

# When Doesn't Cokriging Outperform Kriging?

Hao Zhang and Wenxiang Cai

*Abstract.* Although cokriging in theory should yield smaller or equal prediction variance than kriging, this outperformance sometimes is hard to see in practice. This should motivate theoretical studies on cokriging. In general, there is a lack of theoretical results for cokriging. In this work, we provide some theoretical results to compare cokriging with kriging by examining some explicit models and specific sampling schemes.

*Key words and phrases:* Cokriging, equivalence of probability measures, infill asymptotics, kriging.

Genton and Kleiber (2015) provided an excellent review of recent development in the multivariate covariance functions. In many situations, the ultimate objective of modeling the multivariate covariance function is to obtain superior prediction through cokriging. In theory, cokriging should have a prediction variance no larger than that of the kriging prediction. However, as the authors point out in the paper, sometimes the improvement of cokriging is very little or none. In this note, we try to shed some light through some theoretical investigations.

For univariate Gaussian stationary processes, we now have a good understanding of the properties of kriging and statistical inferences. For example, theoretical results have been established to justify (i) that two different covariance functions may yield asymptotically equally optimal prediction (Stein, 1999), and (ii) some parameters are not consistently estimable if the spatial domain is bounded (Zhang, 2004). We know the conditions under which a misspecified covariance function yields an asymptotically right prediction and can exploit this fact to simplify computations (Zhang, 2004; Du, Zhang and Mandrekar, 2009).

---

Hao Zhang is Professor, Department of Statistics and Department of Forestry and Natural Resources, Purdue University, West Lafayette, Indiana 47907, USA (e-mail: zhanghao@purdue.edu). Wenxiang Cai is Graduate Student, School of International Trade and Economics, The University of International Business and Economics, Beijing, China.

We lack the analogous understanding for the multivariate spatial models. There are no explicit theoretical results to answer the following questions:

- How important is the cross-covariance function? Specifically, could two different multivariate covariance functions yield an asymptotically equally optimal prediction?
- Which parameters are important to cokriging? We know which parameters are important to kriging.
- How much improvement does cokriging have over kriging?

One particular concept that has been shown useful in the study of kriging is the equivalence of probability measures due to a theorem established by Blackwell and Dubins (1962). Let  $\mathbf{s}_i$ ,  $i = 1, \dots, n$  be sampling sites on a fixed domain (area) where the process  $Y(\mathbf{s})$  is observed, and  $\{\mathbf{s}_i, i > n\}$  be a set of sites on the same domain where  $Y$  is to be predicted. If the two Gaussian measures  $P_1$  and  $P_2$  are equivalent on the  $\sigma$ -algebra generated by  $Y(\mathbf{s}_i), i = 1, 2, \dots$ , then with  $P_1$ -probability one,

$$\begin{aligned} & \sup |P_1\{A|Y(\mathbf{s}_i), i = 1, \dots, n\} \\ & \quad - P_2\{A|Y(\mathbf{s}_i), i = 1, \dots, n\}| \\ & \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

where the supremum is taken over  $A \in \sigma\{Y(\mathbf{s}_i), i > n\}$ . The above result implies that the linear predictions under the two measures are asymptotically equally optimal (Stein, 1999).

This result can be readily extended to the multivariate spatial process and therefore implies two cokriging predictors are asymptotically equally optimal under the two probability measures if the two Gaussian measures are equivalent. However, unlike in the univariate case, there are very limited results on equivalence of probability measures. Ruiz-Medina and Porcu (2015) gave some general conditions for equivalent measures for multivariate Gaussian processes though there is still a lack of explicit examples where equivalent measures occur.

