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APPROXIMATE FIBRATIONS

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1. **Introduction and basic lemmas.** The several concepts of a fibration have been important tools in the study of maps. Thus, the defining property, called the homotopy lifting property, is a valuable property for a map to have. The works of Lacher [11], [12], and also Armentrout and Price [1], and Kozlowski [10], suggest that an approximate homotopy lifting property might be almost as valuable while applying to a larger class of maps. This paper intends to show that this is indeed the case. In particular, we prove analogues to the following theorems about fibrations: the existence of a path lifting function (Proposition 1.3), the property that point inverses are absolute neighborhood retracts (Corollary 2.5), the homotopy equivalence of point inverses (Theorem 2.12), and the exact homotopy sequence of a fibration (Corollary 3.5). Our conclusions, of course, are weaker in that they give shape theoretic, rather than homotopy theoretic, information.

We use the following terminology and notation. If $A \subset X$, a topological space, a *neighborhood* of A is a set containing A in its interior; Int A denotes the *interior* of A; Cl A denotes the *closure* of A; Fr A denotes the *frontier*, or topological boundary, of A. On the other hand, the *boundary* of a topological manifold M is denoted Bd M. A *map* is a continuous function. For any positive integer q, I^q is the *q-fold product of intervals*; $I^1 = I$; and other intervals are denoted by $[a, b]$. If $H : X \times I \to Y$ is a homotopy, then $H_t : X \to Y$ is the map defined by $H_t(x) = H(x, t)$. For all metric spaces, $d(x, y)$ is the distance between points x and y, and $N(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$. Let $f : X \to Y$ and $g : X \to Y$ be maps and δ be a cover of Y. We say that f and g are *δ-close* if for each $x \in X$ $f(x)$ and $g(x)$ are contained in some member of δ. Also f and g are *δ-homotopic* if f and g are homotopic by a homotopy h such that $h(\{x\} \times I)$ is contained in some member of δ for each $x \in X$. Such a homotopy is called a δ-homotopy. If δ is a positive number, f and g are δ-close (δ-homotopic) if they are close (homotopic) relative to the cover $\{N(y, \delta/2) \mid y \in Y\}$. If Y is an *ANR* (that is, an absolute neighborhood retract for metric spaces), then for every cover ϵ of Y there is a cover δ such that any two δ-close maps from a metric

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space into Y are ϵ-homotopic by a homotopy which is fixed on the set where $f = g$ [9]. The symbol $\cong$ stands for isomorphism or homeomorphism depending on the context; $\simeq$ stands for homotopy. The homotopy class of a map is denoted $[f]$. The q-th homotopy group of a based space or pair of spaces is denoted π_q.

Definition. A surjective map $p : E \to B$ between metric spaces has the *approximate homotopy lifting property* with respect to a space X provided that, given a cover ϵ of B and maps $g : X \to E$ and $H : X \times I \to B$ such that $pg = H_0$, there exists a map $G : X \times I \to E$ such that $G_0 = g$ and pG and H are ϵ-close. The map G is said to be an *ϵ-lift* of H. If ϵ is a number, we define an ϵ-lift as above. Furthermore, G is *stationary* with H if for each $x_0 \in X$ such that $H(x_0, t)$ is constant as a function of t, the function $G(x_0, t)$ of t is constant also. The map $p : E \to B$ is said to have the *regular* approximate homotopy lifting property with respect to X if the ϵ-lift G can always be chosen to be stationary with H.

The above definition, of course, generalizes the usual homotopy lifting property, the definition of which is the same except that $pG = H$ is required rather than that pG, H be ϵ-close. Thus, the approximate homotopy lifting property holds for a larger set of maps. However, we should point out that, although shape theory is used in the latter parts of this paper, this generalization of the homotopy lifting property is probably not an appropriate generalization to the shape category. The reason for this, roughly stated, is that two maps in the shape category are "similar" if they are merely homotopic in some set, whereas we require that they be close. This paper shows that much of the theory of Hurewicz fibrations (maps with the homotopy lifting property with respect to all spaces) carries over, little changed, to this larger set of maps. We will also make some reference to the theory of Serre, or weak, fibrations (maps with the homotopy lifting property with respect to all n-cells, or equivalently, all polyhedra). The first proposition provides a tool for constructing examples of maps which satisfy the approximate homotopy lifting property with respect to metric spaces, but which do not have the homotopy lifting property for some space.

Proposition 1.1. *Let E and B be ANR's. Suppose $p : E \to B$ is a surjection with the property that for each cover δ of B there is a map $p_\delta : E \to B$ such that p_δ is δ-close to p and p_δ has the homotopy lifting property with respect to a metric space X. Then p has the approximate homotopy lifting property with respect to X.*

PROOF. Given ϵ, g, H as in the definition, choose δ such that any two δ-close maps into B are η-homotopic, where η twice star refines ϵ. Let p_δ be the map given by the hypothesis, and let $K: X \times [-1, 0] \to B$ be an η-homotopy such that $K(x, -1) = p_\delta g(x)$ and $K(x, 0) = pg(x)$. Define a homotopy Φ by $\Phi(x, t) = K(x, t)$ if $t \leqq 0$, and $\Phi(x, t) = H(x, t)$ if $t \geqq 0$. There exists a homotopy $\psi : X \times [-1, 1] \to E$ such that $\psi(x, -1) = g(x)$ and $p_\delta \psi = \Phi$. Now choose a map $q : X \to (0, 1)$ such that $H(\{x\} \times [0, q(x)])$ is contained in some member of η for each x. Define

$$G(x, t) = \begin{cases} \psi(x, 2t/q(x) - 1), & \text{if } 0 \leqq t \leqq q(x)/2, \\ \psi(x, 2t - q(x)), & \text{if } q(x)/2 \leqq t \leqq q(x), \\ \psi(x, t), & \text{if } q(x) \leqq t \leqq 1. \end{cases}$$

