ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 8, Numbers 1 and 2,
Winter and Spring 1978

ON THE SOLUTION OF A FUNCTIONAL EQUATION†

HELENE AIRAULT

0. **Introduction.** The quantum-mechanical problems of n mass points on the line interacting pairwise under the influence of a potential proportional to the inverse square of the distance or to the square of the distance were solved explicitly by F. Calogero [1]. This led him to conjecture that the classical problems would be integrable. This was established in [2] for the three-body problem. Then J. Moser [3] introduced matrices L and B, and writing the equations in P. Lax's form [4], he solved the classical n-particle system on the line with the inverse square potential. He successfully applied the method to the potential $\sin^{-2}x$ and to the Toda lattice. This method was further extended by M. Adler [5] to potentials of the form $x^{-2} + \alpha x^2$. The question arose, to which potentials could this method be applied. In the case of the classical n-body problem characterized by the Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + \sum_{j>k=1}^{n} V(x_j - x_k),$$

F. Calogero [6] considered potentials of the form $V(x) = \alpha(x)\alpha(-x) + \text{const.}$ Writing P. Lax's condition with

$$L_{jk} = \delta_{jk}p_j + (1 - \delta_{jk})\,\alpha(x_j - x_k)$$

and

$$B_{jk} = \delta_{jk} \sum_{\substack{\ell=1 \\ \ell\neq j}}^{N} \beta(x_j - x_\ell)$$

$$- (1 - \delta_{jk})\,\alpha'(x_j - x_k)$$

he was led to solve the equation (related equations appear in [7, 8]).

$$\alpha'(y)\alpha(z) - \alpha(y)\alpha'(z) = \alpha(y + z)[\beta(y) - \beta(z)]. \tag{1}$$

Functions such that $\alpha_1(x) = bdn(ax)/sn(ax)$ and $\alpha_2(x) = bcn(ax)/sn(ax)$ are solutions of (1) and they yield the same potential $V(x) = \lambda P(x) + \mu$, where λ and μ are two constants and P is the Weierstrass P-function. In

†Supported in part by a grant from the National Science Foundation under the United States/French Exchange Program.

particular, when the two periods of P are infinite, one recovers the x^{-2} potential, and when one of the periods is finite and the other infinite, one finds the $\sin^{-2}x$ or the $sh^{-2}x$ potential.

In the following, we prove that if α and β are two meromorphic functions which satisfy (1), then $\alpha(x)\alpha(-x)$ must be equal to $\lambda P(x) + \mu$. [When this proof was shown to F. Calogero at the Mathematical Congress on Solitons (Tucson, January 1976), he said that he had a different proof and he pointed out the work by P. P. Kulish [9] and mentioned that another proof was going to appear in Doklady.] In fact (1) is simply an addition formula for Weierstrassian functions. If one defines α_λ by $\alpha_\lambda{}^2(z) = P(z) - e_\lambda$ where $e_\lambda = P(\omega_\lambda)$ and $\{\omega_\lambda\}$ is an irreducible set of zeros of $P'(z)$ ($\lambda = 1, 2, 3$), then α_λ is a solution of (1) and β is computed to be equal to $-P(y) +$ const.

Now the special form of L and B considered above seems related to the motion of three particles. In the case of three mass points interacting by means of potentials related by the addition formula

$$\begin{pmatrix} 1 & V_1(y) & V_1'(y) \\ 1 & V_3(u) & V_3'(u) \\ 1 & V_2(u+y) & -V_2'(u+y) \end{pmatrix} = 0 \tag{2'}$$

the equations of motion

$$\ddot{z}_1 = -V_3'(z_1 - z_2) - V_2'(z_1 - z_3)$$
$$\ddot{z}_2 = V_3'(z_1 - z_2) - V_1'(z_2 - z_3)$$
$$\ddot{z}_3 = V_2'(z_1 - z_3) + V_1'(z_2 - z_3)$$

may be written $dL/dt = [L, B]$. (The L and B defined in this case are slightly different from the ones defined in [6]). This permits us to include the case of the exponential potential with nearest neighbor interaction (Toda lattice).

