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A GEOMETRIC METHOD OF STUDYING TWO POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER SYSTEMS

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1. **Introduction.** In this paper, we are concerned with the question of existence of solutions to two-point boundary value problems for second order systems

$$x'' = f(t, x, x'). \tag{1}$$

The basic technique employed in obtaining existence results consists of the definition of a function between Euclidean spaces for which the zeroes yield solutions of the boundary value problem and the calculation of the Brouwer degree of this function relative to zero. We present a series of conditions on f which make this technique applicable and obtain thereby a variety of existence theorems. The last theorem presented generalizes some results appearing in recent papers.

During the last fifteen years, there have been a number of papers written on the subject of two-point boundary value problems for systems. These papers have used either the method of modified functions or various function space methods in attacking the problem. The first method was introduced for this subject in [3] and has been used successfully in numerous subsequent papers (see, for example, [9]). More recently, Leray-Schauder degree theory and other notions of topological degree in function spaces have proven to be powerful tools for obtaining existence results in this area (see [10], [11]).

The method presented in this paper avoids the use of both modified functions and function spaces and is quite intuitive since it relies on the analysis of solution trajectories. It is well-known that similar techniques are effective for scalar equations (see [5], [6], [7]), although the scalar case is more elementary since continuity arguments suffice without the use of Brouwer degree.

2. **The basic technique.** We denote d-dimensional Euclidean space by $\mathbf{R}^d$ and let the components of $x \in \mathbf{R}^d$ be x^i ($i = 1, \cdots, d$). Also, $x \cdot y$ is the Euclidean inner product for x and y in $\mathbf{R}^d$, while $\|x\|$ ($x \in \mathbf{R}^d$) represents an arbitrary norm on $\mathbf{R}^d$.

For the sake of simplicity, we begin by considering the second order system

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$$x'' = f(t, x, x'), \tag{1}$$

where $f: [0, b] \times \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$ is continuous, $b > 0$, together with the homogeneous boundary conditions

$$x(0) = x(b) = 0. \tag{2}$$

All solutions of (1) mentioned below will be assumed defined on their maximal intervals of existence.

Let D be an open set in the topology of $[0, b] \times \mathbf{R}^d$ with boundary ∂D in that topology. A point $(t_0, x_0) \in \partial D$ is said to be an "egress point" of D relative to (1) if there exists a solution $x(t)$ of (1) and a $\gamma > 0$ so that $x(t_0) = x_0$, and $(t, x(t), x'(t))$ exists and $(t, x(t)) \in D$ for $t_0 - \gamma < t < t_0$, and $x(t)$ is said to "egress" from D at (t_0, x_0). An egress point (t_0, x_0) will be called "strict" if either $t_0 = b$ or, for every $x(t)$ which egresses from D at (t_0, x_0), there exists a $\delta > 0$ such that $(t, x(t)) \notin \bar{D}$ for $t_0 < t < t_0 + \delta$.

For the moment, we assume that all egress points of D relative to (1) are strict and that D contains the segment $\{(t, 0) \in [0, b] \times \mathbf{R}^d\}$. We also suppose that initial value problems for (1) have unique solutions and that the following hypothesis is satisfied:

(H1) There exists a number $\mu' > 0$ such that if $x(t)$ is a solution of (1) with $x(0) = 0$ and $\|x'(0)\| \leqq \mu'$, then there is a $t_0 > 0$ so that $(t, x(t), x'(t))$ exists for $0 \leqq t \leqq t_0$, and $t_0 = b$ or $(t_0, x(t_0)) \in \partial D$.

We now define a mapping T from the set $S_{\mu'} = \{y \in \mathbf{R}^d : \|y\| \leqq \mu'\}$ into $\mathbf{R}^d$. Let $P \in S_{\mu'}$ and let $x(t)$ be the unique solution of (1) with $x(0) = 0$ and $x'(0) = P$. If $(t, x(t))$ intersects ∂D, let t_0 be the t-coordinate of the first intersection point. Otherwise, take $t_0 = b$. Define $T(P) = x(t_0)$.

First, we show that T is continuous. Let $P \in S_{\mu'}$ and let $\{P_n\}_{n=1}^{\infty}$ be a sequence in $S_{\mu'}$ such that $P_n \to P$ as $n \to \infty$. Let $x(t)$ and $x_n(t)$ be the solutions of (1) with $x(0) = x_n(0) = 0$, $x'(0) = P$ and $x_n'(o) = P_n$ $(n = 1, 2, \cdots)$.

Suppose the trajectory $(t, x(t)) \in D$ for $0 \leqq t \leqq b$. By the standard convergence theorem, $(t, x_n(t)) \in D$ for $0 \leqq t \leqq b$ if n is sufficiently large. Since $x_n(b) \to x(b)$ as $n \to \infty$, $T(P_n) \to T(P)$ as $n \to \infty$ in this case.