Then $G(x, 0) = \psi(x, -1) = g(x)$. If $0 \leqq t \leqq q(x)/2$, then $y = 2t/q(x) - 1 \in [-1, 0]$, so $pG(x, t) = p\psi(x, y)$. Then $pG(x, t)$ is η-close to $p_\delta \psi(x, y)$ which is η-close to $\Phi(x, 0) = H(x, 0)$ which is in turn η-close to $H(x, t)$ by our choice of $q(x)$. Since η twice star refines ϵ, $pG(x, t)$ is ϵ-close to $H(x, t)$. The case $q(x)/2 \leqq t \leqq q(x)$ is verified similarly.

A slight change in the above proof also gives

PROPOSITION 1.2. *Let $p : E \to B$ be a map between ANR's. If p has the approximate homotopy lifting property with respect to a space X, then for any cover ϵ of B there is a cover δ of B such that whenever $g : X \to E$ and $H : X \times I \to B$ are maps such that g is a δ-lift of H_0, there is a map $G : X \times I \to E$ such that $G_0 = g$ and G is an ϵ-lift of H.*

EXAMPLE. Let W be the "Warsaw circle" in R^2; that is, $W = W_1 \cup B$, where $W_1 = \{(0, t) \mid -1 \leqq t \leqq 1\} \cup \{(x, \text{Sin}\, \pi/x)\}$ and B is an arc which meets W_1 only in its endpoints (0,0) and (1,0). Let x_0 be a base point in the 1-sphere S^1, and let $\pi : S^1 \times S^1 \to S^1$ be the projection map onto the second factor. Clearly there is a compactum $A \subset S^1 \times S^1$ such that A is homeomorphic to W and such that there is a homeomorphism

$$h : (S^1 \times S^1) - A \to S^1 \times (S^1 - \{x_0\}).$$

Then the map $p : S^1 \times S^1 \to S^1$ given by

$$p(x) = \begin{cases} \pi h(x), & x \in (S^1 \times S^1) - A \\ x_0, & x \in A \end{cases}$$

is continuous, and p has the property that $p^{-1}(x_0) = A$ and $p^{-1}(y)$ is an essential copy of S^1 in $S^1 \times S^1$ for each $y \neq x_0$.

We claim that p can be uniformly approximated by fibrations. To see this, let $\epsilon > 0$ be given and let U be an open interval in S^1 such that $x_0 \in U \subset N(x_0, (1/2)\epsilon)$. Then, since $p^{-1}(S^1 - U)$ and $p^{-1}(\mathrm{Cl}\, U)$ are homeomorphic to $S^1 \times I$, it is easy to see that $p \mid p^{-1}(S^1 - U)$ extends to a map $p_\epsilon : S^1 \times S^1 \to S^1$ which is topologically equivalent to π. Thus p_ϵ is a fibration, and $d(p, p_\epsilon) < \epsilon$. By 1.1, p satisfies the approximate homotopy lifting property with respect to all metric spaces.

On the other hand, p is not even a Serre fibration. For suppose that p has the homotopy lifting property with respect to S^1. Let $x \neq x_0$ be a point in S^1, let ω be a path in S^1 from x to x_0, let $H : S^1 \times I \to S^1$ be the homotopy given by $H(x, t) = \omega(t)$, and let $g : S^1 \to p^{-1}(x)$ be a homeomorphism. By assumption H lifts to a homotopy $G : S^1 \times I \to S^1 \times S^1$ such that $G(x, 0) = g(x)$ and $G(x, 1) \in A$ for each $x \in S^1$. This is a contradiction since g is essential and each map of S^1 to A is null-homotopic.

Many of the fundamental lemmas in the development of the theory of Hurewicz and Serre fibrations can be modified for maps with the approximate homotopy lifting property. The rest of this section is devoted to four such propositions which are used in this paper and which illustrate the kind of changes needed. The first three propositions follow Dugundji's treatment [6]. Note that the assumption that E and B are metric spaces is implicitly included in the hypothesis of all four propositions.

Let $p : E \to B$ be a surjection between metric spaces. Define $D = \{(e, \omega) \in E \times B^I \mid p(e) = \omega(0)\}$ with the topology induced by the given topology on E and the compact-open topology on B^I (the space of all paths in B). We say that $p : E \to B$ has *approximate path lifting functions* if for every cover ϵ of B, there is a map $\lambda : D \to E^I$ such that $\lambda(e, \omega)(0) = e$ and $\lambda(e, \omega)$ is an ϵ-lift of ω. Furthermore, we say that λ is *regular* provided $\lambda(e, \omega)$ is the constant path at e whenever ω is the constant path at $p(e)$.

PROPOSITION 1.3. (See [6], Ch. XX, Th. 2.2). *The map $p : E \to B$ has the (regular) approximate homotopy lifting property with respect to all spaces if and only if p has (regular) approximate path lifting functions.*

PROPOSITION 1.4. (See [6], Ch. XX, Cor. 2.3). *If $p : E \to B$ has the approximate homotopy lifting property for metric spaces, then p has the approximate homotopy lifting property for all spaces.*

This proposition, together with Proposition 1.1, shows that a mapping which has Hurewicz fibrations arbitrarily close to it has the approximate homotopy lifting property with respect to all spaces.

Note that this comment applies to the mapping in the example.

