INFINITE GROUPS WITH NORMALITY CONDITIONS ON INFINITE SUBGROUPS

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I. Introduction. In the papers [2, §§ 2 and 3], and [3] S. N. Černikov studies infinite groups G with the property that every infinite subgroup H of G , $H \neq G$, is distinct from its normalizer in G . Groups with this property are called *IN* groups. Another way of stating the *IN* property is "every infinite subgroup of G is ascendant in G ".

IN groups are of two types:

(i) *N*-groups (i.e. groups with the normalizer condition) or (ii) *IN* groups with a self normalizing (necessarily finite) proper subgroup.

Obviously, the classes (i) and (ii) exhaust all *IN* groups. In [2, Theorems 2.1 and 2.2], Černikov gives a detailed description of the structure of locally finite groups of type (ii). In the later paper [3], it is shown that non-torsion *IN* groups are *N* groups.

In this paper, we give an exposition of these results. Our proofs are quite different and, in most cases, considerably shorter than those given in [2] and [3]. The papers [2] and [3], especially [2], make use of several results not easily accessible to non-Russian mathematicians. We have attempted here to use techniques that, for the most part, can be found in the books [14], or [18].

Also, we have put the results of [2] in a somewhat wider context. This has been achieved by replacing the notion of an ascendant subgroup by a serial subgroup. Thus, Černikov's Theorems 2.2 and 3.4 of [2] are consequences of our Theorem D (§ III). We also indicate how to construct groups satisfying Černikov's structure theorem for groups of type (ii) (Theorem F in § V).

Finally in § VI, we give a shorter proof of the results of [3], and note some related problems that remain unsolved.

II. Notation on Basic Concepts. The class of nilpotent groups is denoted by $\mathfrak{N}$, while the class of finite groups is given by $\mathfrak{F}$. If Σ is any class of groups, $L\Sigma$ is the class of groups having a local system of Σ subgroups [18; p. 5]. Thus $L\mathfrak{N}$ and $L\mathfrak{F}$ denote locally nilpotent and locally finite groups respectively.

H is a serial subgroup of G (H ser G) if there is a series of G that contains H as a member [18; p. 9]. The notions of ascendant (H asc G),

descendant ($H \text{ desc } G$) and subnormal ($H \text{ sn } G$) subgroups are specializations of serial subgroups. We let $\mathfrak{R}(s)$, $\mathfrak{R}(a)$, $\mathfrak{R}(d)$, $\mathfrak{R}(\text{sn})$ denote the classes of all groups in which all subgroups are respectively serial, ascendant, descendant or subnormal. Similarly, $\mathcal{I}\mathfrak{R}(s)$, $\mathcal{I}\mathfrak{R}(a)$, $\mathcal{I}\mathfrak{R}(d)$, and $\mathcal{I}\mathfrak{R}(\text{sn})$ denote the classes of *infinite* groups in which all infinite subgroups are serial, ascendant, descendant or subnormal. We note that several of the above classes are denoted by other symbols in the literature; e.g. $\mathfrak{R}(s) = \tilde{N}$ [14; p. 221], $\mathfrak{R}(a) = N =$ normalizer condition [14; p. 220], $\mathfrak{R}(\text{sn}) = N_0$, [15; p. 319].

2.1. Serial Subgroups. The following characterization of serial subgroups in $L\mathfrak{F}$ groups plays an essential role in what follows.

2.1.1 [11] or [16]. If $G \in L\mathfrak{F}$, $H \text{ ser } G$ if and only if for every finite subgroup L of G , $H \cap L \text{ sn } L$.

We call a class Σ of groups a radical class if

- (i) $L(\Sigma) \subset \Sigma$,
- (ii) subgroups of Σ groups are Σ groups,
- (iii) in any group, the join $\Sigma(G)$ of all the normal Σ subgroups of G is a Σ group.

The following fact is implicit in the work of Plotkin [17], and is not difficult to prove directly.

2.1.2. If Σ is a radical class, $G \in L\mathfrak{F}$ and H is a serial Σ subgroup of G , then $H^G \in \Sigma$.

Classes Σ for which we use 2.1.2. include Π groups, Π a set of primes, and $L\mathfrak{R}$ groups.

2.2. Černikov Groups. A group G is a Černikov group (or an extremal group) if G is a finite extension of an abelian group with the minimum condition on subgroups ($= \min$).