Otherwise $(t, x(t))$ intersects ∂D, and there is a first intersection point (t_0, x_0). Suppose $t_0 < b$. Let $\delta > 0$ be a number such that $(t, x(t)) \notin \bar{D}$ for $t_0 < t \leqq t_0 + \delta$, and fix $\lambda \in (0, \delta)$. Define a metric ρ on $\mathbf{R}^1 \times \mathbf{R}^d$ by $\rho((s_1, y_1), (s_2, y_2)) = |s_1 - s_2| + \|y_1 - y_2\|$ for $s_1, s_2 \in \mathbf{R}^1$ and $y_1, y_2 \in \mathbf{R}^d$. Then the distance between the compact set

$\{(t, x(t)) : t \in [0, t_0 - \lambda] \cup [t_0 + \lambda, t_0 + \delta]\}$ and ∂D is some positive number $\epsilon(\lambda)$. For n sufficiently large, $\|x_n(t) - x(t)\| < \epsilon(\lambda)/2$ for $0 \leqq t \leqq t_0 + \delta$, so the distance between $\{(t, x_n(t)) : t \in [0, t_0 - \lambda] \cup [t_0 + \lambda, t_0 + \delta]\}$ and ∂D exceeds $\epsilon(\lambda)/2$, and hence $(t, x_n(t))$ intersects ∂D first in the region $\{(t, x) : t_0 - \lambda < t < t_0 + \lambda, \ \|x - x(t)\| < \epsilon(\lambda)/2\}$. Since $x(t)$ is continuous and $\epsilon(\lambda) \to 0$ as $\lambda \to 0$, the diameter of this region approaches 0 as $\lambda \to 0$. Thus $T(P_n) \to T(P)$ as $n \to \infty$.

Finally, in case $t_0 = b$ above, it is easy to modify the above argument to conclude again that $\lim_{n\to\infty} T(P_n) = T(P)$. Thus T is continuous on $S_{\mu'}$. In order to calculate the Brouwer degree of T, we add another hypothesis:

(H2) There exists a number $\mu \in (0, \mu']$ so that if $x(t)$ satisfies (1), $x(0) = 0$, $\|x'(0)\| = \mu$, $(t, x(t)) \in D$ for $0 \leqq t < t_0$ and either $t_0 = b$ or $(t_0, x(t_0)) \in \partial D$, then $x(t_0) \notin \{-\lambda x'(0) : \lambda > 0\}$.

Note that (H2) implies $T(x'(0)) \neq -\lambda x'(0)$ for every $\lambda > 0$ whenever $x(t)$ satisfies (1), $x(0) = 0$ and $\|x'(0)\| = \mu$.

Let $S_\mu = \{y \in R^d : \|y\| \leqq \mu\}$ and let S°_μ be its interior. If the boundary value problem (1), (2) does not have a solution $x(t)$ with $\|x'(0)\| = \mu$ and $(t, x(t)) \in D$ for $0 \leqq t \leqq b$, then T does not vanish on ∂S_μ. Thus the Brouwer degree $d[T, S^\circ_\mu, 0]$ is defined (see [2]). Furthermore, by (H2) the mappings $\alpha T + (1 - \alpha)I$, where I is the identity and $0 \leqq \alpha \leqq 1$, do not vanish on ∂S_μ. Since Brouwer degree is invariant under homotopies, we have $d[T, S^\circ_\mu, 0] = d[I, S^\circ_\mu, 0] = 1$. Thus there exists a $y \in S^\circ_\mu$ so that $T(y) = 0$. It follows that (1), (2) has a solution $x(t)$ with $\|x'(0)\| \leqq \mu$ and $(t, x(t)) \in D$ for $0 \leqq t \leqq b$. We have proven the following theorem.

THEOREM 1. *Let D be a relatively open set in $[0, b] \times R^d$ containing $\{(t, 0) \in [0, b] \times R^d\}$, and suppose all egress points of D relative to* (1) *are strict. Assume initial value problems for* (1) *have unique solutions and that hypotheses* (H1) *and* (H2) *are satisfied. Then the boundary value problem* (1), (2) *has a solution $x(t)$ with $\|x'(0)\| \leqq \mu$ and $(t, x(t)) \in D$ for $0 \leqq t \leqq b$.*

We note that another existence theorem similar to Theorem 1 can be proved by "shooting" from the boundary point at $t = b$ back toward $t = 0$ if the notion of "egress" is replaced by a suitable one of "ingress" and obvious adjustments are made in (H1) and (H2).

3. **Concerning (H1) and (H2).** There are two main questions to be resolved concerning the application of Theorem 1. First, what conditions on f are sufficient to ensure that hypotheses (H1) and (H2) are

satisfied? Second, how does one find a region D for which egress points relative to (1) are strict? We present a series of lemmas in this section which give some answers to the first question and consider the second question in § 4. Implicit in the statements of these lemmas are the assumptions that initial value problems for (1) have unique solutions and that there is a set D for (1) having the properties listed in Theorem 1.