PROPOSITION 1.5. (See [6], Ch. XX, Cor. 2.4). *If* $p : E \to B$ *has the approximate homotopy lifting property with respect to all spaces, then* p *has the regular approximate homotopy lifting property with respect to all spaces.*

The next proposition follows from Proposition 1.5 and the technique of proof of Theorem 2.8.10 of [16].

PROPOSITION 1.6. *If* $p : E \to B$ *has the approximate homotopy lifting property with respect to* $X \times I$ *for some space* X*, then given maps* $F_0, F_1 : X \times I \to E$*, homotopies* H *between* pF_0 *and* pF_1 *and* G *between* $F_0 \mid X \times \{0\}$ *and* $F_1 \mid X \times \{0\}$ *such that* $H(x, 0, t) = pG(x, 0, t)$*, and a cover* ϵ *of* B*, there is a map* $G' : X \times I \times I \to E$ *such that* G' *is a homotopy between* F_0 and F_1*,* G' *extends* G*, and* G' *is an* ϵ*-lift of* H*.*

2. **Properties of point inverses.** For the remainder of this paper we assume that E and B are locally compact ANR's and $p : E \to B$ is a proper map. We make this assumption so that we can state our conclusions shape theoretically. We will also be assuming that p has the regular approximate homotopy lifting property for all spaces. It would suffice to require that p satisfy the approximate homotopy lifting property for metric spaces by Propositions 1.3 through 1.5. We summarize these assumptions by saying that $p : E \to B$ is an *approximate fibration.*

Since we are interested in shape theoretic information about the point inverses of p, it would be useful to know that the shapes of compacta in E are determined by their neighborhood systems in E. In particular, we would like for each compactum X in E to satisfy

(2.1) for each neighborhood U of X there is a compact ANR $M \subset U$ with $X \subset \operatorname{int} M$.

Following Moszyńska [15A], we say that E is a *convenient ANR* if every compactum X in E satisfies 2.1. Clearly every polyhedron is convenient, and it follows easily from Chapman's triangulation theorem [4A] that every Q-manifold is convenient (Q the Hilbert cube).

PROPOSITION 2.2. *If* E *is a locally compact ANR,* $E \times Q$ *is convenient.*

PROOF. If X is a compactum in $E \times Q$, X is contained in a neighborhood of the form $V \times Q$, where V is a separable open subset of E. By R. Edwards [6A], $V \times Q$ is a Q-manifold. Thus X satisfies 2.1, and $E \times Q$ is convenient.

PROPOSITION 2.3. *If $p : E \to B$ is an approximate fibration, then*
(a) $p\pi : E \times Q \to B$, *given by* $p\pi(e, q) = p(e)$ *is an approximate fibration, and*
(b) $p \times 1 : E \times Q \to B \times Q$, given by $(p \times 1)\ (e,q) = (p(e), q)$ *is an approximate fibration.*

PROOF. Part (a) follows from the easily proven fact that a composition of approximate fibrations is an approximate fibration. To prove (b), let ϵ be a cover of $B \times Q$, let X be a space, and let $g : X \to E \times Q$, $H : X \times I \to B \times Q$ be maps such that $(p \times 1)g = H \mid X \times \{0\}$. Define a cover ϵ' of B as follows. For each point $b \in B$, cover $\{b\} \times Q$ with a finite number of open sets $V_1{}^b \times Q_1{}^b, \cdots, V^{n}{}_{n_b} \times Q^{b}{}_{n_b}$ such that each $V_i{}^b \times Q_i{}^b$ is contained in some member of ϵ. Let $V_b = \bigcap_{i=1} V_i{}^b$, and let $\epsilon' = \{V_b\}_{b \in B}$. The V_b have the property that (x, q) and (y, q) are contained in some member of ϵ whenever $x, y \in V_b$ and $q \in Q$. Let $g_E : X \to E$, $g_Q : X \to Q$ be defined by $g(x) = (g_E(x), g_Q(x))$ and define H_B, H_Q similarly. Then $pg_E = H_B \mid X \times \{0\}$ and $pg_Q = H_Q \mid X \times \{0\}$. Let $G_E : X \times I \to E$ be an ϵ'-lift of H_B which extends g_E. It is easy to check that $G : X \times I \to E \times Q$, defined by $G(x, t) = (G_E(x, t), H_Q(x, t))$ is an ϵ-list of H which extends g.

The upshot of 2.2 and 2.3 is that if we are only concerned with the shape theoretic properties of the point inverses of an approximate fibration or the homotopy properties of an approximate fibration, there is no loss of generality in assuming that E and B are convenient.

We use the *ANR* system approach to shape theory developed by S. Mardesić and J. Segal [14]. In particular, given a compact set F in a convenient *ANR* E, we associate with F an *ANR* sequence, $\underline{F}$, as follows: choose a sequence of neighborhoods $\{U_i\}$ such that each U_i is a compact *ANR*, $U_i \subset \operatorname{Int} U_{i-1}$, and $\bigcap U_i = F$; and use the inclusions as bonding maps. In this paper we say that $\underline{F} = \{U_i\}$ is an *ANR sequence associated with F by inclusion.* By [14, Cor. 1] any other *ANR* sequence associated with F has the same homotopy type as $\underline{F}$.

This section is concerned with the shape-theoretic properties of the point-inverses of an approximate fibration, p. The point-inverse $p^{-1}(b)$ is denoted F_b and is referred to as the *fiber* over b.

THEOREM 2.4. *If $p : E \to B$ is an approximate fibration and $b \in B$, then given any neighborhood U of F_b, there is a neighborhood V of F_b in U such that for any neighborhood W of F_b in V, there is a neighborhood W_0 of F_b in W and a homotopy $G : E \times I \to E$ such that*
(1) $G_0 = 1$,
(2) $G_t \mid \mathrm{Cl}(E - U) \cup W_0 = 1$ *for all t,*
(3) $G_t(V) \subset U$ *for all t, and*
(4) $G_1(V) \subset W$.