If G is an infinite Černikov group and $d(G)$ is the intersection of the subgroups of finite index in G , then $d(G)$ has finite index in G [14; p. 230], and $d(G)$ is a periodic divisible abelian group with $\min$ (i.e., $d(G)$ is a direct sum of a finite number of quasi-cyclic groups).

If H is an arbitrary abelian group, p a prime and n a positive integer, $H(p^n)$ denotes the subgroup of H generated by the elements of order p^n .

We require the following facts relating to Černikov groups.

2.2.1 [1], or [18; p. 84]. If Y is a periodic group of automorphisms of a Černikov group, then Y is a Černikov group.

2.2.2 [18; p. 84]. If H is a divisible abelian group and F is a finite group of automorphisms of H then $H = C_H(F)[H, F]$.

2.2.3 [18; p. 230]. A Černikov p -group G is hypercentral. Consequently $G \in \mathfrak{N}(a)$.

2.2.4 [18; p. 69]. If H and K are Černikov groups, and G is an extension of H by K , then G is a Černikov group. Also, (and quite simply) subgroups of Černikov groups are Černikov groups.

2.3. Some Remarks on Injective Modules. Our presentation here is simplified by appealing to recent results of Hartley and McDougall [12]. These results concern modules for integral group rings of certain groups.

We recall that if the abelian group M is a module for the group G , then M may be viewed, in a natural way, as an $R = ZG$ module; here ZG is the integral group ring of G . Specifically, let $m \in M$, $g \in G$ and denote by m^g the image of m under g . Let $\alpha = \sum n_g g \in ZG$, where the g 's are elements of G , $n_g \in \mathbb{Z}$, and all but a finite number of the n_g 's = 0. We define $m\alpha$ by $m\alpha = \sum n_g(m^g)$. It is easy to check that this product makes M an R module.

The following is a simplified version of Lemma 2.3 of [12].

2.3.1. Suppose M is an abelian p group (p a prime) and that M is a module for the finite p' group Q . Let $R = ZQ$. Then M is an injective R module if and only if M is a divisible group.

We also need.

2.3.2 [12; Lemma 2.1]. Let R be a ring with 1 and V be an R module (unitary). Suppose that $V = \bigoplus \{V_\lambda \mid \lambda \in \Lambda\}$ (module direct sum) and that Λ is finite. Let $\bar{V}$ be an R -injective hull of V . Then $\bar{V} = \bigoplus \{\bar{V}_\lambda \mid \lambda \in \Lambda\}$ where $\bar{V}_\lambda$ is an injective hull of V_λ .

The necessary definitions and theorems on injective modules may be found in the book [6; pp. 385–392]. In particular, we assume both the existence and the various characterizations of injective hulls of modules.

2.4. A Schur-Zassenhaus Property. The following version of the Schur-Zassenhaus property appears in the early work of Černikov [4]. A proof is also indicated in [7; p. 22].

2.4.1. Suppose G is a $L\mathfrak{F}$, locally solvable and countable group. Then for every prime p , there is a Sylow p subgroup P and a Sylow p' subgroup Q such that $G = PQ$.

We also require the following fact, which is of a much simpler nature.

2.4.2. Let $G \in L\mathfrak{F}$ have a normal Sylow p subgroup P of finite index: then P has a complement K . Any such complement is a Sylow p' subgroup of G . Further, any Sylow p' subgroup L of G is conjugate to K .

PROOF. Let T be a transversal of P in G . Then $H = \langle T \rangle$ is finite and $G = PH$. Further, $P \cap H$ is a normal Sylow p subgroup of H . It now follows from the Schur-Zassenhaus-Feit-Thompson theorem for finite groups [20; p. 224] that $P \cap H$ has a complement K in H . Evidently K is a p' group, and $G = PH = PK$. Thus K is a complement of P . An easy argument shows that K is a Sylow p' subgroup of G .

Let L be another Sylow p' subgroup of G . Then $LP/P \cong L/L \cap P \cong L$, so that L is finite. Since $G \in L\mathfrak{F}$, $Y = \langle L, K \rangle$ is finite. Observe also that Y has the normal Sylow p -subgroup $Y \cap P$. Again using the Schur-Zassenhaus-Feit-Thompson theorem [20; p. 227] we deduce that L and K are conjugate in Y .

2.5. **A Remark on Locally Solvable Groups.** We note the theorem of Mal'cev which states that every locally solvable group is an SI group. This result is proved in the book [19; p. 97] and also in [14; p. 183]. From this and the Corollary of [16; p. 348], we deduce

2.5.1. If G is a locally solvable group and $1 \neq x \in G$, then $(x^G)' < x^G$.

III. **The Structure of Locally Finite Groups in $\mathfrak{IN}(s) - \mathfrak{N}(s)$.** We need several lemmas before proceeding with the main result.

