ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 8, Number 4, Fall 1978

CLASSES OF NONABELIAN, NONCOMPACT, LOCALLY COMPACT GROUPS

T. W. PALMER*

1. **Introduction.** A great deal is known about locally compact abelian groups and about compact groups. Frequently the same result has been proved in both cases. Thus it is natural to look for a common generalization of these two quite different hypotheses—abelian and compact. This article surveys the literature on this idea. In some respects it may be regarded as an extension and updating of the second part of the important paper by Grosser and Moskowitz [23]. However, it is only meant to provide orientation in this subject, and thus in order to simplify the presentation we frequently do not quote results in their maximum generality. We will also omit most proofs. The extensive bibliography and detailed references to it, will allow the reader to find these when he wishes. At the same time we will try to explain enough to keep the formal prerequisites to a minimum. An acquaintance with the simplest facts about locally compact groups and convolution multiplication in their L^1-group algebras and about operator algebras on Hilbert space is all that is needed. We will use some standard results from [31] § 5 without comment.

In this article locally compact groups are always assumed to satisfy the Hausdorff separation axiom. The identity element of a group is usually denoted by e. We use $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{T}$ to denote the sets of integers, real numbers, complex numbers, and complex numbers of modulus 1 respectively, with their usual structures as topological groups or rings, etc.

The paper is organized as follows. § 2 contains definitions together with sufficient comments to orient the reader. More complete comments and more detailed references for all these matters are contained in § 4. § 3 contains four diagrams summarizing the known inclusions among the twenty classes which we discuss fully. These diagrams give

*The author wishes to thank the National Science Foundation for support under grant number MCS-76-07272. He also wishes to thank H. Leptin, C. C. Moore, R. D. Mosak, J. M. Rosenblatt and K. A. Ross for useful letters or conversations relating to this work. Finally he wishes to thank the University of California at Berkeley for hospitality during the final stages of preparation of this paper.

Received by the editors on February 10, 1976.

AMS (MOS) Classification Numbers: 22D02, 22D05, 22D10, 22D15, 22D20, 43A02, 43A20.

the inclusions which hold when considering: (1) general locally compact groups, (2) almost connected groups, (3) connected groups and, (4) discrete groups. The diagram for compactly generated groups is also described. § 4 contains detailed references and comments about the twenty classes considered. § 5 contains miscellaneous comments about other classes which have been considered, duality theorems and related matters.

The bibliography is fairly comprehensive for papers dealing with conditions on locally compact groups which are common to compact and abelian groups. The main exceptions are some papers cited in Greenleaf [19], Grosser and Moskowitz [23] or Montgomery and Zippin [87]. These are omitted since it is assumed that these sources will be at hand when the present notes are consulted. Also a number of related papers which focus mainly on Lie groups, or Mackey's theory of induced representations have been omitted.

Index of Classes

2. **Definitions of the classes.** Discrete groups are locally compact groups according to the definition, but obviously the topology of a discrete group offers no help in studying the group. Thus if we are going to successfully use topological considerations to study locally compact groups we must make some restriction on them. The most powerful restriction is to assume that the group is connected (as a topological space). This obviously rules out discrete groups (except the trivial group) and it also obviously rules out the existence of proper open subgroups. (If G contains such a subgroup H then the left cosets of H include different components.) Note that a group has a proper open normal subgroup if and only if it has a nontrivial discrete quotient group.

A weaker condition than connectedness is usually enough to ensure that the topology plays a significant role in the structure of a group. Given a locally compact group G the connected component of G containing the identity of G is easily seen to be a closed normal subgroup which we will always denote by G_0. The quotient group G/G_0 is totally disconnected, (i.e., all its connected components are singletons). A group is called *almost connected* if G/G_0 is compact. This rules out all discrete groups except finite ones. It is also obvious that any open subgroup of an almost connected group must have finite index and any discrete quotient group of an almost connected group must be finite.

A group G is said to be *compactly generated* if there is some compact set K which generates G in the sense that G satisfies $G = \bigcup_{n=1}^{\infty} K^n$ where $K^n = \{x_1x_2 \cdots x_n : x_j \in K\}$. If G is an almost connected locally compact group and W is a compact neighborhood of the identity then $H = \bigcup_{n=1}^{\infty} (W \cup W^{-1})^n$ has nonvoid interior and hence is an open subgroup. Hence the compact group G/G_0 can be written as the union of finitely many left cosets of H/G_0. Therefore G is generated by the compact set $(\bigcup_{k=1}^{n} x_kW) \cup (\bigcup_{k=1}^{n} x_kW)^{-1}$ for a suitable choice of $\{x_1, x_2, \cdots, x_n\}$. Hence "compactly generated" may be considered as a very weak type of connectedness condition.

We will need the concept of group extensions. If N and H are groups, the Cartesian product $N \times H$ is again a group under pointwise operations. A group G is isomorphic to $N \times H$ iff it contains normal subgroups $\tilde{N}$ and $\tilde{H}$ isomorphic to N and H respectively and satisfying $G = \tilde{N}\tilde{H}$ and $\tilde{N} \cap \tilde{H} = \{e\}$. Such a group G is called a *direct product* of N and H. More generally if φ is a homomorphism of H into the group of automorphisms of N, then the set $N \times H$ becomes a group denoted by $N \times_\varphi H$ when multiplication is defined by

$$(n_1, h_1)(n_2, h_2) = (n_1\varphi(h_1)(n_2), h_1h_2). \qquad \begin{array}{l} \forall n_1, n_2 \in N; \\ \forall h_1, h_2 \in H. \end{array}$$

A group G is isomorphic to $N \times_\varphi H$ for some homomorphism φ iff it contains a normal subgroup $\tilde{N}$ and a subgroup $\tilde{H}$ isomorphic to N and H respectively and satisfying $G = \tilde{N}\tilde{H}$ and $\tilde{N} \cap \tilde{H} = \{e\}$. In this case φ is defined by

$$\varphi(h)(n) = hnh^{-1} \quad \forall n \in \tilde{N}; \ \forall h \in \tilde{H}.$$

Such a group G is called the *semidirect product* of N and H defined by φ. Both of these examples are special cases of an extension of a group N by a group H and in fact semidirect products are sometimes called split extensions. A group G is defined to be an *extension of N by H* iff it contains a normal subgroup $\tilde{N}$ isomorphic to N and such that $G/\tilde{N}$ is isomorphic to H. Thus G is an extension of N by H iff there is a short exact sequence.

$$\{e\} \to N \to G \to H \to \{e\}.$$

We denote such an extension by writing $G \simeq N^H$ where the notation is obviously not completely explicit. (Some authors would call this an extension of H by N, but we will use the indicated terminology.) All these notions apply to topological groups when obvious topological conditions are added to their statement. In particular we require $\tilde{N}$ and $\tilde{H}$ to be closed and require the maps in the short exact sequence of an extension to be continuous and open onto their images. See [153] for further details on semidirect products.

There is a deep connection between locally compact groups and Lie groups which we wish to mention briefly. Consider first a connected topological group G which has a neighborhood of the identity which is homeomorphic to an open set W in $\mathbf{R}^n$. Left translation by y (i.e., $x \to yx$) is a homeomorphism which maps the identity element onto y. Thus any point in G has a neighborhood homeomorphic to $W \subseteq \mathbf{R}^n$. This is the defining property of a locally Euclidean space or a topological manifold. Topological manifolds are often described by giving a set M, a family $\{M_\alpha : \alpha \in A\}$ of subsets of M which cover M, and a collection of injections $c_\alpha : M_\alpha \to \mathbf{R}^n$ so that $c_\alpha(M_\alpha)$ is open for each $\alpha \in A$ and the map $c_\beta \circ c_\alpha^{-1}$ is a homeomorphism on its domain for each $\alpha, \beta \in A$. Then a topology on M can be defined by declaring a subset $N \subseteq M$ to be open iff $c_\alpha(N \cap M_\alpha)$ is open for each $\alpha \in A$. A function $f : M \to \mathbf{R}$ will be continuous iff $f \circ c_\alpha^{-1}$ is continuous for each α. It makes sense to ask whether all the maps $c_\beta \circ c_\alpha^{-1}$ are differentiable ($= C^1$), or C^∞, or analytic (i.e., have convergent Taylor series around each point). When one of these conditions holds we say that M is a C^1-manifold, a C^∞-manifold or an analytic manifold, respectively. We will get a valuable theory if we define $f : M \to \mathbf{R}$ to be differentiable, C^∞ or analytic iff $f \circ c_\alpha^{-1}$ has the corresponding property for each $\alpha \in A$.

A modern version of Hilbert's fifth problem asks whether any locally Euclidean connected topological group is in fact an analytic manifold in which multiplication and inversion (i.e., $x \to x^{-1}$) are analytic functions (in an obvious sense which we will not bother to explain). Such an object is called a *connected Lie group*. A topological group G will be called a *Lie group* if G_0 is open and G_0 is a connected Lie group. Thus every Lie group is the extension of a connected Lie group by a discrete group. In some ways Lie groups are the nicest kind of locally compact groups.

The truth of this version of Hilbert's fifth problem was first proved by the joint work of Gleason, Montgomery and Zippin in 1952 (see Montgomery and Zippin [87] and also the article of Gluškov [17]). A key to the proof was the idea of locally compact groups without small subgroups. A group is said to have *no small subgroups* iff there is a neighborhood of its identity, e, which includes no subgroup except $\{e\}$. Gleason (1952) and Yamabe (1953) showed that a connected locally compact group is a Lie group if and only if it has no small subgroups.

Yamabe (1953) also proved that if G is an almost connected locally compact group then every neighborhood of the identity contains a compact normal subgroup K such that G/K is a Lie group. Thus the set $\mathscr{S}$ of compact normal subgroups of G such that G/K is a Lie group is directed by reverse inclusion. Hence we may write G as a projective limit of Lie groups

$$G = \varprojlim_{K \in \mathscr{S}} G/K$$

(i.e., if we let $\varphi_K : G \to G/K$ and $\varphi_{KH} : G/K \to G/H$ be the natural maps for $H, K \in \mathscr{S}$ with $H \supseteq K$ then the map $x \to (\cdots, \varphi_K(x), \cdots)$

(for all $x \in G$) defines a homeomorphic isomorphism of G onto the subgroup $P = \varprojlim_{K \in \mathscr{S}} G/K$ of $\prod_{K \in \mathscr{S}} G/K$ defined by the condition $\varphi_{KH}(y_K) = y_H$ for $y = (\cdots, y_K, \cdots) \in \prod_{K \in \mathscr{S}} G/K$ and for $H, K \in \mathscr{S}$ with $H \supseteq K$).

We will give other conditions below that ensure that a locally compact group is the projective limit of Lie groups. Many properties of a projective limit of Lie groups are shared by, or inherited from (depending on one's viewpoint), all its approximating Lie groups G/K with $K \in \mathscr{S}$. Hence Lie group theory can be brought to bear on questions about locally compact groups.

Before we leave the discussion of Lie groups we will mention the idea of the radical of a group which plays so important a role in Lie group theory. Iwasawa (1948) [44] showed that any connected locally compact group G contains a connected normal solvable subgroup which includes all other connected normal solvable subgroups. This subgroup is called the *radical of* G and denoted by G_{rad}. A group is called *semi-simple* iff its radical is trivial and *radical* iff it equals its radical. The radical of a locally compact group G is always closed and G/G_{rad} is always semi-simple. Moskowitz [93] has shown that a connected locally compact group is semi-simple if and only if it is the projectice limit of connected semi-simple Lie groups. Similarly $G_{\text{rad}} = (G_Z)_0$ (where G_Z is the center of G) if and only if G is the projective limit of connected reductive Lie groups.

We will now describe some nice conditions satisfied by both compact groups and locally compact abelian groups. From now on we make the *convention* that *"group" means "locally compact group"*. Left Haar measure on the group will be denoted by λ. A group is said to *unimodular* iff its left Haar measure equals its right Haar measure. The class of such groups is denoted by [Um].

The following conditions relate to structural properties of the groups. In each case we give a symbol for the class of groups satisfying the given condition. Each of the properties which have symbols containing an F and a $(^-)$ is the generalization of a finiteness condition on non-topological groups obtained by replacing the condition that a set be finite by the condition that the closure of the set be compact. A little thought will show that all compact and all abelian groups have these properties.

[Z] *Central groups* = Groups G such that G/G_Z is compact where G_Z is the center of G.

$[FD]^-$ *Topologically Finite Derived subgroup groups* = Groups G with G'^- compact where G' is the derived subgroups, i.e., G' = group generated by $\{xyx^{-1}y^{-1} \cdot x, y \in G\}$.

[FIA]$^-$ *Topologically Finite Inner Automorphism groups* = Groups such that the closure of the set of inner automorphisms in the set of all homeomorphic automorphisms with its natural topology is compact. (See [125] and [157] concerning the topology of Aut(G).)

[FC]$^-$ *Topologically Finite Conjugacy class groups* = Groups such that the closure of each conjugacy class is compact.

[IN] *Invariant Neighborhood groups* = Groups having a compact neighborhood of the identity which is invariant (as a set) under all inner automorphisms.

[SIN] *Small Invariant Neighborhood groups* = Groups such that every neighborhood of the identity contains a compact neighborhood which is invariant under all inner automorphisms. (This condition is equivalent to saying that the left and right uniform structures of G coincide (cf. [23] p. 9).

Notice that discrete groups belong to [SIN] and [IN] since $\{e\}$ is a compact invariant neighborhood of the identity. The equality [FIA]$^-$ = [FC]$^-$ $\cap$ [SIN] is a particularly fundamental result of Grosser and Moskowitz ([21], Theorem 4.1) which may be regarded as a specialized Ascoli-Arzela theorem.

An element in a group G is called an FC$^-$-*element* if the closure of its conjugacy class is compact. It is easy to see that the set G_{FC} of FC$^-$-elements is always a normal subgroup of G. As we shall note below a group G belongs to [IN] if and only if G_{FC} is open. An element in a group G is said to be *periodic* if the closed subgroup it generates is compact. For $G \in$ [FC]$^-$ the set G_p of periodic elements is a closed characteristic subgroup ([23], Theorem 3.16).

The next properties are defined in terms of representations of a group. See Hewitt and Ross [31], Chapter 5 for further details.

Let G be a locally compact group and let $L^1(G)$ be the Banach space of absolutely integrable complex valued functions on G relative to left invariant Haar measure. Then $L^1(G)$ becomes a Banach *-algebra relative to convolution multiplication and a suitable involution. A continuous unitary representation of G (hereafter called simply a *representation* of G) is a continuous group homomorphism of G into the group $\mathcal{U}(\mathcal{H})$ of unitary operators on some Hilbert space $\mathcal{H}$ when $\mathcal{U}(\mathcal{H})$ is provided with the weak operator topology. It is easy to define an integral so that for any unitary representation U of G the map

$$f \xrightarrow{\tilde{U}} \int f(x)U(x)\,d\lambda(x) \qquad f \in L^1(G)$$

is a linear ring homomorphism satisfying $\tilde{U}(f^*) = \tilde{U}(f)^*$ of $L^1(G)$ into the Banach *-algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on $\mathscr{H}$. Furthermore $\{\tilde{U}(f)\xi : f \in L^1(G), \xi \in \mathscr{H}\}$ is dense in $\mathscr{H}$. Such a map is called an essential *-representation of $L^1(G)$ (hereafter called simply a *representation* of $L^1(G)$). The map $U \to \tilde{U}$ is a bijection of the class of representations of G onto the class of representations of $L^1(G)$. Moreover U is irreducible if and only if $\tilde{U}$ is. (Here *irreducible* means topologically irreducible, i.e., there are no *closed* subspaces which are invariant under all the operators in $\{U(x) : x \in G\}$ or in $\{\tilde{U}(f) : f \in L^1(G)\}$ respectively.) Furthermore the von Neumann algebra generated by $U(G)$ always equals the von Neumann algebra generated by $\tilde{U}(L^1(G))$.

Like any Banach *-algebra with a faithful *-representation, $L^1(G)$ has a norm γ which is maximum among those submultiplicative norms satisfying the B^*-condition (now sometimes called the C^*-condition)

$$\gamma(a^*a) = \gamma(a)^2 \quad \forall a$$

(cf. [110], Corollary 4.6.10). The completion of $L^1(G)$ relative to this norm is usually denoted by $C^*(G)$ and called the *C^*-algebra of G*. It is immediate that any representation $\tilde{U}$ of $L^1(G)$ can be extended by continuity to a unique (essential *-) representation $\overline{U}$ of $C^*(G)$ and that $\overline{U}$ is irreducible if and only if $\tilde{U}$ is. In fact if $\overline{U}$ is topologically irreducible then it is algebraically irreducible and hence its kernel is a primitive ideal of $C^*(G)$ ([110], Theorem 4.9.10). We will denote the *space of primitive ideals of* $C^*(G)$ together with its hull-kernel topology, ([110],Definition 2.6.2) by Prim(G).

There is a natural notion of the equivalence of two representations of G. Two representations $U_1 : G \to \mathscr{U}(\mathscr{H}^1)$ and $U_2 : G \to \mathscr{U}(\mathscr{H}^2)$ are called *equivalent* if there is a unitary map U (i.e., a Hilbert space isomorphism) of $\mathscr{H}^1$ onto $\mathscr{H}^2$ such that $U^{-1}U_2(x)U = U_1(x)$ holds for all $x \in G$. Equivalence of representations of $L^1(G)$ and $C^*(G)$ is defined similarly and it is immediate that U_1 and U_2 are equivalent iff $\tilde{U}_1$ and $\tilde{U}_2$ are equivalent and iff $\overline{U}_1$ and $\overline{U}_2$ are equivalent. The set of equivalence classes of irreducible representations of G or equivalently of $L^1(G)$ or $C^*(G)$ is denoted by $\hat{G}$ The map $U \to \operatorname{Ker}(U)$ induces a map of $\hat{G}$ onto Prim(G). When $\hat{G}$ is given the topology of Prim(G) pulled back through this map it is called the *dual space of* G. (If G is abelian then every irreducible representation of G is 1-dimensional and hence is equivalent to a unique continuous group homomorphism of G into $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ where $\mathbf{T}$ is regarded as $\mathscr{U}(\mathbf{C})$. Thus in this case the dual $\hat{G}$ just defined, is naturally homeomorphic to the group of continuous characters of G which is the usual meaning of $\hat{G}$ when G is abelian.)

Under certain circumstances there is a measure μ on $\hat{G}$ satisfying

$$\int_G |f|^2 \, d\lambda = \int_{\hat{G}} \mathrm{Tr}(\overline{U}_f \overline{U}_f^*) \, d\mu(U)$$

$$\forall f \in L^1(G) \cap L^2(G).$$

Such a measure is unique and is called a *Plancherel measure.* Although its existence is important, it is much more useful when an explicit expression for the measure can be given.

The class of groups with all their irreducible (continuous unitary) representations finite dimensional is denoted by [Moore] and these groups are called *Moore groups.* The class of groups G for which all the irreducible representations of $L^1(G)$ take their values in the ideal of compact operators is denoted by [CCR]. The class of groups such that the von Neumann algebra generated by any representation is Type I is denoted by [Type I] (see Dixmier [11]). Any abelian or compact group belongs to [Moore] and we have the inclusions [Moore] $\subseteq$ [CCR] $\subseteq$ [Type I]. The condition Type I is tremendously important, but its importance is quite technical. It is essentially correct to say that $\hat{G}$ is tractable and that G can be understood in terms of $\hat{G}$ exactly when G is Type I. Many important classes of groups are Type I (e.g., semi-simple or nilpotent connected real Lie groups). (See Dixmier [11], Lipsman [77] and Auslander and Moore [4] for more details on this theory.) However other important classes of groups are not Type I. For them a dual object more subtle than $\hat{G}$ is needed. The proper dual object is not yet clear in this case and this is currently an area of active research. See in particular [124].

As a first (but definitely not final) step in the direction of finding a more general dual object one might substitute Prim(G) for $\hat{G}$. If G is Type I then $\hat{G}$ and Prim(G) are homeomorphic, and the converse is true at least for 2^{nd} countable groups (for which $C^*(G)$ is separable). (References for this and the following results will be found under [CCR] in § 4 below.) Since Prim(G) always satisfies the T_0 separation axiom, $\hat{G}$ is T_0 if and (at least for 2^{nd} countable G) only if G is Type I. It is known that a group is CCR if and only if it is Type I and Prim(G) is T_1. The class of groups for which Prim(G) is T_1 will be denoted by [T_1]. Pukanszky [106] has shown that for connected Lie groups Prim(G) can be identified with the space of normalized characters on G. The normalized characters are in turn in bijective correspondence to the quasi-equivalence classes of certain (so called, normal) factor representations.

Recall that an almost connected group G can be written as a projective limit of Lie groups

$$G = \varprojlim_{K \in \mathscr{S}} G/K$$

where $\mathscr{S}$ is the collection of all compact normal subgroups for which G/K is a Lie group. Lipsman ([76], Theorems 2.1 and 5.4) has shown that $\hat{G}$ can be written as the corresponding injective limit

$$\hat{G} = \varinjlim_{K \in \mathscr{S}} \widehat{G/K}.$$

which is in fact a union after suitable identifications. See also Moore [88] Proposition 2.2 and a very simple proof by Miličić [85]. This is useful since for instance if G/K is a Type I Lie group $\widehat{G/K}$ is essentially a manifold. For Type I unimodular groups G, Lipsman also shows that the Plancherel measure on $\hat{G}$ is the injective limit of the Plancherel measures on $\widehat{G/K}$.

