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A CONNECTED ABSOLUTELY FLAT SCHEME

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The underlying topological space of an absolutely flat scheme has an open cover by totally disconnected compact spaces, which are conveniently referred to as Boolean spaces. (In fact, Hochster has shown that this is a characterization [1, p. 59, Thm. 9]. Please note that we are not assuming that schemes are separated as he does! Our schemes coincide with his notion of preschemes.) Can an absolutely flat scheme have a non-trivial connected underlying space? In light of the characterization we have noted, this is actually a purely topological question. The answer is known to be negative for certain types of schemes, notably separated schemes — the most comprehensive result in this direction is an (unpublished) theorem and proof of Olivier, which we now reproduce:

THEOREM. *Let X be a connected absolutely flat scheme. If X admits a locally finite cover $\{U_i \mid i \in I\}$ by separated (Hausdorff) open sets, then X is a point.*

PROOF. Let V be a non-empty open set which intersects only a finite number of U_i. Let $J \subset I$ be a subset, maximal with respect to the property that $V \cap \bigcap_{j \in J} U_j \neq \emptyset$. ($J$ is clearly finite.) Let W be a non-empty affine open set contained in this intersection; as W is compact and U_j is separated, W is closed in U_j for each $j \in J$; $W \cap U_i = \emptyset$ for $i \in I - J$, so W is closed in X. Finally, as X is connected, $W = X$. Thus X is a point.

Nonetheless, such a non-trivial connected space does exist and we shall now present a suitable construction.

CONSTRUCTION. Let S be a perfect separable Boolean space, e.g., the Cantor set. Let $\{x_i\}$ denote a countable dense subset of S. Set $S_n = S - \{x_1, \cdots, x_n\}$ and note that S_n is locally compact and actually has a basis of compact open sets. Because of this latter fact, the one-point compactification of S_n will be a Boolean space. Call this compactification Y_n and label the new point y_n. Y_n is so defined for every positive integer n; we set $Y_0 = S$.

Now we are ready to construct the desired space X. Let the underlying set, $|X|$, be $|S| \cup \{y_i\} = \cup\, |Y_i|$. We define $V \subset |X|$ to be open

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if and only if $V \cap Y_m$ is open in Y_m for every m. This is obviously a topology.

We would also like to know that if U is an open subset of Y_n, then U is open in X. This is true because (for $m \neq n$) $U \cap Y_m = U \cap S \cap Y_m$. Now, for U to be open in Y_n, $U \cap S$ must be open in S, implying $U \cap S \cap Y_m$ is also open in S (as only finitely many points are removed). This guarantees that $U \cap Y_m$ is also open in Y_m. Hence X induces the desired subspace topology on Y_n, with each Y_n being open in X. This means $X = \bigcup Y_i$ is a topological space.

Main Theorem. *The space X constructed above admits an open cover by Boolean spaces but is also connected* (*and of course is not a point*).

Proof. We have already seen that the Boolean spaces Y_n are an open cover for X. Unless X is connected, it must have a proper nonempty clopen set V. We may assume $x_1 \in V$ (or else use the complement V^C). Noting that $S = (V^C \cap S) \cup (V \cap S \cap Y_n) \cup \{x_i \mid 1 \leqq i \leqq n\}$ and that S is perfect and V^C is closed, we quickly deduce that x_1 is a limit point for $V \cap S \cap Y_n$. Thus, for $n \geqq 1$, $V \cap S \cap Y_n$ is not closed in S, and hence not compact. As $V \cap Y_n$ is compact, $V \cap S \cap Y_n \neq V \cap Y_n$; so $y_n \in V$ for all n.

If we were to make the additional assumption that $x_j \in V^C$ for some j, we could repeat the preceding argument and deduce the absurdity that $y_n \in V^C$ for all $n \geqq j$. Hence $\{x_i\} \cap V^C = \emptyset$. But $V^C \cap S$ is open in S and $\{x_i\}$ is dense in S. So $V^C \cap S = \emptyset$. Further, as each $y_n \notin V^C$, $V^C = \emptyset$, contradicting our assumption of a nontrivial partition of X.

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Reference

1. M. Hochster, *Prime ideal structure in commutative rings*, Trans. AMS **142** (1969), 43–60.

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