A BRIEF HISTORY OF THE QUANTUM SOLITON WITH NEW RESULTS ON THE QUANTIZATION OF THE TODA LATTICE*

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I. **Brief History.** When I first learned of this conference, and saw the wide range of interests represented by the participants, I was certain that finally a common definition of that most intriguing concept—the soliton—would emerge. And I was anxious that this definition be broad enough to encompass solitons in both their classical and quantum versions. For recent work had convinced me that the general technique known as Bethe's ansatz for solving diverse one-dimensional quantum problems in fact was nothing more nor less than the quantum soliton.

However, from conversations with conference participants, I discovered that many were not aware of this accumulated work on exactly soluble quantum systems. I even gained the impression that some feel quantum mechanics to be much harder than classical mechanics—an unnecessary complication. My own feeling, on the other hand, is that in many cases quantum mechanics clarifies matters. It forces one to immediately face problems that would eventually have to be faced in the corresponding classical case, problems such as the counting of states, to determine how important solitons really are.

Anyway, the history of the quantum soliton is all to the greater glory of the soliton concept. And quite a respectable history it is. I would date the beginning at 1931 with a paper of H. Bethe [1] on magnetism; that is a total span of 45 years. Quantum mechanics itself has not been around all that much longer!

It now seems my hopes for a consensus on a definition of the soliton were premature, and things are still fermenting. Nevertheless, here is my contribution to the conference in the form of a brief history for general interest, and a more original work on the quantization of the Toda lattice. Perhaps next time there will be some consolidation, and we can then decide just how particular and how general a concept is the soliton.

1. We begin this historical account with the classic paper of H. Bethe [1], which introduced first the many-body wave function crucial

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to all later investigations. Bethe considered the one-dimensional version of a model for magnetism first introduced by Heisenberg.

As such spin models often appear in the literature of manybody physics, the most prominent case being the Ising model, let us take a few lines to explain the notation and the physics. First, we imagine a lattice of N sites, in most cases the line, square, or cubic lattice in 1, 2 or 3 dimensions respectively. Then at each site we place a spin, which may either point up (z-component $+1$), or down (z-component -1). These two configurations at a site provide a basis for an individual spin,

$$\text{up} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \text{ down} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and thus the total configuration of the system is a 2^N component vector.

Various operations on a spin may be accomplished by additional operators, such as σ^+ which changes a down spin into an up spin, σ^- which changes an up spin into a down spin, or σ^z which measures the z-component of spin. An alternative set of operators, the Pauli Spin operators $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$, is defined in equation (2) below. The addition of a subscript to an operator σ gives the position of the site on which the particular spin is located on which the operator operates.

Written in terms of Pauli spin operators $\vec{\sigma}$, the Heisenberg Hamiltonian is taken to be

$$(1) \qquad\qquad H = \pm \sum_{j=1}^{N} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}.$$

We impose periodic boundary conditions so that $N + 1 = 1$. The positive sign favor spins which point in opposite directions (antiferromagnetism), while the negative sign favors aligned spins (ferromagnetism).

The rotational invariance of this system is obvious, and thus both the total spin and the total z-component of spin are conserved. A convenient representation for the Pauli spin operators is:

$$\sigma^+ = (\sigma^x + i\sigma^y)/2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$(2) \qquad \sigma^- = (\sigma^x - i\sigma^y)/2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Hamiltonian then becomes

$$(3) \qquad H = \pm \sum_{j=1}^{N} [\sigma_j^z \sigma_{j+1}^z + 2(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+)].$$

One might now interpret the first term of the Hamiltonian as an inter-action between the z-components of individual spins, while the second term is a "hopping" term which moves overturned spins about from one neighboring lattice site to another.

Let us begin with a state in which all spins are aligned in the positive z-direction; there is only one such state:

$$(4) \qquad \psi_0 = \cdots \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdots.$$

This state is obviously an eigenstate, and

$$(5) \qquad H\psi_0 = \pm N\psi_0.$$

For the case of ferromagnetism, this state would be one of a set of degenerate ground states.