We continue our list of desirable properties shared by compact and abelian groups. A group is called *maximally almost periodic* iff it has enough finite dimensional irreducible representations to separate points. The class of such groups is denoted by [MAP]. The Gelfand-Raĭkov theorem shows the inclusion [MAP] $\supseteq$ [Moore]. It is easy to show that compact groups belong to [Moore] and it follows easily that a group belongs to [MAP] if and only if it is continuously isomorphic to a subgroup of a compact group. The terminology arises from the fact that a group belongs to [MAP] iff there are enough continuous almost periodic functions on it to separate points (cf. Dixmier [11] § 16).

An additional class is defined in terms of [MAP]. The groups in [Tak] $=$ [MAP] $\cap$ [FD]$^-$ are called *Takahashi groups* because they were the groups for which Takahashi's famous duality theorem was originally stated (1952)). For a recent interesting discussion and extension of this duality theory see Poguntke [101]. (Compare also the exposition of the Chu duality theory by Heyer in [32].)

A group is called *hermitian* iff the Banach *-algebra $L^1(G)$ is hermitian. (Since the equivalence of the conditions hermitian and symmetric was shown only recently, many authors use the term symmetric here.) This class of groups is denoted by [Her]. A group G is called *completely regular* or *strongly semi-simple* iff $L^1(G)$ has the corresponding property. See Rickart [110] for definitions of these terms. These classes are denoted by [CR] and [SSS], respectively. (Any (locally compact) group is of course a completely regular topological space so this terminology should not cause confusion.)

Let us clarify the relationship between the various spaces of ideals associated with a group. For any Banach $*$-algebra $\mathcal{U}$ let $MM(\mathcal{U})$, $\mathrm{Prim}(\mathcal{U})$, $\mathrm{Prim}_*(\mathcal{U})$ and $F_*(\mathcal{U})$ denote the spaces of maximal modular ideals, primitive ideals, kernels of topologically irreducible $*$-representations, and kernels of irreducible finite dimensional $*$-representations, respectively. Since each of these is a collection of prime ideals we will endow each with the hull-kernel topology. Rickart [110] calls the first and second spaces the strong structure space and the structure space of $\mathcal{U}$ respectively. It is clear that $F_*(\mathcal{U}) \subseteq MM(\mathcal{U} \subseteq \mathrm{Prim}(\mathcal{U})$ and $F_*(\mathcal{U}) \subseteq \mathrm{Prim}_*(\mathcal{U})$ hold. If $\mathcal{U}$ is hermitian then $\mathrm{Prim}(\mathcal{U}) \subseteq \mathrm{Prim}_*(\mathcal{U})$ holds also (see [96] and [69] p. 39).

There are two Banach $*$-algebras associated with a group: $L^1(G)$ and $C^*(G)$. Any C^*-algebra such as $C^*(G)$ satisfies $\mathrm{Prim}(C^*(G)) = \mathrm{Prim}_*(C^*(G))$ and we have denoted this set by $\mathrm{Prim}(G)$. Results in [161] and [162] show $\mathrm{Prim}(G) = \mathrm{Prim}(C^*(G)) = \mathrm{Prim}_*(C^*(G))$ may be identified with $\mathrm{Prim}_*(L^1(G))$ at least for groups G in $[PG]$. Similarly we identify $F_*L^1(G))$ with $F_*(C^*(G))$ and denote this set by $F_*(G)$. Hence for any group G with this property we have

$$\mathrm{Prim}L^1(G)) \supseteq MM(L^1(G)) \supseteq F_*(G) \subseteq MM(C^*(G)) \subseteq \mathrm{Prim}(G)$$

and if G is also hermitian we have

$$\begin{array}{ccc} & & MM(C^*(G)) \subseteq \mathrm{Prim}(G) \\ F_*(G) & \begin{array}{c} \rotatebox{45}{$\subseteq$} \\ \rotatebox{-45}{$\subseteq$} \end{array} & \qquad\qquad\qquad \rotatebox{90}{$\subseteq$} \\ & & MM(L^1(G)) \subseteq \mathrm{Prim}(L^1(G)). \end{array}$$

The possible existence of primitive ideals in $L^1(G)$ which are not the kernels of $*$-representations creates difficulty when studying non-hermitian groups. For $G \in [\mathrm{SIN}]$, $L^1(G)$ and $C^*(G)$ have central approximate identities [90] so that each primitive ideal is modular since a primitive Banach algebra with a non-trivial center has an identity element ([110], 2.4.5). Hence a group G in $[\mathrm{Her}] \cap [\mathrm{PG}] \cap [\mathrm{SIN}]$ satisfies $MM(C^*(G)) \supseteq MM(L^1(G))$ since $C^*(G)/\mathcal{F}$ may be identified with the completion of $L^1(G)/(\mathcal{F} \cap L^1(G))$ in a suitable operator norm which is a Q-norm by [96]. Also $G \in [\mathrm{SIN}]$ belongs to $[T_1]$ if and only if it satisfies $\mathrm{Prim}(G) = MM(C^*(G))$. Hence $G \in [\mathrm{Her}] \cap [\mathrm{SIN}] \cap [T_1] \cap [PG]$ satisfies $MM(C^*(G)) = \mathrm{Prim}(G) = MM(L^1(G)) = \mathrm{Prim}L^1(G))$. Furthermore Corollary 5.3 of [91] shows that for $G \in [\mathrm{FIA}]^- \subseteq [\mathrm{Her}] \cap [\mathrm{SIN}] \cap [T_1] \cap [\mathrm{PG}]$, $\mathrm{Prim}(G)$ is a Hausdorff locally compact space.

For $G \in [\mathrm{Moore}] \subseteq [\mathrm{Her}]$ we see directly that each element of $\mathrm{Prim}(G)$ is maximal modular whether regarded as an ideal in $L^1(G)$ or $C^*(G)$ so in this case we conclude

$$F_*(G) = MM(L^1(G)) = \mathrm{Prim}(L^1(G))$$
$$= \mathrm{Prim}(G) = MM(L^1(G)).$$

Also Prim(G) is T_2 for a Moore group G if and only if $G \in [\mathrm{FC}]^-$ (at least if G is σ-compact). This is true since the main result of [81] shows that σ-compact $[\mathrm{FC}]^-$ groups have Prim(G) T_2. Conversely for $G \in [\mathrm{Moore}] \subseteq [\mathrm{IN}]$, G/G_{FC} is a discrete Moore group and hence has the form A^F for an abelian group A and a finite group F. Therefore Theorem 1 of [63] shows $G \in [\mathrm{FC}]^-$ if $\mathrm{Prim}(G) = MM(L^1(G))$ is Hausdorff.

We remark again that if G is Type I, then $\hat{G}$ can be identified with Prim(G) and the space $\hat{G}_F$ of equivalence classes of finite dimensional irreducible representations can be identified with $F_*(G)$. Furthermore any $G \in [\mathrm{CCR}]$ obviously satisfies $F_*(G) = MM(C^*(G))$. Finally any $G \in [\mathrm{Her}] \cap [\mathrm{CCR}]$ satisfies $F_*(G) = MM(L^1(G))$ since $L^1(G)/\mathscr{F}$ (for $\mathscr{F} \in MM(L^1(G))$ may be identified with a dense *-subalgebra with identity in the ideal of compact operators on some Hilbert space.

This may be an appropriate place to remark that $G \mapsto L^1(G)$ and $G \mapsto C^*(G)$ are far from being functors. The injection map $H \to G$ of an open subgroup H into a group G and the quotient map $G \to G/N$ of a group onto its quotient modulo a closed normal subgroup induce maps $L^1(H) \to L^1(G)$, $C^*(H) \to C^*(G)$, $L^1(G) \to L^1(G/N)$ and $C^*(G) \to C^*(G/N)$ but these are essentially the only cases. The existence of these maps can be derived from noting that Haar measure on H is merely the restriction of Haar measure on G and [109] 3.5.3. If $G \to H$ is any continuous group homomorphism then there is a homomorphism $M(G) \to M(H)$ between the measure algebras associated with it functorially. This of course induces a map $L^1(G) \to M(H)$. If we regard $M(H)$ as the double centralizer algebra $D(L^1(H))$ of $L^1(H)$ this is particularly easily understood, and suggests the correct guess that there is a map of $C^*(G)$ into $D(C^*(H))$. For further details see § 4 of M. A. Rieffel [143].

A group is called *amenable* if $C(G)$ (the algebra of bounded continuous complex valued functions on G) has a left invariant mean. A left invariant mean on $C(G)$ is a linear functional $m: C(G) \to \mathbf{C}$ satisfying $m(1) = 1$, $m(f^-) = m(f)^-$, $|m(f)| \leqq \|f\|_\infty$ and $m({}_xf) = m(f)$ for all $f \in C(G)$ where ${}_xf$ is defined by ${}_xf(y) = f(x^{-1}y)$ for all $x, y \in G$. The tremendous importance of amenability is not obvious from the definition. The reader should consult the excellent small book by Greenleaf [19] for the rich theory of this class of groups. We mention only one from the long list of equivalent conditions. Any group G acts on $L^2(G)$ by left translation and this is a unitary representation called the *left regular representation*. For a compact group this representation contains all irreducible representations in the sense that they all arise as restrictions of left translation to certain finite dimensional translation invariant subspaces. For abelian groups the left regular representation contains all irreducible representations in a weaker sense (cf. [11], 18.1 for a definition of this notion of weak containment). A group is amenable if and only if the left regular representation contains every irreducible representation in this weaker sense. The class of amenable groups is denoted by [Am].

We now mention two slow growth conditions. A group is said to be *exponentially bounded* iff any compact neighborhood W of the identity satisfies

$$\lambda(W^n) = 0(t^n) \quad \forall t > 1$$

(here W^n means $\{x_1x_2 \cdots x_n : x_j \in W\}$). A group G is said to satisfy the *polynomial growth condition* if for each compact neighborhood W of the identity there is an integer p such that

$$\lambda(W^n) = 0(n^p).$$

These two classes of groups are denoted by [EB] and [PG] respectively. The inclusion [PG] $\subseteq$ [EB] is obvious. For discrete groups Adel'son-Vel'skiĭ and Sreĭder [1] showed that [EB] implies amenability.

For any group G, the *left regular representation* of $L^1(G)$ on $L^2(G)$ is the representation denoted by L and defined by $L_f(g) = f^*g$ for all $f \in L^1(G)$ and all $g \in L^2(G)$. This is the representation of $L^1(G)$ related to the left regular representation of G mentioned above. Let ρ and γ be the spectral radius and the largest B^*-norm on $L^1(G)$ respectively (see Rickart [110] pp. 30 and 226 for definitions). Then any $f \in L^1(G)$ satisfies

$$\|L_f\| \leqq \gamma(f) \leqq \rho(f^*f)^{1/2}.$$

Equality on the left holds for all $f \in L^1(G)$ iff the left regular representation of G weakly contains all irreducible representations and hence iff G is amenable. Equality on the right holds for all $f \in L^1(G)$ iff G is hermitian. This is Raĭkov's criterion ([110], Theorem 4.7.21). Hulanicki [35] and Jenkins [46] each showed that slow growth conditions stronger than [PG] on a discrete group G would imply $\|L_f\| = \rho(f^*f)^{1/2}$ for all $f \in L^1(G)$. Hence a discrete group satisfying such a condition would be both hermitian and amenable. Anusiak [2] used a similar argument to show that certain nondiscrete groups are hermitian and amenable. A different stronger slow growth condition was shown by Emerson and Greenleaf [15] to imply amenability in arbitrary locally compact groups. In the same paper it was shown that abelian groups satisfy this stronger condition and it was conjectured that this stronger condition is equivalent to [EB] and [PG] in connected groups. Jenkins [52] verified the equivalence of [EB] and [PG] for connected groups. (Actually in [52] the groups were required to be second countable but this is unnecessary.) In the same paper and in Rosenblatt [116] several other conditions were shown to be equivalent to [EB] and [PG] in connected groups including important conditions on the approximating Lie groups. By an argument, which we will give later, these conditions imply that a connected (in fact an almost connected) group in [PG] is amenable. In [49] and [51] it was conjectured that all these conditions are equivalent to [Her] for connected groups. This conjecture was recently proved false by Leptin [74] who showed that the affine group of the real line

$$\left\{ \begin{pmatrix} e^x & y \\ 0 & 1 \end{pmatrix} : x, y \in \mathbf{R} \right\}$$

(the so called $ax + b$-group) is hermitian although it is easily seen not to be exponentially bounded. Very recent results of J. Ludwig [83] show [PG] $\subseteq$ [Her] for connected groups. Jenkins [45] gives an example of an amenable group (in fact a discrete solvable group) which is not hermitian.

A subset S of a group G is said to be *uniformly discrete* iff there is a neighborhood W of the identity in G such that the family of sets $\{sW : s \in S\}$ is pairwise disjoint. The class of groups containing no uniformly discrete free semigroup on two generators is denoted by [NF]. Jenkins [52] shows that [NF] is equivalent to [EB] and [PG] for connected groups. If a group G contains a uniformly discrete free semigroup S on two generators x, y and if W is a compact neighborhood of the identity such that $\{sW : s \in S\}$ is pairwise disjoint then it is easy to check the estimate $\lambda((W \cup xW \cup yW)^n) \geqq (2^{n+1} - 1)\lambda(W)$. Hence for any group [EB] implies [NF]. For discrete groups, Jenkins [46] has shown that [Her] implies [NF]. Jenkins [50] shows that any group containing a free semigroup on two generators which satisfies a condition stronger than being uniformly discrete must be non hermitian. In the same paper an example of a Hermitian group which contains a closed free semigroup on two generators is given.

Each of the above classes includes all compact and all abelian groups. We now list a few important classes which do not:

[K] = Compact groups
[A] = Abelian groups
[Nil] = Nilpotent groups
[V] = Vector groups
[D] = Discrete groups
[F] = Finite groups

We define a (locally compact) group G to be nilpotent if the upper central series $\{e\}, G_Z, G_{Z,2}, G_{Z,3}, \cdots$ terminates at G after a finite number of steps where $G_{Z,n+1}$ is the closed characteristic subgroup of G defined by the requirement that $G_{Z,n+1}/G_{Z,n}$ be the center of $G/G_{Z,n}$. This is equivalent to requiring that the lower central series $G, G_{(1)}, G_{(2)}, \cdots$ terminate at $\{e\}$ after a finite number of steps where $G_{(n+1)}$ is the (necessarily characteristic) subgroup generated by $\{xyx^{-1}y^{-1} : x \in G, y \in G_{(n)}\}$. Since $[G, H^-] \subseteq [G, H]^-$ holds (where $[G, L]$ is the subgroup generated by $\{xyx^{-1}y^{-1} : x \in G, y \in L\}$) this is also equivalent to the termination at $\{e\}$ after a finite number of steps of the topological lower central series $G, G_{(1)}^-, G_{(2)}^-, \cdots$. Of course every nilpotent group is solvable. (A group G is *solvable* iff $G, G^{(1)}, G^{(2)}, \cdots$ terminates at $\{e\}$ after a finite number of steps where $G^{(n+1)} = (G^n)'$.) Clearly the class of nilpotent groups is stable under the formation of (closed) subgroups and homomorphic images, and a finite direct product is nilpotent if and only if each actor is nilpotent.

These are the properties which we will discuss. In order to simplify the presentation we are omitting consideration of a very useful and important generalization which applies to each class which may be defined in terms of the group of inner automorphisms. For instance the class [IN] is the class of groups containing a compact neighborhood of $\{e\}$ which is stable as a set under the action of the inner automorph ism group. If B is some other subgroup of the automorphism group of a group G we could ask whether there is a compact neighborhood of the identity stable under B. If so we would say that G belongs to $[\mathrm{IN}]_B$. Similar alterations can be made in the definition of several other classes. Consideration of all the classes depending on the different subgroups B (usually required to contain the inner automorphisms) can effect considerable economy in proofs. However exploration of all the results of this type which have been obtained would considerably complicate our discussion and we are confident that after the orientation provided by the present survey the reader can easily absorb this more general viewpoint.

3. **Diagrams.** We will now summarize the inclusions which are known to hold between these classes, in four diagrams. The arrows indicate implications. In a very few cases (which will be pointed out in the subsequent discussion) implications have only been proved for second countable groups. We believe this hypothesis is not actually needed in any of these cases although the detailed proof may in some cases be complicated. We hope that these diagrams contain no false implications, and that they record all the implications which have been pre-

viously noted or which are trivial consequence of known results. As we have already remarked, a number of additional implications have been conjectured. Verification of some of these conjectures would considerably change and simplify these diagrams.

On these diagrams we record some structural characterizations by a short-hand notation which will be clear from the following examples. The symbols $[V \times K]$, $[\mathrm{Tak}^F]$, and $[K^{V \times FC}]$ denote respectively the classes of all direct products of vector groups and compact groups, of all finite extensions of Takahashi groups, and of all extensions of a compact group by the direct product of a vector group and a discrete finite conjugacy class group. In general V, K, A, D, F, etc. will denote re-

Diagram 1

General Locally Compact Groups

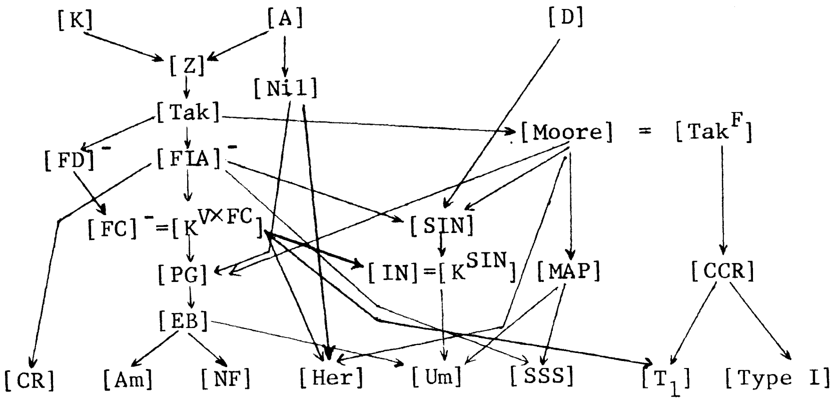


Diagram 2

Almost Connected Locally Compact Groups

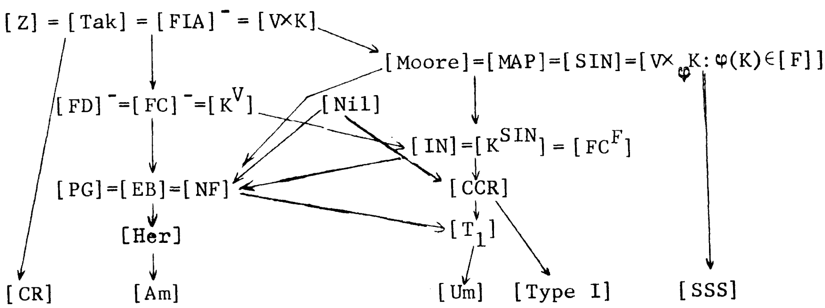


Diagram 3

Connected Locally Compact Groups

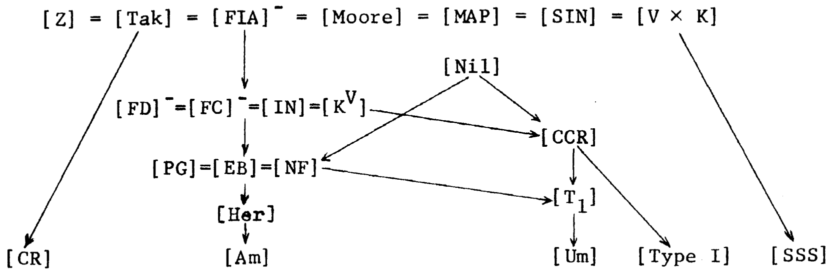
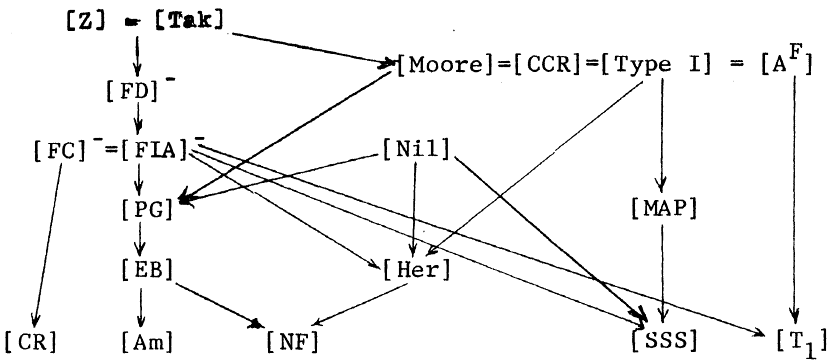


Diagram 4

Discrete Groups



All discrete groups belong to [SIN], [IN], and [Um].

spectively a vector group, a compact group, an abelian group, a discrete group, a finite group, etc.

We have not included diagrams for compactly generated groups nor for totally disconnected groups. In the latter case the diagram would be identical to Diagram 1 for arbitrary groups. In the former case, the only difference from Diagram 1 would be: (1) the equation $[FC]^- = [FD]^-$; (2) the implication $[MAP] \Rightarrow [SIN]$, and the consequent removal of the redundant arrow indicating $[MAP] \Rightarrow [Um]$. References to these facts will appear in the next section.

4. **References, proofs and counterexamples.** We shall now give systematic references to the major results concerning the classes of groups introduced above. For each class we will first give references to the inclusions of that class in other classes which are shown on the diagrams and to structure theorems for groups in the class. Second we will mention results on the stability of the class. Third we will mention diverse results concerning groups in the class including complicated hypotheses which imply membership in the class. Finally we will mention examples of groups in the class which are not in more restrictive classes.