PROOF. First we show

(*) If $p \in B$ and C is any ANR neighborhood of b, there is a compact neighborhood D of b such that for each ANR neighborhood A of b in D there is a homotopy $H : B \times I \to B$ and a neighborhood A_0 of b such that

(1) $H_0 = 1$
(2) $H_t \mid \mathrm{Cl}(B - C) \cup A_0 = 1$ for all t,
(3) $H_t(D) \subset C$ for all t, and
(4) $H_1(D) \subset A$.

Given C, let D be a compact neighborhood of b which contracts to b in C. There is a homotopy $F : D \times I \to C$ such that $F_0 = 1_D$, $F_1(D) = b$, and $F_t(b) = b$ for all t [6, Ch. XV, Th. 6.5]. Given A, choose a neighborhood A_0 of b which is so small that $F \mid A_0 \times I$ is homotopic in A to the map $i(a,t) = a$ by a homotopy ϕ such that $\phi(a,0,s) = a$, $\phi(a,t,0) = F(a,t)$ and $\phi(a,t,1) = i(a,t)$ for all $s, t \in I, a \in A_0$. Let $\psi : (D \times \{1\} \times \{0\}) \cup (A_0 \times \{1\} \times I) \to A$ be defined by

$$\psi(x,1,s) = \begin{cases} F(x,1), s = 0 \\ \phi(a,1,s), a \in A_0. \end{cases}$$

By [3A] ψ extends to a homotopy $\bar{\psi} : D \times \{1\} \times I \to A$. Now define $\psi' : (D \times \{0,1\} \times I) \cup (A_0 \times I \times I) \cup (D \times I \times \{0\}) \to C$ by

$$\psi'(x,t,s) = \begin{cases} \bar{\psi}(x,t,s), & t = 1 \\ x, & t = 0 \\ F(x,t), & s = 0 \\ \phi(x,t,s), & x \in A_0. \end{cases}$$

Extend to $\psi'' : D \times I \times I \to C$ by [3A].

Define $H' : D \times I \to C$ by $H'(x,t) = \psi''(x,t,1)$. Then $H_0' = 1_D$, $H_1'(D) \subset A$ and $H_t' \mid_{A_0} = 1_{A_0}$ for all t. Let L be an open neighborhood of D in C whose closure is contained in C, and let $J = C - L$.

Define $H'' : C \times \{0\} \cup ((J \cup D) \times I) \to C$ by

$$H''(x,t) = \begin{cases} x, & t = 0 \text{ or } x \in J \\ H'(x,t), & \text{otherwise.} \end{cases}$$

Extend H'' to a homotopy $H''' : C \times I \to C$ by [3A] again. Finally, extend over $(B - C) \times I$ by the identity to get the desired H.

Now, given U, let C be an ANR neighborhood of b such that $p^{-1}(C) \subset U$. Let D be given by (*) and let $V = p^{-1}(D)$. Given W, choose an ANR neighborhood A of b such that $p^{-1}(A) \subset W$. Let A_0, H

be as in (*), and let $W_0 = p^{-1}(A_0)$.

Define $K : E \times I \to B$ by $K(e, t) = H(p(e), t)$, for $e \in E$, $t \in I$. Let $\epsilon = \min\{d(H(D \times I), \operatorname{Cl}(B - C)), d(H_1(D), \operatorname{Cl}(B - A))\}$. By hypothesis there is a map $G : E \times I \to E$ such that $G_0 = 1$, $d(pG, K) < \epsilon$, and G is stationary when K is. It is easy to check that G satisfies the conclusion of the theorem.

If $\underline{F}_b = \{U_i, \alpha_{ij}\}$ is an ANR sequence associated with F_b by inclusion, it follows immediately from 2.4 that $\underline{F}_b$ is strongly movable in the sense of Mardešić [13]; i.e., for such i there is an i', $i' \geqq i$ such that for each i'', $i'' \geqq i$, there is a j, $j \geqq i'$ and $j \geqq i''$, and a map $r^{i'i''} : U_{i'} \to U_{i''}$ satisfying

$$\alpha_{ii''} r^{i'i''} \simeq \alpha_{ii'}$$

and

$$r^{i'i''} \alpha_{i'j} \simeq \alpha_{i''j}.$$

It is proved in [4] and [13] that a metric compactum X has a strongly movable associated ANR system if and only if X is a fundamental absolute neighborhood retract; that is, X is an absolute neighborhood retract in the sense of shape theory (see [4] for a precise definition). We summarize these remarks in the following corollary to 2.4, using Propositions 2.2, 2.3.

COROLLARY 2.5. *If $p : E \to B$ is an approximate fibration and $b \in B$, then $\underline{F}_b$ is strongly movable and thus F_b is a fundamental absolute neighborhood retract.*

REMARK. Other conditions on the neighborhoods of a point $b \in B$ such as local homotopy connectedness, LC^n, can be used to get corresponding conditions on the fiber F_b. The same argument also shows that a Hurewicz fibration between manifolds has the property that each fiber is an absolute neighborhood retract.

We now turn to the question of how the fibers are embedded assuming the spaces are manifolds. It is too much to expect a fiber to have 1-ULC complements. For example, any map of S^n onto itself whose only nondegenerate point inverse is a wild, cellular arc is an approximate fibration. However, we do get the following property. A compact set F of a manifold satisfies the *small loops condition* [5], [7], [8] if for each neighborhood U of F, there is a neighborhood V of F in U and an $\epsilon > 0$ such that each loop in V-F of diameter less than ϵ which is null-homologous in V-F is null-homotopic in U-F.