Suppose we now overturn one spin on site j; this state we label (j). It is not an eigenstate. However, we can now consider an eigenstate ψ_1 with components $\psi_1(j)$ in the basis of states (j). We then find for the eigenvalue equation

$$(6) \qquad \begin{aligned} H\psi_1(j) &= \pm (N - 4)\psi_1(j) \\ &\pm 2[\psi_1(j + 1) + \psi_1(j - 1)] = E\psi_1(j). \end{aligned}$$

This difference equation is easily solved by $\psi_1(j) = e^{ikj}$, and we find for the energy

$$(7) \qquad E = \pm [N - 4 - 4\cos k].$$

Imposing the periodic boundary condition, we find

$$k = 2\pi n/N; \; n = 1, 2, \cdots, N.$$

Such a collective excitation in a magnet is traditionally called a spin wave, and it is these spin waves we wish to identify as examples of quantum solitons.

We now arrive at the basic contribution of Bethe to the investigation of this system. For the antiferromagnet, Bethe proposed that if we consider the subspace of M overturned spins, and choose basis vectors with the overturned spins located at $x_1 < x_2 < \cdots < x_M$, then the coefficients of the wave function will be of the form

(8) $$\psi_M(x_1, \cdots, x_M) = \sum_P A(P) \exp \left[i \sum_{j=1}^{M} x_j k_{Pj} \right].$$

The first sum over P is a sum over the $M!$ permutations $(P1, P2, \cdots, PM)$ of the integers 1 to M. $A(P)$ and k_j are to be chosen so as to satisfy $H\psi = E\psi$. It is this basic form for the wave-function which we call "Bethe's ansatz."

First, if the overturned spins are well separated so that no two are nearest neighbors, then the energy will simply be given by

(9) $$E = N - 4M + 4 \sum_{j=1}^{M} \cos k_j.$$

Second, if we consider now the situation when exactly two spins j and $j + 1$ are nearest neighbors at x_j and $x_{j+1} = x_j + 1$, we find

(10)
$$4\psi(\cdots x_j, x_{j+1} \cdots) = 2[\psi(\cdots x_j, x_j \cdots)$$
$$+ \psi(\cdots x_{j+1}, x_{j+1} \cdots)].$$

(This equation makes sense as a continuation of the particular form, Bethe's ansatz, of equation (8) in the variables x_j.) Let us try to satisfy this equation by considering pairs of terms of equation (8). Thus we consider two permutations $P = (P1, \cdots, Pj, Pj + 1, \cdots, PM)$ and $P' = (P1, \cdots, Pj + 1, Pj, \cdots, PM)$. If $k_{Pj} = k$ and $k_{Pj+1} = k'$, we then have

(11) $$2[A(P)e^{ik'} + A(P')e^{ik}] = [A(P) + A(P')][1 + e^{i(k+k')}].$$

Or finally,

(12) $$A(P)/A(P') = - \frac{1 + e^{i(k+k')} - 2e^{ik}}{1 + e^{i(k+k')} - 2e^{ik'}} = -e^{i\delta(k,k')}.$$

Thus to summarize, coefficients for the exchange of two k's (momenta) are related by the two-body S-matrix. For indeed, the right-hand side of equation (12) would be the same no matter what the total number of overturned spins. We simply state, without derivation, that the periodic boundary conditions will once again serve to determine the allowed values of the k's.

This discussion of course has not demonstrated the consistency of our whole procedure. In fact, it is consistent, and the proof relies in large part on hard work by the reader. This brief description should be augmented by the original literature, and pencil and paper.

We now wish at this point to emphasize the important physical content of Bethe's ansatz for the wave function. It is a scattering state, and

the incoming arrangment of k's is the permutation $P = (k_1 > k_2 > \cdots > k_M)$. The $M!$ terms of the wave function correspond to successive rearrangements of the k's by two-body collisions. But it is always the same k's involved, i.e., there is no diffraction. We now remark that this is exactly the same situation found for the scattering of the solitons of the classical system.

In fact, this single property of no diffraction—i.e., "what comes out is exactly what goes in"—is enough to demand that the asymptotic wave function, at least, is exactly of Bethe's form given in equation (8), where the coefficients are related by the two-body phase shift δ as in equation (10). The energy of course will still be given by equation (9).