Obviously any property which holds for all groups in a given class holds for all groups in a more restrictive class. We will only mention results in the maximum generality for which a proof is known to us. In particular we will usually omit references to results which have been subsequently proved in greater generality. Thus, for instance, in order to find all known properties of central groups one must read all the results mentioned as holding for any of the classes.

[Z]: This class of groups has been extensively studied by Grosser and Moskowitz. All of the following references refer to their works. The fairly difficult inclusion [Z] $\subseteq$ $[\mathrm{FD}]^-$ is Corollary 1 of Theorem 4.4 of [**21**]. The considerably easier inclusion [Z] $\subseteq$ [Moore] is Theorem 2.1 of [**22**]. Since the Gelfand-Raĭkov theorem implies [Moore] $\subseteq$ [MAP] these results give [Z] $\subseteq$ [Tak]. Theorem 3.16 of [**23**] shows that if $G \in [\mathrm{FC}]^-$ then G_P is closed characteristic subgroup. Theorem 5.4 of [21] shows that for $G \in [\mathrm{Z}]$, G/G_P is the direct product of a vector group and a discrete torsion free abelian group. Finally Lemma 4, p. 329 and Corollary 1, p. 336 of this same reference show that an almost connected group is central if and only if it is the direct product of a vector group and a compact group. The compact group must be chosen as G_P which is a maximum compact subgroup in this case. Corollary 1 mentioned above actually states that for a compactly generated central group G, G/G_P is the direct product of a vector group (i.e., $\mathbf{R}^n$ for some nonnegative integer n) and a group of the form Z^m for some nonnegative integer m. Of course a central group is the extension A^K of an abelian group by a compact group but the converse is false.

The class [Z] is stable under continuous homomorphic images. For connected groups it is stable under local isomorphisms. Furthermore a direct product of groups is central if and only if all the factors are. A closed subgroup H of a central group G is central if (a) H is open, (b) H is connected, (c) G_Z is open, or (d) H is σ-compact and HG_Z is closed. Finally a connected extension of a discrete central group by a connected group is central. All these results can be found in [**21**] Theorem 2.1 and Corollaries 1 and 2 of Theorem 4.3. We note that [Z] is not closed under finite extensions since there are non central discrete Moore groups which must be finite extensions of abelian groups.

Of course the direct product of any abelian group with any (non-abelian) compact group is central and this shows that the inclusions [A] $\subsetneq$ [Z] and [K] $\subsetneq$ [Z] are proper. The following example from p. 337 of [21] shows that not all central groups are such direct products and that the extension $G = G_P{}^{V \times \mathbf{Z}^n}$ satisfied by any compactly generated central group need not split even for a finitely generated discrete central group.

EXAMPLE 1. Let $\mathbf{Z}_2$ be the two element multiplicative group $\{\pm 1\}$ and let $\mathbf{Z}^2 = \{n : n = (n_1, n_2) \text{ with } n_1, n_2 \in \mathbf{Z}\}$ be the usual additive group. Let $G = \mathbf{Z}_2 \times_\varphi \mathbf{Z}^2$ with multiplication:

$$(\epsilon, n)(\delta, m) = (\epsilon\delta(-1)^{n_1 m_2}, n + m).$$

Easy calculations give

$$G_Z = \{(\epsilon, n) : n_1 \text{ and } n_2 \text{ are even}\}$$

so G is central and

$$G_P = G' = \mathbf{Z}_2 \times \{0\}.$$

Either direct calculation or consideration of $(n, m) \longrightarrow (-1)^{n_1 m_2}$ as a non-symmetric 2-cocycle which is therefore not a coboundary shows that the extension $G = \mathbf{Z}_2{}^{\mathbf{Z}^2}$ does not split.

An explicit Plancherel formula for central groups is given in Theorem 4.1 of [24]. Reference [20] contains an extensive duality theorem (Theorem 2.3) relating properties of G and $\hat{G}$, a Fourier inversion theorem (Theorem 3.3), a Poisson summation formula (Corollary 3.4) and a type of Wiener Tauberian theorem (Theorem 3.8). The last result has been generalized to $[\mathrm{FIA}]^-$ groups under which heading it will be discussed.

[Tak]: Recall that [Tak] is defined to be $[\mathrm{FD}]^- \cap [\mathrm{MAP}]$. Robertson [113] gave an outline for a proof of the inclusion [Tak] $\subseteq$ [Moore] and a detailed proof was published by Kaniuth [61] p. 233. The inclusion [Tak] $\subseteq [\mathrm{FIA}]^-$ follows easily from [Moore] $\subseteq$ [SIN], $[\mathrm{FD}]^- \subseteq [\mathrm{FC}]^-$, and $[\mathrm{FIA}]^- = [\mathrm{SIN}] \cap [\mathrm{FC}]^-$. A structure theorem stated for Takahashi groups by Robertson [113] was strengthened and generalized by Robertson and Wilcox [114]. They show that a group in $[\mathrm{FC}]^- \cap [\mathrm{MAP}] \supseteq$ [Tak] belongs to [SIN] and has a direct sum decomposition $V \times H$ where $H \in [\mathrm{FC}]^- \cap [\mathrm{MAP}]$ has an open compact subgroup. Grosser and Moskowitz [23] Theorem 4.6 show that any group in $[\mathrm{FD}]^- \cap [\mathrm{SIN}] \supseteq$ [Tak] is the direct sum of a vector group and the extension of a compact group by a discrete abelian group.

Let G be a discrete Takahashi group. Then G is a discrete Moore group so that there is a homomorphism φ of G onto a finite group F with $A = \mathrm{Ker}(\varphi)$ abelian. For each $v \in F$ choose $\tilde{v} \in G$ satisfying $\varphi(\tilde{v}) = v$ and let $\tilde{F}$ denote the set of these elements. For each $u \in A$ let $\mathrm{ad}_u : F \to G'$ be the map $\mathrm{ad}_u(v) = [u, \tilde{v}] = u\tilde{v}u^{-1}\tilde{v}^{-1}$. Since F and G' are finite there are only finitely many distinct maps of the form ad_v. Furthermore $\mathrm{ad}_v = \mathrm{ad}_w$ is equivalent to v and w lying in the same coset of G_z. Hence G s a central group.

The class of Takahashi groups is clearly stable under the formation of continuous homomorphic images (since the image would be in $[\mathrm{FD}]^- \cap [\mathrm{Moore}]$) and closed subgroups (since in $[\mathrm{FD}]^- \cap [\mathrm{MAP}]$). A direct product is Takahashi if and only if each factor is. The class of Takahashi groups is not closed under finite extensions since every Moore group is a finite extension of a Takahashi group.

Poguntke [**101**] has given an interesting discussion of the Takahashi duality theorem which points out that every group G has a unique (in an obvious sense) continuous homomorphism into a Takahashi group G_T through which any other such homomorphism factors. As a consequence a group G is a Takahashi group if and only if $G \cong G_T$ and if and only if G can be continuously, injectively, imbedded into the direct product of a compact and abelian group. Heyer's discussion of Chu duality [**32**], is in many respects very similar but shows that the latter theory is more generally applicable. For still another view on such duality theorems see p. 61 of M. A. Rieffel, [**144**].

The following example given by Moore [**88**] p. 409 is a Takahashi group which is not a finite extension of a central group. This is of particular interest since in the same article it is shown that a Lie group in [Moore] must be a finite extension of a central group.

EXAMPLE 2. Let $A = \prod_{n=1}^{\infty} \mathbf{T}$ with the product topology and let $H = \sum_{n=1}^{\infty} \mathbf{Z}_2$ (i.e., the discrete weak direct sum of $\mathbf{Z}_2 = \{\pm 1\}$ in which each element is a sequence which is almost always equal to $+1$). Define $\varphi : H \to \mathrm{Aut}(A)$ by

$$(\varphi_\beta(\alpha))_n = (\alpha_n)^{\beta_n} \qquad \forall \alpha \in A; \ \ \forall \beta \in H.$$

Let $G = A \times_\varphi H$. Considering $\mathbf{Z}_2$ (written multiplicatively as $\{\pm 1\}$ again) as a subgroup of $\mathbf{T}$ we see $G_Z = \prod_{n=1}^{\infty} \mathbf{Z}_2 \times \{1\}$ and $G' = \{(\alpha, 1) : \alpha_n = 1 \text{ for almost all } n\}$. Hence G/G_Z is not compact but $G'^- = A \times \{1\}$ is, so G satisfies $G \in [\mathrm{FD}]^-$ and $G \notin [\mathrm{Z}]$. All the representations of G are finite dimensional with dimension a power of 2 as shown by references given in [**88**]. It is easy to see that no subgroup of finite index is a central group. Note also that G equals G_P.

The following example is discussed by Liukkonen in [**79**] p. 104.

EXAMPLE 3. Let $\mathbf{Z}_2 = \{0, 1\}$ be written additively and let $A = \prod_{n=1}^{\infty} \mathbf{Z}_2 \times \sum_{n=1}^{\infty} \mathbf{Z}_2$ where the notation is as explained in the last example and the first product carries the (compact, totally disconnected) product topology while the second sum carries the discrete topology. G is a semidirect product $A \times_{\varphi} \mathbf{Z}_2$ (with this $\mathbf{Z}_2 = \{0, 1\}$ written additively) where $\varphi(\epsilon)(\alpha, \beta) = (\alpha + \epsilon\beta, \beta)$. Since G has an abelian subgroup of index 2 it is a Moore group (indeed all irreducible representations are of degree less than or equal to 2) and since $G'^{-} = G_Z = \prod_{n=1}^{\infty} \mathbf{Z}_2 \times \{0\} \times \{0\}$ is compact we have $G \in$ [Tak], but $G \notin$ [Z]. Furthermore G is totally disconnected, is a finite extension of an abelian group and is nilpotent of class 2.

EXAMPLE 4. Example 4.11 of Grosser and Moskowitz [**23**] ((b) in their tables) is a considerably more complicated example of a compactly generated Takahashi group which is not central. It is a semidirect product of a compact abelian group with $\mathbf{Z}$.

[FIA]$^-$: We have already mentioned the fundamental result [FIA]$^-$ = [FC]$^-$ $\cap$ [SIN] ([**21**], Theorem 4.1). The inclusion [FIA]$^-$ $\subseteq$ [CR] was obtained in different ways by Liukkonen and Mosak [**82**], Corollary 2.5 and Kaniuth and Steiner [**65**], Corollary p. 324. Mosak ([**91**], Corollary 5.3) shows that $G \in$ [FIA]$^-$ satisfied $G \in [T_1]$ and in fact $\mathrm{Prim}(G) = MM(C^*(G))$ is locally compact Hausdorff (cf. [**81**]).

The inclusion [FIA]$^-$ $\subset$ [SSS] was obtained by Kaniuth and Schlichting [**64**], Satz 2 and Mosak [**91**], Corollary 1.7. Wilcox [**134**] has shown that a group $G \in$ [SIN] belongs to [FIA]$^-$ if and only if it is the direct product $V \times K^{FC}$ of a vector group and the extension of a compact group by a discrete [FC]$^-$ group (i.e., an FC-group). Since any discrete group belongs to [SIN], we conclude [FIA]$^-$ = [FC]$^-$ for discrete groups. Grosser and Moskowitz [**23**] Proposition 4.4 shows that an almost connected [FIA]$^-$ group is the direct product of a vector group and a compact group and hence is central. Finally since a compactly generated [FC]$^-$ group belongs to [FD]$^-$([**23**], Theorem 3.20) we have [FD]$^-$ $\supseteq$ [FIA]$^-$ in this case.

The equation [FIA]$^-$ = [FC]$^-$ $\cap$ [SIN] makes it clear that the class [FIA]$^-$ is stable under taking open subgroups and quotients and that a direct product of groups is in [FIA]$^-$ if and only if each factor is. An easy direct calculation shows that $H \in$ [FIA]$^-$ and A abelian imply $H \times_{\varphi} A \in$ [FIA]$^-$ if $\varphi(A)$ is compact.

Grosser and Moskowitz [**23**], Proposition 4.5 show that a compactly generated [FIA]$^-$ group is an extension $(V \times K)^{\mathbf{Z}^p}$ of the direct product of a vector group and a compact group by $\mathbf{Z}^p$. In Theorem 4.6 of the same article they show that a group $G \in$ [SIN] $\cap$ [FD]$^-$ $\subseteq$ [FIA]$^-$ has the form $V \times K^D$ where D is discrete abelian. Wilcox [134] has proved a slightly weaker form of this structure theorem for all groups in $[\mathrm{FC}]^-$.

Kaniuth and Steiner]65], Theorem 4 show that when G is an $[\mathrm{FIA}]^-$ group, then every proper closed ideal of $L^1(G)$ is included in a maximal modular ideal. We will call groups with this property *Tauberian* and denote the class of such groups by [Taub]. Note that this form of the Wiener Tauberian theorem is trivial for discrete groups. Liukkonen and Mosak [82] prove a number of properties of the center of $L^1(G)$ for an arbitrary group by reducing first to the case of an $[\mathrm{FIA}]^-$-group.

The following example is (d) in the Grosser and Moskowitz tables [23].

EXAMPLE 5. Let G be the discrete weak direct sum of a countably infinite number of copies of a finite nonabelian simple group H. Then G is discrete and obviously belongs to $[\mathrm{FC}]^-$ and hence to $[\mathrm{FIA}]^-$. Furthermore G clearly belongs to [MAP] (consider representing each factor separately). Since G' is the weak direct sum of countably many copies of H', G does not belong to $[\mathrm{FD}]^-$ and since G is discrete and not a finite extension of an abelian group it is not Type I. By choosing H nilpotent or not we may have G nilpotent or not. These remarks determine the membership of G in each of the classes of our diagrams.

This example shows that the inclusion $[\mathrm{Tak}] \subseteq [\mathrm{FIA}]^- \cap [\mathrm{MAP}] = [\mathrm{FC}]^- \cap [\mathrm{MAP}]$ is proper. Later we will see that $[\mathrm{FIA}]^- \cap [\mathrm{MAP}] \subseteq [\mathrm{FIA}]^-$ and $[\mathrm{FIA}]^- \cap [\mathrm{MAP}] \subseteq [\mathrm{MAP}]$ are also both proper.

EXAMPLE 6. Let G be the group described in 5.10 of Grosser and Moskowitz [23] which is example (a) in their tables. This group is discrete and G' is finite. Hence we conclude $G \in [\mathrm{FD}]^- \cap [\mathrm{FIA}]^-$. However G is not maximally almost periodic and hence not Type I (since [Type I] $\subseteq$ [MAP] for discrete groups) nor Takahashi. Thus the inclusion $[\mathrm{Tak}] \subseteq [\mathrm{FD}]^- \cap [\mathrm{FIA}]^-$ is proper. This group is nilpotent of class 2 and hence in [Her]. Thus the membership of G in each class of our diagrams is determined.

The next example is due to T. Sund and was communicated to the author by R. D. Mosak. An example with the same properties was attributed to L. Robertson and occurs as example (i) in the tables of Grosser and Moskowitz [23], but the paper which was said to contain it apparently never appeared and the present author has been unable to learn the nature of this example.

EXAMPLE 7. Let G be the central extension

$$\{e\} \to \mathbf{T} \to G \to \mathbf{Z}^2 \to \{e\}$$

defined by a totally skew normalized multiplier of $\mathbf{Z}^2$ such as

$$\omega((m_1, m_2), (n_1, n_2)) = e^{im_1 n_2}.$$

Then as a set G is $\mathbf{T} \times \mathbf{Z}^2$ and multiplication is given by

$$(\zeta, m_1, m_2)(\xi, n_1, n_2) = (\zeta\xi e^{im_1 n_2}, m_1 + n_1, m_2 + n_2).$$

Then G' is $e^{i\mathbf{Z}}$, G'^- is $\mathbf{T}$ so G belongs to [FD]$^-$. The sets $\{e^{it} : |t| < \epsilon\} \times \{0\}$ for $\epsilon > 0$ and $\mathbf{T} \times \{0, \pm 1\} \times \{0, \pm 1\}$ show that G belongs to [SIN] (and hence to [FIA]$^-$) and is compactly generated. Furthermore [7] shows that G is not Type I since ω is not a type I multiplier. Hence G does not belong to [Tak] and therefore not to [MAP]. This determines the membership of G in all the classes of our diagrams.

These examples show that our diagrams display all possible inclusions of [FIA]$^-$ in larger classes.

[FD]$^-$: The inclusion $\{yxy^{-1} : y \in F\}^- \subseteq (G'x)^-$ which holds for any $x \in G$ implies [FD]$^- \subseteq$ [FC]$^-$.

It is easy to see that [FD]$^-$ is stable under forming continuous homomorphic images, closed subgroups and extensions by abelian groups and that a direct product of groups belongs to [FD]$^-$ if and only if each factor does. Furthermore the extension of a compact group by an [FD]$^-$ group is [FD]$^-$ by Hewitt and Ross [31] 5.24 (*a*).

Liukkonen and Mosak ([82], Proposition 3.1) show that the center of $L^1(G)$ satisfies the Wiener-Ditkin condition when G belongs to [FD]$^-$ but that the obvious semidirect product of $\mathbf{R}^3$ with a discrete copy of $SO(3)$ does not have this property.

Schlichting [119] shows that if G is a locally compact group with G' finite then G belongs to [MAP] (and hence to [Tak]) if and only if it is a finite extension of a central group.

Notice that Example 6 is a discrete [FD]$^-$ group which is not Type I nor MAP.

The following two examples are (e) and (k) in the tables in Grosser and Moskowitz [23].

EXAMPLE 8. Let H be the Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{R} \right\}$$

and let N be the cental normal subgroup

$$N = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbf{Z} \right\} .$$

Let G be the quotient H/N. Then

$$G_Z = G' = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + N : z \in \mathbf{R} \right\}$$

and hence is topologically isomorphic to $\mathbf{T}$. Thus G is a connected group in $[\mathrm{FD}]^- \cap [\mathrm{Nil}]$ and hence in [CCR] and [Her]. Since $[\mathrm{SIN}] = [\mathrm{MAP}] = [\mathrm{Z}\}$ for connected groups, G does not belong to [SIN] nor [MAP].

The dual of the Heisenberg group H is worked out on p. 442 of Warner's book [**130**], and this allows one to calculate explicitly the dual of $G = H/N$. Since $G \in [\mathrm{Her}]$ we have $MM(L^1(G)) \subseteq \mathrm{Prim}(L^1(G)) \subseteq \mathrm{Prim}*(L^1)G))$ and since $G \in [\mathrm{CCR}]$ no ideal in $\mathrm{Prim}*(L^1(G))$ is modular unless it is the kernel of a finite dimensional representation. Hence we see that $G \in [\mathrm{CR}]$ and $G \notin [\mathrm{SSS}]$. This determines the membership of G in each class in our diagrams.

Example 9. Let $K = \prod_{n=-\infty}^{\infty} \mathbf{Z}_2$ with $\mathbf{Z}_2 = \{\pm 1\}$ written multiplicatively and the compact, totally disconnected, product topology on K. Let $G = K \times_\varphi \mathbf{Z}$ be the semidirect product defined by shifting coordinates, i.e., $(\varphi(n)\alpha)(m) = \alpha(m - n)$. Hence G is both compactly generated (say by $K \times \{0, \pm 1\}$) and totally disconnected. A simple calculation gives $G' = K \times \{0\}$ and $G_Z = \{(\alpha, 0) : \alpha \in K \text{ is constant}\}$ from which we conclude that G is a solvable but not nilpotent $[\mathrm{FD}]^-$ group. There can be no invariant neighborhood of the identity smaller than $K \times \{0\}$ because of the translation action of $\varphi(Z)$. Therefore G does not belong to [SIN] and hence not to [MAP] (since otherwise it would be a Takahashi group).

We could refer to 4.5.2.1 and 4.5.7.2. (cf. 4.5.7.3) in Warner's book [**130**] to see that G is a [CCR]-group, but we need to explicitly calculate $\hat{G}$ to settle the other properties of G anyway. Hewitt and Ross [**31**] 23.21 shows that $\hat{K}$ may be identified with the discrete weak direct sum $\sum_{n=-\infty}^{\infty} \mathbf{Z}_2$ of copies of $\mathbf{Z}_2 = \{0, 1\}$ written additively by

$$\alpha(\beta) = \prod_{n=-\infty}^{\infty} \beta_n{}^{\alpha_n} \qquad \alpha \in \sum \mathbf{Z}_2;\ \beta \in K$$

where the factors are equal to $+1$ at all but a finite number of indices. Hence we may choose $C \cup \{0\}$ where $C = \{\alpha \in \sum_{n=-\infty}^{\infty} \mathbf{Z}_2 : \alpha(n) = 0$ for and $\alpha(0) = 1\}$ as an explicit Borel cross section of $\hat{H}$ under the action of $\varphi(Z)$. Since no element of C is fixed under any element of $\varphi(Z)$ we see by Mackey's little group method ([84], p. 647)

$$\hat{G} = \{U^{\xi} : \xi \in T\} \dot{\cup} \{U^{\alpha} : \alpha \in C\}$$

where $U^{\xi}_{(\beta,m)} = \xi^m$ is one dimensional for each $\xi \in T$ and U^{α} is the infinite dimensional induced representation corresponding to α which can be described by

$$U^{\alpha}_{(\beta,m)}f(n) = \prod_{k=-\infty}^{\infty} \beta_{k-n}^{\alpha_k} f(n+m)$$

for $f \in \ell^2(\mathrm{Z})$.

