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PRIME IDEAL POSETS IN NOETHERIAN RINGS

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The prime ideals of a noetherian ring with the inclusion relation form a partially ordered set, which shall be called an N-poset. How can one tell if a given poset is an N-poset? Surprisingly little is known about this question. In [2, pp. 66–68], Hochster treats it briefly and offers a partial list of axioms dealing with the equivalent question of classifying the spectral topologies of noetherian rings. He also offers a question due to Kaplansky which provided the initial impetus for the work presented here. The original question is, "Must two primes, P, P' of height greater than one in a noetherian domain necessarily have a nonzero prime Q in their intersection?" Alternatively, it is equivalent to asking if the poset of nonzero primes in a noetherian domain can ever be decomposed into an (ordered) disjoint union of proper subsets. (Of course, we exclude the trivial case of height one maximal ideals.) More recently, it has been shown by McAdam [4], among others, that the answer is yes. Following this line of thought, we would like to know more about this decomposition. Primarily, what kind of component pieces can be used and how many (finite or infinite) of them can there be? To this particular aspect of the problem, this paper is addressed.

The crux of this paper is a technique which enables us to intersect certain collections of noetherian domains and obtain a new domain which is again noetherian. The poset of nonzero primes in this new domain decomposes into the disjoint union of the initial posets of nonzero primes. While the procedure is not completely arbitrary, the collection may be infinite and entirely new types of examples of N-posets can be formed. In § 2, we proceed to build some examples which seem particularly enlightening. Many more are possible. It is hoped that, in an area which has suffered from a paucity of examples, both the results and the construction leading to it will provide some relief.

NOTATION. K will be a fixed field. The symbols X, Y, Z unsubscripted or with one subscript will denote sets of indeterminates. A single indeterminate will always carry *two* subscripts. Cardinalities of sets will be denoted $|K|$, etc. The rings we shall employ in the construction will be localizations of polynomial rings; the letters f, g will be reserved for polynomials. $\mathfrak{D}$ will be an index set for a collection

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of rings and Greek letters will denote members of $\mathfrak{D}$. Finally, i, j will denote other indexes — the index set will be clear from the context. In dealing with a specific ring R_α, it might seem proper to subscript everything pertaining to it. However, technical precision will be sacrificed to make the notation less cumbersome; subscripts will be used only when they serve the cause of clarity.

Definition. A domain R will be called a *G-ring* provided there are sets of indeterminates X, Z such that:

1. R is a localization of $K(X)[Z]_{(Z)}$.
2. R is noetherian.
3. The infinite cardinal $|X| \geqq |K|, |Z|$.

Definition. A set $\{R_\alpha \mid \alpha \in \mathfrak{D}\}$ of G-rings is *compatible* provided $|X_\alpha|$ is independent of α and $|X_\alpha| \geqq |\mathfrak{D}|$.

The ensuing construction will enable us to "paste together" the posets of a compatible set of G-rings. It should be remarked here that G-rings are quite numerous and hence this construction will have considerable applicability. (For example, let K be countable, X infinite, and Z finite. Then choose R to be any ring satisfying condition (1).) On the other hand, the first condition does restrict us somewhat. R must be integrally closed among other things — but it need not be local!

Theorem 1. *Let $\{R_\alpha\}$ be a compatible set of G-rings. Then these rings may be embedded in a common quotient field in such a way that $R = \bigcap e(R_\alpha)$ will be noetherian.*

Proof. The cardinality restrictions force the quotient fields to be isomorphic; our common quotient field will be $K(Y)$, where $|Y| = |X_\alpha|$. Unless care is taken in the definition of the embeddings, R will not usually be noetherian. To obtain this result, we will control the prime ideals, making sure we end up with a set of nonzero primes which is naturally isomorphic to the disjoint union of the sets of nonzero primes of the R_α's. Toward this end, when we have defined $e(R_\alpha)$, the image of R_α in $K(Y)$, we want every prime of $e(R_\alpha)$ to contain a set of indeterminates unique to that prime. We also want each indeterminate in Y which is not a unit in $e(R_\alpha)$ to be a unit in $e(R_\beta)$ for each $\beta \neq \alpha$. Then, if $K[Y] \subset R$, the primes cannot coincide. We will now describe the rather technical embeddings.