LEMMA 1. *If f is bounded on $D \times \mathbf{R}^d$, then* (H1) *and* (H2) *are satisfied.*

PROOF. Let K be a bound for $\|f\|$ on $D \times \mathbf{R}^d$ and take $\mu' = \mu = Kb/2$. Then (H1) is true by the standard extension theorem for ordinary differential equations.

Suppose there were a solution $x(t)$ of (1) with $x(0) = 0$, $\|x'(0)\| = \mu$, $(t, x(t)) \in D$ for $0 \leqq t < p$ $(p \leqq b)$ and $x(p) = -\lambda x'(0)$ $(\lambda > 0)$. By Taylor's Theorem,

$$x(p) - x(0) = px'(0) + \int_0^p (p - s)x''(s)\, ds,$$

so

$$(\lambda + p)\|x'(0)\| \leqq \int_0^p (p - s)\|x''(s)\|\, ds,$$

and it follows that

$$\mu \leqq \frac{Kp^2}{2(\lambda + p)} \leqq \frac{Kb^2}{2(\lambda + b)} < \frac{Kb}{2},$$

a contradiction. Thus (H2) is satisfied. Q.E.D.

By choosing $D = [0, b] \times \mathbf{R}^d$ in Lemma 1, we obtain from Theorem 1 the classical existence theorem for bounded f that is usually proven by an application of the Schauder Fixed Point Theorem (see [4, p. 424]). In this case, the assumption that initial value problems for (1) have unique solutions is easily removed by the standard approximation argument.

LEMMA 2. *Suppose there are nonnegative constants L_1, L_2 and L_3 such that*

$$\|f(t, x, y)\| \leqq L_1 + L_2\|x\| + L_3\|y\| \tag{3}$$

for $(t, x) \in D$, $y \in \mathbf{R}^d$. Then (H1) *and* (H2) *are satisfied if b is sufficiently small.*

PROOF. If $L_2 = L_3 = 0$, the result follows from Lemma 1. Suppose either $L_2 \neq 0$ or $L_3 \neq 0$, and define $C = (L_3 + (L_3^2 + 4L_2)^{1/2})/2$. For a given $\mu > 0$, let $x(t)$ be a solution of (1) with $x(0) = 0$, $\|x'(0)\| = \mu$ and $(t, x(t)) \in D$ for $0 \leqq t < p \leqq b$. From the theory of differential inequalities (see [12, Chapter 2]), we have for $0 \leqq t < p$,

$$\begin{aligned} D^-(L_1 + L_2\|x\| + C\|x'\|) &\leqq L_2\|x'\| + C(L_1 + L_2\|x\| + L_3\|x'\|) \\ &= C(L_1 + L_2\|x\| + C\|x'\|) \end{aligned}$$

since $C^2 - CL_3 - L_2 = 0$ and (3) is satisfied, so

$$L_1 + L_2\|x(t)\| + C\|x'(t)\| \leqq (L_1 + C\mu)_e{}^{Ct}, \tag{4}$$

for $0 \leqq t < p$. Thus (H1) is satisfied for any value of $\mu' > 0$.

Now suppose moreover that $x(p) = -\lambda x'(0)$ $(\lambda > 0)$. By Taylor's Theorem,

$$x(p) - x(0) = px'(0) + \int_0^p (p - s)x''(s)\, ds,$$

so

$$\begin{aligned} (\lambda + p)\mu &\leqq \int_0^p (p - s)\|f(s, x, x')\|\, ds \\ &\leqq \int_0^p (p - s)[L_1 + L_2\|x(s)\| + L_3\|x'(s)\|]\, ds \\ &\leqq \int_0^p (p - s)(L_1 + C\mu)e^{Cs}\, ds \qquad \text{(using (4))} \\ &= \frac{L_1 + C\mu}{C^2}(e^{Cp} - Cp - 1), \end{aligned}$$

so

$$\frac{\mu C}{L_1 + C\mu} \leqq \frac{e^{Cp} - Cp - 1}{C(\lambda + p)} < \frac{e^{Cp} - Cp - 1}{Cp}.$$

Choose b small enough that $e^{Cb} - 2Cb - 1 < 0$. Then the function $e^{Cs} - 2Cs - 1$ is negative for $0 < s \leqq b$, so $e^{Cp} - 2Cp - 1 < 0$. Thus $1 > (e^{Cp} - Cp - 1)/Cp$, so for μ sufficiently large we have,

$$\frac{\mu C}{L_1 + C\mu} > \frac{e^{Cp} - Cp - 1}{Cp},$$

a contradiction. Thus if $e^{Cb} - 2Cb - 1 < 0$, (H2) is satisfied for μ sufficiently large. Q.E.D.