LEMMA 3.1. $\mathfrak{N}(s) \cap L\mathfrak{F} = (L\mathfrak{N}) \cap L\mathfrak{F}$.

PROOF. We need only note that $L\mathfrak{N} \subset \mathfrak{N}(s)$ [14; p. 222].

LEMMA 3.2. If $G \in \mathfrak{IN}(s) \cap L\mathfrak{F}$, Σ is a radical class, and H is maximal Σ subgroup with infinite normalizer, then $H \triangleleft G$.

PROOF. From the hypotheses, H is in G . Now use 2.1.2.

LEMMA 3.3. If $G \in \mathfrak{IN}(s) \cap L\mathfrak{F}$ and for some finite normal subgroup F , $G/F \in \mathfrak{N}(s)$, then $G \in \mathfrak{N}(s)$.

PROOF. By Lemma 3.1, it suffices to show that $G \in L\mathfrak{N}$. We do this by showing that all Sylow p subgroups are normal. In view of Lemma 3.2, we need only show that finite Sylow p subgroups have infinite normalizers. Let K be a finite Sylow p -subgroup of G . Then KF/F is a Sylow p subgroup of G/F and hence $KF \triangleleft G$. This implies that $N(K)$ has finite index in G . By Lemma 3.2, $K \triangleleft G$.

LEMMA 3.4. If $G \in \mathfrak{N}(s) \cap L\mathfrak{F}$ and some collection $\Gamma \neq \emptyset$ of infinite normal subgroups has the property $\cap \Gamma = 1$, then $G \in \mathfrak{N}(s)$.

PROOF. Let x and y be elements of coprime order in G , then (using Lemma 3.1) $[x, y] \in H$ for each $H \in \Gamma$. Thus $[x, y] \in \cap \Gamma = 1$ so that x and y commute. It follows that $G \in L\mathfrak{N}$.

LEMMA 3.5. If $G \in \mathfrak{N}(s) \cap L\mathfrak{F}$ and $G = P]R$ ($] =$ split extension), where P is a p group and R is a finite p' group, then for all infinite subgroups $Y \leq P$ with $Y \triangleleft G$, we have $YR \triangleleft G$ and $[P, R] \leq Y$.

PROOF. By Lemma 2.4.2, YR/Y is a Sylow p' subgroup of G/Y . Since, $G/Y \in \mathfrak{N}(s) \cap L\mathfrak{F} \leq L\mathfrak{N}$, both results follow.

THEOREM A. If $G \in (\mathfrak{N}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F}$, then G has an infinite Sylow p subgroup P such that $G = P]K$, where $N(K)$ is a finite group and K is a non-identity nilpotent group.

PROOF. Let A be the intersection of all the infinite normal subgroups of G . A is infinite by Lemma 3.3 and 3.4. Thus A has an infinite abelian subgroup Y ([10] or [18; p. 95]). It follows that $A = Y^G$ is $L\mathfrak{N}$ (2.1.2). Thus, $A = \Pi\{Q_i \mid i \in I\}$, where Q_i is a non-identity Sylow q_i subgroup of A . For each i , $Q_i \triangleleft G$. Thus, if $|I| > 1$, all Q_i 's are finite, and since A is infinite, we conclude that $|I|$ is infinite. But this yields a proper infinite subgroup of A that is normal in G . Hence $|I| = 1$, and A is a p group.

Let P be a Sylow p -subgroup of G containing A ; since P is infinite, we have $P \triangleleft G$. We now show that G/P is finite. Suppose that G/P is infinite. Let x and y be p' elements of G and $U \leq G$ such that $x, y \in U$ and U/P is countably infinite. Let T be a transversal of P in U and $T_1 = \langle T \rangle$. Now $P \cap T_1$ is a normal Sylow p subgroup of T_1 , and T_1 is countable. By 2.4.1, there is a p' group $Q \leq T_1$ such that $T_1 = (P \cap T_1)Q$. Then $U = T_1P = Q(P \cap T_1)P = QP$. Evidently, Q is an infinite p' subgroup of U . By 2.1.2, Q^U is a p' group. Thus $U = Q^U \times P$ and Q^U is a normal Sylow p' subgroup of U . Thus, $x, y \in Q^U$, so that $\langle x, y \rangle$ is a p' group. We conclude that $S = \{x \in G \mid (|x|, p) = 1\}$ is a p' subgroup of G . S is clearly infinite. Lemma 3.2 now yields $S \triangleleft G$, and consequently we have $G = P \times S$. This contradicts Lemma 3.4. Thus, G/P is finite. Further, since P is infinite, $G/P \in \mathfrak{N}(s)$; hence, G/P is nilpotent.