We now provide some sufficient conditions for the equivalent of Gaussian measures for a particular bivariate model. Let  $\mathbf{Y}(\mathbf{s}) = (Y_1(\mathbf{s}), Y_2(\mathbf{s}))'$  be a stationary bivariate Gaussian process with the following bivariate covariance function under the probability measure  $P_k$ ,  $k = 1, 2$ , such that

$$C_{ij}(\mathbf{h}) = \text{Cov}(Y_i(\mathbf{s}), Y_j(\mathbf{s} + \mathbf{h})) \\ = M(|\mathbf{h}|, \sigma_{ij,k}, \alpha_k, \nu), \quad i, j = 1, 2,$$

where  $M(\cdot, \sigma^2, \alpha, \nu)$  denotes the Matérn covariance function with variance  $\sigma^2$ , scale parameter  $\alpha$  and the smoothness parameter  $\nu$ . The following are sufficient conditions for the two measures  $P_k$  to be equivalent on the  $\sigma$ -algebra generated by  $\{Y_i(\mathbf{s}), \mathbf{s} \in D, i = 1, 2\}$  for some bounded set  $D \in \mathcal{R}^d$ ,  $d \leq 3$ :

$$(1) \quad \sigma_{ii,1}^2 \alpha_1^{2\nu} = \sigma_{ii,2}^2 \alpha_2^{2\nu}, \\ \sigma_{12,1} / \sqrt{\sigma_{11,1} \sigma_{22,1}} = \sigma_{12,2} / \sqrt{\sigma_{11,2} \sigma_{22,2}}.$$

To prove this claim, we employ the Karhunen–Loève expansion under measure  $P_1$ . Since the two processes  $\{Y_i(\mathbf{s}) / \sqrt{\sigma_{ii,1}}, i = 1, 2\}$  have the same covariance function  $M(|\mathbf{h}|, \alpha, \alpha, \nu)$  and therefore possess the same Karhunen–Loève expansion under measure  $P_1$ ,

$$\frac{Y_i(\mathbf{s})}{\sqrt{\sigma_{ii,1}}} = \sum_{l=1}^{\infty} \sqrt{\lambda_l} f_l(\mathbf{s}) Z_{il},$$

where for  $i = 1, 2$ ,  $\{Z_{il}, l = 1, \dots\}$  consists of i.i.d. standard normal random variables under measures  $P_1$ . Clearly, the eigenvalues  $\lambda_l$  and eigenfunctions  $f_l(\mathbf{s})$  only depend on the correlation function and hence do not depend on  $i$ . In addition,

$$Z_{il} = \frac{1}{\sqrt{\lambda_l \sigma_{ii,1}}} \int_D Y_i(\mathbf{s}) f_l(\mathbf{s}) ds.$$

Using the above expression, it is not hard to show that

$$(2) \quad E_1(Z_{1l} Z_{2m}) = r \delta_{l,m} \\ \text{for } r = \sigma_{12,1} / \sqrt{\sigma_{11,1} \sigma_{22,1}},$$

$$(3) \quad E_2(Z_{1l} Z_{2m}) = r E_2(Z_{1l} Z_{1m}).$$

The Karhunen–Loève expansion implies that  $\{Z_{il}, l = 1, 2, \dots, \infty\}$  is a basis of the Hilbert space generated by  $\{Y_i(\mathbf{s}), \mathbf{s} \in D\}$  with respect to measure  $P_1$ . Hence,  $\{Z_{1l}, Z_{2l}, l = 1, 2, \dots\}$  is a basis of the Hilbert space generated by the two processes  $\{Y_i(\mathbf{s}), i = 1, 2, \mathbf{s} \in D\}$ . The two measures are equivalent on the Hilbert space if and only if they are so on  $\{Z_{1i}, Z_{2i}, i = 1, 2, \dots\}$  (Ibragimov and Rozanov, 1978, page 72). To show the equivalence of the two measures, we only need to verify (Stein, 1999, page 129)

$$(4) \quad \sum_{i=1}^2 \sum_{j=1}^2 \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} (E_1(Z_{il} Z_{jm}) - E_2(Z_{il} Z_{jm}))^2 \\ < \infty.$$

Because conditions (1) imply that the two measures are equivalent on  $\{Y_i(\mathbf{s}), \mathbf{s} \in D\}$  (Zhang, 2004), we must have

$$\sum_{l=1}^{\infty} \sum_{m=1}^{\infty} (E_1(Z_{il} Z_{im}) - E_2(Z_{il} Z_{im}))^2 < \infty, \quad i = 1, 2.$$

