver Collection which, it seems, Dodgson had originally intended to include in *Pillow Problems* since it is annotated "Thought out, Mar/85" – and to which he returned on Dec. 27, 1890, again unsuccessfully – presents another opportunity for treatment as a Markov chain, this time an absorbing one:

> A, B, C, throw a guinea in rotation, till it gives 3 consecutive "heads," when he who threw the last of them, wins it. What is the expectation of each?

The solution attempts on both dates, contained in jottings on a single page, are labelled by Dodgson as "doubtful"; indeed, the problem seems difficult by elementary methods. An annotation states: "[there is a] chance that the throwing shall go on forever, without 3 consecutive heads ever occurring" which hearkens back to Dodgson's letter (also of March 1885) at the beginning of the *Educational Times* controversy. It is important to the tenor of the present article because, even now when we teach finite Markov chain theory to students with only basic probability, since it requires only (Dodgson's own) skills of finite sample space probability and matrix methods but is intuitively attractive, the explanation of *why* "absorption occurs with probability one at some finite time" presents the same intuitive difficulty as confronted Dodgson.

Let the state space consist of the triples HHH, HHT, HTH, HTT, THH, THT, TTH, TTT, relabelled consecutively 0 to 7. Take the initial time point 0 as that after the third toss (the fact that C can win on the third toss will be accounted for separately), time 1 as that after the fourth toss, etc. The state represents the result of the last three tosses. The nonzero transition probabilities are $p_{00} = 1$, $p_{12} = p_{13} = p_{24} = p_{25} = p_{36} = p_{37} = p_{40} = p_{41} = p_{52} = p_{53} = p_{64} = p_{65} = p_{76} = p_{77} = 1/2$. The state 0 is absorbing; the states $(1, \ldots,$

7) are a self-communicating set of transient states (denote the substochastic submatrix of transitions between them by Q), with access to 0 through state 4. Thus from any transient state absorption is certain in finite time; the game ends on absorption into state 0.

If $f_{i0}^{(n)}$, $n \geq 1$, is the probability of absorption at time n beginning with state $i = 1, \ldots, n$ the probabilities of A, B, C winning are respectively:

$$(1/8) \sum_{n=0}^{\infty} \sum_{i=1}^{7} f_{i0}^{(3n+1)}, \quad (1/8) \sum_{n=0}^{\infty} \sum_{i=1}^{7} f_{i0}^{(3n+2)},$$

$$(1/8)\left(1 + \sum_{n=1}^{\infty} \sum_{i=1}^{7} f_{i0}^{(3n)}\right).$$

In matrix terms these are respectively

$$(1/8)\{\, \underset{\sim}{1}'(I + Q + Q^2)^{-1} \underset{\sim}{1} \,\} = 32/103,$$

$$(1/8)\{\, \underset{\sim}{1}'Q(I + Q + Q^2)^{-1} \underset{\sim}{1} \,\} = 30/103,$$

$$(1/8)\{1 + \underset{\sim}{1}'Q^2(I + Q + Q^2)^{-1} \underset{\sim}{1} \,\} = 41/103.$$

The expected number of tosses until the game ends is

$$3 + (1/8)\, \underset{\sim}{1}'(I - Q)^{-1} \underset{\sim}{1} = 14.$$

Although one can in no way give Dodgson any credit in connection with the beginning of Markov chains, which do not make their appearance in Markov's writings until 1906, it is curious that Dodgson's chains above cover two important structural cases.

We conclude with another problem from the Weaver collection intended for publication: it carries the notations "thought out Dec. 11/89" and "Press 28/8/90." I have not found where (or if) it was published.

> Four friends are in a room, one having 6 shillings, another 6 sixpences, a third 6 four-penny-bits, and the last 6 three-penny-bits. A very stupid boy is blindfolded, and sent in, with an empty purse in his hand, to play blind-man's buff, on the understanding that any friend touched for the *first* time, puts *one* coin into the purse – for the *second* time, *two* – for the *third* time, *three*. The boy comes out with 6 coins in the purse, but does not the least know how many touches have occurred. What is the "expectation," as to the contents of the purse?

Dodgson begins well by saying that to result in 6 coins, either one person was touched three times (3 touches), or one person was touched two times and another two times (4 touches), or one person was touched twice and the three remaining persons once each (5 touches). However, he then has trouble with the prior probabilities, as one might well do here, and his labelling is strange: he labels "A" to be the first person touched, "B" the second person, and so on. His (conditional) prior probabilities for the probabilities of 3, 4, 5 touches (4/64, 9/64, 12/64) do not add to unity, so he rescales them to get (4/25, 9/25, 12/25).

If T is the total in the purse, in pence (a shilling is 12 pence):

$$E(T|3 \text{ touches, 6 coins}) = 6(3 + 4 + 6 + 12)/4 = 150/4 = 75/2;$$

$$E(T|4 \text{ touches, 6 coins}) = 3(7 + 9 + 10 + 15 + 16 + 18)/6 = 75/2;$$

$$E(T|5 \text{ touches, 6 coins}) = 150/4 = 75/2.$$

Because he forgets that in the second case each of the two persons touched must pay a total of 3 coins, Dodgson obtains 25/2, with a consequent final answer of 28.5 pence. I have included this problem because of its use of conditional argument and expectations and because of the curious property, which would have pleased Lewis Carroll: that the prior distribution over the possible number of touches given 6 coins is irrelevant to the final answer (75/2).

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