Each indeterminate has its own purpose. To designate that purpose, we suggestively index Y. First partition Y into $|\mathfrak{D}|$ sets, each of cardinality $|Y|$, to be designated Y_α for each $\alpha \in \mathfrak{D}$ (possible since $|\mathfrak{D}| \leqq |Y|$). Next, each Y_α is divided into two sets — the first of cardinality

$|Y|$, to be placed in 1-1 correspondence with the nonzero polynomials of $K[X_\alpha, Z_\alpha]$ (denoted $K[X_\alpha, Z_\alpha])^*$), and the second of cardinality $|Z_\alpha|$, to be placed in 1-1 correspondence with Z_α. Now we have $Y_\alpha = \{Y_{\alpha f}\} \cup \{Y_{\alpha i}\}$, where the f range over $K[X_\alpha, Z_\alpha]^*$ and $\{Y_{\alpha i}\} \sim \{Z_{\alpha i}\} = Z_\alpha$.

We now describe a K-algebra map $e : R_\alpha \to K(Y)$ by describing the action of e on the indeterminates of R_α. The set of images $\{e(X_\alpha), e(Z_\alpha)\}$ will be algebraically independent, thus insuring that e is an embedding. First set $e(Z_{\alpha i}) = Y_{\alpha i}$. Defining e on X_α requires additional machinery.

Let $\mathcal{O}$ be the least ordinal of cardinality $|Y|$. (Proper initial segments of $\mathcal{O}$ have smaller cardinality than $\mathcal{O}$ and by ordering sets in the following via correspondence with $\mathcal{O}$, we needn't worry about prematurely exhausting our sets.) Order $K[X_\alpha, Z_\alpha]^*$ via correspondence with $\mathcal{O}$. Devise a well-ordering $X_\alpha = \{X_{\alpha i}\}$ together with an order-preserving injection ϕ_1 from $K[X_\alpha, Z_\alpha]^*$ to X_α satisfying the following restrictions: (i) for $f \in K[X_\alpha, Z_\alpha]^*$, $\phi_1(f) = X_{\alpha i}$ implies $i > j$ for each $X_{\alpha j}$ appearing in f. (ii) if $f, g \in K[X_\alpha, Z_\alpha]^*$ and f and g are adjacent in the well-ordering of $K[X_\alpha, Z_\alpha]^*$, then there exists a non-zero finite number of indeterminates in X_α between $\phi_1(f)$ and $\phi_1(g)$. (As we do not intend a 1-1 correspondence, this step is easily performed.) In the second restriction, the finiteness condition prevents premature exhaustion of the set and the non-empty condition says there will be $|Y|$ indeterminates not corresponding to any polynomial. Let ϕ_2 be a 1-1 correspondence from $\bigcup_{\beta \neq \alpha} Y_\beta$ to the set of left-over indeterminates. Then $\phi = \phi_1 \cup \phi_2 : K[X_\alpha, Z_\alpha]^* \cup \bigcup_{\beta \neq \alpha} Y_\beta \to X_\alpha$ is a 1-1 correspondence. Now we inductively define $e(X_{\alpha i})$ beginning with the least $X_{\alpha i}$ in the well-ordering of X_α.