For  $i \neq j$ , equations (2) and (3) imply

$$\sum_{l=1}^{\infty} \sum_{m=1}^{\infty} (E_1(Z_{1l} Z_{2m}) - E_2(Z_{1l} Z_{2m}))^2 \\ = r^2 \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} (E_1(Z_{1l} Z_{1m}) - E_2(Z_{1l} Z_{1m}))^2 < \infty.$$

Therefore, (4) is proved and so is the sufficiency of the conditions. We now have an explicit example where two different bivariate covariance functions yield asymptotically equal cokriging results.

Next, we will try to explain why sometimes it is hard to see the improvement of cokriging over the kriging prediction. Consider a bivariate Gaussian process with mean 0 and exponential covariance functions such that

$$(5) \quad C_{ij}(\mathbf{h}) = \text{Cov}(Y_i(\mathbf{s}), Y_j(\mathbf{s} + \mathbf{h})) \\ = \sigma_{ij} \exp(-\alpha |\mathbf{h}|), \quad i, j = 1, 2.$$

Assume the two processes are observed at  $n$  points  $\mathbf{s}_i, i = 1, \dots, n$ , and predict  $Y_1(0)$ . Write  $\mathbf{Y}_1 = (Y_1(\mathbf{s}_i), i = 1, \dots, n)'$ ,  $\mathbf{Y}_2 = (Y_2(\mathbf{s}_i), i = 1, \dots, n)'$ . It is known that in this case the cokriging predictor is identical to the kriging predictor. To see this, let  $R$  denote the correlation matrix of  $\mathbf{Y}_1$ , which is also the correlation matrix of  $\mathbf{Y}_2$ . Then

$$\text{Cov}(\mathbf{Y}_i, \mathbf{Y}_j) = \sigma_{ij} R.$$

Let  $V$  be the matrix with  $(i, j)$ th element  $\sigma_{ij}$ . Then the covariance matrix of  $(\mathbf{Y}_1, \mathbf{Y}_2)$  is  $V \otimes R$ . Let  $\mathbf{k}$  denote

the vector of correlation coefficients between  $Y_1(\mathbf{s})$ , the variable to be predicted, and  $\mathbf{Y}_1$ . Then

$$\begin{aligned}
 (6) \quad & E(Y_1(\mathbf{s})|\mathbf{Y}_1, \mathbf{Y}_2) \\
 &= ((\sigma_{11}, \sigma_{22}) \otimes \mathbf{k}')(V^{-1} \otimes R^{-1})\mathbf{Y} \\
 (7) \quad &= ((\mathbf{k}', 0) \otimes R^{-1})\mathbf{Y} = \mathbf{k}'R^{-1}\mathbf{Y}_1 \\
 &= E(Y_1(\mathbf{s})|\mathbf{Y}_1).
 \end{aligned}$$

Therefore, cokriging is identical to kriging and we should not expect any improvement of cokriging over kriging. We can also show that they are identical if  $Y_2(s)$  is observed at a subset of locations where  $Y_1$  is observed.

One scenario where cokriging might outperform kriging is when the auxiliary variable is observed at more locations than the predicted variable. In the next example, we will examine analytically what variables affect the improvement of cokriging over kriging. We assume the same bivariate model (5) and  $Y_2(s)$  are observed at  $s \in O = \{i/n, i = \pm 1, \pm 2, \dots, \pm n\}$ , but  $Y_1(s)$  is observed at half of the points  $s \in O_1 = \{2i/n, i = \pm 1, \pm 2, \dots, \pm n/2\}$  where  $n$  is an even integer. Denote the kriging predictor and cokriging predictor of  $Y_1(0)$  by

$$\begin{aligned}
 (8) \quad & \hat{Y}_1(0) = E(Y_1(0)|Y_1(s), s \in O_1), \\
 (9) \quad & \tilde{Y}_1(0) = E(Y_1(0)|Y_1(s), s \in O_1, Y_2(t), t \in O).
 \end{aligned}$$

We will derive the following asymptotic relative efficiency of kriging to cokriging:

$$(10) \quad \lim_{n \rightarrow \infty} \frac{E(Y_1(0) - \tilde{Y}_1(0))^2}{E(Y_1(0) - \hat{Y}_1(0))^2} = 1 - r^2/2,$$

where  $r$  is the correlation coefficient of  $Y_1(s)$  and  $Y_2(s)$ .