The infinite dimensional representations U^{α} when restricted to $K \times \{0\}$ become direct sums $\oplus\{\gamma : \gamma \in \hat{K}$ is in the $\varphi(\mathrm{Z})$ orbit of α *in* $\hat{K}\}$. Hence we see directly that K is a large compact subgroup and G is CCR. Since G is hermitian it satisfies the inclusions $MM(L^1(G)) \subseteq \mathrm{Prim}(L^1(G)) \subseteq \mathrm{Prim}(G)$. Since G is CCR the elements of $MM(L^1(G))$ are exactly the kernels of finite dimensional representations. Hence G satisfies, $G \in$ [CR] and $G \notin$[SSS]. Thus we have determined all of the properties of our diagrams for this example.

A slight modification of the last example was discussed by Grosser and Moskowitz [23], p. 30 as an example of a compactly generated [FD]$^-$ group which is not the projective limit of Lie groups. This time let K be the compact connected product $\prod_{n=-\infty}^{\infty} \mathbf{T}$ and let $\varphi : \mathbf{Z} \to \mathrm{Aut}(K)$ be as in the last example. Then $G = K \times_{\varphi} \mathbf{Z}$ is compactly generated and the expressions for G_Z and G' are the same as those given in Example 9. We leave to the reader the rest of the analysis of this example.

Examples 8 and 9 show that in both the connected and totally disconnected case, [FD]$^-$ does not imply [SSS] and hence does not imply [FIA]$^-$. However we know of no example of an [FD]$^-$ group (or even an [FC]$^-$ group) which is not in [CR]. Example 7 is a compactly generated [FD]$^-$ group which is not Type I.

[FC]$^-$: The fundamental structural characterization of [FC]$^-$-groups was announced by Robertson [113] and a proof was published as Theorem 2.2 in Liukkonen [79]. This theorem asserts that a group belongs to [FC]$^-$ if and only if it is the extension $K^{V \times FC}$ of a compact group by the direct product of a vector group and a discrete group in [FC]$^-$ (i.e., an FC-group). From this result, the inclusion [FC]$^-$ $\subseteq$ [IN] follows easily and in fact Liukkonen shows ([79], Corollary to 2.2) that G belongs to [IN] if and only if G_{FC} is open. A proof of this last result, based on a result of Tits that G_{FC} is always closed in a projective limit of Lie groups, is given in Theorem 1 of Wu and Yu [137].

Another important structure theorem for $[FC]^-$ is Theorem 3.16 of Grosser and Moskowitz [23]. For $G \in [FC]^-$ this asserts that G_P is a closed characteristic $[FC]^-$-subgroup which is the intersection of all compact normal subgroups and that G/G_P is the direct product of a vector group and a discrete torsion free abelian group. This does not characterize $[FC]^-$ groups as is pointed out on p. 22 of the same reference. As a corollary (Corollary 3.18, ibid.) one concludes that $(G')^-$ is periodic if $G \in [FC]^-$. Theorem 3.20 of [23] shows that G is a compactly generated $[FC]^-$-group if and only if G_P is a compact normal subgroup and G/G_P is the direct product of a vector group and a finitely generated discrete torsion free abelian group. Thus for compactly generated groups $[FC]^-$ equals $[FD]^-$. From this it follows that an almost connected group is in $[FC]^-$ (or equivalently in $[FD]^-$) if and only if it is the extension K^V of a compact group by a vector group. See [79], Theorem 3.7 for another proof.

Wang [126], Theorem 4.6 characterizes $[FC]^-$ groups as those groups G such that (1) G_0 has a unique maximal compact subgroup K with G_0/K a vector group in the center of G/K and (2) for each $x \in G$ there is a closed normal subgroup N of G with $x \in G_0N$ and with N either compact or the semidirect product of a compact group with **Z**. In the same paper, Theorem 4.5 shows that a connected group G is in $[FC]^-$ iff G_{FC} is dense. Similarly Wu and Yu [137] Corollary 2, p. 302 shows that if G_{FC} is dense in G then $G \in [FC]$ is equivalent to $G \in [IN]$. This paper and Wang [127] also give other results for groups G satisfying $G_{FC}{}^- = G$. Peters [97] characterizes $[FC]^-$-groups in terms of positive definite functions.

The inclusion $[FC]^- \subseteq [Her]$ is due to Anusiak [2]. Since the extension of a compact group by a [PG] group is in [PG], and the direct product of two [PG]-groups is in [PG], Robertson's structure theorem reduces the inclusion $[FC]^- \subset [PG]$ to the discrete case. However Hulanicki shows in the proof of Proposition 2 of [35] that a discrete $[FC]^-$ group is in [PG]. At least for σ-compact $[FC]^-$-groups Liukkonen and Mosak [81] show that Prim(G) is T_2 so in particular $[FC]^- \subseteq [T_1]$.

Of course Grosser and Moskowitz' Ascoli-Arzela theorem shows $[FC]^- = [FIA]^-$ for discrete groups since they all belong to [SIN].

The class $[FC]^-$ is obviously stable under formation of closed subgroups and continuous homomorphic images, and it is also clear that a direct product of groups is $[FC]^-$ if and only if each factor is $[FC]^-$.

Similarly the extension of a compact group by an $[\mathrm{FC}]^-$ group is $[\mathrm{FC}]^-$ (use Hewitt and Ross [31] 5.24 (a)). The fact that the obvious semidirect products $\mathbf{R} \times_\varphi \mathbf{Z}_2$ and $\mathbf{Z} \times_\varphi \mathbf{Z}_2$ are not $[\mathrm{FC}]^-$ excludes a number of otherwise plausible stability results.

Kaniuth and Steiner ([65], Theorem 5) show that a discrete nilpotent group G belongs to $[\mathrm{FC}]^-$ if and only if $MM(L^1(G))$ is T_2. Liukkonen ([79], Theorems 3.1 and 3.6) shows that a Type I $[\mathrm{FC}]^-$-group belongs to $[\mathrm{FD}]^- \cap [\mathrm{CCR}]$ and has $\hat{G}$ Hausdorff. Schlichting [119] considers locally compact groups for which the conjugacy classes are actually finite. In [118] he studies the set G_f of elements with finite conjugacy classes. Among other things he notes that G/G_f and G_f' are both finite if and only if G_{vN} is finite and G/G_{vN} has the form A^F where $G_{vN} = \bigcap_{U \in \hat{G}_F} \mathrm{Ker}(U)$ is the von Neumann kernel of G.

Example 5 is a discrete $[\mathrm{FC}]^-$ group not in $[\mathrm{FD}]^-$ and Examples 8 and 9 are both $[\mathrm{FC}]^-$ groups not in $[\mathrm{FIA}]^-$ (hence not in [SIN]) with the first connected and the second totally disconnected. Example 6 is a discrete $[\mathrm{FC}]^-$ group which is not in [MAP] and hence not Type I and Example 8 is a connected $[\mathrm{FC}]^-$ group not in [SSS] and hence not in [MAP]. As remarked in our discussion of $[\mathrm{FD}]^-$ groups we know of no $[\mathrm{FC}]^-$-groups which is not in [CR].

[Moore]: The fundamental article by Moore [88] was evidently available in preprint long before its appearance in print since it is referred to in [23] and [113] (submitted in 1968) and elsewhere. Theorem 2 in [88] shows that a Lie group belongs to [Moore] if and only if it is a finite extension of a central group. Example 2 shows that there are Moore groups which are not Lie groups and fail to satisfy this condition. Robertson outlined a proof in [113] and Kaniuth supplies a detailed proof in [61], p. 233 that a group is a Moore group if and only if it is a finite extension of a Takahashi group. Both of these results may be considered as generalizations of Thoma's earlier discovery [123] that a discrete group is a Moore group if and only if it is a finite extension of an abelian group. Indeed this reference contains a proof that a discrete group is a Moore group if and only if it is Type I (a formally weaker condition) and if and only if it has a uniform bound for the finite dimensions of its irreducible representations (a formally stronger condition). Moore [88] extended to general locally compact groups the equality of the class of finite extensions of abelian groups and the class of groups each of which has a finite bound for the degrees of all of its irreducible representations. In another direction Kaniuth [61] generalized to [SIN] groups Thoma's result that discrete groups are Moore groups if and only if they are Type I. He also notes that this occurs if and only if the regular representation is Type I. (As an auxiliary result he shows (Lemma 2) that a 2-step nilpotent group with open center is a Moore group.) Robertson [**113**] characterizes Moore groups as those semi-direct products $V \times_\varphi B$ where V is a vector group and $B \in$ [Moore] is such that $B_0 \in [K]$ is a finite extension of an open normal subgroup N for which the semidirect product reduces to a direct product $V \times N$.

The inclusion [Moore] $\subseteq$ [CCR] is obvious since all finite dimensional operators are compact and the inclusion [Moore] $\subseteq$ [MAP] is a direct consequence of the Gelfand Raĭkov theorem. Since [PG] $\supseteq$ [Tak] is stable under compact extensions and both [SIN] $\supseteq$ [Tak] and [Her] $\supseteq$ [Tak] are closed under finite extensions we conclude [Moore] $\subseteq$ [PG] $\cap$ [SIN] $\cap$ [Her] from [Moore] $=$ [Tak^F].

Robertson [**113**] notes that the structural results above show that [Moore] is stable under taking closed subgroups, quotients, inverse limits, and finite extensions. The closure under finite extensions was a key result of Moore [**88**].

The next two examples are essentially the discrete and almost connected form of the same example. Example 11, the almost connected form, is example (c) and (f) in the tables of Grosser and Moskowitz [**23**]. Together these examples show that our four diagrams together with the diagram for compactly generated groups described earlier contain all possible inclusions of [Moore] in any of the other classes.

EXAMPLE 10. Let G be the semidirect product $\mathbf{Z} \times_\varphi \mathbf{Z}_2$ where $\mathbf{Z}_2 = \{\pm 1\}$ written multiplicatively acts by $\varphi(\epsilon)n = \epsilon n$. This is a finitely generated discrete group with $G_Z = \{e\}$, $G' = 2\mathbf{Z} \times \{1\}$ and $G_{FC} = \mathbf{Z} \times \{1\}$. Hence G is solvable but not nilpotent, and a finite extension of an abelian group and hence in [Moore] but not in [FC]$^-$. Thus the implications in Diagram 4 settle the membership of G in all the classes listed except [CR]. We will now give an explicit calculation to show that G does not belong to [CR].

For $f \in \ell^1(G) = \mathcal{U}$ denote $f(n, 1)$ and $f(n, -1)$ by $f_+(n)$ and $f_-(n)$ respectively. Easy calculations show $(f * g)_+ = f_+ * g_+ + f_- * \tilde{g}_-$ and $(f * g)_- = f_+ * g_- + f_- * \tilde{g}_+$ where $\tilde{h}(n) = h(-n)$ and the convolution on the right is in $\ell^1(Z)$. (If $\mathcal{F}$ is an ideal in $\mathcal{U}$ then $\mathcal{F}_+ = \{f_+ : f \in \mathcal{F}\}$ is an ideal in $\ell^1(Z)$ and $\mathcal{F}_+ = \{f_- : f \in \mathcal{F}\}$. For any ideal $\mathcal{F}$ in $\ell^1(Z)$ $\mathcal{F} = \{f : f_+, f_- \in \mathcal{F}\}$ is an ideal in $\ell^1(G)$ and every maximal ideal of $\ell^1(G)$ must either have this form with $\mathcal{F}_+$ maximal in $\ell^1(Z)$ or satisfy $\mathcal{F}_+ = \ell^1(Z)$.)

Clearly G has four group characters, namely:

$$\chi_{00}(n, \epsilon) = 1 \quad \chi_{01}(n, \epsilon) = \epsilon$$

$$\chi_{10}(n, \epsilon) = (-1)^n \quad \chi_{11}(n, \epsilon) = (-1)^n \epsilon.$$

Furthermore for each $\zeta \in \mathbf{T} \sim \{\pm 1\}$ we get a 2-dimensional irreducible representation

$$U_\zeta(n, 1) = \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{-n} \end{pmatrix} \; U_\zeta(n, -1) = \begin{pmatrix} 0 & \zeta^n \\ \zeta^{-n} & 0 \end{pmatrix}.$$

Denote the maximal modular ideals of $\ell^1(G)$ corresponding to these irreducible representations by

$$\begin{aligned}
\mathscr{F}_{00} &= \{f : \Sigma f_+(n) + \Sigma f_-(n) = 0\} \\
\mathscr{F}_{01} &= \{f : \Sigma f_+(n) - \Sigma f_-(n) = 0\} \\
\mathscr{F}_{10} &= \{f : \Sigma(-1)^n f_+(n) + \Sigma(-1)^n f_-(n) = 0\} \\
\mathscr{F}_{11} &= \{f : \Sigma(-1)^n f_+(n) - \Sigma(-1)^n f_-(n) = 0\} \\
\mathscr{F}_\zeta &= \{f : \hat{f}_+(\zeta) = \hat{f}_+(\bar{\zeta}) = 0, \hat{f}_-(\zeta) = \hat{f}_-(\bar{\zeta}) = 0\}
\end{aligned}$$

respectively where $\hat{g}$ is the Fourier transform of $g \in \ell^1(Z)$ and we need only consider ζ with $\operatorname{Im}(\zeta) > 0$. Hence

$$\begin{aligned}
\mathscr{F}_{00} \cap \mathscr{F}_{11} &= \{f : \hat{f}_+(1) = \hat{f}_-(1) = 0\} \\
\mathscr{F}_{10} \cap \mathscr{F}_{11} &= \{f : \hat{f}_+(-1) = \hat{f}_-(-1) = 0\}
\end{aligned}$$

so that if $\{\zeta_n\}_{n \in N}$ is any sequence in $\mathbf{T}$ convergent to $+1$ then $\mathscr{F}_{\zeta_n}$ converges to both $\mathscr{F}_{00}$ and $\mathscr{F}_{01}$ and similarly if the sequence in $\mathbf{T}$ converges to -1 then $\mathscr{F}_{\zeta_n}$ converges to both $\mathscr{F}_{10}$ and $\mathscr{F}_{11}$.

Thus the space of maximal modular ideals of $\mathscr{U}$ is not even Hausdorff. Hence G does not belong to [CR]. Another proof that $G \notin$ [CR] can be based on the work of Peters [**97**]. Since $G \in$ [Moore] $\subseteq$ [Her], every primitive idealof $\ell^1(G)$ is the kernel of some finite dimensional irreducible representation of G and hence is maximal modular. Hence $MM(G)$ can be identified with Prim(G) and thus Corollary 3 of [97] and the remarks following it show $G \notin$ [CR].

EXAMPLE 11. Let G be the semidirect product $\mathbf{R} \times_\varphi \mathbf{Z}_2$ where $\varphi(\epsilon)t = \epsilon t$ with $\mathbf{Z}_2 = \{\pm 1\}$ written multiplicatively. An easy calculation gives $G_Z = \{e\}$, $G_{FC} = G' = \mathbf{R} \times \{1\}$. Hence G is an almost connected finite extension of an abelian group and hence a Moore group which is not in [Nil] nor $[FC]^-$. All the properties of G are now determined by Diagram 2 except [CR]. Either of the arguments used in the last example will show $G \notin$ [CR]. In particular if χ_0 and χ_1 are the characters defined by $\chi_0(t, \epsilon) = 1$ and $\chi_1(t, \epsilon) = \epsilon$ then $\operatorname{Ker}(\chi_0) \cap \operatorname{Ker}(\chi_1) = \{f \in L^1(G) : \hat{f}_+(0) = \hat{f}_-(0) = 0\}$ so that both these ideals are the limit of $\operatorname{Ker}(U_r)$ as r approaches zero where

$$U_r(t, 1) = \begin{pmatrix} e^{irt} & 0 \\ 0 & e^{-irt} \end{pmatrix}$$

$$U_r(t, -1) = \begin{pmatrix} 0 & e^{irt} \\ e^{-irt} & 0 \end{pmatrix}.$$

[MAP]: The inclusion [MAP] $\subseteq$ [SSS] is established in Theorem 3.1 of Grosser and Moskowitz [24]. The inclusion [MAP] $\subseteq$ [Um] was first proved by Leptin and Robertson [75]. However a much simpler proof was subsequently discovered by Mosak and Robertson independently. We give it here with their kind permission. Let G be a [MAP]-group and let x be a point in G. Choose a compact symmetric neighborhood W of e containing x and let H be the open subgroup $\cup_{n=1}^{\infty} W^n$. Then the modular functions satisfy $\Delta_G(x) = \Delta_H(x)$ since the Haar measures agree. Furthermore H is a compactly generated [MAP] group and hence a [SIN]-group. We conclude $\Delta_G(x) = \Delta_H(x) = 1$ so G is unimodular and [MAP] $\subseteq$ [Um]. The inclusion [MAP] $\subseteq$ [SIN] for compactly generated groups is an easy special case of Dixmier [11] 17.3.7 (cf. Theorem 12.2 (iv) of Hofmann and Mostert [34].)

The classical (1936, 1940) Freudenthal-Weil theorem which asserts that for connected groups [MAP] = [SIN] = [$V \times K$] = [Moore] = [Z] etc. is proved in Dixmier [11] 16.4.6 and Grosser and Moskowitz [21] Theorem 4.3. Dixmier's problem (ibid. 16.5.4) of finding a proof of this theorem independent of Lie theory seems to be still open.

Grosser and Moskowitz ([23], Theorem 2.9) give a proof of an extended Fruedenthal-Weil theorem which shows that for almost connected groups [MAP] equals [SIN] and both are equivalent to the existence of a semidirect product decomposition $V \times_{\varphi} K$ in which V is a vector group, K is a compact group and $\varphi(K)$ is finite. Theorem 2.18 in the same reference states that for almost connected groups the above conditions are equivalent to [Moore] and Corollary 2.17 states that they are also equivalent to the compactness of the quotient of G modulo the centralizer of G_0. Corollary 2.10 asserts that an almost connected group G belongs to [MAP] if and only if G_0 does. These results have a long history going back in part to Kuranishi (1950) and Murakami [95].

It is obvious that [MAP] is stable under taking closed subgroups and that a direct product is [MAP] if and only if each factor is (consider faithful injectability into a compact group). Murakami [95] Lemma 4 shows that a finite extension of a [MAP] group is [MAP]. Leptin and Robertson [75] show that the quotient of a [MAP] group by a compact group or by its center is again [MAP].

Hofmann and Mostert [34] Proposition 12.2 show [MAP] $\cap$ [IN] = [MAP] $\cap$ [SIN]. Robertson and Wilcox [114] Theorem 2 note that if G

is a [MAP] group then the decomposition $G_0 = V \times K$ guaranteed by the Freudenthal-Weil theorem can be chosen so that V is normal in G Theorem 3 of the same paper shows that groups $G \in$ [MAP] $\cap$ $[\mathrm{FC}]^- \subseteq$ [SIN] can be written as $V \times K^D$ where K^D is also in the smaller set and as usual V is a vector group, K is a compact group and D is discrete. Other results on compactly generated [MAP] groups are given.

EXAMPLE 12. Let G be the discrete free group on two generators. Then G is finitely generated. For each element x of G different from e there is a normal subgroup N of G of finite index not containing x. (For a proof see Hewitt and Ross [31] 4.21 (e).) Hence G belongs to [MAP]. However G is clearly not contained in [NF] (and hence not in [Her]) and the argument used in Example 1.2.3 of Greenleaf's book [19] shows $G \notin$ [Am]. Furthermore Moore and Rosenberg([89], Proposition 6.1) show that G is not in $[T_1]$. Applying Theorem 1 of [61] to $G/[G, G']$ we see that this quotient and hence G does not belong to [CR]. This determines the membership of G in all the classes of diagram 4.

This example shows that even for finitely generated groups [MAP] $\cap$ [SIN] is not included in $[T_1]$, [Am], [NF], [Her], [EB], [PG], etc.

EXAMPLE 13. Let $\mathbf{Z}_2 = \{\pm 1\}$ be written multiplicatively. Define A and K by $A = \sum_{n=1}^{\infty} \mathbf{Z}$ and $K = \prod_{n=1}^{\infty} \mathbf{Z}_2$ and give them the discrete topology and the product topology respectively. Define $\varphi : K \longrightarrow \mathrm{Aut}(A)$ by $(\varphi(\alpha)f)_n = \alpha(n)f(n)$ and let G be $A \times_\varphi K$. Then G is totally disconnected. On p. 385 of [22] it is shown that $G \in$ [MAP], and the argument given there together with Theorem 2.11 (2) of [23] shows $G \notin$ [IN]. Since the derived group is $(\sum_{n=1}^{\infty} 2\mathbf{Z}) \times \{1\}$, G is solvable but not nilpotent. Hence G is amenable. Either [16] or the Corollary to Theorem 4 of [73] show $G \in$ [Her]. Similarly $G \in$ [PG] by Theorem 1.4 of [26] and $G \in$ [CCR] by Proposition 4.3 of [145]. Next we will show that G does not belong to [CR] and this will complete the determination of its membership in all of the classes of our diagram.