What now follows in the rest of this first section is a fast catalogue of various quantum systems for which a wave function of Bethe's form provides a solution. As this is not a review we do not apportion credit for particular contributions; in most cases, the most general model represents the cumulative achievement of several clever investigations by various authors over a span of several years. We do include a most recent reference to enable the interested reader to begin a backward entry into the literature. The excellent book by Lieb and Mattis [2] will provide a review of work up to 1966.

2. Let us first consider the class of systems obtained by generalizing the original one-dimensional Heisenberg antiferromagnet. To date, the most general spin systems to be solved exactly are either of the form:
Class A [3];

(13A)
$$H = - \sum_{n.n.} \left[e^{i\theta}\sigma_+\sigma_-' + e^{-i\theta}\sigma_-\sigma_+' + \frac{\Delta}{2}\sigma_z\sigma_z' \right] - h\sum \sigma_z,$$

or,

Class B [4];

(13B) $$H = - \sum_{n.n} [A\sigma_x\sigma_x' + B\sigma_y\sigma_y' + C\sigma_z\sigma_z'].$$

The quantities θ, Δ, h, A, B, C are all free constants, and the common ground of these problems is $\theta = h = 0$, $A = B$. The notation $n.n.$ indicates a summation over nearest neighbor pairs.

We now ask: What exactly is meant by a "solution"? In most cases the exact solutions answer questions about the spectrum-ground state energy per particle, thermodynamics, excitations near the ground state,

existence and calculation of energy gaps, etc. In some cases one can vary the boundary conditions. In addition, Baxter has a very strong conjecture for the long-range-order in the ground state of class B. Finally, we remark that very likely the low-lying collective excitations also scatter without diffraction.

One further direction for the generalization of spin problems becomes apparent if we consider the Pauli spin matrices as generators of the continuous group $SU(2)$. Then a natural generalization of the Heisenberg magnets to groups $SU(N)$ would be [5]

$$(14) \qquad\qquad H = \pm \sum_{n.n.} \sum_{\alpha=1}^{N^2-1} F_\alpha F_\alpha';$$

where F_α are the $N^2 - 1$ generators of $SU(N)$, and the quadratic form is chosen to be $SU(N)$ invariant. This multi-component one-dimensional problem has a solution given by Bethe's ansatz.

And finally, the one-dimensional Hubbard model has been solved exactly by Bethe's ansatz [6]. This is a model for a two-component fermion system with both nearest-neighbor "hopping" terms and an interaction between particles on the same lattice site. The other spin systems also have alternative interpretations as lattice gas problems.

3. Based on these spin systems are certain two-dimensional lattice models for order-disorder transitions. The connection is not at all direct, nor are the models easily motivated in a few words. Yet certainly the unexpected behavior of the exact solutions of these models represents one of the most exciting recent developments in the theory of critical phenomena.

Lattice models based on the first spin system, Class A, obey the so-called ice rule (for a review, see [7]); while models based on the second class, B, are of the eight-vertex type [8]. Both models exhibit a range of unusual, i.e., non-Ising-like, behavior. We also mention as a point of general interest, that Baxter's method of solving the eight-vertex problem utilizes in a fundamental way the idea of an iso-spectral transformation.

4. Bethe's ansatz has also been found useful, either exactly or asymptotically, for the wave functions of a variety of one-dimensional continuum quantum systems. This line of development began with the exact treatment of the one-dimensional δ-function boson problem [9] with Hamiltonian

$$(15) \qquad\qquad H = - \sum_{j=1}^{N} \partial^2/\partial x_j{}^2 + 2C \sum_{i<j} \delta(x_i - x_j).$$

It was found that in the repulsive case, the wave function was given by Bethe's ansatz. Thus most questions regarding the spectrum can be rather completely answered. In particular, the low-lying excitations are found to be of two kinds, reminiscent of a particle-hole spectrum for fermions, or a particle-phonon spectrum for bosons. This two-branch form is quite characteristic of those systems for which Bethe's ansatz gives a solution.

We here propose that for such a system, which in addition possesses a non-trivial classical limit, the particle branch of the excitation spectrum is to be identified with the soliton mode of the classical problem. Recent and direct evidence for this in the case of the Toda lattice will be presented in the second section of this talk.

To proceed with subsequent generalizations, this original δ-function problem has been solved for mixtures of particles with various statistics, and for attractive interaction when the thermodynamic limit exists.