The next two lemmas apply to bounded regions D where f satisfies some type of "Nagumo condition."

LEMMA 3. *Assume D is bounded. For $i = 1, 2, \cdots, d$, suppose ϕ_i is a positive, non-decreasing, continuous function on $[0, \infty)$ such that $\int^\infty sds/\phi_i(s) = \infty$ and $|f^i(t, x, y)| \leqq \phi_i(|y^i|)$ for $(t, x) \in D$ and all $y \in \mathbf{R}^d$. Then* (H1) *and* (H2) *are satisfied.*

PROOF. It follows from Lemma 5.1 of [4, p. 428] that any solution $x(t)$ of (1) is extensible as long as $(t, x(t)) \in D$, so (H1) is satisfied for any $\mu' > 0$.

Let R be a number so that $|x^i| \leqq R\,(i = 1, \cdots, d)$ whenever $(t, x) \in D$. Choose μ large enough so that if $v \in \mathbf{R}^d$ and $\|v\| = \mu$, then some component v^i of v satisfies $\int_0^{|v^i|} sds/\phi_i(s) > R$. Let $x(t)$ satisfy (1), $x(0) = 0$ and $\|x'(0)\| = \mu$, and let $x^{i'}(0)$ be the component of $x'(0)$ such that $\int_0^{|x^{i'}(0)|} sds/\phi_i(s) > R$. Suppose $x^{i'}(0) > 0$ and that there is a $p > 0$ so that $(t, x(t)) \in D$ and $x^{i'}(t) > 0$ for $0 \leqq t < p$ and $x^{i'}(p) = 0$. Then

$$\left| \int_0^p \frac{x^{i'}(t)x^{i''}(t)}{\phi_i(x^{i'}(t))}\, dt \right| \leqq \int_0^p x^{i'}(t)\, dt \leqq R.$$

But if we let $s = x^{i'}(t)$, then also

$$\left| \int_0^p \frac{x^{i'}(t)x^{i''}(t)}{\phi_i(x^{i'}(t))}\, dt \right| = \int_0^{x^{i'}(0)} \frac{sds}{\phi_i(s)} > R,$$

a contradiction. Thus in case $x^{i'}(0) > 0$, $x^i(t_0)$ is nonnegative as long as $(t, x(t)) \in D$ (for $0 \leqq t < t_0$). In a similar way, one can show that, if $x^{i'}(0) < 0$, then $x^i(t_0)$ is nonpositive as long as $(t, x(t)) \in D$ for $0 \leqq t < t_0$. Thus (H2) is satisfied. Q.E.D.

LEMMA 4. *Assume D is bounded. Suppose ϕ is a positive, non-decreasing, continuous function on $(0, \infty)$ such that $s^2/\phi(s) \to \infty$ as $s \to \infty$ and $\|f(t, x, y)\| \leqq \phi(\|y\|)$ for $(t, x) \in D$ and all $y \in \mathbf{R}^d$. Then* (H1) *and* (H2) *are satisfied.*

PROOF. Let R be a number such that $\|x\| \leqq R$ whenever $(t, x) \in D$. Choose Q so that $s^2/\phi(s) > 4R$ for $s > Q$. Now if one inspects the proof of Lemma 2.1 in [11], one finds that the result is true for any interval $[0, p]$, and if $x(t)$ is a solution of (1) with $(t, x(t)) \in \bar{D}$ for $0 \leqq t \leqq p$, then a bound for $\|x'(t)\|$ on $[0, p]$ is $\max\{Q, 8R/p\}$. Thus (H1) holds for any $\mu' > 0$.

Let $\epsilon = \inf\{\|x\| : (t, x) \in \partial D\}$, and choose $Q' \geqq Q$ so that $s^2/\phi(s) \geqq 32R^2/\epsilon$ for $s \geqq Q'$. We will show that (H2) is satisfied for $\mu > \max\{Q', 8R/b\}$.

Let $x(t)$ satisfy (1), $x(0) = 0$, $\|x'(0)\| = \mu$ and $(t, x(t)) \in D$ for $0 \leqq t < p$. By our choice of μ and the above comments, $p < b$. Suppose that $(p, x(p)) \in \partial D$ and $x(p) = -\lambda x'(0)$. Note that $\lambda = \|x(p)\|/\|x'(0)\| \geqq \epsilon/\mu$. By Taylor's Theorem,

$$x(p) - x(0) = px'(0) + \int_0^p (p - s)x''(s)\, ds,$$

so

$$\begin{aligned}(\lambda + p)\mu &\leqq \int_0^p (p - s)\|x''(s)\|\, ds \\ &\leqq \int_0^p (p - s)\phi(\|x'(s)\|)\, ds \\ &\leqq \frac{p^2}{2}\phi(\max\{\|x'(s)\| : 0 \leqq s \leqq p\}).\end{aligned}$$