We are now permitted to apply 2.4.2 to produce a complement K of P . If $N(K)$ were infinite, we would have $K \triangleleft G$ by Lemma 3.2; this implies $G = PK \in L\mathfrak{N}$, a contradiction.

THEOREM B. If $G \in (\mathfrak{N}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F}$, then G is a Černikov group.

PROOF. Write $G = P]K$, where P and K are as in Theorem A, and let A be the intersection of the infinite normal subgroups of G . Then $A \leq P$ and A is infinite (Lemma 3.4). We show that A' is finite. If A is infinite, then $A = A'$. Since G is locally solvable, $x \in A$ implies $x^G < A$ (2.5.1). Thus for each $x \in A$, x^G is finite, and $G/C(x^G)$ is finite. It follows that $A/(A \cap C(x^G))$ is finite and this implies that $A \leq C(x)$. Hence $A' = 1$, a contradiction. Thus A' is finite.

Let $G_1 = G/A'$ and $A_1 = A/A'$. By Lemma 3.3, $G_1 \in (\mathcal{IN}(s) \cap \mathcal{L}\mathcal{R})$; further A_1 contains no proper normal infinite subgroup of G_1 . Now $A_1(p) \triangleleft G_1$. If $A_1(p)$ is infinite, then $A_1(p) = A_1$. Then A_1 is residually finite and consequently A_1K_1 is residually finite, where $K_1 = KA'/A'$. By Lemma 3.4, $A_1K_1 \in \mathcal{IN}(s)$. Thus $A_1 \leq C(K_1)$ and by Lemma 3.2, $K_1 \triangleleft G_1$; also K_1 is a nilpotent p' group and $G_1 = P/A' \times K_1$. Thus G_1 , as a direct product of two $\mathcal{L}\mathcal{R}$ groups, is in $\mathcal{L}\mathcal{R} \cap \mathcal{L}\mathcal{R}$. Thus $G_1 \in \mathcal{IN}(s)$, a contradiction. We conclude that $A_1(p)$ is finite.

Since $A_1(p)$ is finite A_1 and consequently A , has min. Further, $C(A) \leq C(K)$. Note that K is a Sylow p' subgroup of G (2.4.2), and that K is nilpotent (Theorem A). Suppose for the moment that $N(K)$ is infinite. Lemma 3.2 then forces $K \triangleleft G$. At this stage, we have $G = P \times K$. Then G , as the direct product of two $\mathcal{L}\mathcal{R}$ groups, is $\mathcal{L}\mathcal{R}$. This violates the assumption that $G \notin \mathcal{IN}(s)$. We conclude then that $N(K)$, and hence $C(K)$, is finite. The finiteness of $C(K)$ implies that $C(AK)$ is finite. By Lemma 3.5, $N(AK) = G$. By 2.2.1, $G/C(AK)$ is a Černikov group. Now G is an extension of a finite group by a Černikov group. It now follows from 2.2.4 that G is a Černikov group.

THEOREM C. Let $G \in (\mathcal{IN}(s) - \mathcal{IN}(s)) \cap \mathcal{L}\mathcal{R}$ and write $G = P]K$ as in Theorem A. If $x \in K$ normalizes some proper infinite subgroup $d(P)$, then $x \in C(P)$.

PROOF. Let $x \in K$, and H be an infinite proper subgroup of $d(P)$ such that $H^{(x)} = H$. We now show that $C(x) \cap d(P)$ is infinite.

If $C(x) \cap d(P)$ is finite, then $[d(P), \langle x \rangle]$ has finite index in $d(P)$ (2.2.2). But we also have $[d(P), \langle x \rangle] < H$ by Lemma 3.5 (applied to the $\mathcal{IN}(s)$ group $d(P)\langle x \rangle$). Since H has infinite index in $d(P)$, we have a contradiction. Hence, $C(x)$ is infinite and by Lemma 3.2, we have $\langle x \rangle \triangleleft P\langle x \rangle$. Thus $x \in C(P)$.