Also, Bethe's ansatz provides an asymptotic solution to similar one-dimensional problems with two-body potentials of the following forms [10, 11, 12, 13]

(16) $V(r) = g/r^2, \ = g/\sin^2 (r\pi/L), \ = g/\sinh^2 (r\pi/L).$

The final example in fact includes the Toda lattice as a limiting case, and will be discussed in detail in the second section of this talk.

5. To summarize then: It has been found that for a very wide variety of physically motivated one-dimensional quantum systems, Bethe's ansatz provides us either exactly or asymptotically with a wave function. The essential content of Bethe's ansatz is that there is no diffraction—what comes out is what went in. This I would propose is the essence of the soliton phenomena.

II. **New Results.** In this second section we wish to present a method which allows the exact quantization of a class of one-dimensional systems; the class includes the limiting cases of the inverse square potential and the Toda lattice. Further, the method for the first time makes a connection between two very active areas of many-body physics: the exact solution of one-dimensional quantum systems by a wave function of the form known as Bethe's ansatz, and the investigation of soliton solutions of various non-linear classical one-dimensional systems. For the Toda lattice in particular, we show that the particle and hole excitations of the quantum problem reduce in the classical limit to solitons and cnoidal waves respectively.

6. We wish to quantize a class of one-dimensional N-particle systems governed by the Hamiltonian:

$$(17) \qquad H = - \frac{\hbar^2}{2m} \sum_{j=1}^{N} \partial^2/\partial x_j^2 + g \sum_{j<k}^{N} \left[\sinh \left(\frac{x_j - x_k}{a} \right) \right]^{-2}$$

In particular, we shall be concerned with the thermodynamic limit $N, L \to \infty$ $d \equiv N/L$ finite; here L is the box size.

The potential energy has the following limiting forms:

$$(18) \qquad g \sum_{j<k} \left[\sinh \frac{(x_j - x_k)}{a} \right]^{-2}$$

$$\begin{cases} \to a^2 g \sum_{j<k} (x_j - x_k)^{-2}, |x_j - x_k| \to 0 \\ \to 4g \sum_{j} e^{-2 \frac{(x_{j+1} - x_j)}{a}}, |x_j - x_k| \to \infty. \end{cases}$$

In the second expression, the x's are ordered: $x_1 < x_2 < \cdots < x_N$. Thus the first limit of high density corresponds to the inverse square potential [10, 11, 12, 13] while the second limit of low density corresponds to the Toda lattice [14]. The inverse square problem has previously been quantized by the author.

Before beginning the actual calculation, it is convenient to simplify the notation by a judicious choice of units. First, let us scale all lengths by a; that is, we set $a = 1$. Second, let us set the mass so that $\hbar^2 = 2m$. Finally, we define $2S(S + 1) = g$, $S \geqq -1$.

7. As the starting point we take the results of a recent preprint of F. Calogero, C. Marchioro and O. Ragnisco [15]. In this work, it is shown that the problem we are considering has N conserved quantities. In the event that the particles are far apart, so that the potential energy is negligible, then as might be expected the N conserved quantities are simply the momenta of the isolated particles.

But it is recognized that if the momenta of the isolated particles are uniquely specified, then they will be the same for any arrangement of particles. And in particular the momenta of isolated particles will simply be rearrangements of the incoming momenta k_j ($j = 1, \cdots, N$). Thus the asymptotic wave function will be of the form

$$(19) \qquad \psi(x_1, \cdots, x_N) \sim \sum_{P} A(P) \prod_{j=1}^{N} e^{ik_{Pj} x_j}.$$

The symbol P represents one of the $N!$ permutations of the k's given by $(P1, P2, \cdots, PN)$. This wave function we recognize to be Bethe's ansatz [16], as discussed in the first section.

Further, the coefficients $A(P)$ must be simply given as products depending only on the two-body phase shift $\delta(k)$. For if one constructs wave packets arranged so that three particles are never simultaneously near, then the k's will be rearranged in pairs, and the phase difference between corresponding coefficients will be

(20)
$$A(\cdots k', k \cdots)/A(\cdots k, k' \cdots)$$
$$= -e^{-i\delta(k-k')}.$$

But if one obtained a different result as one allowed three packets to overlap, then this difference would be diffraction and would produce k's different from the original k's or their permutations. And this, by the results of Calogero, et al., does not occur.