1. **The solutions of (1).** Assume that α and β are two meromorphic functions which satisfy the equation (1). Consider two points x and y and write

$$\beta(y) - \beta(-x-y) + \beta(-x-y) - \beta(x) = \beta(y) - \beta(x).$$

Multiplying by $\alpha(-x)\alpha(-y)\alpha(x+y)$, one obtains

$$[\alpha'(y)\alpha(-x-y) - \alpha(y)\alpha'(-x-y)]\alpha(-y)\alpha(x+y)$$
$$+ [\alpha'(-x-y)\alpha(x) - \alpha(-x-y)\alpha'(x)]\alpha(-x)\alpha(x+y)$$
$$= [\alpha'(y)\alpha(x) - \alpha(y)\alpha'(x)]\alpha(-x)\alpha(-y).$$

Using the fact that $V(x) = \alpha(x)\alpha(-x)$, gives

$$\begin{aligned} &V(x+y)[\alpha'(y)\alpha(-y) - \alpha'(x)\alpha(-x)] \\ &\quad - V(y)[\alpha'(-x-y)\alpha(x+y) - \alpha'(x)\alpha(-x)] \\ &\quad + V(x)[\alpha'(-x-y)\alpha(x+y) - \alpha'(y)\alpha(-y)] = 0. \end{aligned}$$

Rewriting the same relation with $-y$ instead of y and $-x$ instead of x, and subtracting the second relation from the first, one obtains

$$\begin{pmatrix} 1 & V(x) & V'(x) \\ 1 & V(y) & V'(y) \\ 1 & V(x+y) & -V'(x+y) \end{pmatrix} = 0. \tag{2}$$

The functions $V(x) = \lambda P(x) + \mu$, where P is the Weierstrass function and λ and μ are two constants, are solutions of (2)(see [11]) and they are the only meromorphic ones. A proof of this last fact follows.

If V has no pole at 0, and verifies (2), one may suppose $V(0) = 0$ and write

$$\begin{pmatrix} 1 & V(x) & V'(x) \\ 1 & 0 & V'(0) \\ 1 & V(x) & -V'(x) \end{pmatrix} = 0$$

which implies $2V(x)V'(x) = 0$ which means V is identically zero. So, if V is not a constant, it must have a pole at zero. Writing $V(z) = az^{-n} + V_2(z)$ one sees that the pole has to be of order 2 and V has to be even. One may suppose $V_2(0) = 0$ and $a = 1$. Then, write $V(\epsilon) = \epsilon^{-2} + V_2(\epsilon)$ and make ϵ tend to zero in the following equation

$$\begin{pmatrix} 1 & V(u) & V'(u) \\ 1 & 1/\epsilon^2 + V_2(\epsilon) & -2/\epsilon^3 + V_2'(\epsilon) \\ 1 & V(u+\epsilon) & -V'(u+\epsilon) \end{pmatrix} = 0$$

or

$$\begin{pmatrix} 1 & V(u) & V'(u) \\ 0 & 1/\epsilon^2 & -2/\epsilon^3 \\ 1 & V(u+\epsilon) & -V'(u+\epsilon) \end{pmatrix} + \begin{pmatrix} 1 & V(u) & V'(u) \\ 1 & V_2(\epsilon) & V_2'(\epsilon) \\ 1 & V(u+\epsilon) & -V'(u+\epsilon) \end{pmatrix} = 0.$$

One obtains

$$2V(u)V'(u) = -\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\begin{pmatrix} 1 & V(u) & V'(u) \\ 0 & 1 & -2/\epsilon \\ 1 & V(u+\epsilon) & -V'(u+\epsilon)\end{pmatrix}$$
$$= \frac{1}{6}V'''(u).$$

This is the differential equation for the Weierstrass P function.

Consider the case where $V(z) = P(z) - e_\lambda$ ($\lambda = 1, 2, 3$) where P is the Weierstrass function, and as usual, $e_\lambda = P(\omega_\lambda)$ where ω_λ ($\lambda = 1, 2, 3$) is an irreducible set of zeros of $P'(z)$. One can compute β in (1) using the additional theorems for the Weierstrass sigma-functions [10; 11].