THEOREM D. An infinite locally finite group G is in $\mathcal{IN}(s) - \mathcal{IN}(s)$ if and only if G has an infinite normal Sylow p subgroup P such that

- (i) P is a Černikov p group,
- (ii) $G = P]K$ where $K \neq 1$, and $G/d(P)$ is a finite nilpotent group,
- (iii) if $x \in K$ normalizes an infinite proper subgroup of $d(P)$, then $x \in C(P)$,
- (iv) there is an $x \in K$ such that $[x, P] \neq 1$.

PROOF. The necessity of conditions (i)–(iv) is incorporated in Theorems A–C. We show here that in any group G with properties (i)–(iv), every infinite subgroup is ascendant.

Let H be an infinite subgroup of G . If $d(P) \leq H$, then $H \text{ sn } G$ since $G/d(P)$ is nilpotent. Suppose $d(P) \not\leq H$. Then $H = (H \cap P)L$, $H \cap P$ being a normal Sylow p subgroup of H and L a p' group (2.4.2). By 2.4.2, L is conjugate to a subgroup L^t of K .

Now, $H^t \cap d(P)$ is a proper infinite subgroup of $d(P)$ and is normalized by L^t . By condition (iii) $L^t \leq C(P)$ and so $L \leq C(P)$. We now have $H = (H \cap P) \times L \text{ asc } P \times L$ (2.2.3). Since $P \times L \text{ sn } G$, we have $H \text{ asc } G$.

The condition (iv) insures that $G \notin \mathfrak{N}(s)$.

Later in § IV, we give a somewhat different statement of Theorem D (Theorem D'). To do this, we shall need

COROLLARY 3.1. *Let $G \in (\mathfrak{IN}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F}$. Then, in the notation of Theorem D, $d(P) = [P, K]$ and $[d(P), K] = d(P)$.*

PROOF. It suffices to prove $[d(P), K] = d(P)$. Suppose $[d(P), K] < d(P)$. Using 2.2.2, we see that $C_{d(P)}(K)$ is infinite, as $d(P)/[d(P), K]$ is infinite. Then, Theorem D (iii) yields $K \leq C(P)$ which violates part (iv) of Theorem D. Thus, $[d(P), K] = d(P)$.

As a consequence of Theorem D, we have

COROLLARY 3.2. $(\mathfrak{IN}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F} = (\mathfrak{IN}(a) - \mathfrak{N}(a)) \cap L\mathfrak{F}$.

PROOF. Suppose $G \in (\mathfrak{IN}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F}$. By Theorem D, G has min. This implies that $G \in \mathfrak{IN}(a)$. Evidently, $G \notin \mathfrak{N}(a)$. This yields $(\mathfrak{IN}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F} \leq (\mathfrak{IN}(a) - \mathfrak{N}(a)) \cap L\mathfrak{F}$. For the other inclusion, let $G \in (\mathfrak{IN}(a) - \mathfrak{N}(a)) \cap L\mathfrak{F}$. Clearly $G \in \mathfrak{IN}(s)$. Further it is not hard to show the existence of a finite subgroup F of G with $N(F) = F$. In view of 2.1.1, F is not a serial subgroup. This establishes $(\mathfrak{IN}(a) - \mathfrak{N}(a)) \cap L\mathfrak{F} \leq (\mathfrak{IN}(s) - \mathfrak{N}(s)) \cap L\mathfrak{F}$ and the corollary is proved.

Also of interest is

COROLLARY 3.3. *An uncountable $\mathfrak{IN}(s) \cap L\mathfrak{F}$ ($\mathfrak{IN}(a) \cap L\mathfrak{F}$) group is an $\mathfrak{N}(s)$ ($\mathfrak{N}(a)$) group.*

Corollary 3.3 is proved by observing that the groups occurring in Theorem D are countable.

IV. Strongly Irreducible Automorphisms of Divisible Abelian Groups.

DEFINITION. Let P be a divisible abelian p group of finite rank (i.e. the p -rank of the abelian group P , [19; p. 35]) and A a periodic group of automorphisms of P . A is called strongly irreducible ($S-I$) if for each $1 \neq a \in A$, and each infinite subgroup $H \leq P$, $H^{(a)} = P$.

These are the types of automorphisms that arise in Theorem D (i). In the section, we prove

THEOREM E. Let P be a divisible abelian p -group of finite rank, and A be a nilpotent p' group of $S-I$ automorphisms of P . Then

- (i) A is cyclic,
- (ii) if $1 \neq a \in A$, then $\langle a \rangle$ acts irreducibly on $P(p)$,
- (iii) if $n = \text{rank}(P)$, and q is the smallest prime dividing $|A|$, then $n = \text{the order of } p \pmod{q}$; and so $n \mid q - 1$.