Given the asymptotic wave function of the form of equations (3) and (4), one may immediately apply the results of the author [10, 11, 12, 13] to obtain the complete thermodynamics, including at zero temperature, and the excited states. This will be done in section 4; first we must derive the two-body phase shift $\delta(k)$.

8. If one considers the problem with $N = 2$, it is convenient to work with center of mass coordinates. Let $r = x_2 - x_1$; then the Schrödinger equation becomes:

(21)
$$-\frac{d^2\psi}{dr^2} + \frac{S(S + 1)\psi}{\sinh^2 r} = \frac{E}{2}\psi.$$

We write $k^2 = E/2$. Thus asymptotically

(22)
$$\psi \underset{r\to\infty}{\sim} e^{-ikr} - e^{-i\delta(2k)+ikr}.$$

We further require that near the origin,

(23)
$$\psi \underset{r\to0}{\sim} r^{1+S}.$$

This second condition restricts ψ to a particular solution, and thus determines $\delta(k)$.

The differential equation is not particularly difficult, and one may readily find a solution in terms of hypergeometric functions; see Landau and Lifshitz [17], for example. Then by standard manipulations of appropriate asymptotic expressions, one finds

(24) $\delta(k) = 2[\arg \Gamma(1 + S + ik/2) - \arg \Gamma (1 + ik/2)].$

The symbol $\arg \Gamma$ is the argument of the gamma function. We shall use exclusively the derivative of $\delta(k)$, $\delta'(k)$. Note the following special cases:

a. S = integer,

(25a)
$$\delta'(2k) = \sum_{N=1}^{S} \frac{n}{n^2 + k^2} \; ;$$

b. $k \gg S$, inverse square potential,

(25b)
$$\delta'(k) = 2\pi S \, \delta(k);$$

(the right hand side of the equation is Dirac's delta function);

c. $S, k \to \infty$, classical limit,

(25c)
$$\delta'(2k) = \tfrac{1}{2} \ln(1 + (S/k)^2).$$

As a remark of interest, the S-matrix $(e^{-i\delta(k)})$ generally has poles in the upper half complex k-plane, although in no case do they correspond to bound states. In particular, if $S = 1$, the phase shift is identical to that of the attractive delta-function potential, the only difference being the existence or non-existence of the bound state. Thus our potential is of the type first remarked upon by Bargmann [18]. And for $S = 1$, our results will then be the analytic continuation to negative coupling constant of the corresponding results of the repulsive delta-function potential.

9. Having determined the two-body phase shift, one may apply periodic boundary conditions to the asymptotic wave-function, and hence derive N equations for the k's:

(26)
$$e^{ikL} = (-1)^{N+1} \exp\left\{ i \sum_{k'} \delta(k - k') \right\}.$$

(compare with [19], [20]). Taking the logarithm of the above equation, one obtains:

(27)
$$k = \frac{2\pi}{L} \, I(k) + \frac{1}{L} \sum_{k'} \delta(k - k').$$

$2\pi \, I(k)$ is equal to $\log(\pm 1)$, and serves as the quantum number for the states.

Generally, in the thermodynamic limit, the k's will distribute themselves with a density $\rho(k)$. For instance, in the ground state,

(28)
$$\frac{1}{2\pi} = \rho(k) + \frac{1}{2\pi} \int_{-B}^{B} \delta'(k - k')\rho(k') \, dk'$$

with

$$(29) \qquad E_o/L = \int_{-B}^{B} k^2 \rho(k)\, dk$$

$$d = N/L = \int_{-B}^{B} \rho(k)\, dk.$$

For excitations near the ground state, the dispersion curve of energy ΔE and momentum ΔK is given parametrically by

$$(31) \qquad \Delta K(k) = 2\pi \int_{B}^{k} \rho(k')\, dk',$$

$$(32) \qquad \Delta E(k) = |\epsilon(k) - \epsilon(B)|.$$

Here ρ is determined by (12), while ϵ is determined by

$$(33) \qquad -\mu + k^2 = \epsilon(k) + \frac{1}{2\pi} \int_{-B}^{B} \delta'(k - k')\epsilon(k')\, dk'.$$

Here μ is the chemical potential which at zero temperature is given by

$$(34) \qquad \mu = \partial E_0/\partial N$$

The corresponding equations for finite temperature are given in [21]. The explicit evaluation will be presented in a subsequent paper.