THEOREM 2.6. *If $p : E \to B$ is an approximate fibration, and if E and B are manifolds, then for every $b \in B$, F_b satisfies the small loops condition.*

Proof. Given a neighborhood U of F_b, choose a cell neighborhood C of b such that $p^{-1}(C) \subset U$. Choose V such that $F_b \subset V \subset p^{-1}(C)$ and choose $\epsilon > 0$ such that each loop of diameter less than ϵ in V is null-homotopic in $p^{-1}(C)$. Now suppose that $\omega : \mathrm{Bd}\, I^2 \to V - F_b$ is a loop of diameter less than ϵ which is null-homologous in $V - F_b$. Then there is a map $g : I^2 \to p^{-1}(C)$ such that $g \mid \mathrm{Bd}\, I^2 = \omega$. Since $C - p$ has the homotopy type of a sphere and $p\omega$ is null-homologous in $C - p$, $p\omega$ is null-homotopic in $C - p$. Therefore, by using the fact that C is a cell, pg extends to a map $H : I^2 \times I \to C$ such that $b \notin H_1(I^2)$ and $H_t \mid \mathrm{Bd}\, I^2 = p\omega$ for all t. Let $\eta = \min\ \{d(H_1(I^2), \{b\}), d(p(E - U), C)\}$. By hypothesis there is an η-lift G of H which extends g and is stationary with H. Then $G_1 : I^2 \to U - F_b$ and $G_1 \mid \mathrm{Bd}\, I^2 = \omega$.

We complete this section by showing that if $p : E \to B$ is an approximate fibration and B is path-connected, then any two fibers have the same shape. This is analogous to the homotopy equivalence of fibers of a Hurewicz fibration [16], and the proof has some similarities.

In Propositions 2.7 through 2.11 we suppose that $p : E \to B$ is an approximate fibration and that E is a convenient *ANR*. Let $a, b \in B$, let $\{Ui\}$ be an *ANR* sequence associated with F_a by inclusion, and let $\{V_i\}$ be an *ANR* sequence associated with F_b by inclusion. In case $a = b$, take $U_i = V_i$ for each i. Let $\{\epsilon_i\}$ be a sequence of positive numbers converging to zero such that $p^{-1}(N(b, 2\epsilon_i)) \subset V_i$ for each i. Let $\omega : I \to B$ be a path with $a = \omega(0)$, $b = \omega(1)$. There is another sequence of numbers $\{\delta_i\}$ such that, for every i, $0 < \delta_{i-1} < \delta_i < (1/2)\epsilon_i$ and any two $2\delta_i$-close maps into $N(\omega, \epsilon_i)$ are $(1/2)\epsilon_i$-homotopic in B. An $(\{\epsilon_i\}, \{\delta_i\})$-*covering* of ω is an increasing function G on the positive integers together with a sequence of maps $G_i : U_{G(i)} \times I \to E$ such that $G_i(x, 0) = x$ for all x, and for each i, $d(pG_i(x, t), \omega(t)) < \delta_i$ for all x, t.

Proposition 2.7. *Given any sequences $\{\epsilon_i\}$, $\{\delta_i\}$ satisfying the conditions above and any path ω, there is an $(\{\epsilon_i\}, \{\delta_i\})$-covering of ω.*

Proof. For each i, choose $G_i' : F_a \times I \to E$ by the regular approximate homotopy lifting property such that $d(pG_i'(x, t), \omega(t)) < (1/2)\delta_i$ and $G_i'(x, 0) = x$ for all $x \in F_a$, $t \in I$. By [3A, Theorem 8.1], G_i' extends to $G_i'' : E \times I \to E$ such that $G_i''(x, 0) = x$ for all $x \in E$. By a compactness argument, there is an integer $G(i)$ such that $d(pG_i''(x, t), \omega(t)) < \delta_i$ for all $x \in U_{G(i)}$, $t \in I$. The proof is completed by choosing G to be increasing and setting $G_i = G_i'' \mid U_{G(i)} \times I$.

Proposition 2.8. *If $(G, \{G_i\})$ is an $(\{\epsilon_i\})$-covering of ω, and $g_i : U_{G(i)} \to V_i$ is defined by $g_i(x) = G_i(x, 1)$, then $\underline{g} = (G, \{g_i\})$ is a map of ANR-sequences* [14].

PROOF. If $x \in U_{G(i)}$, then $g_i(x) \in V_i$ since $d(pg_i(x), \omega(1)) < \delta_i < 2\epsilon_i$. Given $j \geqq i$, we will show that $g_i \mid U_{G(j)} \simeq g_j$ in V_i. Since $d(pG_i(x, t), \omega(t)) < \delta_i$ and $d(pG_j(x, t), \omega(t)) < \delta_j < \delta_i$ for every $x \in U_{G(j)}$, $t \in I$, we have $d(pG_i \mid U_{G(j)} \times I, pG_j) < 2\delta_i$. Therefore $pG_i \mid U_{G(j)} \times I$ is $(1/2)\epsilon_i$-homotopic to pG_j by means of a homotopy $H : U_{G(j)} \times I \times I \to B$ such that $H(x, t, 0) = pG_i(x, t)$, $H(x, t, 1) = pG_j(x, t)$, $d(H(x, t, s), \omega(t)) < \epsilon_i$, and $H(x, 0, s) = pg_i(x, 0) = pG_j(x, 0)$ for all $x \in U_{G(j)}$, $t \in I$, $s \in I$. By Proposition 1.6, there is a homotopy $K: U_{G(j)} \times I \times I \to E$ between G_j and $G_i \mid U_{G(j)} \times I$ such that $d(pK(x, t, s), H(x, t, s)) < \epsilon_i$. The homotopy defined by $K(x, 1, s)$ is the desired one since $H(x, 1, s) \in N(b, \epsilon_i)$ and $p^{-1}(N(b, 2\epsilon_i)) \subset V_i$.