The asymptotic relative efficiency of kriging prediction does not depend on the scale parameter  $\alpha$ . Intuitively this is understandable. However, for a finite sample size  $n$ ,  $\alpha$  may affect the efficiency. We now present a simulation study to see how  $\alpha$  and  $r$  affect the relative efficiency of kriging prediction. We consider the exponential covariance model with  $\sigma_{11} = \sigma_{22} = 1$  and  $r = 0.2$  and  $0.5$ , and  $\alpha = 2, 4$  and  $8$ . The auxiliary variable  $Y_2$  is observed at  $\pm i/n, i = 1, \dots, n$ , but the primary variable  $Y_1$  is observed at  $\pm i/n$  for even integers  $0 < i \leq n$ . We calculate the prediction variance for predicting  $Y_1(0)$  using both kriging and cokriging and obtain the relative efficiency of kriging for different  $n, \alpha$  and  $r$ .

Figure 1 plots the relative efficiency for different  $r, \alpha$  and  $n$ . We see that the relative efficiency of kriging decreases as  $n$  increases, which means that it is more likely to see the outperformance of cokriging over kriging when  $n$  is larger. When the spatial autocorrelation is strong (i.e.,  $\alpha$  smaller), the asymptotic efficiency is achieved relatively faster (i.e., with  $n$  not too larger). This agrees with many other infill asymptotic results.

We now prove (10). We first note a Markovian property of the exponential model established by Du, Zhang

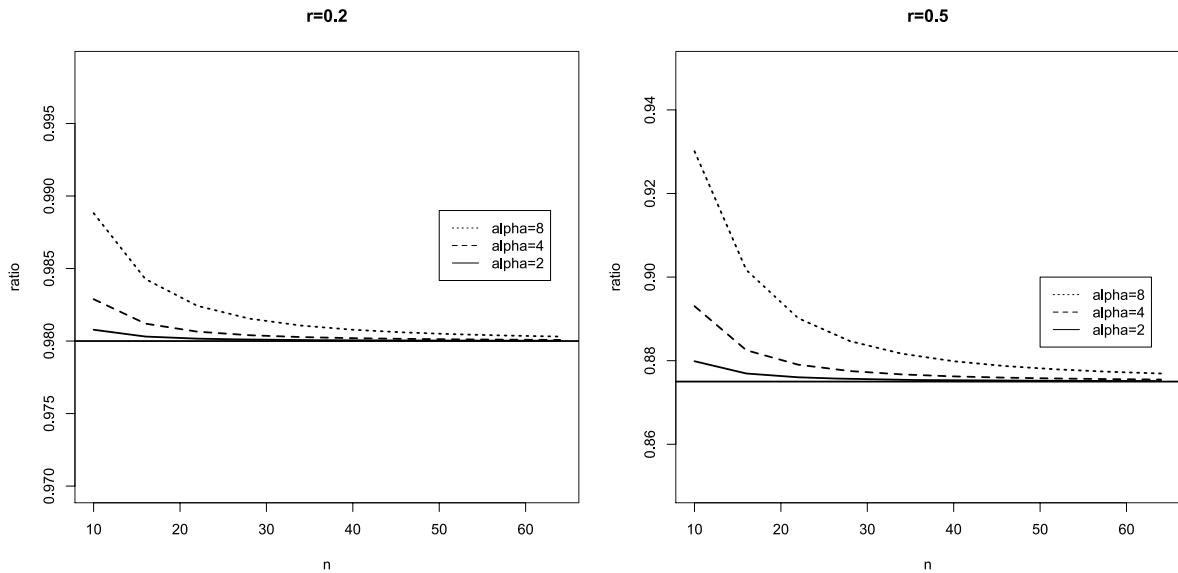


FIG. 1. Relative efficiency of kriging to cokriging for different  $r, \alpha$  and  $n$ . The solid horizontal line is the asymptotic relative efficiency  $1 - r^2/2$ .