For $0 \leqq s \leqq p$, $\|x'(s)\| \leqq \max\{Q, 8R/p\}$. Since $\|x'(0)\| = \mu$, we have $8R/p \geqq \mu \geqq Q' \geqq Q$, so

$$\begin{aligned}(\lambda + p)\mu &\leqq \frac{p^2}{2}\phi\left(\frac{8R}{p}\right) \\ &\leqq \frac{p^2}{2}\frac{\epsilon(8R/p)^2}{32R^2} = \epsilon,\end{aligned}$$

so

$$\mu \leqq \frac{\epsilon}{\lambda + p} < \epsilon\frac{\mu}{\epsilon} = \mu,$$

a contradiction. Thus (H2) is satisfied. Q.E.D.

The next theorem now follows immediately from Theorem 1 and the lemmas.

THEOREM 2. *Let D be a relatively open set in $[0, b] \times \mathrm{R}^d$ containing $\{(t, 0) \in [0, b] \times \mathrm{R}^d\}$, and suppose all egress points of D relative to* (1) *are strict. Assume initial value problems for* (1) *have unique solutions and that the hypotheses of Lemma* 1, 2, 3 *or* 4 *are satisfied. Then* (1), (2) *has a solution $x(t)$ with $(t, x(t)) \in D$ for $0 \leqq t \leqq b$, where b must be sufficiently small if the hypotheses of Lemma* 2 *are assumed.*

Now we generalize Theorem 2 by extending it to include the nonhomogeneous boundary conditions

$$x(0) = A, \ x(b) = B, \tag{5}$$

where $A, B \in \mathbf{R}^d$.

THEOREM 3. *Let D be a relatively open set in $[0, b] \times \mathbf{R}^d$ containing $\{(t, g(t)) : 0 \leqq t \leqq b\}$, where g is of class C^2 on $[0, b]$ and satisfies* (5), *and suppose all egress points of D relative to* (1) *are strict. Assume initial value problems for* (1) *have unique solutions and that the hypotheses of Lemma* 1, 2, 3, *or* 4 *are satisfied. Then* (1), (5) *has a solution $x(t)$ with $(t, x(t)) \in D$ for $0 \leqq t \leqq b$, where b must be sufficiently small if the hypotheses of Lemma* 2 *are assumed.*

PROOF. Make the change of variable $z = x - g(t)$ for $(t, x) \in [0, b] \times \mathbf{R}^d$. Then D is transformed into a relatively open set D^* containing $\{(t, 0) \in [0, b] \times \mathbf{R}^d\}$. The boundary value problem (1), (3) becomes

$$z'' = f(t, z + g, z' + g') - g''(t), \tag{6}$$

$$z(0) = z(b) = 0. \tag{7}$$

Egress points of D^* relative to (6) are strict, and initial value problems for (6) have unique solutions.

The hypotheses of Lemmas 1 and 2 are satisfied for (6) relative to D^* if they are satisfied for (1). Suppose the hypotheses of Lemma 3 are satisfied for (1). For $i = 1, \cdots, d$, define $\psi_i(s) = \phi_i(s + N_{i1}) + N_{i2}$ ($s \in [0, \infty)$), where $N_{ij} = \max\{|g^{i(j)}(t)| : t \in [0, b]\}$ $(j = 1, 2)$. Then

$$\begin{aligned} |f^i(t, z + g, z' + g') - g^{i''}(t)| &\leqq \phi_i(|z^{i\prime} + g^{i\prime}|) + N_{i2} \\ &\leqq \psi_i(|z^{i\prime}|). \end{aligned}$$

Since each $\int^\infty s\,ds/\phi_i(s) = \infty$, by the limit comparison test for improper integrals (see [1, p. 140]) we have each $\int^\infty s\,ds/\psi_i(s) = \infty$. Then the hypotheses of Lemma 3 hold for (6) with ϕ_i replaced by ψ_i for $i = 1, \cdots, d$. Finally, if the hypotheses of Lemma 4 hold for (1), then they also hold for (6) when D is replaced by D^* and $\phi(s)$ by $\psi(s) = \phi(s + N_1) + N_2$, where $N_j = \max\{\|g^{(j)}(t)\| : 0 \leqq t \leqq b\}$ $(j = 1, 2)$.

By Theorem 2, (6), (7) has a solution $z(t)$ with $(t, z(t)) \in D^*$ for $0 \leqq t \leqq b$, so (1), (5) has a solution $x(t)$ with $(t, x(t)) \in D$ for $0 \leqq t \leqq b$. Q.E.D.

4. **Egress points.** In this section we give some conditions on f which yield a region D for which egress points relative to (1) are strict.