We say that the $\underline{g}$ of Proposition 2.8 is a map of *ANR*-sequences *induced* by ω, but note that $\underline{g}$ also depends on $\{\epsilon_i\}$, $\{\delta_i\}$, and $(G, \{G_i\})$.

PROPOSITION 2.9. *If ω and ω' are paths from a to b, $\omega \simeq \omega'$ rel $\{a, b\}$, and $\underline{g}$ and $\underline{g}'$ are maps of ANR-sequences induced by ω and ω' respectively, then $\underline{g} \simeq \underline{g}'$.*

PROOF. Let $\underline{g} = (G, \{g_i\})$ where $g_i(x) = G_i(x, 1)$ for some $(\{\epsilon_i\}, \{\delta_i\})$-covering $(G, \{G_i\})$ of ω, and let $\underline{g}' = (G', \{g_i\})$ where $g_i'(x) = G_i'(x, 1)$ for some $(\{\epsilon_i'\}, \{\delta_i'\})$-covering $(G', \{G_i'\})$ of ω'. Given i, let $j = \max\{G(i), G'(i)\}$. Now define a homotopy $H : U_j \times I \times I \to B$ as follows. Since $d(pG_i(x, t), \omega(t)) < \delta_i$ for $x \in U_j$, $t \in I$, H can be defined on $U_j \times I \times [0, 1/3]$ to be a $(1/2)\epsilon_i$-homotopy between $H_0 = pG_i$ and $H_{1/3}(x, t) = \omega(t)$. Similarly $H \mid U_j \times I \times [2/3, 1]$ is a $(1/2)\epsilon_i'$-homotopy between $H_{2/3}(x, t) = \omega'(t)$ and $H_1 = pG_i'$. Finally let $H \mid U_j \times I \times [1/3, 2/3]$ be defined by the hypothesized homotopy between ω and ω'. Applying Proposition 1.6 as above shows that $g_i \mid U_j \simeq g_i' \mid U_j$ in V_i.

We now say that the homotopy class of maps of *ANR*-sequences $[\underline{g}]$ is *induced* by the homotopy class rel end points $[\omega]$. By Proposition 2.9, $[\underline{g}]$ depends only on $[\omega]$.

PROPOSITION 2.10. *If ω is the constant path at $b \in B$, and $[\underline{g}]$ is induced by $[\omega]$, then $[\underline{g}] = [1_{\{V_i\}}]$.*

PROOF. Choose the sequences $\{\epsilon_i\}$, $\{\delta_i\}$ as required for an $(\{\epsilon_i\}, \{\delta_i\})$-covering of ω. Pick G such that for each i, $p(V_{G(i)}) \subset N(b, \delta_i)$. Define $G_i : V_{G(i)} \times I \to E$ by $G_i(x, t) = x$ for all $x \in V_{G(i)}$, $t \in I$. Then $(G, \{G_i\})$ is an $(\{\epsilon_i\}, \{\delta_i\})$-covering of ω; and if $g_i(x) = G_i(x, 1) = x$, then $\underline{g} = (G, \{g_i\})$ is homotopic to the identity map of the *ANR*-sequence $\{V_i\}$.

PROPOSITION 2.11. *If ω is a path from a to b, ω' is a path from b to c, $[\underline{g}]$ is induced by $[\omega]$, and $[\underline{g}']$ is induced by $[\omega']$, then $[\underline{g}'\underline{g}]$ is induced by $[\omega * \omega']$.*

PROOF. Let $\{W_i\}$ be an ANR sequence associated with F_c by inclusion. If $a = c$, take $W_i = U_i$, and if $b = c$, take $W_i = V_i$. Let $\{\epsilon_i'\}$ be a sequence of positive numbers converging to zero such that $p^{-1}(N(c, 2\epsilon_i')) \subset W_i$. Choose a decreasing sequence of numbers $\{\delta_i'\}$ such that $0 < \delta_i' < (1/2)\epsilon_i'$ and any two $2\delta_i'$-close maps into $N(\omega', \epsilon_i)$ are $(1/2)\epsilon_i'$-homotopic in B. Choose an $(\{\epsilon_i'\}, \{\delta_i'\})$-covering $(G', \{G_i'\})$ of ω'. Now select another sequence of positive numbers $\{\epsilon_i\}$, converging to zero, such that $p^{-1}(N(b, 2\epsilon_i)) \subset V_{G'(i)}$. Choose a monotone decreasing sequence of positive numbers $\{\delta_i\}$ such that $\delta_i \leqq \delta_i'$, $\delta_i \leqq (1/2)\epsilon_i$, and any two $2\delta_i$-close maps into $N(\omega, \epsilon_i)$ are $1/2\epsilon_i$-homotopic in B. Finally, let $(G, \{G_i\})$ be an $(\{\epsilon_i\}, \{\delta_i\})$-covering of ω. Define G'' by $G''(i) = G(G'(i))$ and $G_i'' : U_{G''(i)} \times I \to E$ by

$$G_i''(x, t) = \begin{cases} G_{G'(i)}(x, 2t), & \text{if } 0 \leqq t \leqq 1/2 \\ G'_i(g_{G'(i)}(x), 2t - 1), & \text{if } 1/2 \leqq t \leqq 1. \end{cases}$$

It is easy to check that $(G'', \{G_i''\})$ is an $(\{\epsilon_i'\}, \{\delta_i\}$-covering of $\omega * \omega'$. Since $G_i''(x, 1) = g_i'(g_{G'(i)}(x))$, we conclude that $[\underline{g}'\underline{g}]$ is induced by $[\omega * \omega']$.