Note that the kernel of the integral equation, $1/2\pi\, \delta'(k)$, is well-behaved. It is symmetric, and the eigenvalues lie within the range of the Fourier transform: 0 to S. However, a simple iteration will in general not converge if $S > 1$.

10. We shall here be content with recovering the classical results for the Toda lattice. In the classical limit, expression (25c) applies, and for the Toda limit $k \rightarrow 0$ (low density). Hence the equation (28) for $\rho(k)$ becomes

$$(35) \qquad \frac{1}{2\pi} = \rho(k) + \frac{1}{4\pi} \int_{-B}^{B} \ln \left[\frac{2S}{(k - k')} \right]^2 \rho(k')\, dk'$$

One sees that the first term on the right side of the equation is negligible compared with the second for large k, and hence large B. Thus the integral equation reduces to

$$(36) \qquad 1 = d \ln (2S) - \int_{-B}^{B} \ln |k - k'| \rho(k')\, dk'$$

Upon taking the derivative of (20) with respect to k, one finds

$$(37) \qquad 0 = \int_{-B}^{B} \frac{\rho(k')\,dk'}{k - k'} .$$

This is a finite Hilbert transform of a particularly simple kind, and the solution is given by

$$(38) \qquad \rho(k) = \frac{C}{(B^2 - k^2)^{1/2}} .$$

The constant C may be evaluated with the help of (30) as $C = d/\pi$. Then, upon substituting (38) into (36), we find

$$(39) \quad \text{or} \qquad \begin{aligned} &1 = d \ln (2S) - d \ln B + d \ln 2 \\ &B = 4S e^{-1/d}. \end{aligned}$$

The energy density is given by (29) as

$$(40) \qquad E_0/L = dB^2/2.$$

Elimination of B between (39) and (40), yields the energy per particle

$$(41) \qquad E_0/N = 8S^2 e^{-2/d} = 4g\, e^{-2/d}.$$

From (18), this is the correct value for the Toda lattice.

11. Next, we consider the excitations in the same limit of a classical Toda lattice. In the same way as before, (33) reduces to

$$(42) \quad \begin{aligned} -\mu + k^2 &= \frac{\ln (2S)}{2\pi} \int_{-B}^{B} \epsilon(k)\,dk \\ &\quad - \frac{1}{2\pi} \int_{-B}^{B} \epsilon(k')\,dk' \ln|k - k'|. \end{aligned}$$

Upon taking the derivative, we again have the finite Hilbert transform

$$(43) \qquad 2k = \frac{1}{2\pi} \int_{-B}^{B} \frac{\epsilon(k')\,dk'}{k' - k} .$$

The solution of this problem once again is rather simple, and we find

$$(44) \qquad \epsilon(k) = -4\,(B^2 - k^2)^{1/2}.$$

Having thus determined $\rho(k)$ and $\epsilon(k)$ for $|k| < B$, we employ equations (31) and (32) to determine the first branch of the dispersion curve as;

Type I:

(45)
$$\Delta K = 2d \cos^{-1} (k/B) = 2d\,\theta$$

$$\Delta E = 4\,(B^2 - k^2)^{1/2} = 4B|\sin\,\theta|$$

Equation (45) is taken as the definition of θ. The group velocity is given as

(46)
$$\frac{d(\Delta E)}{d(\Delta K)} \equiv v = \frac{2B}{d}\,\cos\,\theta,\, 0 \leqq \theta \leqq \pi.$$

We must now determine $\rho(k)$ and $\epsilon(k)$ for $|k| > B$. This is done by taking the basic equations (28) and (33), and using the above solutions within the integrals. Thus upon continuing k outside the region of integration $|k| < B$, we find [21]