Our first result employs certain auxiliary functions introduced in [10]. For $i = 1, \cdots, N$, let $r_i(t, x)$ be of class C^2 on $[0, b] \times \mathbf{R}^d$, $u_i(t, x)$ the gradient vector of r_i, $v_i(t, x)$ the gradient vector of $\partial r_i/\partial t$, where the gradients are taken with respect to the components of x only in both cases, and let $P_i(t, x)$ be the Hessian of r_i with respect to x. Let the first and second derivatives of r_i with respect to (1) be denoted by

$$r_i' = \frac{\partial r_i}{\partial t} + u_i \cdot x',$$

$$r''_{if} = \frac{\partial^2 r_i}{\partial t^2} + 2v_i \cdot x' + x'P_i \cdot x' + u_i \cdot f,$$

for $i = 1, \cdots, N$.

Lemma 5. *For $i = 1, \cdots, N$, let $r_i(t, x)$ be as described above and*

$$r''_{if} > 0 \text{ when } r_i = 0 \text{ and } r_i' = 0. \tag{8}$$

Then egress points of $D = \{(t, x) \in [0, b] \times \mathbf{R}^d : r_i(t, x) < 0,\ i = 1, \cdots, N\}$ relative to (1) *are strict.*

Proof. Suppose (t_0, x_0) is an egress point of D relative to (1) with $t_0 < b$. Then there is a solution $x(t)$ of (1) and a $\gamma > 0$ so that $x(t_0) = x_0$ and $(t, x(t)) \in D$ for $t_0 - \gamma < t < t_0$. Let i be an integer such that $r_i(t_0, x_0) = 0$.

Let $s(t) = r_i(t, x(t))$ for all t in the domain of $x(t)$. Then $s(t)$ is of class C^2 on its domain and $s'(t) = r_i'(t, x(t))$, $s''(t) = r''_{if}(t, x(t))$. Since $s(t) < 0$ for $t_0 - \gamma < t < t_0$ and $s(t_0) = 0$, we have $s'(t_0) \geqq 0$. If $s'(t_0) > 0$, then $s(t) > 0$ on some interval to the right of t_0, and it follows that (t_0, x_0) is a strict egress point in this case. If $s'(t_0) = 0$, then (8) implies that $s''(t_0) > 0$. Again, $s(t) > 0$ on some interval to the right of t_0, and we conclude that (t_0, x_0) is strict. Q.E.D.

In our final lemma, we obtain a result very close to Lemma 5 using, instead of auxiliary functions, an outer normal condition discussed in [11].

Lemma 6. *Let Ω be an open set in $\mathbf{R}^d$ such that for each $x \in \partial\Omega$, there exists an $n(x) \in \mathbf{R}^d$ for which*

$$\bar{\Omega} \subseteq \{y \in \mathbf{R}^d : n(x) \cdot (y - x) \leqq 0\}. \tag{9}$$

Let $D = [0, b] \times \Omega$, and suppose that

$$n(x) \cdot f(t, x, y) > 0 \text{ when } x \in \partial\Omega,\ n(x) \cdot y = 0. \tag{10}$$

Then egress points of D relative to (1) *are strict.*

PROOF. Suppose $x(t)$ is a solution of (1) with $x(t) \in \Omega$ for $t_0 - \gamma < t < t_0 < b$, where $\gamma > 0$, and $x(t_0) \in \partial\Omega$. By (9) we have $n(x(t_0)) \cdot (x(t) - x(t_0))/(t - t_0) \geqq 0$ for $t_0 - \gamma < t < t_0$, so $n(x(t_0)) \cdot x'(t_0) \geqq 0$. If $n(x(t_0)) \cdot x'(t_0) > 0$, then for $t - t_0$ sufficently small but positive, $n(x(t_0)) \cdot (x(t) - x(t_0)) > 0$, so $x(t) \notin \bar{\Omega}$ by (9). Thus $(t_0, x(t_0))$ is a strict egress point of D in this case.

If $n(x(t_0)) \cdot x'(t_0) = 0$, then (10) implies that $n(x(t_0)) \cdot x''(t_0) > 0$. There is a $t_1 > t_0$ so that

$$0 < n(x(t_0)) \cdot (x'(t) - x'(t_0)) = n(x(t_0)) \cdot x'(t),$$

for $t_0 < t \leqq t_1$. Integrating, we have for $t_0 < t \leqq t_1$,

$$0 < \int_{t_0}^{t} n(x(t_0)) \cdot x'(s)\, ds = n(x(t_0)) \cdot (x(t) - x(t_0)),$$

and (9) implies that $(t_0, x(t_0))$ is a strict egress point of D. Q.E.D.

Putting together Theorem 3 with Lemmas 5 and 6, we have the next theorem.