Set

$$e(X_{\alpha i}) = \begin{cases} Y_{\beta j} & \text{if } \phi(Y_{\beta j}) = X_{\alpha i} \\ \dfrac{e(f)}{Y_{\alpha f}} & \text{if } \phi(f) = X_{\alpha i}. \end{cases}$$

Note that the manner in which the order was chosen guarantees that $e(f)$ will have been defined when it is needed. For convenience, denote $e(f)/Y_{\alpha f}$ by $Y'_{\alpha f}$, set $Y_\alpha{}' = \{Y'_{\alpha f} \mid f \in K[X_\alpha, Z_\alpha]^*\}$ and $Y' = UY_\alpha{}' : \alpha \in \mathfrak{D}$. Now e is a 1-1 correspondence from X_α to $Y_\alpha{}' \cup \bigcup_{\beta \neq \alpha} Y_\beta$ and from Z_α to $\{Y_{\alpha i}\}$.

Next we wish to see that the image of the set of indeterminates is algebraically independent. Because relations of algebraic dependence involve only finitely many elements, it will suffice to show that we can extend K to our quotient field by adjoining elements from the image set one at a time and obtain a transcendental extension at each step. This will certainly be true if each single extension has the form $K(W) \to K(W)(Y_{\beta j})$ where W is a subset of Y and $Y_{\beta j} \in Y - W$. Perform the extensions in the same order that we defined the mapping. Then, at each step, we adjoin some $Y_{\beta j}$ and inductively the extension has the desired form, or we adjoin some $e(f)/Y_{\alpha f}$. As a field extension, because $e(f) \in K(W)$, we are exactly adjoining $Y_{\alpha f}$. Hence e is an embedding. In this fashion, we obtain a family of embeddings. It will sometimes be convenient to denote the embedding of R_α by e_α. To avoid the necessity of frequently referring back to the construction, we shall state those immediately apparent properties which shall be needed later.

LEMMA 1. (i) $e : K(X_\alpha)[Z_\alpha]_{(Z_\alpha)} \to K(U_{\beta \neq \alpha} Y_\beta, Y_\alpha')[\{Y_{\alpha i}\}]_{(\{Y_{\alpha i}\})}$ *is a natural isomorphism.*

(ii) $Y, Y' \subset e(R_\alpha)$.

(iii) *Every prime in* $e(R_\alpha)$ *contains some* $Y_{\alpha f}$.

PROOF. Statement (i) is obvious from the definition of e. For (ii), (i) handles all cases except $\{Y_{\alpha f}\}$ and Y_β' for $\beta \neq \alpha$. As $Y_{\alpha f} = (1/Y'_{\alpha f})\, e(f)$, the first set presents no difficulty. Also $Y'_{\beta f} = 1/Y_{\beta f}\, e_\beta(f)$ and because $Y_{\beta f}$ is a unit for $\beta \neq \alpha$, $Y'_{\beta f}$ is in $e(R_\alpha)$ if $e_\beta(f)$ is. Next observe that $e_\beta(f)$ will be a polynomial in $\{Y_{\gamma j}, Y'_{\beta g} \mid \gamma \neq \beta, Y'_{\beta g} < Y'_{\beta f}\}$. As $Y_\gamma \subset e(R_\alpha)$, we find that $Y'_{\beta f} \in e(R_\alpha)$ if all $Y'_{\beta g} < Y'_{\beta f}$ are. By induction, we have the desired result. Finally, for (iii), note that every prime of R_α contains some polynomial f. Consequently, every prime of $e(R_\alpha)$ contains some $e(f)$. As $Y'_{\alpha f}$ is a unit in $e(R_\alpha)$, such a prime also contains the corresponding $Y_{\alpha f}$.

As suggested in the statement of the theorem, define $R = \bigcap e(R_\alpha)$. It remains to show that R is noetherian. A theorem of Heinzer and Ohm [1, p. 295, Corollary 18] asserts that if R is the intersection of a family of flat R-algebras, each of which is noetherian, such that every ideal of R generates a proper ideal in at least one but only finitely many of the algebras, then R is noetherian. R is clearly the intersection of a family of noetherian R-algebras $\{e(R_\alpha)\}$. Recalling that localizations are flat extensions, we will show that each $e(R_\alpha)$ is a localization of R in lemma 2. The remainder of the Heinzer-Ohm hypothesis will be verified in lemmas 3 and 4, thereby completing the proof of theorem 1.