and Mandrekar (2009), which says  $E(Y_1(s)|Y_1(s), s \in B)$  only depends on the two nearest neighbors of  $s$  in a finite set  $B$  such that  $s$  is between the minimum and the maximum elements of  $B$  (Du, Zhang and Mandrekar, 2009, Lemma 1). Also from the lemma, we obtain

$$E(Y_1(0) - \hat{Y}_1(0))^2 = 2\sigma_{11}^2\alpha/n + o(n^{-2}).$$

In the extreme case when  $r = 1$ , we can view the process  $Y_1(s)$  being observed at  $O$ . Then in this extreme case, the above equation implies

$$E(Y_1(0) - \tilde{Y}_1(0))^2 = \sigma_{11}^2\alpha/n + o(n^{-2}).$$

The ratio in (10) is clearly 1/2. Hence, we have verified (10) for this extreme case. On the other hand, when  $r = 0$ , the two predictors  $\hat{Y}_1(0)$  and  $\tilde{Y}_1(0)$  are identical and (10) is obviously true.

We are going to show that

$$(11) \quad \begin{aligned} \tilde{Y}_1(0) &= b_1Y_1(-2/n) + b_2Y_1(2/n) \\ &+ b_3Y_2(-2/n) + b_4Y_2(-1/n) \\ &+ b_5Y_2(1/n) + b_6Y_2(2/n), \end{aligned}$$

where

$$(12) \quad b_1 = b_2 = \frac{e^{-2\alpha/n}}{e^{-4\alpha/n} + 1},$$

$$b_3 = b_6 = -\frac{re^{-2\alpha/n}}{e^{-4\alpha/n} + 1},$$

$$(13) \quad b_4 = b_5 = \frac{re^{-\alpha/n}}{e^{-2\alpha/n} + 1}.$$

Some straightforward calculation yields

$$\begin{aligned} &E(Y_1(0) - \tilde{Y}_1(0))^2 \\ &= -\sigma_{11}^2(-2e^{-4\alpha/n}r^2 + e^{-6\alpha/n} + 2e^{-2\alpha/n}r^2 \\ &\quad + e^{-4\alpha/n} - e^{-2\alpha/n} - 1) \\ &\quad /((e^{-4\alpha/n} + 1)(e^{-2\alpha/n} + 1)) \\ &= \sigma_{11}^2(2 - r^2)\alpha/n + o(n^{-2}). \end{aligned}$$

Then (10) immediately follows. Hence, it is sufficient to show (11). It is possible to show that  $Y_1(0) - \tilde{Y}_1(0)$  is uncorrelated with any  $Y_1(s), s \in O_1$  and with any  $Y_2(t), t \in O$ . Hence,  $\tilde{Y}_1(0)$  must be the best linear prediction. Here we take an alternative but more intuitive approach. We will apply the Markovian property of the Gaussian exponential model to show that  $\tilde{Y}_1(0)$  only depends on  $Y_1(-2/n), Y_1(2/n), Y_2(-2/n), Y_2(-1/n), Y_2(1/n)$  and  $Y_2(2/n)$ . Consequently, the coefficients  $b_i$ 's in (12) and (13) can be found by solving linear equations.

For any odd integer  $i$  between  $-n$  and  $n$ ,

$$(14) \quad \begin{aligned} &E(Y_2(i/n)|Y_1(s), s \in O_1, Y_2(t), t \in O, t \neq i/n) \\ &= E\{E(Y_2(i/n)|Y_1(t), Y_2(t), t \in O, t \neq i/n)| \\ &\quad Y_1(s), s \in O_1, t \in O, t \neq i/n\} \\ &= E\{E(Y_2(i/n)|Y_2(t), t \in O, t \neq i/n)| \\ &\quad Y_1(s), s \in O_1, t \in O, t \neq i/n\} \\ &= E\{Y_2(i/n)|Y_2(t_{i-}), Y_2(t_{i+})\}, \end{aligned}$$

where  $t_{i-}$  and  $t_{i+}$  are the two nearest neighbors of  $i/n$  in  $O$ . For example, for  $i = -1$ ,  $t_{i-} = -2/n$  and  $t_{i+} = 1/n$ .