THEOREM 4. *Suppose that the hypotheses of Lemma* 5 *or* 6 *are satisfied and that D contains $\{(t, g(t)) : 0 \leqq t \leqq b\}$, where g is of class C^2 on* $[0, b]$ *and satisfies* (5). *Assume initial value problems for* (1) *have unique solutions and that the hypotheses of Lemma* 1, 2, 3 *or* 4 *are satisfied. Then* (1), (5) *has a solution $x(t)$ with $(t, x(t)) \in D$ for $0 \leqq t \leqq b$, where b must be sufficiently small if the hypotheses of Lemma* 2 *are assumed.*

The assumptions that initial value problems have unique solutions and that the inequalities in (8) and (10) are strict can be removed, although in some cases additional hypotheses are needed. Rather than attempting to state and prove the most general result possible, we give one special result which indicates what can be done in the other cases.

THEOREM 5. *Suppose that the hypotheses of Lemma* 5 *are satisfied, except that* (8) *is replaced by*

$$(11) \qquad r''_{if} \geqq 0 \quad when \quad r_i = 0 \quad and \quad r_i' = 0,$$

and that D contains $\{(t, g(t)) : 0 \leqq t \leqq b\}$, where g is of class C^2 on $[0, b]$ *and satisfies* (5). *For each $i = 1, \cdots, N$, assume $u_i \neq 0$ when $r_i = 0$ and that one of the following is true*:

(a) $\partial r_i/\partial t\,(t, x) = 0$ *for* $(t, x) \in [0, b] \times \mathbf{R}^d$,

(b) $v_i(t, x) = 0$ *for* $(t, x) \in [0, b] \times \mathbf{R}^d$ *and $P_i(t, x)$ is nonnegative definite when* $r_i(t, x) = 0$,

(c) *$P_i(t, x)$ is uniformly positive definite for $r_i(t, x) = 0$, $\|y\| = 1$.*
If the hypotheses of Lemma 4 are satisfied, then (1), (5) *has a solution $x(t)$ with $(t, x(t)) \in \bar{D}$ for $0 \leqq t \leqq b$.*

Proof. We begin by assuming that (8) holds for each r_i, $i = 1, \cdots, N$. Suppose that for some r_i we have $\partial r_i/\partial t\ (t, x) = 0$ *for* $(t, x) \in [0, b] \times \mathbf{R}^d$. We will show in this case that $yP_i(t, x) \cdot y \geqq 0$ when $(t, x) \in \bar{D}$, $r_i(t, x) = 0$ and $u_i(t, x) \cdot y = 0$.

Suppose on the contrary that there exists a $(t, x) \in \bar{D}$ with $r_i(t, x) = 0$ and a $y \in \mathbf{R}^d$ with $\|y\| = 1$, $u_i(t, x) \cdot y = 0$ and $yP_i(t, x) \cdot y < 0$. For all constants c, $r_i'(t, x, cy) = u_i \cdot cy = 0$, so by (8) we have

$$c^2 yP_i(t, x) \cdot y + u_i(t, x) \cdot f(t, x, cy) > 0. \tag{12}$$

Choose c large enough that $c^2/\phi(c) > d^2\|u_i(t, x)\|/|yP_i(t, x) \cdot y|$, where d is a constant such that if $\|z\|_E$ is the Euclidean norm of $z \in \mathbf{R}^d$, then $\|z\|_E \leqq d\|z\|$ for all $z \in \mathbf{R}^d$. Then

$$\begin{aligned} |u_i(t, x) \cdot f(t, x, cy)| &\leqq d^2\|u_i(t, x)\|\ \|f(t, x, cy)\| \\ &\leqq d^2\|u_i(t, x)\|\phi(c) \\ &\leqq c^2|yP_i(t, x) \cdot y|, \end{aligned}$$

a contradiction of (12).

From the proof of Lemma 4, we know that there is an $\mu_1 > 0$ such that if $x(t)$ satisfies $\|x''\| \leqq \phi(\|x'\|)$ and $(t, x(t)) \in D$ for $t \in [0, b]$, then $\|x'(t)\| \leqq \mu_1$ on $[0, b]$. Also, for $i = 1, \cdots, N$ we can choose μ_2 so that (for all cases (a), (b) and (c))

$$zP_i(t, x) \cdot z \geqq |2v_i(t, x) \cdot z| \tag{13}$$

when $(t, x) \in \bar{D}$, $r_i(t, x) = 0$, $r_i'(t, x, z) = 0$ and $\|z\| \geqq \mu_2$. Let $\mu = \max\{\mu_1, \mu_2\}$.