Let $P(z) - e_\lambda = \alpha_\lambda{}^2(z)$ where $\alpha_\lambda(z) = \sigma_\lambda(z)/\sigma(z)$ ($\lambda = 1, 2, 3$). Recall that $\sigma_\lambda(z) = \sigma(z+\omega_\lambda)/\sigma(\omega_\lambda)\exp(-s\eta_\lambda)$ where

$$\eta_\lambda = \zeta(\omega_\lambda).$$

Rewrite (1),

$$\frac{\alpha'(y)}{\alpha(y)} - \frac{\alpha'(z)}{\alpha(z)} = [\beta(y) - \beta(z)]\frac{\alpha(y+z)}{\alpha(y)\alpha(z)}.$$

Using [10, p. 29],

$$\frac{\alpha_\lambda'(y)}{\alpha_\lambda(y)} = \frac{d}{dy}\log\frac{\sigma_\lambda(y)}{\sigma(y)}$$
$$= \frac{1}{2}\frac{P'(y)}{P(y)-e_\lambda} = -\frac{\sigma_\mu(y)\sigma_\nu(y)}{\sigma_\lambda(y)\sigma(y)}$$

where $\{\mu, \nu, \lambda\} = \{1, 2, 3\}$. Then

$$\frac{\alpha_\lambda'(y)}{\alpha_\lambda(y)} - \frac{\alpha_\lambda'(z)}{\alpha_\lambda(z)} = -\frac{\sigma_\mu(y)\sigma_\nu(y)}{\sigma_\lambda(y)\sigma(y)} + \frac{\sigma_\mu(z)\sigma_\nu(z)}{\sigma_\lambda(z)\sigma(z)}.$$

Now reduce to the same denominator and use [10, D-7, p. 51]

$$-\sigma_\mu(y)\sigma_\nu(y)\sigma_\lambda(z)\sigma(z) + \sigma_\mu(z)\sigma_\nu(z)\sigma_\lambda(y) = \sigma_\lambda(y+z)\sigma(y-z).$$

So, one has to prove

$$\frac{\sigma(y+z)\sigma(y-z)}{\sigma^2(y)\sigma^2(z)} = [\beta(y) - \beta(z)].$$

Use [10, D-I, p. 51],

$$\sigma(z+y)\sigma(y-z) = \sigma^2(y)\sigma_\lambda{}^2(z) - \sigma_\lambda{}^2(y)\sigma^2(z).$$

Dividing by $\sigma^2(y)\sigma^2(z)$, one gets

$$\frac{\sigma_\lambda{}^2(y)}{\sigma^2(y)} - \frac{\sigma_\lambda{}^2(z)}{\sigma^2(z)} = -\beta(y) + \beta(z),$$

then

$$\beta(y) = -\alpha_\lambda{}^2(y) \quad (\lambda = 1, 2, 3).$$

As β is determined up to an additive constant, one may take $\beta(y) = -P(y)$.

2. **The case of three mass points.** Consider now the motion of three particles, under the action of three potentials. Denote by z_1, z_2, z_3 the positions and by p_1, p_2, p_3 the momenta. Between z_k and z_i, the potential V_j acts, where $i \neq j \neq k$ and $\{i, j, k\} = \{1, 2, 3\}$. Let $V_k'(z)$, $k = 1, 2, 3$, denote the derivative of V_k. The equations of motion are

$$\begin{aligned}
\ddot{z}_1 &= -V_3'(z_1 - z_2) - V_2'(z_1 - z_3)\\
\ddot{z}_2 &= V_3'(z_1 - z_2) - V_1'(z_2 - z_3)\\
\ddot{z}_3 &= V_2'(z_1 - z_3) + V_1'(z_2 - z_3).
\end{aligned}$$

The potential function is

$$\begin{aligned}
U(z_1, z_2, z_3) = V_3(z_1 - z_2) &+ V_1(z_2 - z_3)\\
&+ V_2(z_1 - z_3).
\end{aligned}$$

One defines $\alpha_1, \alpha_2, \alpha_3$ by $V_k(z) = \alpha_k{}^2(z) + \lambda$ where λ is a constant, $k = 1, 2, 3$. Let

$$L = \begin{pmatrix} p_1 & i\alpha_3(z_1 - z_2) & i\alpha_2(z_1 - z_3)\\ -i\alpha_3(z_1 - z_2) & p_2 & i\alpha_1(z_2 - z_3)\\ -i\alpha_2(z_1 - z_3) & -i\alpha_1(z_2 - z_3) & p_3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} K_1 & i\alpha_3'(z_1 - z_2) & i\alpha_2'(z_1 - z_3)\\ i\alpha_3'(z_1 - z_2) & K_2 & i\alpha_1'(z_2 - z_3)\\ i\alpha_2'(z_1 - z_3) & i\alpha_1'(z_2 - z_3) & K_3 \end{pmatrix}$$