The dual space of A is $\hat{A} = \prod_{n=1}^{\infty} \mathbf{T}$. The set $C = \prod_{n=1}^{\infty} \mathbf{T}_+$ with $\mathbf{T}_+ = \{\zeta \in \mathbf{T} : \mathrm{Im}(\zeta) \geqq 0\}$ is a set intersecting each orbit of the action of K on A in a singleton. The representations associated with a point $\zeta = (\zeta_1, \zeta_2, \cdots)$ in C will be finite dimensional (of dimension 2^n) iff $\zeta_j \in \{\pm 1\}$ holds except in a finite number (n) of cases. Denote the corresponding subset of C by C_F. Then the points of $\hat{G}_F$ (which can be identified with $MM(L^1(G))$ since $G \in$ [CCR] $\cap$ [Her] holds) are parameterized by the points in $G_F \times \prod_{n=1}^{\infty} \mathbf{Z}_2$ (where the indexing in the second product should be suitably adjusted). The topology on $\hat{G}_F$ is described in terms of this parameterization by Baggett in [5] Theorem

3.3. However the use of Theorem 2.5 B together with the Corollary to Theorem 7.3 of [5] shows already that $\hat{G}_F \cong MM(L^1(G))$ is not Hausdorff. We conclude $G \notin$ [CR].

EXAMPLE 14. Let G be the integer Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} : m, n, p \in \mathbf{Z} \right\}.$$

We write the generic element above at (m, n, p) so we have $(m, n, p)(h, j, k) = (m + h, n + j, p + k + mj)$. Then we find $G_Z = G_{FC} = G' = \{(0, 0, p) : p \in \mathbf{Z}\}$ so G is nilpotent but not in $[\mathrm{FC}]^-$. The equation $(m, n, p) = (0, 1, 0)^n(1, 0, 0)^m(0, 0, 1)^p$ shows that G is finitely generated. The normal subgroups $G_n = \{(nh, nj, nk) : h, j, k \in \mathbf{Z}\}$ show that G is in [MAP]. Clearly G is not a finite extension of an abelian group so G does not belong to [Moore]. Moore and Rosenberg [89], Theorem 5 shows $G \in [T_1]$ and Kaniuth and Steiner [65], Theorem 5 shows $G \notin$ [CR]. This determines the membership of G in each of our classes.

Example 12 is enough to show that there are no inclusions of [MAP] in any other class missing from our four diagrams or from the diagram for totally disconnected groups and for compactly generated groups. The other examples merely add further details of patterns of memberships.

The following example is due to Jenkins [45]. It was the first example of an amenable nonhermitian group discovered.

EXAMPLE 15. Let G be the discrete group generated by

$$z = \begin{pmatrix} 1/2 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } w = \begin{pmatrix} -1/2 & 1 \\ 0 & 1 \end{pmatrix}.$$

Jenkins shows that z and w generate a free subsemi-group but that they satisfy the relation $z^{-1}w = w^{-1}z$ in G. Thus G does not belong to [NF] and hence does not belong to [Her]. This fact together with Theorem 5 of Moore and Rosenberg [89] shows that G does not belong to $[T_1]$.

Let us denote the matrix

$$\begin{pmatrix} s & r \\ 0 & 1 \end{pmatrix}$$

by (r, s) we have $(r, s)(u, v) = (r + su, sv)$. We see easily that if (r, s) belongs to G then s has the form $\epsilon 2^{-n}$ for some $\epsilon \in \mathbf{Z}_2 = \{\pm 1\}$ and some $n \in \mathbf{Z}$ and r belongs to the set $\mathbf{D}$ of dyadic rationals (i.e., r has the form $p2^{-n}$ for some $p \in \mathbf{Z} \sim 2\mathbf{Z}$ and some $n \in \mathbf{Z}$). Clearly G' is the normal abelian subgroup $\{(r, 1) \in G\} \cong \mathbf{D}$ and $G_Z = G_{FC}$ equals $\{(0, 1)\}$. Hence G is solvable but not nilpotent and is therefore amenable. It is also obvious that G/G' is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}$ and that the isomorphism is given by $(r, \epsilon 2^{-n})G' \leftrightarrow (\epsilon, n)$ and in terms of the generators we may interpret $n \in \mathbf{Z}$ as the algebraic length of expression (i.e., $\sum_{i=1}^{n} p_j + \sum_{i=1}^{n} q_j$ for $z^{p_1}w^{q_1}z^{p_2} \cdots z^{p_n}w^{q_n}G'$) and $\mathbf{Z}_2$ as the parity of the w-length (i.e., $\sum_{i=1}^{n} q_j$ (mod 2) in the above case). Hence an element belongs to G' if and only if it has algebraic length zero and its w-length (hence also its z-length) has parity zero.

The set $N = \{(r, \pm 1) \in G\}$ is a normal subgroup of G containing G' and isomorphic to the natural semi-direct product of the discrete additive group $\mathbf{D}$ of dyadic rationals with $\mathbf{Z}_2 = \{\pm 1\}$. The set $A = \{(0, s) \in G\} = \{(0, \epsilon 2^{-n}) : \epsilon \in \mathbf{Z}_2, n \in \mathbf{Z}\}$ is a subgroup isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}$ and generated by $w^{-1}z = (0, -1)$ and $wzw^{-1} = (0, 1/2)$. Furthermore $Z = \{(0, 2^{-n}) : n \in \mathbf{Z}\}$ is a subgroup isomorphic to $\mathbf{Z}$ and generated by wzw^{-1}. Clearly G is the semi-direct product of G' and A and also of N and Z.

Theorem 5 of Wilcox [133] shows that G belongs to [MAP] if and only if the set F of elements of the dual group $(\hat{G}') = \hat{\mathbf{D}}$ of [MAP] which have finite orbits under the action of A is dense in $(\hat{G}')$ (or equivalently iff F separates points on G). Formula (2) on page 404 of Hewitt and Ross [31] gives the dual of $\mathbf{D}$ as the compact 2-adic solenoid Σ_2 which is described quite concretely in 10.15 of the same reference. A typical element of Σ_2 has the form $(t, \epsilon) = (t; \epsilon_0, \epsilon_1, \epsilon_2, \cdots)$ where $t \in [0, 1)$ and $\epsilon_n \in \mathbf{Z}_2 = \{0, 1\}$ and the duality is given by

$$(p2^{-n}, (t, \epsilon)) = \exp\left\{ 2\pi i p 2^{-n} \left(t - \sum_{k=0}^{n-1} \epsilon_k 2^k \right)\right\}.$$

The action of $wzw^{-1} \in A$ on $(t, \epsilon) \in (\hat{G}')$ is given by

$$(t, \epsilon_0, \epsilon_1, \epsilon_2, \cdots) \to \begin{cases} (2t; 0, \epsilon_0, \epsilon_1, \cdots) \text{ if } 2t < 1 \\ (2t - 1; \overbrace{1, \cdots, 1}^{n+1}, 0, \epsilon_{n+1}, \epsilon_{n+2}, \cdots) \text{ if } 2t \geqq 1. \end{cases}$$

and ϵ_n is the first nonzero ϵ_k.

By using this action it is easy to see that for any rational number $p/q \in [0, 1)$ with p and q relatively prime and q odd there is an eventually periodic sequence $\epsilon = (\epsilon_0, \epsilon_1, \epsilon_2, \cdots)$ such that $(p/q, \epsilon)$ belongs to F. For instance we have $(1/3, 1, [1, 0,] \cdots)$, $(2/3, 0, 1, [1, 0,] \cdots)$, $(1/5, 1, [0, 1, 1, 0,] \cdots)$ and $(1/7, 1, [1, 1, 0,] \cdots)$ in F where the period is enclosed in square brackets. Using the topology of Σ_2 described in 10.15 of [31] we find that F is dense in $(G')\hat{} \simeq \Sigma_2$. Hence G belongs to [MAP].

Thus this group is a finitely generated solvable discrete group in [MAP], [SIN], [IN], [Um], [Am] and [SSS] and not in any of our other classes except possibly [CR] (its membership in which remains undetermined).

[SIN]: The inclusion [SIN] $\subseteq$ [IN] is obvious for locally compact groups. For connected groups, the equality [SIN] $=$ [MAP] $=$ $[V \times K]$ is the classical Freudenthal-Weil theorem (cf. Dixmier [11], 16.4.6 and Grosser and Moskowitz [21], Theorem 4.3.) Grosser and Moskowitz [23], Theorem 2.9 and Theorem 2.18 show that for almost connected groups [SIN] $=$ [MAP] $=$ [Moore] $=$ $[V \times_\varphi K(\varphi(K) \in [F])]$ where the last symbol denotes the class of semidirect products with $\varphi(K)$ a finite subgroup of Aut(V). Theorem 2.11 of the same reference shows that any neighborhood of the identity in a [SIN] group G contains a compact normal subgroup N with G/N a Lie group so that [SIN]-groups are all projective limits of Lie groups. Another structural characterization (Corollary 2.17 of [23]) shows that a group G belongs to [SIN] if and only if the quotient of G modulo the centralizer of G_0 is compact. Theorem 2.13 of [23] characterizes [SIN] groups as those groups G which are discrete extensions $(V \times K)^D$ of direct products of vector groups V and compact groups K where $V \times K$ is particularly well behaved relative to the inner automorphisms of G.

It is easy to see that [SIN] is stable under taking open subgroups, quotients and finite extensions. (To see the last, consider $\{e\} \to N \to G \to F \to \{e\}$ with F finite and $N \in$ [SIN]. If U is a neighborhood of $\{e\}$ in G then there is some compact N-invariant neighborhood W of $\{e\}$ in $N \cap U$ (since N is open). Then the intersection of the finitely many conjugates of W in G is a G-invariant neighborhood of $\{e\}$ in U.) Grosser and Moskowitz [21], Theorem 2.2 show that [SIN] is stable under local isomorphisms, if the groups are connected.

Wilcox [134] shows that a group $G \in$ [SIN] belongs to $[\mathrm{FIA}]^-$ if and only if it has the form $V \times K^{FC}$ where FC denotes a discrete group with finite conjugacy classes. The same reference contains a proof that a compactly generated [SIN]-group has the form $V \times K^{\mathbf{Z}^m}$ for some nonnegative integer m.

Liukkonen ([79], Corollary to Theorem 4.6 and Theorem 5.3) shows that (at least in the second countable case) (1): a Type I, [SIN] group is a Moore groups and (2) a [SIN] group G has a T_2 dual $\hat{G}$ if and only if it is the projective limit of central groups (and hence is Moore).

In [62] Kaniuth gives several duality results on the connection between properties of G and topological properties of $\hat{G}$ for groups in [SIN], [SIN] $\cap [T_1]$ (Theorem 2) and [SIN] $\cap$ [Am] (Theorem 3). Hauenschild [27] shows (Lemma 6, p. 272) for $G \in$ [SIN] $\cap$ [Am] that Prim(G) is T_2 if and only if $G \in [\mathrm{FIA}]^-$. Theorem 3 of the same reference shows for $G \in$ [SIN] $\cap$ [Am] that $\hat{G}$ contains a nonempty open T_2 subset if and only if $G \in [\mathrm{FD}]^-$ is a finite extension of G_{FC} and that $\hat{G}$ contains an open dense T_2 subset if and only if G is Type I. An example shows that amenability (or a similar condition) is needed in the hypothesis of the first of these results.

[SIN]-groups can be characterized among unimodular groups as those groups G for which the von Neumann algebra $VN(G)$ generated by the left regular representation is finite ([11], 13.10.5). Kaniuth [61] shows that for $G \in$ [SIN], $VN(G)$ is Type I iff $G \in$ [Moore] and $VN(G)$ is Type II_1 iff either G/G_{FC} (which is discrete) is infinite or $G_{FC}{}'^-$ is noncompact.

Mosak [90] notes that $G \in$ [SIN] is equivalent to the existence of a bounded approximate identity in the center of $L^1(G)$. Johnson [56], Proposition 4.1 has noted that a certain cohomology group vanishes for [SIN] groups along with amenable groups and various examples of semi-simple Lie groups.

Example 11 shows that Diagrams 2 and 3 contain all possible inclusions of [SIN] in other classes. Examples 12 and 17 below show that for compactly generated groups we have all inclusions of [SIN] in other classes except possibly [SSS]. Of course discrete groups are [SIN]-groups, and Examples 10, 12 and 14 show again that there are no missing inclusions of [SIN] in other classes except possibly [SSS].

The following example is due to J. B. Fountain et al. [142].

EXAMPLE 16. Let G be the discrete groups with generators $\{x_i : i \in \mathrm{N}\}$ and relations $x_i{}^2 = 1$, $x_i x_k x_j x_k = x_k x_j x_k x_i$ for $i, j < k$. It is shown in the reference cited above that G is locally finite and hence belongs to [PG], [Am], [NF], etc., but that G does not belong to [Her]. Hence it does not belong to [FC] nor to [Type I], etc. It is also shown that G belongs to [MAP] and hence to [SSS], but its membership in $[T_1]$ and [GR] remain undetermined.

The following example is (j) in the tables of Grosser and Moskowitz [23] and is also discussed on p. 518 of [149].

EXAMPLE 17. Let G be the semidirect product $\mathbf{C} \times_{\varphi} \mathbf{Z}$ where $\varphi(n)(\alpha) = e^{in}\alpha$ (or $\varphi(1)$ can be any other rotation through an angle which is an irrational multiple of 2π). Clearly G is a compactly generated solvable [SIN]-groups since $K = \{\alpha : |\alpha| \leqq 1\} \times \{0, \pm 1\}$ is a compact generating set and $\{\alpha : |\alpha| \leqq \epsilon\} \times \{0\}$ is a compact invariant neighborhood of e for each $\epsilon > 0$. However Theorem 5.4 of [23] shows that G is not a [MAP]-group. Since the powers of the compact generating neighborhood given above satisfy $\lambda(K^n) = (2n + 1)n^2\pi$, G belongs to [PG]. Since $G_{FC} = \mathbf{C} \times \{0\}$ holds, G is not an $[\mathrm{FC}]^-$-groups. Applying the Corollary on p. 110 of Auslander and Moore [4] we find that G is not Type I. At the present time we have not determined whether this group belongs to $[T_1]$, [Her], [CR] or [SSS].

[IN]: The theory of [IN]-groups is quite similar to the theory of [SIN]-groups. This reflects the fact that a group belongs to [IN] if and only if it is the extension of a compact group by a [SIN]-group. In fact the intersection of all the compact invariant neighborhoods of the identity in an [IN]-group G is a compact normal subgroup, K. The fact that G/K belongs to [SIN] (which was known to several earlier authors) is proved as Theorem 2.5 of Grosser and Moskowitz [23] and on p. 149 of Liukkonen and Mosak [82]. The converse follows from the fact, which we prove below, that any extension of a compact group by an [IN]-group is an [IN]-group.

Wu and Yu ([137], Theorem 1) and Liukkonen ([79], Corollary 2.2) both prove the beautiful result that a group G belongs to [IN] if and only if G_{FC} is open. Grosser and Moskowitz ([23], Corollary 2.8) show that a connected [IN]-group is the extension K^V of a compact group by a vector group. Since a group K^V obviously belongs to $[\mathrm{FD}]^-$ we conclude $[K^V] = [\mathrm{FD}]^- = [\mathrm{FC}]^- = [\mathrm{IN}]$ for connected groups.

An easy argument with the characteristic function of a compact invariant neighborhood of e establishes the inclusion $[\mathrm{IN}] \subseteq [\mathrm{Um}]$. Since [IN] is closed under finite extensions and since G_{FC} is open for $G \in [\mathrm{IN}]$, an almost connected group belongs to [IN] if and only if G/G_{FC} is finite. Hence for almost connected groups the inclusions $[\mathrm{IN}] \subseteq [\mathrm{Her}]$ and $[\mathrm{IN}] \subseteq [\mathrm{PG}]$ follow from the inclusions $[\mathrm{FC}] \subseteq [\mathrm{Her}]$ and $[\mathrm{FC}] \subseteq [\mathrm{PG}]$ and from the stability of [Her] and [PG] under finite extensions.

The inclusion $[\mathrm{IN}] \subseteq [\mathrm{CCR}]$ for second countable, almost connected groups is part of Theorem 4.6 of Liukkonen [79].

It is easy to see that [IN] is stable under taking closed subgroups, quotients, and finite extensions. (For the last, note that $\{e\} \to N \xrightarrow{\varphi} G \xrightarrow{\psi} F \to \{e\}$ implies that $\sum_{j=1}^{n} x_j^{-1}\varphi(U)x_j$ is a compact invariant neighborhood of e in G if U has the same property in N and $\{\psi(x_j) : j = 1, 2, \cdots, n\} = F$.) Furthermore 5.24(a) of [31] shows that the extension of a compact group by an [IN] group belongs to [IN].

Theorem 2.11 of Grosser and Moskowitz [23] states that any $G \in$ [IN] has a compact normal subgroup N with G/N a Lie group. Liukkonen ([79], Proposition 4.1) uses Thoma's theorem on discrete Type I groups to show that a Type I [IN]-group has G/G_{FC} finite. Theorem 5.2 of the same reference shows that the following conditions are equivalent for a second countable [IN]-group: (1) $\hat{G}$ is T_2, (2) $G \in [\mathrm{FC}]^-$, (3) $G \in$ [Type I], (4) $G \in$ [CCR]. The Corollary on p. 273 of Hauenschild [27] contains essentially the same result.

It is easy to see that the center $L^1(G)_Z$ of $L^1(G)$ is nonempty if and only if $G \in$ [IN]. This was first exploited by Liukkonen and Mosak [82] who showed that $L^1(G)_Z$ is a completely regular, Tauberian, hermitian, Banach *-algebra. Some of these results were proved for groups of the form $K \times A$ by Willcox [135] in 1956.

Example 8 shows that a connected [IN]-group need not be in [SSS] and hence not in [SIN] nor [MAP] but we know no connected [IN]-group which is not in [CR]. Example 11 shows that an almost connected [IN]-group need not be in [CR] nor in $[\mathrm{FC}]^-$. Example 12 shows that a finitely generated discrete [IN]-group need not be in [Am], [NF], [Her], [Type I], $[T_1]$, etc. Example 9 shows that a totally disconnected [IN]-group need not be in [MAP] nor [SIN]. Examples 10, 12 and 14 show that a discrete (necessarily [IN]) group need not be in [CR]. Hence the only inclusion which might be missing from our diagrams are $[\mathrm{IN}] = [\mathrm{FC}]^- \subseteq [\mathrm{CR}]$ for connected groups and $[\mathrm{D}] \subseteq [\mathrm{SSS}]$ both of which have been noted previously.

[PG]: The inclusion $[\mathrm{PG}] \subseteq [\mathrm{EB}]$ is obvious from the definition. The longstanding conjecture that $[\mathrm{PG}] \subseteq [\mathrm{Her}]$ holds for almost connected groups has recently been settled affirmatively [83], [163].

The class [PG] is clearly stable under taking quotients. Furthermore if K is a compact normal subgroup of G, then G has polynomial growth if and only if G/K does. Also if H is a closed normal subgroup with G/H compact then G has polynomial growth if and only if H does. These results are Theorems 1.3 and 1.4 of Guivarc'h [26] (cf. Hulanicki [38] Theorem 3.2 and 3.4). Guivarc'h's paper contains many related results. Some of these are subsumed under the inclusion $[\mathrm{Nil}] \subseteq [\mathrm{PG}]$ which was brought to the writer's attention by R. D. Mosak. To prove this inclusion note that the proof of Proposition 5 in Hulanicki [35] shows that any nilpotent (indeed any locally nilpotent) discrete group belongs to [PG] and the proof of Theorem 3.1 in Hulanicki [38] shows that this implies [Nil] $\subseteq$ [PG] for any group.

EXAMPLE 18. Let G be the Heisenberg group

$$\left[\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{R}\right].$$

Denoting the above element by (x, y, z) we get the multiplication rule $(x, y, z)(u, v, w) = (x + u, y + v, z + w + xv)$. Hence $G' = G_Z = G_{FC} = \{(0, 0, w) : w \in \mathbf{R}\}$. Clearly this is a connected, simply connected, nilpotent Lie group with a central series of length 2. Hence it belongs to [PG] and [EB] by our remarks above, to [CCR] by Dixmier [13], to [Her] by Leptin [71] or Poguntke [104] and hence also to [EB], [NF], [Am] and [Um]. As pointed out previously $MM(L^1(G)) = F*(G)$ holds for $G \in$ [CCR] $\cap$ [Her]. Hence the explicit calculation of the topology of $\hat{G}$ obtained by Dixmier [10] allows us to show below that G belongs to [CR] but not to [SSS]. Since G is connected and not in $[FC]^-$ it is in no other class. Hence we have determined the membership of G in all of our classes.

We will now show the membership of G in [CR] and its nonmembership in [SSS]. It is easy to see that the extension $G = (G')^{\mathbf{R}^2}$ does not split. However G is the semidirect product $A \times_\varphi H$ with $A = \{(0, y, z) : y, z \in \mathbf{R}\}$ and $H = \{(x, 0, 0) : x \in \mathbf{R}\}$. From this (following [130], p. 442) we find that the continuous unitary irreducible representations of G are the one dimensional representations

$$U^{s,t}_{(x,y,z)} = e^{-i(sx+ty)} \quad \forall s, t \in \mathbf{R}$$

and the ∞-dimensional representations on $L^2(\mathbf{R})$ given by

$$U^r_{(x,y,z)} h(t) = e^{ir(z+ty)} h(t + x) \begin{cases} \forall r \in \mathbf{R} \sim \{0\}; \\ \forall h \in L^2(\mathbf{R}). \end{cases}$$

Haar measure on G is merely three dimensional Lebesgue measure. Hence for $f \in L^1(G)$ we have

$$\tilde{U}_f^{s,t} = \hat{f}(s, t, 0)$$

where $(\hat{\ })$ indicates the Fourier transform. Similarly

$$\tilde{U}_f^r h(t) = \int \int \int e^{ir(z+ty)} f(x, y, z) h(t + x)\, dx dy dz$$

$$= \int \hat{f}_x(-rt, -r) h(t + x)\, dx$$

for $h \in L^2(\mathbf{R})$, $r \in \mathbf{R} \sim \{0\}$ where $f_x(y, z) = f(x, y, z)$. Now Dixmier's calculation of $\hat{G}$ shows that $MM(L^1(G))$ is homeomorphic to $\mathbf{R}^2$ *via* $(s, t) \mapsto \mathrm{Ker}(\tilde{U}^{s,t})$ and our remarks on [CR] and [SSS] follow easily. We remark that $\hat{G}$ is not Hausdorff since any net $\{U^{r_\alpha}\}_{\alpha\in A}$ with $\{r_\alpha\}_{\alpha\in A}$ converging to zero, converges to each point of $\{U^{s,t} : s, t \in \mathbf{R}\}$.