Define for all $(t, x) \in [0, b] \times \mathbf{R}^d$,

$$F(t, x, y) = \begin{cases} f(t, x, y), & \|y\| \leqq \mu, \\ f\left(t, x, \mu\dfrac{y}{\|y\|}\right), & \|y\| > \mu. \end{cases}$$

Fix i and (t, x, y) so that $(t, x) \in \bar{D}$, $r_i(t, x) = 0$ and $r_i'(t, x, y) = 0$. If $\|y\| \leqq \mu$, then $r''_{iF} > 0$ since $r''_{if} > 0$. If $\|y\| > \mu$, we have

$$\begin{aligned} r''_{iF} &= \frac{\partial^2 r_i}{\partial t^2} + 2v_i \cdot y + yP_i \cdot y + u_i \cdot f\left(t, x, \frac{\mu y}{\|y\|}\right) \\ &= \frac{\partial^2 r_i}{\partial t^2} + \frac{\|y\|}{\mu}\left(2v_i \cdot \frac{\mu y}{\|y\|}\right) \end{aligned}$$

$$+ \frac{\|y\|^2}{\mu^2}\left(\frac{\mu y}{\|y\|} P_i \cdot \frac{\mu y}{\|y\|}\right)$$

$$+ u_i \cdot f\left(t, x, \frac{\mu y}{\|y\|}\right)$$

$$\geqq \frac{\partial^2 r_i}{\partial t^2} + 2v_i \cdot \frac{\mu y}{\|y\|}$$

$$+ \frac{\mu y}{\|y\|} P \cdot \frac{\mu y}{\|y\|} + u_i \cdot f\left(t, x, \frac{\mu y}{\|y\|}\right) > 0,$$

where we have used (13), $\|y\|/\mu > 1$ and $r''_{if} > 0$. Thus F satisfies (8) for each i in all cases (a), (b) and (c).

Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of functions of class C^1 on $[0, b] \times \mathbf{R}^d \times \mathbf{R}^d$ which converges uniformly to F on $\bar{D} \times \mathbf{R}^d$. It follows from (13) and the compactness of $\{(t, x) : r_i(t, x) = 0\}$ $(i = 1, \cdots, N)$ that F_n satisfies (8) for n sufficiently large. Also, for n sufficiently large, F_n satisfies the hypotheses of Lemma 4 if we replace ϕ by $\psi(s) = \phi(s) + \epsilon$ $(\epsilon > 0)$. Since initial value problems for

$$x'' = F_n(t, x, x') \tag{14}$$

have unique solutions, we can apply Theorem 4 and conclude that $(14)_n$, (5) has a solution $x_n(t)$ for n sufficiently large with $(t, x_n(t)) \in D$ on $[0, b]$. Some subsequence of $\{x_n(t)\}_{n=1}^{\infty}$ converges to a solution $x(t)$ of $x'' = F(t, x, x')$ and (5) with $(t, x(t)) \in \bar{D}$ for $0 \leqq t \leqq b$. Since F satisfies $\|F(t, x, y)\| \leqq \phi(\|y\|)$ for $(t, x) \in D$ and $y \in \mathbf{R}^d$, we have $\|x'(t)\| \leqq \mu$ for $0 \leqq t \leqq b$. From the definition of F, $x(t)$ satisfies (1) and (5).

Finally, since $u_i \neq 0$ when $r_i = 0$, the proof can be completed by assuming (11) and using an approximation argument like the last paragraph of Theorem 1.1 in [9]. Q.E.D.

A familiar special case of Theorem 5 is obtained by putting $r_i(t, x) = x^i - \beta^i(t)$, $r_{d+i}(t, x) = -x^i + \alpha^i(t)$, for $i = 1, \cdots, d$, where $\alpha^i(t)$, $\beta^i(t)$ are of class C^2 on $[0, b]$, $\beta^i(t) > \alpha^i(t)$ for $0 \leqq t \leqq b$ and $\alpha^i(0) < A^i < \beta^i(0)$, $\alpha^i(b) < B^i < \beta^i(b)$, $i = 1, \cdots, d$. Then each $v_i = 0$ and $P_i = 0$ so that (b) holds for each i, and condition (11) becomes

$$f^i(t, x, x') \geqq \beta^{i''}, \quad \text{when} \quad x^i = \beta^j(t) \quad \text{and} \quad x^{i\prime} = \beta^{i\prime}(t),$$

$$f^i(t, x, x') \leqq \alpha^{i''}, \quad \text{when} \quad x^i = \alpha^i(t) \quad \text{and} \quad x^{i\prime} = \alpha^{i\prime}(t),$$

for $i = 1, \cdots, d$.

We have used a slightly different type of Nagumo condition than that used in [9] and [10], but otherwise the hypotheses of Theorem 5 are less restrictive than those of Theorem 1.1 in [9] and of Theorem 6.1 in [10]. It is interesting to note that we have had to use the method of modified functions only in removing the hypothesis that initial value problems have unique solutions. It seems possible that the extra hypotheses needed in our proof of Theorem 5 are necessitated by our use of this method rather than being really necessary for the validity of the result.

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