LEMMA 2. *Suppose* $\{M_{\alpha j}\}$ *is the set of maximal ideals of* R_α. *Denote* $e(M_{\alpha j}) \cap R$ *by* $\tilde{M}_{\alpha j}$. *Then* $R_{\cup \tilde{M}_{\alpha j}} = e(R_\alpha)$.

PROOF. By lemma 1(ii), R contains $K[Y, Y']$. By lemma 1(i), we can see that $e(R_\alpha)$ is a localization of $K[Y, Y']$ and therefore a localization of R. As the units of $e(R_\alpha)$ are outside $\cup \tilde{M}_{\alpha j}$, they will be units in $R_{\cup \tilde{M}_{\alpha j}}$. Hence $e(R_\alpha) \subseteq R_{\cup \tilde{M}_{\alpha j}}$. The reverse inclusion is obvious.

LEMMA 3. *Every element of R is a non-unit in at most finitely many* $e(R_\alpha)$.

PROOF. Each $r \in K(Y)$ can be written using only finitely many indeterminates from Y. If this expression for r involves no indeterminates from the set Y_β (and this must be true for all but finitely many β), then lemma 1(i) asserts that r is a unit in $e(R_\beta)$.

LEMMA 4. *If P is prime in R, then for exactly one* α, $P \cdot e(R_\alpha) \neq e(R_\alpha)$.

PROOF. Choose an arbitrary element $r \in P$. Let $\{P_{\beta i} \mid \text{both subscripts vary}\}$ be the set of height one primes of $e(R_\beta)$ which contain r. By lemma 3, the set contains primes for only finitely many β; as R_β is noetherian, the set contains only finitely many for each β. Thus, the set is finite. Now, if $s \in P_{\beta i} \cap R$ for every prime in the set, a power of s will be a multiple of r. (To see this, simply note that it is true in each $e(R_\beta)$ and a uniform power may be selected because r is almost always a unit.) Therefore, s is in the radical of r and consequently $s \in P$. Because P contains a finite intersection of primes, it contains one of them. Thus, we have shown that $P = \bigcup (P_{\gamma i} \cap R)$, the union of those primes which it contains.

Next we claim that $P_{\alpha i} \cap R \subset P$ for only one $\alpha \in \mathfrak{D}$. Suppose not and let $P_{\alpha i} \cap R$, $P_{\gamma i} \cap R \subset P$ for $\alpha \neq \gamma$. By lemma 1(iii), there exists f, g such that $Y_{\alpha f}, Y_{\gamma g} \in P$. Therefore $Y_{\alpha f} + Y_{\gamma g} \in P$. But this element is invertible in $e(R_\beta)$ for all $\beta \neq \alpha, \gamma$. In $e(R_\alpha)$ (a symmetric argument works in $e(R_\gamma)$.),

$$
\begin{aligned}
Y_{\alpha f} + Y_{\gamma g} &= \frac{1}{Y'_{\alpha f}} e_\alpha(f) + Y_{\gamma g} \\
&= \frac{1}{Y'_{\alpha f}} (e_\alpha(f) + Y_{\gamma g} Y'_{\alpha f}) \\
&= \frac{1}{Y'_{\alpha f}} (e_\alpha(f + X_{\alpha i} X_{\alpha j}))
\end{aligned}
$$

where $X_{\alpha i}, X_{\alpha j} = e_\alpha^{-1}(Y_{\gamma g}), e_\alpha^{-1}(Y'_{\alpha f})$ respectively. Because $Y_{\alpha f}$ is not invertible in $e(R_\alpha)$, f can't be invertible in R_α. Hence, from the definition (condition 1) of G-ring, f must have zero constant term when considered as a polynomial in Z_α (with coefficients in $K[X_\alpha]$). The same can't also be true of $f + X_{\alpha i}X_{\alpha j}$ and so this element is invertible in R_α. Thus $Y_{\alpha f} + Y_{\gamma g}$ is the product of two units in $e(R_\alpha)$. Thus $(Y_{\alpha f} + Y_{\gamma g})^{-1} \in e(R_\beta)$ for all β. This is a contradiction, verifying the claim.