There is a non-nilpotent group which belongs to exactly the same classes as the Heisenberg group. This is $\mathbf{C} \times_\varphi \mathbf{T}$ where $\varphi(\zeta)\mu = \zeta\mu$ for $\zeta \in \mathbf{T}$ and $\mu \in \mathbf{C}$. Since it is a compact abelian extension of an abelian group it belongs to [PG] and [Her] by the stability properties discussed under these headings. The discussion of its dual given on p. 70 of [77] shows one quickly that it belongs to [CR] but not to [SSS].

The next example is usually called the Mautner group since it was discovered by Mautner.

EXAMPLE 19. Let $G = \mathbf{C}^2 \times_\varphi \mathbf{R}$ where $\varphi(t)(\mu, \nu) = (e^{2\pi it}\mu, e^{it}\nu)$. Then G is a connected solvable Type R Lie group and hence belongs to $[T_1]$ by Theorem 2, p. 171 of Auslander and Moore [4] but by the Corollary on p. 110 of the same reference G is not Type I. Hence G is not in [CCR] nor [IN] etc., but it is in [PG], [Am], [Her], [NF] and [Um]. Its membership in [CR] and [SSS] remains undetermined.

Among connected [PG]-groups, Examples 8 and 18 are not in [SSS] and Example 19 is not Type I. Example 11 is an almost connected [PG]-group which is not in [CR] but we have no connected example of this phenomenon. We know no compactly generated [PG]-group which is not $[T_1]$. Among discrete [PG]-group, Examples 10 and 14 are not in [CR], Example 6 is not in [MAP] and Example 16 is not in [Her]. We know of no discrete [PG]-groups which are not in [SSS] or $[T_1]$. In fact we know of no [PG]-group at all which is not in $[T_1]$.

[EB]: The theory of exponentially bounded groups is quite similar to the theory of groups with polynomial growth and the two classes coincide under many (possibly all) additional hypotheses. The inclusion [EB] $\subseteq$ [Um] can be seen by noting that if $x \in G$ satisfies $\Delta(x) > 1$ and U is a compact neighborhood of x then $\lambda(U^{n+1}) \geqq \lambda(Ux^n) = \Delta(x)^n\lambda(U)$. The inclusion [EB] $\subseteq$ [NF] is immediate since if x and y generate a free subsemigroup S and U is a compact neighborhood of the identity with $uU \cap vU = \varnothing$ for $u, v \in S$, $u \neq v$, then $\lambda((xU \cup yU)^n) \geqq 2^n\lambda(U)$ since $(xU \cup yU)^n \supseteq \bigcup\{uU : u \in S, \text{ length } (u) = n\}$. The inclusion [EB] $\subseteq$ [Am] follows from Theorem 3.6.2 of Greenleaf's book [19] and the following argument which shows that an exponentially bounded group satisfies the topological Folner condition. For any compact set K let U be a compact neighborhood containing $K \cup K^{-1}$. Then for any positive integer n, and any $x \in G$ we have

$$\begin{aligned} 0 &\leqq \lambda(xU^n \Delta U^n) = \lambda(xU^n \sim U^n) + \lambda(U^n \sim xU^n) \\ &= \lambda(xU^n \sim U^n) + \lambda(x^{-1}U^n \sim U^n) \leqq 2\lambda(U^{n+1} \sim U^n) \\ &= 2(\lambda(U^{n+1}) - \lambda(U^n)) \end{aligned}$$

which implies

$$\liminf \frac{\lambda(xU^n \Delta U^n)}{\lambda(U^n)} \leqq \liminf 2 \left(\frac{\lambda(U^{n+1})}{\lambda(U^n)} - 1 \right) = 0.$$

Hence for any $\epsilon > 0$, U^n will satisfy the topological Folner condition for sufficiently large n. (Adel'son-Vel'skiĭ and Šreĭder in [1] seem to have been the first authors to consider the condition [EB]. For discrete groups they showed that it implies amenability. Later Hulanicki in [35] showed that for discrete groups [EB] implies that all functions f with finite support in $\ell^1(G)$ satisfy $\gamma(f) = \rho(f)$.)

The known stability properties of the class [EB] are exactly the same as those of [PG] and the references are the same also.

Milnor [86] and Wolf [136] have shown that a finitely generated discrete solvable group is exponentially bounded if and only if the group is a finite extension of a nilpotent group. A number of properties of exponentially bounded groups and groups with polynomial growth are given in Jenkins [53], [54], [55] and Hulanicki [40], [41], [42] and Hulanicki, Jenkins, Leptin and Pytlik [43].

The counterexamples for groups in [EB] are exactly the same as for groups in [PG], and the main question concerns the existence of compactly generated, or even arbitrary, groups in [EB] but not in [PG].

[CCR]: A C^*-algebra is said to be CCR if the range of each of its irreducible $*$-representations is exactly the ideal of compact operators on the representation space. Since $*$-representations of C^*-algebras are always continuous, it is obvious from the definition given earlier that G belongs to [CCR] if and only if $C^*(G)$ is CCR. Now the inclusions [CCR] $\subseteq$ [Type I] and [CCR] $\subseteq$ [T_1] follow from Theorems 5.5.2 and 4.1.11 in Dixmier's book [11]. The fact that [CCR] $=$ [Type I] $\cap$ [T_1] is considerably more difficult to prove. Theorem 4.6.4 in Sakai's book [117] shows the difficult fact that the range $T_{\mathcal{U}}$ of each irreducible $*$-representation T of a Type I C^*-algebra $\mathcal{U}$ contains a nonzero compact operator (for a separable C^*-algebra a more lucid proof is given as Theorem 9.1 in [11]). Theorem 4.3.7 of [11] thus shows that $T_{\mathcal{U}}$ includes the ideal of all compact operators on the representation space. However if Prim($\mathcal{U}$) is T_1 then $T_{\mathcal{U}}$ is topologically simple so $T_{\mathcal{U}}$ equals the ideal of all compact operators. Applying this to $C^*(G)$ we see [CCR] $=$ [Type

I] $\cap$ $[T_1]$. Theorem 9.1 in Dixmier's book [11] shows that a second countable group G (for which $C^*(G)$ is necessarily separable) with $\hat{G}$ T_0 is Type I. Hence a second countable group with $\hat{G}$ T_1 is CCR ([11], 4.7.15).

The class [CCR] is stable under taking quotients and open subgroups by [11] 4.2.4 and the existence of maps between the C^*-algebras in these cases. Also [CCR] is stable under compact extensions (see, e.g., [145] Proposition 4.3). Lipsman ([76], Theorem 3.2) and Moore ([88], remark following Proposition 2.2) have shown that a projective limit of CCR groups is CCR.

It has been known for some time that motion groups (i.e., semi-direct products $A \times_\varphi K$ with A abelian and K compact), connected semi-simple Lie groups and nilpotent Lie groups are CCR. For proofs see 4.5.2.1, 4.5.2.11 and 4.5.7.2 (or 4.5.7.3) in Warner's book [130], 15.5.6 in Dixmier's book [11], Dixmier [13] and Kirillov [66]. Since [CCR] is stable under projective limits, almost connected groups satisfy [Nil] $\subseteq$ [CCR] ([76], p. 461). Moore in Chapter 5 of [4] shows that a connected solvable Lie group which is in [CCR] must be Type R (Theorem 1′, p. 174) and that a Type R connected, simply connected, solvable Lie groups is in [CCR]. (A Lie group is said to be Type R if all the eigenvalues of its adjoint representation on its Lie algebra are of absolute value one, or equivalently the roots of its Lie algebra are purely imaginary.)

Recent results of Lipsman [78], Pukanszky [107], and Moore and Rosenberg [89] have given considerable new insight into the class [CCR]. In particular the results of Pukanszky's Theorem 2 are definitive for when a connected and simply connected Lie group is in [CCR]. However these results are too complicated for us to state here.

The above results give many examples. Before singling out one characteristic example we note that Examples 10 and 11 are respectively discrete and almost connected [CCR]-groups which are not in [SSS]. We know of no connected [CCR]-group which is not in [CR] nor of any [CCR]-group which is not unimodular. An almost connected example of the latter phenomenon would be most interesting.

EXAMPLE 20. Let G be the group $SL(2, \mathbf{R})$ of 2×2 real matrices of determinant 1. This is a noncompact, semi-simple Lie group with finite center $\{\pm I\}$. Modulo its center it is simple. Hence it belongs to [CCR] but not to [Am] ([19], Theorem 3.3.2) nor to [Her]([50], Proposition 4.5). These results already determine its membership in all of the classes except [CR] and [SSS] and we do not know whether it belongs to either of these classes. Since G is not hermitian $L^1(G)$ may have maximal modular ideals unrelated to representations of G. Since it is well known that $SL(2, \mathbf{R})$ has no finite dimensional representations (and since it is [CCR]) there must be many such ideals in order for G to belong to [SSS] or fail to belong to [CR].

[Type I]: The only inclusions of this class in larger classes shown on our diagrams are in Diagram 4. These results all follow easily from Thoma's theorem [**123**] that a discrete Type I group must be a finite extension of an abelian group.

The class [Type I] is stable under taking open subgroups ([**11**], 4.3.5, Liukkonen [**79**] Proposition 4.5 or Kallman [**58**] proposition 2.4), continuous homomorphic images (compose the representations of the image with the homomorphism) and extensions by compact groups (see Lipsman [**77**], p. 79 for this result of Mackey). Furthermore Kallman [**58**] has shown that if G is almost connected (Corollary 6.4) or a Lie group (Theorem 6.1) and G contains a closed Type I subgroup H with G/H compact then G is Type I. (Errors in the published version of this important result have apparently now been removed.) The class [Type I] is also stable under certain regular extensions (see [**84**] and [**4**]). The same references given under [CCR] show that the projective limit of Type I groups is Type I.

Lipsman ([**76**], Theorem 4.1) notes that if an almost connected group G has G_{rad} nilpotent then G is Type I. Liukkonen ([**79**], Proposition 4.1 and Theorem 4.6) shows that a Type I [IN]-group G is CCR and has G/G_{FC} finite. Also Proposition 3.1 and Theorem 3.6 show that a Type I $[\mathrm{FC}]^-$-group is $[\mathrm{FD}]^-$, has $\hat{G}$ T_2 and can be written as $K^{V \times A}$ where A is discrete abelian. Finally the Corollary to 4.6 of the same reference deduces that a Type I [SIN]-group is in [Moore] (cf. [**91**], Theorem 5.12 and [**61**] Satz 1).

Example 20 shows that a connected Type I group need not be hermitian nor amenable. Examples 8, 18, (and 21 below) show that a connected Type I group need not be strongly semi-simple. We know no example of a connected Type I group which is not completely regular, but Example 11 is almost connected and satisfies these conditions. Examples 21 and 22 below also show that a connected Type I group need not be unimodular nor $[T_1]$. Thus the only inclusion of [Type I] in any other class which might be missing from our diagrams is [Type I] $\subseteq$ [CR] for connected groups, and we doubt that this is true.

EXAMPLE 21. This example is the affine group of the real line or the "$ax + b$ group". It can be represented as the group of matrices

$$\left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x, y \in \mathbf{R};\ x > 0 \right\}$$

but we prefer to represent it as the semi-direct product $\mathbf{R} \times_\varphi \mathbf{R}$ with multiplication

$$(x, y)(u, v) = (x + e^y u, y + v).$$

Clearly this group is a solvable, connected Lie group. Since it is not unimodular (Hewitt and Ross [31] 15.17 g) or since it is not Type R (its Lie algebra has a real root) it is in neither [EB], [PG], [NF], nor $[T_1]$ by remarks under [NF] below. On the other hand G is in [Her] by Leptin [74] which destroys the conjecture that [Her] equals [PG] for connected groups. This is the simplest example of a Type I group which is not in [CCR] since it does not belong to $[T_1]$. Since G is solvable it is in [Am]. Mueller-Roemer [94] determines the maximal modular ideals of $L^1(G)$ explicitly and thus shows that G is completely regular but not strongly semi-simple. He also shows that every closed proper ideal of $L^1(G)$ is included in some maximal modular ideal of $L^1(G)$. These results together with the results in Diagram 3 determine the membership of C in each of our classes.

EXAMPLE 22. Let G be the semidirect product $\mathbf{R}^2 \times_\varphi \mathbf{R}$ where $\varphi(t)$ is described by the matrix $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}$. This is a connected, simply connected, solvable Lie group which is not Type R and hence does not belong to [PG], [EB], [NF] nor $[T_1]$. However Haar measure is merely three dimensional Lebesgue measure and G is unimodular. Since it is a regular semidirect product of abelian groups, G is Type I. Furthermore it is hermitian by [74] and amenable since it is solvable. We have recently shown that G belongs to [CR] but not to [SSS].

EXAMPLE 23. Let G be the semidirect product $H \times_\varphi \mathbf{R}$ where H is the Heisenberg group of Example 18, and $\varphi(t)$ is defined by $\varphi(t)(x, y, z) = (e^t x, e^{-t} y, z)$. A typical element (x, y, z, t) with $(x, y, z) \in H$ and $t \in \mathbf{R}$ can be represented by the matrix

$$\begin{pmatrix} e^t & x & e^t z \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{pmatrix}$$

The center of G is just the image of the center of H, $\{(0, 0, z, 0) : z \in \mathbf{R}\}$. Clearly G is a four dimensional connected solvable Lie group. It is shown in [141] that G is neither hermitian nor Wiener. Since the quotient of G modulo its center is the group of Example 22, we see that G is neither in $[T_1]$ nor [CCR]. On the other hand G is amenable since it is solvable, and is Type I since it is an exponential solvable group [77, VI.A]. Furthermore G is unimodular since four dimensional Lebesgue measure is both left and right Haar measure. These remarks determine the membership of G in all the twenty classes except [CR] and [SSS].

$[T_1]$: Moore and Rosenberg [89] in a remark at the beginning of § 7 show that $[T_1]$ is included in [Um] for almost connected groups. Their Theorem 4 together with the slight but crucial extension due to Poguntke [103] shows that an almost connected group belongs to $[T_1]$ if and only if it is the projective limit of Lie groups each of which has a Lie algebra which is the direct sum of a semi-simple Lie algebra and a Lie algebra of Type R. (Such a Lie group is said to be Type R on its radical.) This shows that whether an almost connected Lie group belongs to $[T_1]$ depends only on its Lie algebra (i.e., is invariant under local isomorphism) whereas the Type I property is not determined by the Lie algebra. Essentially the same results for connected Lie groups were obtained by Pukanszky [107] and for connected algebraic groups by Lipsman [78].

In another direction Moore and Rosenberg [89] show that a discrete, finitely generated, amenable group is in $[T_1]$ if and only if it is a finite extension of a nilpotent group. (Our replacement of "solvable" in the hypothesis of their Theorem 5 by "amenable" is justified by their remark after the proof of Theorem 5 and by [103].) It should be noted that these are exactly the groups in [Her] and [PG] at least when considering finitely generated solvable groups.

As already noted $[T_1]$ is stable under local isomorphisms among connected groups. Theorem 3 of [89] and the main result of [103] show that G is stable under taking finite extensions or open subgroups of finite index. Every closed ideal of a C^*-algebra is the intersection of primitive ideals ([110], Theorem 4.9.6) so Prim(G) is T_1 if and only if each primitive ideal is a maximal closed ideal and hence if and only if the range of each irreducible *-representation of $C^*(G)$ is topologically simple. Hence the argument used previously to note that [CCR] is stable under quotients shows that $[T_1]$ is stable under quotients. Thus Lipsman [76] Theorem 2.1 or Moore [88] Proposition 2.2 shows that a projective limit is $[T_1]$ if and only if the approximating groups are $[T_1]$.

Example 20 shows that a connected $[T_1]$-group need not be hermitian nor amenable, and Examples 8 and 18 show that such a group need not be strongly semi-simple. Example 19 is a connected $[T_1]$ group which is not Type I, but we do not have an example of a connected $[T_1]$ group which is not completely regular. Example 11 is an almost connected $[T_1]$-group which is not completely regular. Examples 10 and 14 are discrete $[T_1]$-groups which are not completely regular. We know no discrete $[T_1]$-group which is not in [Am], [NF], [Her], [PG], nor [SSS].

[Um]: Our diagrams show no inclusion of [Um] in a larger class.

An open subgroup or a closed normal subgroup of a unimodular group is unimodular since in the first case the Haar measures agree and in the second case 15.23 of [31] applies. Leptin and Robertson [75] show that if N is a compact normal subgroup of G or if N is G_Z, then G is unimodular if and only if G/N is unimodular. Proposition 8 of Kallman [59] shows that a compact extension of a unimodular group is unimodular. (This result subsumes earlier results in [75] and in [5].)

A number of published theorems have unimodularity in their hypotheses because the proofs depend on Hilbert algebra techniques. (For interesting examples see Peters [98].) For a unimodular group G, $C_{00}(G)$ provided with convolution multiplication and the inner product of $L^2(G)$ is a Hilbert algebra, but if G is not unimodular one only gets a generalized Hilbert algebra or Tomita algebra. (For the theory of such algebras see M. Takesaki, [147].) Since the theory of Tomita algebras is newer and less developed than the theory of Hilbert algebras, it seems likely that the proofs of some (but probably not all) of these results can be generalized.

M. K. Smith [146] notes that if the regular representation of a unimodular group G weakly contains an n-dimensional ($n < \infty$) representation then G/G_{FC} has at most n^2 elements.

Examples 8 and 18 show that a connected unimodular group need not be strongly semi-simple. Examples 19, 20 and 22 show that a connected unimodular group need not belong to [Type I], [CCR], [Her], [NF], [Am] nor [T_1]. We have no example of a connected unimodular group which is not completely regular but Example 11 is an almost connected example.

[Her]: The fact that discrete hermitian groups have no free subsemigroups on two generators was discovered by Jenkins ([46], Theorem 5.1), who also gave an improved proof in [48]. In [50] Jenkins has also shown that no noncompact, connected semi-simple Lie group with finite center can be hermitian. In fact he even shows that a connected reductive Lie group with a noncompact semi-simple component is nonhermitian. We give an argument, based on this result, which shows that any almost connected hermitian group G is amenable. Let K be a compact normal subgroup of G such that $L = G/K$ is an almost connected Lie group. Let H be the connected component of the identity in L/L_{rad}. The stability properties of [Her] given below show that H/H_Z is hermitian. Since it is also a connected semi-simple Lie group with trivial center, Jenkins' result shows that it is compact. The stability properties of the class [Am] now show that G itself is amenable. Barnes

[161] extends Gangolli's result [16] that the semi-direct product $A \times_{\varphi} K$ of an abelian group A by a compact group K is hermitian. Ludwig [163] shows that any compact extension of a nilpotent group is hermitian. This generalizes results in [37], [63] (Lemma 4), [71] and [104]. Leptin [74] proves two main theorems on semi-direct product extensions of vector groups which show the symmetry of many connected solvable Lie groups, including all but one of those of dimension four or less. The exceptional group (Example 23) is not hermitian. Leptin ([72], p. 277) notes that discrete finitely generated solvable groups are hermitian if and only if they are finite extensions of nilpotent groups, and hence if and only if the have poylnomial growth.

Leptin [70] has shown that [Her] is stable under finite extensions. Wichman [131] extends and simplifies the proof. Leptin and Poguntke [141] show that [Her] is stable under forming semi-direct product extensions by compact groups. By considering tensor product algebras, one can show that the direct product of a hermitian group with an abelian group is hermitian. Furthermore [Her] is stable under the formation of open subgroups since a closed *-subalgebra of a hermitian Banach *-algebra is hermitian ([110], 4.1.7 and 4.1.10). Finally we note that [Her] is stable under formation of quotients since the quotient of a hermitian Banach *-algebra is obviously hermitian.

For related results we mention Anusiak [3] (see the Theorem on p. 290 and Proposition 5.5).

Example 21 is a connected hermitian group which is not unimodular nor [NF]. Examples 8, 18 and 21 are connected hermitian groups which are not strongly semi-simple. Example 19 is a connected hermitian group which is not Type I. We know of no connected hermitian group which is not completely regular, but Example 11 is an almost connected example. It seems reasonable to guess that all hermitian groups are amenable. Examples 10 and 14 are discrete hermitian groups which are not completely regular, and Example 6 is a discrete hermitian group which is not [MAP]. However we know of no discrete hermitian group which is not strongly semi-simple and of polynomial growth.

[NF]: Jenkins [52] shows that a connected [NF]-group has Type R approximating Lie groups and hence belongs to [PG]. Consideration of the connected component shows that the same is true for almost connected [NF]-groups which are therefore in $[T_1]$ by Moore and Rosenberg [89]. (The second countability hypothesized in [52] is not needed in the proof.) The very recent work of J. Ludwig [83] shows $[NF] \subseteq [Her]$ for connected groups.

The class [NF] is obviously stable under formation of closed subgroups and continuous homomorphic images (cf. [49], § 1).