THEOREM. *The condition $dL/dt = [L, B]$ is equivalent to the equations of motion if and only if the three potentials V_1, V_2, V_3 satisfy the following identity:*

$$(3)\qquad \begin{pmatrix} 1 & V_1(y) & V_1'(y) \\ 1 & V_3(u) & V_3'(u) \\ 1 & V_2(u+y) & -V_2'(u+y) \end{pmatrix} = 0$$

for all u and y.

PARTICULAR CASES: (1) $V = V_1 = V_2 = V_3$ which gives $V(y) = aP(y) + b$. (2) $V = V_1 = V_3$ and $V_2 = 0$ which implies $V(x) = \lambda e^{rx}$: This case corresponds to a small Toda lattice. (3) $V_1(y) = aP(y) + b$ and $V_2 = V_3 = aP(y + d) + c$.

PROOF. Call $\alpha_3 = \alpha_3(z_1 - z_2)$; $\alpha_2 = \alpha_2(z_1 - z_3)$ and $\alpha_1 = \alpha_1(z_2 - z_3)$. *The condition* $dL/dt = [L, B]$ is equivalent to

$$(4)\qquad \begin{cases} i(K_2 - K_1)\alpha_3 - \alpha_2\alpha_1' - \alpha_1\alpha_2' = 0 \\ i(K_3 - K_1)\alpha_2 + \alpha_1\alpha_3' - \alpha_3\alpha_1' = 0 \\ i(K_3 - K_2)\alpha_1 + \alpha_2\alpha_3' + \alpha_2'\alpha_3 = 0. \end{cases}$$

Multiply each line of (4) respectively by $\alpha_1\alpha_2$, $-\alpha_1\alpha_3$, and $\alpha_2\alpha_3$ and add. Then

$$(-\alpha_1\alpha_1' + \alpha_3\alpha_3')\alpha_2^2 - (\alpha_2\alpha_2' + \alpha_3\alpha_3')\alpha_1^2 + (\alpha_1'\alpha_1 + \alpha_2'\alpha_2)\alpha_3^2 = 0$$

and this is (3).

REFERENCES

1. F. Calogero, *Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials,* J. Math. Phys. **12** (1971), 419–436.

2. F. Calogero and C. Marchioro, *Exact solution of a one-dimensional three-body scattering problem with two-body and/or three-body inverse-square potentials,* J. Math. Phys. **15** (1974), 1425–1430.

3. J. Moser, *Three integrable Hamiltonian systems connected with isospectral deformations,* Advances in Math. **16** (1975), 197–220.

4. P. Lax, *Integrals of nonlinear equations of evolution and solitary waves,* Comm. Pure Appl. Math. **21** (1968), 467–490.

5. M. Adler, *A new integrable system and a conjecture by Calogero,* preprint (1975).

6. F. Calogero, *Exactly solvable one-dimensional many-body problems,* Lett. Nuovo Cimento **13** (1975), 411–416.

7. ———, *One-dimensional many-body problems with pair interactions whose exact ground-state wave function is of product type,* Lett. Nuovo Cimento **13** (1975), 507–511.

8. B. Sutherland, *Exact ground-state wave function for a one-dimensional plasma,* Phys. Rev. Lett. **34** (1975), 1083–1085.

9. P. P. Kulish, *Factorization of the classical and quantum S-matrix and conservation laws,* Serpukhov (IHEP. 75-123) (Refs. 10), **13** (1975).

10. K. Weierstrass, *Formeln und Lehrsätze sum Gebrauche der elliptischen Functionen,* Berlin, Springer, 1893.

11. H. Bateman, *Higher Transcendental Functions,* Vol. 2, New York, McGraw-Hill, 1953.

INSTITUT HENRI POINCARÉ, 11 URE PIERRE ET MARIE CURIE, 75.005-PARIS, FRANCE