(47)
$$\rho(k) = \frac{1 - d\ln(2S)}{2\pi} + \frac{d}{2\pi^2} \int_{-B}^{B} \frac{\ln|k - k'|\,dk'}{(B^2 - k^2)^{1/2}}$$

$$= d|\cosh^{-1}(k/B)|,\, |k| > B$$

and

$$\epsilon(k) = k^2 - \mu + \frac{2}{\pi} \int_{-B}^{B} dk'(B^2 - (k')^2)^{1/2} \ln \left[\frac{2S}{|k - k'|} \right]$$

(48) $= |k|(k^2 - B^2)^{1/2} - B^2|\cosh^{-1}(k/B)|,\, |k| > B.$

Again, upon using the solutions (47) and (48) in (31) and (32), we determine the second branch of the dispersion curve as a function of the parameter ϕ, $k \equiv B \cosh \phi$, to be;

Type II:

(49)
$$\Delta K = dB\,[\phi \cosh \phi - \sinh \phi]$$

$$\Delta E = B^2\,|\cosh \phi \sinh \phi - \phi|.$$

The group velocity is given as

(50)
$$\frac{d(\Delta E)}{d(\Delta K)} \equiv v = \frac{2B}{d}\,\frac{\sinh |\phi|}{\phi}\,.$$

We remark that if we were to restore the original units where $\hbar$ is explicit, we would find that ΔE of the second branch is finite in the classical limit $\hbar \to 0$. On the other hand, ΔE of the first branch is proportional to $\hbar$, and thus vanishes as $\hbar \to 0$. For both types of excitations, the group velocity is finite in the limit $\hbar \to 0$.

12. We now compare our results with the results of Toda as summarized in [14]. But two remarks are in order before we make this comparison. The first observation is that in comparing dispersion curves, the total momentum is not a particularly useful concept when treating classical one-dimensional systems. The reason is that the total system may recoil with a finite momentum, without changing either the total energy, or the particle velocities, appreciably. In contrast, for a quantum system, we have instead only that the spectrum is periodic in the momentum with period $2\pi d$.

We avoid this ambiguity in the total momentum by instead comparing the energy and group velocity of the excitation. Thus the type I excitation obviously has the form of a phonon dispersion, including energy proportional to $\hbar$, and allowing for differences in units, fits Toda's equation (4.8.4). Thus we identify the type I excitation as a small amplitude cnoidal wave, or more simply a phonon.

The second remark concerns the type II excitation. Since the energy is finite in the classical limit, we are obviously tempted to identify this branch with the solitons of the Toda lattice. However, since Toda does not give us the energy of the soliton, we must first make a short calculation. In Toda's expression for the energy given in equation (4.2.7), we substitute the expressions (4.6.6) and (4.6.7), and find

$$
\begin{aligned}
E = \sum_n \{ 1 + \beta^2 \operatorname{sech}{}^2(\alpha n - \beta t) \\
+ \frac{\beta^2}{2} [\tanh{(\alpha(n + 1) - \beta t)} - \tanh(\alpha n - \beta t)]^2 \}
\end{aligned}
$$

(51)

As in Toda, $\beta = \sinh \alpha$. After a few manipulations involving Jacobi theta functions, we find

(52) $E = \beta^2 \coth \alpha = \sinh \alpha \cosh \alpha.$

The velocity of a soliton is given in Toda as (4.6.9),

(53) $v = \beta/\alpha = \dfrac{\sinh \alpha}{\alpha} \; .$

Looking back to (49) and (50), we are very tempted to equate the parameters α and ϕ. However, our energy differs by a term $B^2|\phi|$.

And now we finally come to the second remark: We have determined our excitations at a fixed N and L. However, in the calculation of Toda there is a net compression due to the presence of the soliton. Thus, in fact we should expect the two energies to differ by an amount

equal to the product of the pressure and the change of volume. The change of volume due to the soliton is given by Toda in equation (4.6.10) as 2α. We easily calculate the pressure as dE_0/dL using equation (25), and find $p = B^2$. Thus, allowing for the difference in units, we find that the type II excitation is to be identified with the soliton of the classical Toda Lattice.

13. To summarize: We have presented a method to quantize a variety of one-dimensional quantum systems. This method is based on the property of "no diffraction", which may be independently proven. The class of such systems includes the Toda lattice, and in this case, we are able to show that our calculated quantities reduce to the known classical results. In particular, an identification is made of the excitations with small amplitude cnoidal waves and solitons.

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