In [49] Jenkins gives a number of conditions which imply that a group does not belong to [NF]. Counterexamples for connected or almost connected [NF]-groups are the same as for [PG]-groups. We know of no compactly generated [NF]-groups not in [Am], [Um], [PG], [Her] or [T_1], but Example 14 is not Type I. Examples 10, 14, 16 and 6 are discrete [NF]-groups with the first two not in [CR], the third not in [Her] and the last not in [MAP], but we know of no discrete [NF]-group not in [Am], [PG], [SSS] or [T_1]. In fact [NF] = [PG] may hold for general groups as far as we know.

[CR]: Our diagrams show no inclusion of [CR] in any other class. However, under some strong additional hypotheses [CR] is included in [FC]$^-$. In particular Kaniuth ([63], Corollary 2) shows that if G is a [SIN]-group which is a finite extension of either a closed nilpotent subgroup or G_{FC}, then G is completely regular if and only if G belongs to [FC]$^-$. From Corollary 1 of the same paper one can also conclude that a Moore group is completely regular if and only if it belongs to [FC]$^-$. (This also follows from Corollary 3 of Peters [98].) Finally Theorem 1 of [63] shows that a group in [Am] $\cap$ [SIN] $\cap$ [Her] $\cap$ [CR] must have $(G/G_{FC})_{FC}$ trivial.

The only obvious stability property of [CR] is that the quotient of a completely regular group is completely regular. This follows from [110] 2.7.2 and the canonical homomorphism of $L^1(G)$ onto $L^1(G/N)$. Mueller-Roemer ([139], 2.11) shows that contracting extensions of completely regular groups are completely regular.

Example 21 is a connected completely regular group which is not in [SSS], [NF], [Um] nor [T_1]. We know of no connected or almost connected completely regular group which is not Type I, but Example 7 is a compactly generated example. Among discrete completely regular groups Examples 5 and 6 are not in [FD]$^-$ and [MAP] respectively. We know of no discrete completely regular group not in [NF], [PG], [FC]$^-$, [SSS], or [T_1]. We know of no completely regular group which is not amenable and hermitian. This lack of examples results at least in part from our inability to determine whether various groups belong to [CR]. In particular Examples 15, 16, 17, 19, 20 and 23 might provide counterexamples if all of their properties were determined.

[Am]: Our diagrams indicate no inclusion of [Am] in any larger class. An easy proof that any solvable group is amenable follows from 1.2.1 and 1.2.6 in Greenleaf's book [19]. It has long been conjectured that any discrete nonamenable group must contain a free group on two generators as a subgroup. J. Tits [148] has shown that a finitely generated linear group (i.e., a group which can be faithfully embedded in some matrix group) either is a finite extension of a solvable group (and hence amenable) or contains a free group on two generators. Furthermore Rickert ([**111**], [**112**]) has shown that an almost connected group is amenable if and only if it contains no free group on two generators. The important result that an almost connected group G is amenable if and only if G has G/G_{rad} compact (and hence is the extension of a connected solvable group by a compact group) is 3.3.3 in [**19**].

The stability properties of the class of amenable groups are particularly well understood. § 2.3 of Greenleaf's book [**19**] contains proofs that [Am] is closed under formation of continuous homomorphic images, closed subgroups, extensions by amenable groups and increasing unions.

Many alternative definitions of amenability are known involving the existence of invariant means on various spaces of functions, topological invariance, nets converging to invariance, Reiter's condition, fixed point properties, deforming bounded representations into unitary ones, weak containment of all irreducible representations in the left regular representation (mentioned previously), Følner's condition and ergodic properties. For all of these we refer the reader to Greenleaf's book [**19**]. However we wish to briefly mention two other more recent characterizations. Johnson in [**56**] develops a cohomology theory for Banach algebras which is just Hochschild's cohomology theory for algebras with suitable topological restrictions. He shows that a locally compact group G is amenable if and only if the cohomology groups of $L^1(G)$ all vanish. This suggests calling a Banach algebra *amenable* iff all its cohomology groups vanish. Along the same lines Bunce [**9**] has very recently shown that a discrete group is amenable if and only if the C^*-algebra generated by its left regular representation is amenable. (This incidentally provided the first examples of non-amenable C^*-algebras.) We refer the reader to Section IV of Guivarc'h [**26**] for other results on amenability.

Example 21 is a connected, amenable group which is not in [NF], [Um] nor [SSS] and Example 19 is a connected amenable group which is not Type I. We have no connected amenable group which is not completely regular, but Example 11 is an almost connected example. Example 23 is a connected amenable group which is not hermitian. Finally Examples 6, 10, 14, 15 and 16 provide examples of discrete amenable groups which do not belong to [MAP], [CR], [Her], [NF] nor [T_1] but we have no example which is not strongly semi-simple.

[SSS]: Our diagrams show no inclusions of [SSS] in larger classes. No general stability properties of [SSS] occur to us either.

Kaniuth [**63**] proves some interesting results related to the class [SSS].

Define G_{FC}^1 to be G_{FC} and for $n > 1$ define G_{FC}^n inductively by $G_{FC}^n = \{x \in G : xG_{FC}^{n-1} \in (G/G_{FC}^{n-1})_{FC}\}$. Then G is said to be FC-nilpotent if $G_{FC}^n = G$ for some finite n. Kaniuth shows that FC-nilpotent groups in [SIN] $\cap$ [Her] belong to [SSS]. ($C^*(G)$ is stongly semi-simple even if the hypothesis $G \in$ [Her] is dropped.) Hence if $G \in$ [SIN] contains a closed nilpotent subgroup of finite index or has G/G_{FC} finite then G belongs to [SSS]. Furthermore since $G \in$ [Her] $\cap$ [SIN] $\cap$ [T_1] implies Prim(G) $\subseteq MML^1(G)$, it implies $G \in$ [SSS]. Notice that each of these results generalizes [FIA]$^-$ $\subseteq$[SSS]. Recent results of Müller-Roemer ([139], 2.11) show that contracting extensions never belong to [SSS].

We remark that [SSS] $\cap$ [Her] $\cap$ [CCR] $\subseteq$ [MAP] holds since $MM(L^1(G))$ equals $F_*(G)$ in this case so [24] Theorem 3.1 can be applied. Hence for almost connected groups [In] $\cap$ [SSS] $\subseteq$ [Moore] holds and for connected groups [IN] $\cap$ [SSS] $\subseteq$ [Z] holds. For the same reason a connected nilpotent group in [SSS] must belong to [Z].

The only interesting counterexample we know for connected or almost connected strongly semi-simple groups is Example 11 which is an almost connected strongly semi-simple group which is not completely regular. Among compactly generated strongly semi-simple groups, Example 13 is not in [Am], [NF], [Type I] nor [T_1] and Example 7 is not in [MAP] nor in [Type I]. For discrete strongly semi-simple groups Examples 5, 6, 10, 12, 14, 15 and 16 rule out all additional implications.

5. **Additional comments.** Besides the twenty classes considered in the last section many other classes each of which includes [K] $\cup$ [A] have been or could be considered. By intersecting various classes one gets classes some of which may prove interesting (e.g., [Tak] = [FD]$^-$ $\cap$ [MAP] and [FIA]$^-$ = [FC]$^-$ $\cap$ [SIN]). Results from [114]; [79] and [34] which we have quoted previously show [FIA]$^-$ $\cap$ [FD]$^-$ = [FD]$^-$ $\cap$ [SIN], [FIA]$^-$ $\cap$ [Moore] = [FIA]$^-$ $\cap$ [Type I] = [FD]$^-$ $\cap$ [Moore] = [FC]$^-$ $\cap$ [Moore] = [Tak], [FIA]$^-$ $\cap$ [MAP] = [FC]$^-$ $\cap$ [MAP], [FD]$^-$ $\cap$ [Type I] = [FC]$^-$ $\cap$ [Type I], [MAP] $\cap$[SIN] = [MAP] $\cap$ [IN], [SIN] $\cap$ [Type I] = [Moore], and [IN] $\cap$ [Type I] = [IN] $\cap$ [CCR]. However our twenty three examples are enough to show that there are no nonobvious equations connecting these classes, our previous twenty classes, [FD]$^-$ $\cap$ [SIN] and [MAP] $\cap$ [Type I].

The class [MAP] $\cap$ [SIN] which lies properly between [Moore] and [MAP] has been called [Mur] (for Murakami) in [113]. Heyer [32] has noted that the class [Chu] of groups satisfying the Chu duality theorem lies properly between [Moore] and [MAP] but he provides an example showing [Mur] $\not\subseteq$ [Chu]. He shows that [Chu] is stable under projective limits, finite direct products and finite extensions. Two other classes lying between [Moore] and [MAP] are studied by Poguntke in [**102**].

The additional classes which have received the most intensive consideration deal with the ideal structure of $L^1(G)$ and $C^*(G)$. Attempts to generalize the Wiener Tauberian theorem have led to the consideration of the following classes.

[W] = Wiener groups = Every proper closed ideal of $L^1(G)$ is included in some ideal of Prim$*L^1(G)$) (= Prim(G)).
[Taub] = Tauberian groups = Every proper closed ideal of $L^1(G)$ is included in some ideal of $MM(L^1(G))$.
[WW] = Weakly Wiener groups = Every proper closed ideal of $L^1(G)$ is included in some ideal of Prim($L^1(G)$).

Notice that [Taub] $\subseteq$ [WW] holds and that a discrete group G necessarily belongs to both classes since each proper ideal of $\ell^1(G)$ is modular. Since primitive ideals of $L^1(G)$ belong to Prim(G) for hermitian groups, we have [Taub] $\cap$ [Her] $\subseteq$ [*WW*] $\cap$ [Her] $\subseteq$ [W]. Hence in particular [Her] $\subseteq$ [W] holds for discrete groups. Moreover, Ludwig [**163**] shows that all solvable discrete groups belong to [W]. The same reference shows that [PG] is included in [WW] and that almost connected [PG] groups and arbitrary compact extensions of nilpotent groups belong to [W]. These results imply [FC]$^-$ $\subseteq$ [W] (which was previously proved in [**43**]) and improve various results of [**72**]. In [**72**] Leptin shows that all semi-direct products of abelian groups belong to [W]. Kaniuth and Steiner ([**65**], Theorem 4) and Liukkonen and Mosak ([**82**], Remark 2.8) show [FIA]$^-$ $\subseteq$ [Taub]. He proves that all connected solvable Lie groups of dimension $\leqq 4$ with one exception, and that various other counterexample groups such as our Example 19 (the Mautner group) belong to [W]. The exceptional group, Example 23, shows that not all connected solvable groups belong to [W]. Gangolli [**16**] shows that the semidirect product $A \times_\varphi K$ of an abelian group A and a compact group K belongs to [Her] $\cap$ [WW] $\subseteq$ [W]. In [**94**]Mueller-Roemer shows that the affine group of the real line belongs to [Taub] (and hence to [W] since it is now known to be hermitian). In [**139**] he extends this result to show that a contracting extension N^H belongs to [Taub], [WW] or [W] if and only if H does. Hauenschild and Kaniuth ([**28**], Theorem 5) show [Am] $\cap$ [SIN] $\cap$ [Her] $\cap$ [SSS] $\subseteq$ [Taub] $\cap$ [W]. From this they conclude that a [SIN] group G which is a finite extension of either G_{FC} or a closed nilpotent subgroups is in [Taub] $\cap$ [W]. Similarly Moore groups are in [Taub] $\cap$ [W].

It is obvious that [W], [Taub] and [WW] are closed under taking quotients since $L^1(G/N)$ is a quotient of $L^1(G)$. Leptin ([**72**], Corollary to Theorem 6) shows that a connected group belongs to [W] if and only if its approximating Lie groups do.

Among our twenty-three examples all but Examples 16, 20, and 23 belong to [W]. Example 20 does not belong to [W] since no non-compact semi-simple Lie group does ([72] appendix due to M. Duflo). Examples 1 to 8, 10 to 16 and 21 belong to [Taub] and we have not determined this property for the rest.

Since the T_0 and T_1 separation axioms on $\hat{G}$ lead to such interesting conditions on G it is natural to consider the T_2 separation axiom. In fact we will consider this as a property of Prim(G) instead of $\hat{G}$, and we introduce the following classes defined in terms of the ideal structure of $C^*(G)$.

$[T_2]$ = Groups for which Prim(G) is Hausdorff.
[C*CR] = Groups for which $C^*(G)$ is completely regular.
[C*SSS] = Groups for which $C^*(G)$ is strongly semi-simple.

Liukkonen and Mosak [81] show that $[\mathrm{FC}]^- \subseteq [T_2]$ holds at least for σ-compact groups. Mosak ([91], Corollary 5.3) had previously noted that $[\mathrm{FIA}]^-$-groups belong to $[T_2]$ and satisfy $MM(C^*(G)) = \mathrm{Prim}(G)$ without any countability restriction. Liukkonen ([79], Theorem 3.6 and 3.7) had also shown $[\mathrm{FC}]^- \subseteq [T_2] \cap [\mathrm{CCR}]$ for almost connected or Type I second countable groups. Theorems 5.2 and 5.3 in the same paper note that a Type I [IN]-group in $[T_2]$ belongs to $[\mathrm{FC}]^-$ and a Type I [SIN]-group in $[T_2]$ is even a projective limit of [Z]-groups. The first result in the last sentence is also obtained by Hauenschild ([27], Korollar p. 273) who also considers in the same paper the existence of open T_2 subsets of $\hat{G}$.

For groups in $[\mathrm{SIN}] \cap [T_1]$ we have $[\mathrm{C^*CR}] \subseteq [T_2]$ since $MM(C^*(G)) = \mathrm{Prim}(G)$ holds in this case. For a discrete group in $[T_1]$ we have $[C^*\mathrm{CR}] = [T_2]$ since all ideals are modular. Peters [98] obtains interesting results on a condition closely related to $[C^*\mathrm{CR}]$. In particular Corollary 3 shows $[\mathrm{SIN}] \cap [\mathrm{Am}] \cap [C^*\mathrm{CR}] \subseteq [\mathrm{FIA}]^-$.

Many duality results have been proved for nonabelian groups G which characterize properties of G in terms of properties of various dual objects ($\hat{G}$, Prim(G), etc.) and thus generalize the usual duality results involving the Pontryagin duality theorem for abelian groups. Under various circumstances Prim(G) can be identified with certain spaces of character functions on G ([106]; [60] Satz 2; [91] Theorem 5.2, [65]) and thus duality theories are sometimes stated in terms of these spaces and usually proved by reference to them. The most detailed duality theorem can naturally be proved for [Z] groups, and in Theorem 2.3 of [20] Grosser, Mosak and Moskowitz have given such a theorem. For $[\mathrm{FC}]^-$-groups, Kaniuth ([60], Satz 3) shows that G is compact or discrete if and only if $\hat{G}$ is discrete or compact respectively. Mosak ([91], Corollary 4.3) and Sund [122] obtain similar results for groups in $[\mathrm{FIA}]^-$ and $[\mathrm{FC}]^-$. Kaniuth in [62] gives duality theorems for groups in [SIN] and $[\mathrm{SIN}] \cap [T_1]$. Baggett [6] shows that any second countable group is compact if $\hat{G}$ is discrete but that the nondiscrete group $\mathbf{R} \times_\varphi \mathbf{Z}$ with $\varphi(n)x = e^n x$ has a compact dual. Stern [120] has removed the countability restriction from the result just quoted.

For central groups a Plancherel formula is given in Theorem 4.1 of [24]. For Moore groups a similar formula is given in [28] Satz 4. For related results see [64] Satz 4 and 5 and [76] Theorem 5.4.

Consideration of the center $L^1(G)_Z$, $C^*(G)_Z$, $M(G)_Z$ of $L^1(G)$, $C^*(G)$ and $M(G)$ respectively arises naturally in various contexts and plays a role in a number of the duality results proved above. In [64] Satz 1 Kaniuth and Schlichting show that the carrier space of $L^1(G)_Z$ can be identified with a space of characters when G to $[\mathrm{FC}]^-$. Mosak obtains similar results for $[\mathrm{FIA}]^-$-groups in Theorem 4.1 of [91]. In Corollary 1.6 he notes that $L^1(G)_Z$ is dense in $C^*(G)_Z$ at least for $[\mathrm{FIA}]^-$-groups. In [92] Mosak and Moskowitz show that for [IN]-groups each central idempotent of $M(G)$ is supported on a compact subgroup (cf. [64], Satz 7). Liukkonen and Mosak [82] extend a number of results known for $L^1(G)$ when G is abelian to $L^1(G)_Z$ when G is not abelian.

Finally we wish to describe the class [Type T] of Type T groups introduced by Azencott [150]. Let μ be a regular Borel probability measure on a group G such that not every convolution power of μ is singular with respect to Haar measure. Let H_μ denote the set of functions $f : G \rightarrow \mathbf{C}$ which satisfy

$$f(x) = \int_G f(xy)\, d\mu(y) \qquad \forall x \in G$$

and are uniformly continuous with respect to the left uniformity of G. This set is a commutative C^*-algebra under the multiplication

$$fg(x) = \lim_{n\to\infty} \int_G f(xy)\, g(xy)\, d\mu\,(y) \qquad \forall f, g \in H_\mu$$

where μ^n denotes the n^{th} convolution power of μ. Let Π_μ be the carrier space of H_μ. The group G operates naturally on each Π_μ and we say that G is Type T if it operates transitively on each Π_μ. In the cited reference it is shown on p. 92 that [Type T] is stable under formation of quotients and on p. 112 that Type T groups are unimodular. On p. 93 it is shown that a group G is Type T if and only if it is almost connected and has an approximating Lie group which is Type T and has a finite number of connected components. The combined work of Brown and Guivarc'h [151] and Moore and Rosenberg [156], [89] shows that a Lie group G for which G/G_{rad} has finite centers is Type T if and only if it belongs to $[T_1]$. In [156] the restriction on the center of G/G_{rad} in the above theorem is shown to be essential by considering the covering group of $SL(2, \mathbf{R})$. Hence for almost connected groups [Type T] and $[T_1]$ are closely related but not identical.

BIBLIOGRAPHY

1. G. M. Adel'son-Vel'skiĭ and Yu. A. Šreĭder, *The Banach mean on groups*, Uspehi Mat. Nauk (N.S.) **12** (1957), 131–136. **MR20** #1238.

2. Z. Anusiak, *Symmetry of L_1-group algebras of locally compact groups with relatively compact classes of conjugated elements*, Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys. **18** (1970), 329–332. **MR41** #8590. **MR** 49 #11147.

3. ———, *On generalized Beurling's theorem and symmetry of L_1-group algebras*, Colloq. Math. **23** (1971), 287–297. **MR 49** #11147.

4. L. Auslander and C. C. Moore, *Unitary representations of solvable Lie groups*, Mem. Amer. Math. Soc. **62** (1966) 199 pp **MR 34** #7723.

5. L. Baggett, *A description of the topology on the dual spaces of certain locally compact groups*, Trans. Amer. Math. Soc. **132** (1968), 175–215. **MR 53** #13472.

6. ———, *A separable group having a discrete dual space is compact*, J. Func. Analysis **10** (1972), 131–148. **MR 49** #10816.

7. L. Baggett and A. Kleppner, *Multiplier representations of abelian groups*, J. Func. Analysis **14** (1973), 299–324. **MR 51** #791.

8. R. A. Bonic, *Symmetry in group algebras of discrete groups*, Pacific J. Math. **11** (1961), 73–94. **MR 22** #11281.

9. J. W. Bunce, *Finite operators and amenable C^*-algebras*, Proc. Amer. Math. Soc. **56** (1976), 145–151. **MR 53** #6333.

10. J. Dixmier, *Sur les représentations unitaires des groupes de Lie nilpotents*, VI, Can. J. Math. **12** (1960), 324–352. **MR 22A** #5900b.

11. ———, *Les C^*-algèbres et leurs Représentations*, Gauthier-Villars, Paris 1969. **MR 20** #1234.

12. ———, *Sur les représentations unitaires des groupes de Lie algébriques*, Ann. Inst. Fourier, **7** (1957), 315–328. **MR 20** #582.

13. ———, *Sur les représentations unitaires des groupes de Lie nilpotent*, V, Bull. Soc. Math. Fr. **87** (1959), 65–79. **MR 22** #5900.

14. M. Duflo and C. C. Moore, *On the regular representation of a nonunimodular locally compact group*, J. Func. Analysis, **21** (1976), 209–243.

15. W. R. Emerson and F. P. Greenleaf, *Asymptotic behavior of products $C^P = C + \cdots + C$ in locally compact abelian groups*, Trans. Amer. Math. Soc. **145** (1969), 171–204. **MR 40** #2780.

16. R. Gangolli, *On the symmetry of L_1 algebras of locally compact motion groups and the Wiener Tauberian theorem*, J. Func. Analysis, **25** (1977), 244–252.

17. V. M. Gluškov, *The structure of locally compact groups and Hilbert's fifth problem*, Uspehi Mat. Nauk (N.S.) **12** (1957), 3–41: Translated as A.M.S. Translation (2) **15** (1960), 55–93. **MR 21** #698.

18. E. C. Gootman and R. R. Kallman, *The left regular representations of a p-adic group is Type I.*

19. F. P. Greenleaf, *Invariant Means on Topological Groups*, Van Nostrand, N.Y., 1969. **MR 40** #4776.

20. S. Grosser, R. Mosak and M. Moskowitz, *Duality and harmonic analysis on central topological groups*, I *and* II, Indag. Math. **35** (1973), 65–91. *Correction ibid.* 375. **MR 49** #5225 abc.