Define  $e_i = Y_2(i/n) - E\{Y_2(i/n)|Y_2(t_{i-}), Y_2(t_{i+})\}$  for an odd  $i$ . Then  $e_i$  is independent of  $Y_1(s), s \in O_1$  and  $Y_2(t), t \in O$  and  $t \neq i/n$ . Consequently,

$$(15) \quad \begin{aligned} &E(Y_1(0)|Y_1(s), s \in O_1, Y_2(t), t \in O) \\ &= E(Y_1(0)|Y_1(s), Y_2(s), s \in O_1, e_i, i \text{ odd}) \\ &= E(Y_1(0)|Y_1(s), Y_2(s), s \in O_1) \\ &\quad + E(Y_1(0)|e_i, i \text{ odd}). \end{aligned}$$

The first term in the above equation depends only on  $Y_1(-2/n)$  and  $Y_1(2/n)$  due to the Markovian property. For the second term, because the cross-covariance function is proportional to the covariance function of  $Y_2(t)$ , we have

$$E(Y_1(0)|e_i, i \text{ odd}) = rE(Y_2(0)|e_i, i \text{ odd}).$$

Applying again the property of conditional expectation and the Markovian property, we get

$$\begin{aligned} &E(Y_2(0)|e_i, i \text{ odd}) \\ &= E(E\{Y_2(0)|Y_2(t), t \in O\}|e_i, i \text{ odd}) \\ &= \beta E\{Y_2(-1/n) + Y_2(1/n)|e_i, i \text{ odd}\} \\ &= \beta E\{Y_2(-1/n) + Y_2(1/n)|e_{-1}, e_1\}, \end{aligned}$$

where  $\beta$  is the constant in  $E(Y_2(0)|Y_2(-1/n), Y_2(1/n)) = \beta(Y_2(-1/n) + Y_2(1/n))$ , and the last equation follows the fact that  $e_i$  is independent to  $Y(1/n)$  and  $Y_2(-1/n)$  if  $i \neq 1$  or  $-1$ . Therefore, the second term of (15) is a linear function of  $e_{-1}$  and  $e_1$  and hence a linear function of  $Y_2(i/n), i = -2, -1, 1$  and  $2$ .

### ACKNOWLEDGMENT

Hao Zhang is supported by a Grant from the China Social Science Foundation (11&ZD167) and NSF Grant (IIS-1028291). Wenxiang Cai is supported by the Excellent Dissertation Fund of the University of International Business and Economics.

## REFERENCES

- BLACKWELL, D. and DUBINS, L. (1962). Merging of opinions with increasing information. *Ann. Math. Statist.* **33** 882–886. [MR0149577](#)
- DU, J., ZHANG, H. and MANDREKAR, V. S. (2009). Fixed-domain asymptotic properties of tapered maximum likelihood estimators. *Ann. Statist.* **37** 3330–3361. [MR2549562](#)
- GENTON, M. G. and KLEIBER, W. (2015). Cross-covariance functions for multivariate geostatistics. *Statist. Sci.* **30** 147–163.
- IBRAGIMOV, I. A. and ROZANOV, Y. A. (1978). *Gaussian Random Processes*. Springer, New York. [MR0543837](#)
- RUIZ-MEDINA, M. D. and PORCU, E. (2015). Equivalence of Gaussian measures of multivariate random fields. *Stoch. Environ. Res. Risk Assess.* **29** 325–334.
- STEIN, M. L. (1999). *Interpolation of Spatial Data: Some Theory for Kriging*. Springer, New York. [MR1697409](#)
- ZHANG, H. (2004). Inconsistent estimation and asymptotically equal interpolations in model-based geostatistics. *J. Amer. Statist. Assoc.* **99** 250–261. [MR2054303](#)