THEOREM 2.12. *If $p : E \to B$ is an approximate fibration, $a, b \in B$, and there is a path in B from a to b, then F_a and F_b have the same shape. Hence if B is path connected, any two fibers have the same shape.*

PROOF. By 2.2 and 2.3, the map $p\pi : E \times Q \to B$ is an approximate fibration with convenient total space, so that if there is a path in B from a to b, $(p\pi)^{-1}(a)$ and $(p\pi)^{-1}(b)$ have the same shapes by 2.7–2.11. The proof is completed by observing that for each $x \in B$, $p\pi^{-1}(x)$ is homeomorphic to $F_x \times Q$ which is in turn shape equivalent to F_x.

3. **An exact sequence for a map.** Let F be a compact set in E, $e \in F$, and $\{U_j\}$ be an ANR sequence associated with F by inclusion.

Define $\underline{\pi}_q(F, e) = \lim_{\leftarrow} \pi_q(U_j, e)$ and $\underline{\pi}_q(E, F, e) = \lim_{\leftarrow} \pi_q(E, U_j, e)$, where the bonding maps of the inverse sequences are inclusion induced. It is easy to prove that the resulting groups (or pointed sets) do not depend on the choice of the sequence $\{U_j\}$. For a general reference on these groups, see [15]. An inverse sequence $\{G_j, \alpha^j\}$ of groups and homormorphisms is *constant* provided that $\alpha^j \mid \text{im}\, \alpha^{j+1} : \text{im}\, \alpha^{j+1} \to \text{im}\, \alpha^j$ is an isomorphism for each j. In this situation $\lim_{\leftarrow} G_i \cong \text{im}\, \alpha^1$ under projection.

THEOREM 3.1. *If $p: E \to B$ is an approximate fibration, E is a convenient ANR, $b \in B$, and $e \in F_b$, then there exists an ANR sequence $\{U_j\}$ associated with F_b by inclusion such that the inverse sequences $\{\pi_q(U_j, e), \alpha_*{}^j\}$ and $\{\pi_q(E, U_j, e), \gamma_*{}^j\}$ are constant for all q where $\alpha^j: U_{j+1} \to U_j$ and $\gamma^j: (E, U_{j+1}) \to (E, U_j)$ are inclusion maps.*

PROOF. Let U_1 be any neighborhood of F_b. In Theorem 2.4 set $U = U_1$ and define U_2 to be an ANR neighborhood contained in the V given there. Now apply Theorem 2.4 again: set $U = U_2$ and define U_3 to be an ANR neighborhood contained in the resulting V. To define U_4, apply Theorem 2.4 twice more. First set $U = U_3$ and define $U_4{}'$ to be the resulting V. Then set $U = U_1$, $U_2 = V$, $U_3 = W$ and define $U_4{}''$ to be the resulting W_0. Finally define U_4 to be an ANR neighborhood contained in $U_4{}' \cap U_4{}''$. Continuing in this manner, we obtain a sequence $\{U_j\}$ with the property that for each $j \geqq 1$ there is a homotopy $G: E \times I \to E$ such that $G_0 = 1$, $G_t \mid \mathrm{Cl}(E - U_j) \cup U_{j+3} = 1$ for all t, $G_t(U_{j+1}) \subset U_j$ for all t, and $G_1(U_{j+1}) \subset U_{j+2}$.

To prove $\alpha_*{}^j \mid \mathrm{im}\, \alpha_*{}^{j+1}: \mathrm{im}\, \alpha^{j+1} \to \mathrm{im}\, \alpha_*{}^j$ is an epimorphism, let $[f_j] \in \mathrm{im}\, \alpha_*{}^j$. Then $f_j \simeq \alpha^j f_{j+1}$ for some $[f_{j+1}] \in \pi_q(U_{j+1})$ (we suppress the base point throughout the proof). By choice of $\{U_j\}$, $\alpha^j f_{j+1} \simeq \alpha^j \alpha^{j+1} f_{j+2}$ for some $[f_{j+2}] \in \pi_q(U_{j+2})$. Hence $[f_j] = \alpha_*{}^j[\alpha^{j+1} f_{j+2}]$.

To prove that $\alpha_*{}^j \mid \mathrm{im}\, \alpha_*{}^{j+1}$ is a monomorphism, suppose $\alpha_*{}^j[f_{j+1}] = 0$ for some $[f_{j+1}] \in \alpha_*{}^{j+1}$. (Here we are assuming $q > 0$; the proof for $q = 0$ is similar.) Then $f_{j+1} \simeq \alpha^{j+1} f_{j+2}$, and $\alpha^j \alpha^{j+1} f_{j+2} \simeq 0$ for some $[f_{j+2}] \in \pi_q(U_{j+2})$. Again by the choice of $\{U_j\}$, there is a map of U_j into U_{j+1} whose restriction to U_{j+2} is the identity. Hence $\alpha^{j+1} f_{j+2} \simeq 0$ so $[f_{j+1}] = 0$.