21. S. Grosser and M. Moskowitz, *On central topological groups*, Trans. Amer. Math. Soc. **127** (1967), 317–340. **MR 35** #292.

22. ———, *Representation theory of central topological groups*, Trans. Amer. Math. Soc. **129** (1967), 361–390. **MR 37** #5327.

23. ———, *Compactness conditions in topological groups*, J. Reine Angew. Math. **246** (1971), 1–40. **MR 44** #1766.

24. ———, *Harmonic analysis on central topological groups*, Trans. Amer. Math. Soc. **156** (1971), 419–454. **MR 43** #2165.

25. A. Guichardet, *Sur la cohomologie des groupes topologiques* III, Bull. Sci. Math., (2) **98** (1974), 201–208.

26. Y. Guivarc'h, *Croissance Polynomiale et Périodes des fonctions harmoniques*, Bull. Soc. Math. France **101** (1973), 333–379. **MR 51** #5841.

27. W. Hauenschild, *Zur Darstellungstheorie von SIN-Gruppen*, Math. Ann. **210** (1974), 257–276. **MR 50** #4810.

28. W. Hauenschild and E. Kaniuth, *Harmonische Analyse auf Gruppen mit endlichdimensionalen irreduziblen Darstellungen*, Math. Zeit. **144** (1975), 239–256. **MR 52** #10951.

29. W. Herfort, *Untersuchung zu Kommutator gruppen und Bohrkompaktifizierung in zentralen und Kompakten Gruppen*, Monatsh. Math. **77** (1973), 211–224. **MR 49** #457.

30. ———, *Untersuchungen zu Kommutator gruppe in zentralen und Kompakten topologischen Gruppen*, II, Monatsh. Math. **78** (1974), 341–346. **MR 48** #533.

31. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, I *and* II, Springer, Berlin, 1963 and 1970. **MR 28** #158 and **41** #7378.

32. H Heyer, *Groups with Chu duality*, pp. 181–215 in Probability and Information Theory, II. Springer Lecture notes in Math. #296, 1973, Berlin. **MR 51** #796.

33. K. H. Hofmann, *Über das Nilradikal lokalkompakter Gruppen*, Math. Z. 91 (1966), 206–215. **MR 34** #4405.

34. K. H. Hofmann and P. S. Mostert, *Splitting in topological groups*, Mem. Amer. Math. Soc. **43** (1963). **MR 27** #1529.

35. A. Hulanicki, *On the spectral radius of hermitian elements in group algebras*, Pacific J. Math. **18** (1966), 277–287. **MR 33** #6426.

36. ———, *On the spectral radius in group algebras*, Studia Math. **34** (1970), 209–214. **MR 41** #5984.

37. ———, *On symmetry of group algebras of discrete nilpotent groups*, Studia Math. **35** (1970), 207–219. **MR 43** #3814.

38. ———, *On positive functionals on a group algebra multiplicative on a subalgebra*, Studia Math. **37** (1971), 163–171. **MR 46** #9645.

39. ———, *On the spectrum of convolution operators on groups with polynomial growth*, Invent. Math. **17** (1972), 135–142. **MR 48** #2304.

40. ———, *Subalgebras of $L^1(G)$ associated with the Laplacian on a Lie group*, Colloq. Math. **31** (1974), 259–287. **MR 51** #8743.

41. ———, *On the L_p-spectra of the Laplacian on a Lie group with polynomial growth*, Proc. Amer. Math. Soc. **44** (1974), 482–484. **MR 50** #13378.

42. ———, *On the spectrum of the Laplacian on the affine group of the real line*, Studia Math. **54** (1975), 199–204. **MR 53** #3195.

43. A. Hulanicki, J. W. Jenkins, H. Leptin, T. Pytlik, *A remark on Wiener's Tauberian theorems for groups with polynomial growth*, Colloq. Math. **35** (1976), 293–304. **MR 53** #13469.

44. K. Iwasawa, *Some types of topological groups*, Annals of Math. (2), **50** (1948), 507–558. **MR 10** #679.

45. J. W. Jenkins, *An amenable group with a nonsymmetric group algebra*, Bull. Amer. Math. Soc. **75** (1969), 357–360. **MR 38** #6366.

46. ———, *Symmetry and nonsymmetry in the group algebras of discrete groups*, Pacific J. Math. **32** (1970), 131–145. **MR 41** #842.

47. ———, *Subsemigroups of an amenable group*, Proc. Amer. Math. Soc. **26** (1970), 226–227. **MR 41** #7013.

48. ———, *On the spectral radius of elements in a group algebra*, Ill. J. Math. **15** (1971), 551–554. **MR 44** #4538.

49. ———, *Free semigroups and unitary group representations*, Studia Math. **43** (1972), 27–39. **MR 47** #395.

50. ———, *Nonsymmetric group algebras*, Studia Math. **45** (1973), 295–307. **MR 49** #5855.

51. ———, *A characterization of growth in locally compact groups*, Bull. Amer. Math. Soc. **79** (1973), 103–106. **MR 47** #5172.

52. ———, *Growth of connected locally compact groups*, J. Func. Analysis **12** (1973), 113–127. **MR 50** #2388.

53. ———, *Representations of exponentially bounded groups*, Amer. J. Math. **98** (1976), 29–38.

54. ———, *Følner's condition for exponentially bounded groups*, Math. Scand. **35** (1974), 165–174. **MR 51** #5842.

55. ———, *A fixed point theorem for exponentially bounded groups*, J. Func. Analysis **22** (1976), 346–353. **MR 54** #2866.

56. B. E. Johnson, *Cohomology in Banach Algebras*, Memoirs of Amer. Math. Soc., No. **127**, Providence, 1972. **MR 51** #11130.

57. M. I. Kabenyuk, *Normal subgroups of locally compact groups*, Siberian Math. J. **16** (1975), 767–771. **MR 53** #699.

58. R. R. Kallman, *Certain topological groups are type* I, *Part* II, Advances in Math. **10** (1973), 221–255. **MR 49** #3022.

59. ———, *The existence of invariant measures on certain quotient spaces*, Advances in Math. **11** (1973), 387–391. **MR 48** #8682.

60. E. Kaniuth, *Zur harmonischen Analyse klassenkompakter Gruppen*, Math. Z. **110** (1969), 297–305. **MR 41** #8591.

61. ———, *Die Structur der regularen Darstellung lokalkompakter Gruppen mit invarianter Umgebungsbasis der Eins*, Math. Ann. **194** (1971), 225–248. **MR 45** #2082.

62. ———, *Topology in duals of SIN-groups*, Math. Z. **134** (1973), 67–80. **MR 48** #4197.

63. ———, *On maximal ideals in group algebras of SIN-groups*, Math. Ann. **214** (1975), 167–175. **MR 52** #6325.

64. E. Kaniuth and G. Schlichting, *Zur harmonischen Analyse klassenkompakter Gruppen* II, Invent. Math. **10** (1970), 332–345. **MR 47** #5521.

65. E. Kaniuth and D. Steiner, *On complete regularity of group algebras*, Math. Ann. **204** (1973), 305–329.

66. A. A. Kirillov, *Unitary representations of nilpotent Lie groups* (*in Russian*), Uspekhi Mat. Nauk. **17** (1962), 57–110. **MR 25** #5396.

67. B. A. Kramaver, *The topological* $\overline{FC}$-*groups*, (Translation) Siberian Math. J. **12** (1971), 266–271. **MR 44** #2882.

68. H. Leptin, *Zur harmonischen Analyse klassenkompakter Gruppen*, Invent. Math. **5** (1968), 249–254. **MR 38** #2257.

69. ______, *On group algebras on nilpotent Lie groups,* Studia Math. **47** (1973), 37–49. **MR 48** #9262.

70. ______, *On symmetry of some Banach algebras,* Pacific J. Math. **53** (1974), 203–206. **MR 51** #6432.

71. ______, *Harmonische Analyse auf gewissen nilpotente Lieschen Gruppen,* Studia Math. **48** (1973), 201–205. **MR 50** #2393.

72. ______, *Ideal theory in group algebras of locally compact groups,* Invent. Math. **31** (1976), 259–278. **MR 53** #3139.

73. ______, *Symmetrie in Banachschen Algebren,* Arch. Math. (Basel) **27** (1976), 394–400. **MR 51** #5841.

74. ______, *Lokal kompakte Gruppen mit symmetrischen Algebren,*

75. H. Leptin and L. Robertson, *Every locally compact MAP group is unimodular,* Proc. Amer. Math. Soc. **19** (1968), 1071–1082. **MR 37** #6397.

76. R. L. Lipsman, *Representation theory of almost connected groups,* Pacific J. Math. **42** (1972), 453–467. **MR 48** #6317.

77. ______, *Group representations: A survey of some current topics,* Springer Lecture Notes **388**, 1974. **MR 51** #8333.

78. ______, *The CCR property for algebraic groups,* Am. J. Math. **97** (1975), 741–752.

79. J. R. Liukkonen, *Dual spaces of groups with precompact conjugacy classes,* Trans. Amer. Math. Soc. **180** (1973), 85–108. **MR 47** #6937.

80. ______, *Characters and centroids of* [SIN] *amenable groups,* Mem. Amer. Math. Soc, **148** (1974), 145–151. **MR 51** #5844.

81. J. R. Liukkonen and R. Mosak, *The primitive dual space of* $[\mathrm{FC}]^-$ *groups,* J. Func. Analysis **15** (1974), 275–296. **MR 49** #10814.

82. ______, *Harmonic analysis and centers of group algebras,* Trans. Amer. Math. Soc. **195** (1974), 147–163. **MR 50** #2815.

83. J. Ludwig, *A class of symmetric group algebras,* preprint, 1976.

84. G. W. Mackey, *Infinite dimensional group representations,* Bull. Amer. Math. Soc. **69** (1963), 628–686. **MR 27** #3745.

85. D. Miličić, *Representations of almost connected groups,* Proc. Amer. Math. Soc. **47** (1975), 517–518.

86. J. Milnor, *Growth of finitely generated solvable groups,* J. Diff. Geom. **2** (1968), 447–449. **MR 39** #6212.

87. D. Montgomery and L. Zippin, *Topological Transformation Groups,* Interscience, N.Y., 1955. **MR 17** #383.

88. C. C. Moore, *Groups with finite dimensional irreducible representations,* Trans. Amer. Math. Soc. **166** (1972), 401–410. **MR 46** #1960.

89. C. C. Moore and J. Rosenberg, *Groups with* T_1 *primitive ideal spaces,* J. Func. Anal. **22** (1976), 204–224. **MR 54** #7693.

90. R. D. Mosak, *Central functions in group algebras,* Proc. Amer. Math. Soc. **29** (1971), 613–616. **MR 43** #5323.

91. ______, L^1 *and* C^**-algebras of* $[FIA]_B^-$ *groups and their representations,* Trans. Amer. Math. Soc. **163** (1972), 277–310. **MR 45** #2096.

92. R. D. Mosak and M. Moskowitz, *Central idempotents in measure algebras,* Math. Zeit. **122** (1971), 217–222. **MR 45** #9062.

93. M. Moskowitz, *On pro-reductive groups,* Proc. Camb. Philos. Soc. **76** (1974), 401–406. **MR 49** #10818.

94. P. R. Mueller-Roemer, *A Tauberian group algebra,* Proc. Amer. Math. Soc. **37** (1973), 163–166. **MR 48** #2669.

95. S. Murakami, *Remarks on the structure of Maximally Almost Periodic groups,* Osaka Math. J. **2** (1950), 119–129. **MR 13-12**.

96. T. W. Palmer, *Hermitian Banach *-algebras*, Bull. Amer. Math. Soc. **78** (1972), 522–524. **MR 45** #7481.

97. J. Peters, *Representing positive definite B-invariant functions on $[FC]_B^-$ groups* Monatsh. Math. **80** (1975), 319–324. **MR 54** #8166.

98. ———, *Groups with completely regular primitive dual space,* J. Func. Analysis **20** (1975), 136–148. **MR 52** #652.

99. P. Plaumann, *Erweiterungen kompakter Gruppen durch abelsche Gruppen,* Math. Zeit. 99 (1967), 123–140. **MR 35** #5548.

100. ———, *Klassen zusammenhanger Gruppen,* Archiv der Math. 19 (1968), 113–117. **MR 37** #2894.

101. D. Poguntke, *A Universal property of the Takahashi quasi-dual,* Can. J. Math. **24** (1972), 530–536. **MR 46** #294.

102. ———, *Zwei Klassen lokalkompakter maximal fast periodischen Gruppen,* Monatsh. Math. **81** (1976), 15–40. **MR 54** #12960.

103. ———, *Der Raum der primitiven Ideale von endlichen Erweiterungen lokalkompakter Gruppen,* Arch. Math. (Basel) **28** (1977), 133–138.

104. ———, *Nilpotente Liesche Gruppen haben symmetrische Gruppenalgebren,* Math. Ann. **227** (1977), 51–59.

105. E. Porada, *On the spectral radius in $L_1(G)$,* Colloq. Math. **23** (1971), 279–285. **MR 48** #4651.

106. L. Pulanszky, *Characters of connected Lie groups,* Acta Math. **133** (1974), 81–137. **MR 50** #13383.

107. ———, *Unitary representations of Lie groups with co-compact radical and applications,* Trans. Amer. Math. Soc. **236** (1978), 1–50.

108. T. Pytlik, *On the spectral radius of elements in group algebras,* Bull. Acad. Polon. Sci. Ser. Math. astr. et phys. **21** (1973), 889–902. **MR 48** #6818.

109. H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups,* Oxford Mathematical Monographs, Oxford, 1968. **MR 46** #5933.

110. C. E. Rickart, *General Theory of Banach Algebras,* Van Nostrand, Princeton, 1960. **MR 22** #5903.

111. N. Rickert, *Amenable groups and groups with the fixed point property,* Trans. Amer. Math. Soc. **127** (1967), 221–232. **MR 36** #5260.

112. ———, *Some properties of locally compact groups,* J. Austral. Math. Soc. **7** (1967), 433–454. **MR 36** #2735.

113. L. Robertson, *A note on the structure of Moore groups,* Bull. Amer. Math. Soc. **75** (1969), 594–599. **MR 39** #7027.

114. L. Robertson and T. W. Wilcox, *Splitting in MAP groups,* Proc. Amer. Math. Soc. **33** (1972), 613–618. **MR 45** #5271.

115. J. M. Rosenblatt, *Invariant measures and growth conditions,* Trans. Amer. Math. Soc. **193** (1974), 33–53. **MR 49** #7699.

116. ———, *A distal property of groups and the growth of connected locally compact groups,* preprint.

117. S. Sakai, *C*-algebras and W*-algebras,* Springer-Verlag, New York, 1971.

118. G. Schlichting, *Ein Charakterisierung gewissen diskreter Gruppen durch ihre regulare Darstellung.* Manuscripta Math. 9 (1973), 389–409. **MR 48** #2302.

119. ———, *Topologische gruppen mit endlichen Klassen konjugierter Elemente,* Math. Zeit. **142** (1975), 15–17. **MR 51** #3348.

120. A. I. Stern, *The connection between the topologies of a locally bicompact group and its dual space,* (Translation) Functional Anal. Appl. **5** (1971), 311–317. **MR 45** #3639.

121. ———, *Locally bicompact groups with finite dimensional irreducible representations* (Russian), Mat. Sb. (N.S.), **90** (132) (1973), 86–95. 166. **MR 49** #5221.

122. T. Sund, *Duality theory for locally compact groups with precompact conjugacy classes, I, The character space,* Trans. Amer. Math. Soc. **211** (1975), 185–202. **MR 52** #8332.

123. E. Thoma, *Ein Charakterisierung diskreter Gruppen vom Type* I, Invent. Math. **6** (1968), 190–196. **MR 40** #1540.

124. M. E. Walter, *A duality between locally compact groups and certain Banach algebras,* J. Func. Analysis, **17** (1974), 131–160. **MR 50** #14067.

125. S. P. Wang, *The automorphism group of a locally compact group,* Duke Math. J. **36** (1969), 277–282.

126. ———, *Compactness properties of topological groups,* Trans. Amer. Math. Soc. **154** (1971), 301–314. **MR 42** #6152.

127. ———, *Compactness properties of topological groups* II, Duke Math. J. 39 (1972), 243–252. **MR 45** #5270.

128. ———, *On a compactness property of topological groups,* Trans. Amer. Math. Soc. **187** (1974), 83–88. **MR 48** #8688.

129. ———, *Compactness properties of topological groups,* III, Trans. Amer. Math. Soc. **209** (1975), 399–418. **MR 51** #10529.

130. G. Warner, *Harmonic Analysis on Semi-simple Lie Groups,* I *and* II, Springer, 1972.

131. J. Wichmann, *On the symmetry of matrix algebras,* Proc. Amer. Math. Soc. **54** (1976), 237–240.

132. T. W. Wilcox, *On the structure of maximally almost periodic groups,* Bull. Amer. Math. Soc. **73** (1967), 732–734. **MR 35** #4344.

133. ———, *On the structure of maximally almost periodic groups,* Math. Scand. **23** (1968), 221–232. **MR 40** #588.

134. ———, *A note on groups with relatively compact conjugacy classes,* Proc. Amer. Math. Soc. **42** *(1974), 326–329.* **MR 48** #8685.

135. A. Willcox, *Note on certain group algebras,* Proc. Amer. Math. Soc. **7** (1956), 874–879. **MR 19** #46.

136. J. Wolf, *Growth of finitely generated solvable groups and curvature of Riemannian manifolds,* J. Diff. Geom. **2** (1968), 421–446. **MR 40** #1939.

137. T. S. Wu and Y. K. Yu, *Compactness properties of topological groups,* Michigan Math. J. **19** (1972), 299–313. **MR 47** #6938.

138. ———, *Compactness properties of topological groups,* II, Duke Math. J. **39** (1972), 243–252. **MR 45** #5220.

139. P. R. Müller-Roemer, *Kontrahierende Erweiterungen und kontrahierbare Gruppen,* J. Reine Angew. Math. **283/284** (1976), 238–264. **MR 53** #10977.

140. D. Poguntke, *Chu-Dualität und zwei Klassen maximal fastperiodischer Gruppen,* Monatshefte für Math. **82** (1976), 31–50. **MR 54** #12961.

141. H. Leptin and D. Poguntke, *Symmetry and nonsymmetry for Locally Compact Groups,* preprint, 1977.

142. J. B. Fountain, R. W. Ramsey, and J. H. Williamson, *Functions of Measures on Compact Groups,* Proc. Roy. Irish. Acad. Sect. A **76** (1976), 231–251.

143. M. A. Rieffel, *Induced representations of C^*-algebras,* Advances in Math. **13** (1974), 176–257. **MR 50** #5489.

144. ———, *Morita equivalence for C^*-algebras,* J. Pure Appl. Algebra **5** (1974), 51–96. **MR 51** #3912.

145. I. Schochetman, *Topology and the duals of certain locally compact groups,* Trans. Amer. Math. Soc. **150** (1970), 477–489. **MR 42** #422.

146. M. K. Smith, *Regular representations of unimodular groups,* Math. Ann. **203** (1973), 183–185. **MR 47** #8763.

147. M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and its Applications,* Springer Lecture Notes in Mathematics, Vol. **128** (1970). **MR 42** #5061.

148. J. Tits, *Free subgroups in linear groups,* J. Alg. **20** (1972), 250–270. **MR 45** #4105.

149. T. S. Wu, *On (CA) topological groups,* II, Duke Math. J. **38** (1971), 513–519. **MR 45** #6897.

150. R. Azencott, *Espaces de Poisson des Groupes Localement Compacts,* Lecture Notes in Math. **148**, Springer, Berlin, 1970.

151. I. D. Brown and Y. Guivarc'h, *Espaces de Poisson des groupes de Lie,* Ann. Scient. Ec. Norm. Sup. (4), **7** (1974), 175–180. **MR 55** #570a.

152. G. von Dijk, *On symmetry of group algebras of motion groups,* Math. Ann. **179** (1969), 219–220.

153. K. H. Hofmann, *Sur la décomposition semidirecte des groupes compacts connexes,* Inst. Naz. Alta Math. Symp. Math. **16** (1975), 471–476.

154. J. R. Liukkonen and M. W. Mislove, *Symmetry in Fourier-Stieltjes algebras,* Math. Ann. **217** (1975), 97–112. **MR 54** #8163.

155. J. R. Liukkonen and R. D. Mosak, *Harmonic analysis and centers of Buerling algebras,* Comment. Math. Helvetici, **52** (1977), 297–315.

156. C. C. Moore and J. Rosenberg, *Comments on a paper of Brown and Guivarc'h,* Ann. Scient. Ec. Norm Sup. (4), **8**, (1975), 379–382. **MR 55** #570b.

157. J. Peters and T. Sund, *Automorphisms for locally compact groups,* preprint.

158. K. F. Taylor, *The type structure of the regular representation of a locally compact group,* Math. Ann. **222** (1976), 211–214. **MR 54** #12965.

159. T. Sund, *Duality theory for locally compact groups with precompact conjugacy classes* II. *The dual space,* Trans. Amer. math. Soc. **224** (1976), 313–321.

160. ———, *Isolated points in duals of certain locally compact groups,* Math. Ann. **224** (1976), 33–39. **MR 54** #10477.

161. B. A. Barnes, *Ideal and representation theory of the L^1-algebra of a group with polynomial growth,* Colloq. Math. (to appear).

162. J. Boidal, H. Leptin, J. Schürman, and D. Vahle, *Primitive Ideale räumen,* Gruppen algebren, preprint, 1977.

163. J. Ludwig, *A class of symmetric and a class of Wiener algebras,* preprint, 1977.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403