Now, because $P = \bigcup(P_{\alpha i} \cap R)$ for some fixed α, $P \subseteq \bigcup \tilde{M}_{\alpha i}$ and so $PR_{\cup \tilde{M}_{\alpha i}}$ is a proper ideal of $R_{\cup \tilde{M}_{\alpha i}}$. By lemma 2, this is the required result. This completes the proof of theorem 1.

THEOREM 2. *The set of nonzero primes of R is order-isomorphic to the disjoint union of the sets of the nonzero primes of the T_α's.*

PROOF. Because $e(R_\alpha)$ is a localization of R, the set of primes of R contained in $\bigcup \tilde{M}_{\alpha j}$ is order-idomorphic to the set of primes of R_α. By lemma 4, every nonzero prime of R is contained in $\bigcup \tilde{M}_{\alpha j}$ for exactly one α. This partitions the nonzero primes as desired and it is clear that no inclusion relations involve primes from different sets. Note that Theorem 2 shows that R is exactly the noetherian domain we desire.

2. **Examples.** The reader should have little difficulty constructing new N-posets using this construction. We will present a pair of examples which seem to be of some interest. These, in turn, may suggest others.

The easiest type of example involves the 'pasting together' of the posets of local rings. Our first example uses this idea to answer an infinite analogue of the original Kaplansky question.

EXAMPLE 1. There is a noetherian domain R with infinitely many maximals of arbitrary height such that every nonzero prime is contained in a unique maximal ideal.

PROOF. Let K be a countable field and X a countably infinite set of indeterminates. Then let Z_i be a finite set of indeterminates. (The exact number of indeterminates in Z_i will be the height of a maximal ideal. For each i, we may select a different integer.) Set $R_i = K(X)[Z_i]_{(Z_i)}$; R_i is a G-ring of the height we choose. Further, $\{R_i \mid 1 \leqq i < \infty\}$ is a compatible set of G-rings. The domain R constructed in Theorem 1 with these ingredients clearly has the desired property.

On the other hand, the initial G-rings may be far from local.

EXAMPLE 2. There is a two-dimensional domain R with an infinite set of maximals $\{M_i \mid 0 \leqq i < \infty\}$ such that if $i, j \neq 0$, then $M_i \cap M_j$ contains a nonzero prime but (for the height 2 maximal M_0) $M_0 \cap M_i$ never contains a nonzero prime.

PROOF. Let K be a countable field and X a countably infinite set of indeterminates. Let $Z = \{Z_{11}, Z_{12}, Z_{13}\}$. Then set $R_1 = K(X)[Z]_{(Z)}[Z_{11}^{-1}]$. This ring is a localization of an affine $K(X)$-algebra in three indeterminates and the poset attached to this ring is a subset of the poset of the entire algebra. It has infinitely many maximals, each of height 2, and the intersection of any two contains a nonzero prime. (This last fact is a general fact about affine domains — the proof is straightforward and is left to the reader.)

Next set $R_2 = K(X)[Z_{21}, Z_{22}]_{(Z_{21}, Z_{22})}$, a two-dimensional local domain. $\{R_1, R_2\}$ is a compatible set of $\bar{G}$-rings and so the construction yields a noetherian domain $R = e(R_1) \cap e(R_2)$. Letting $M_0 = e(M(R_2)) \cap R$ represent the maximal coming from the local ring, we see that R has the desired poset of primes.

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