COROLLARY 3.2. *The projections onto the first factors $\alpha_*: \underline{\pi}_q(F_b, e) \to \mathrm{im}\, \alpha_*{}^1$ and $\gamma_*: \underline{\pi}_q(E, F_b, e) \to \mathrm{im}\, \gamma_*{}^1$ are isomorphisms.*

COROLLARY 3.3. *There is an exact sequence*

$$\cdots \to \underline{\pi}_q(F_b, e) \xrightarrow{i_*} \pi_q(E, e) \xrightarrow{\nu_*} \underline{\pi}_q(E, F_b, e) \xrightarrow{\delta} \underline{\pi}_{q-1}(F_b, e) \to \cdots,$$

where i_* is α_* followed by the inclusion induced homomorphism $\pi_q(U_1, e) \to \pi_q(E, e)$, ν_* is the inclusion induced homomorphism $\pi_q(E, e) \to \pi_q(E, U_1, e)$ followed by $\gamma_*{}^{-1}$, and δ is usual boundary operator $\pi_q(E, U_1, e) \to \pi_{q-1}(U_1, e)$ preceded by γ_* and followed by $\alpha_*{}^{-1}$.

PROOF. Using the exactness of the homotopy sequences of the pairs (E, U_1), (E, U_2), it is an easy diagram-chasing argument to prove that

$$\cdots \to \mathrm{im}\, \alpha_*{}^1 \to \pi_q(E) \to \mathrm{im}\, \gamma_*{}^1 \to \mathrm{im}\, \alpha_*{}^1 \to \cdots$$

is exact. Then the second corollary follows from the first.

THEOREM 3.4. *If* $p : E \to B$ *is an approximate fibration,* $b \in B$*, and* $e \in F_b$*, then* p *induces an isomorphism* $\underline{p}_* : \underline{\pi}_q(E, F_b, e) \to \pi_q(B, b)$.

PROOF. Without loss of generality we assume that both E and B are convenient ANR's. (Otherwise we prove the theorem for $p \times 1 : E \times Q \to B \times Q$ using Propositions 2.2 and 2.3 and the theorem for $p : E \to B$ follows since $E \times Q$ and $B \times Q$ are homotopy equivalent to E and B respectively.) Choose an ANR sequence $\{U_i\}$ associated with F_b by inclusion as in Theorem 3.1 and a similar sequence $\{V_i\}$ associated with b. By renumbering if necessary we may assume that $p(U_i) \subset V_i$. Then we have the commutative diagram

$$\begin{array}{ccccccc} \longrightarrow & \pi_q(E, U_{i+1}) & \xrightarrow{\gamma_*{}^i} & \pi_q(E, U_i) & \longrightarrow \\ & \downarrow p_* & & \downarrow p_* & \\ \longrightarrow & \pi_q(B, V_{i+1}) & \xrightarrow{\beta_*{}^i} & \pi_q(B, V_i) & \longrightarrow \end{array}$$

where the lower groups are based at b, the upper groups are based at $e \in F_b$, and β^i is the inclusion. Define $\underline{p}_* : \pi_q(E, f_b, e) \to \pi_q(B, b)$ to be the induced homomorphism on the inverse limits. $\underline{p}_*$ is clearly independent of the sequences $\{U_i\}$, $\{V_i\}$. To see that $\underline{p}_*$ is an isomorphism, it suffices to show that $p_* : (\operatorname{im} \gamma_*{}^i) \to (\operatorname{im} \beta_*{}^i)$ is an isomorphism. To prove that p_* is epic, choose $[f] \in \operatorname{im} \beta_*{}^1$. By local contractibility, we may assume that $f(\operatorname{Bd} I^q) = b$. Let W be a neighborhood of b such that $p^{-1}(W) \subset U_{i+1}$. Let $\epsilon = d(b, B - W)$, and let δ be a cover of B such that δ-close maps are ϵ-homotopic. By hypothesis there is a map $g : (I^q, I^{q-1}) \to (E, e)$ such that pg and f are δ-close. Then $[g] \in \pi_q(E, U_{i+1})$ and $p_*[g] = [f]$ in $\pi_q(B, V_{i+1})$. Thus $p_*\gamma_*{}^i[g] = [f]$.

To show that p_* is monic on $\operatorname{im} \gamma_*{}^i$, suppose $[f] \in \operatorname{im} \gamma_*{}^i$ and $p_*[f] = 0$. Choose $k > 2$ so that $p^{-1}(V_{i+k}) \subset U_i$. By constancy, we may assume that $f : (I^q, \operatorname{Bd} I^q, *) \to (E, U_{i+k+1}, e)$. By constancy again, $p_*[f] = 0$ in $\pi_q(B, V_{i+k}, b)$. By lifting the appropriate homotopy, we conclude that $[f] = 0$ in $\pi_q(E, U_i, e)$.

COROLLARY 3.5. *If* $p : E \to B$ *is an approximate fibration* $b \in B$*, and* $e \in F_b$*, then there is an exact sequence*

$$\cdots \to \underline{\pi}_q(F_b, e) \xrightarrow{i_*} \pi_q(E, e) \xrightarrow{p_*} \pi_q(B, b) \xrightarrow{\partial} \underline{\pi}_{q-1}(F_b, e) \to \cdots,$$

where i_* *was defined in Corollary* 3.3 *and* $\partial = \delta \underline{p}_*{}^{-1}$, δ *from*

Corollary 3.3, $p_* = p_* \nu_*$, p_*^{-1} *from Theorem* 3.4.

REMARK. If E and B are assumed to be polyhedral and $p: E \to B$ is assumed to satisfy the approximate homotopy lifting property for each q-cell I^q, the sequence of Corollary 3.5 can still be obtained in the following manner.

Modify Spanier's treatment of Serre (or weak) fibrations given in [**16**, 7.2.5.–7.2.7.] to cover approximate liftings. Use Proposition 1.2 to keep the induction going for Theorem 7.2.6. Then observe that when a lifting is needed in the proof of Theorems 2.4 and 3.4 under this new hypothesis, it is given by the modifications of Spanier's lemmas.

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