Theorem E will be used to give a different formulation of Theorem D. Before proceeding further, we note that periodic groups of automorphisms of divisible abelian p groups of finite rank are finite [19; p. 84]. Thus, we assume all $S-I$ groups of automorphisms are finite. Also, throughout the remainder of this section, P will denote a divisible abelian p group of finite rank.

LEMMA 4.1. Let $1 \neq A$ be an $S-I$ subgroup of $\text{Aut}(P)$ with $(|A|, p) = 1$. Then for each element $y \in P(p)$, $y \neq 1$, and each element $1 \neq a \in A$, we have $y^{(a)} = P(p)$.

PROOF. Let $1 \neq a \in A$. We view P as a module for the ring $R = Z\langle a \rangle$. By 2.3.1 P is an injective R module, and it is not difficult to show that P is an R injective hull of $P(p)$.

Let $1 \neq y \in P(p)$ and $H = y^{(a)}$. Then H is an R submodule of P . If $H < P(p)$, Maschke's theorem yields a R submodule $0 \neq H_1 \leq P(p)$ such that $P(p) = H \oplus H_1$. We now use 2.3.2 to obtain $P = \overline{P(p)} = \overline{H \oplus H_1} = \overline{H} \oplus \overline{H_1}$; here the $\overline{}$'s denote R injective hulls. Now $\overline{H}$ is an infinite $\langle a \rangle$ -invariant subgroup of P so that $\overline{H} = P$. This forces $\overline{H_1} = 0$ and consequently $H_1 = 0$ (in the module notation). This contradicts the assumption $H < P(p)$. Thus $H = P(p)$ and the proof is complete.

The following easy remark will simplify some of our arguments.

LEMMA 4.2. Let G be any abelian p group and $\alpha \in \text{Aut}(G)$ with finite order and $(|\alpha|, p) = 1$. If α centralizes all elements of order p in G , then $\alpha = 1$.

PROOF. Suppose the lemma false and let $x \in G$ be of least order such that $\alpha \notin C(x)$. Since G is abelian, $[x, \alpha]^p = [x^p, \alpha]$. By the minimality of $|x|$, $[x^p, \alpha] = 1$. Thus $[x, \alpha]^p = 1$ and $[x, \alpha] \in C(\alpha)$. Thus, $1 \neq [x, \alpha]^p = [x, \alpha^p]$. Therefore $\langle \alpha^p \rangle = \langle \alpha \rangle \leq C(x)$; a contradiction.

LEMMA 4.3. *Let $1 \neq A$ be an $S-I$ subgroup of $\text{Aut}(P)$ with $(|A|, p) = 1$. Let $n = \text{rank}(P)$, and $\alpha: A \rightarrow GL(n, p)$ be defined by $a\alpha = a|P(p)$ (the restriction of a to $P(p)$). Then α is an isomorphism and for each $1 \neq a \in A$, $\langle a\alpha \rangle$ is an irreducible subgroup of $GL(n, p)$.*

PROOF. Certainly α is a homomorphism. Lemma 4.2 implies $\ker \alpha = 1$. The irreducibility of $\langle a\alpha \rangle$ follows from Lemma 4.1.

PROOF OF THEOREM E. Lemma 4.3 immediately gives (ii). Let T be an abelian subgroup of A . By Lemma 4.3, T is an irreducible subgroup of $GL(n, p)$. As is well known, abelian irreducible subgroups of $GL(n, p)$ are cyclic [8; p. 65]. Thus T is cyclic. Thus, if q is a prime, $q \neq 2$, then the Sylow q subgroup of A is cyclic [20; p. 252]. Let L be the Sylow 2 subgroup of A , and $s \in L$ be an element of order 2. Since $\langle s \rangle$ acts irreducibly on $P(p)$, we must have $r(P) = 1$. Thus, (again by Lemma 4.3) L is cyclic.

Hence, all Sylow subgroups of A are cyclic. The nilpotence of A implies now that A is cyclic. This establishes (i).

Let q be the smallest prime divisor of A and $a \in A$ be an element of order q . Then $P(p)$ is a faithful and irreducible module for $\langle a \rangle$; i.e. $P(p)$ is irreducible as a $Z_p \langle a \rangle$ module. The possible dimensions of such modules are enumerated in the book [13; II, 3.10]. In particular, we must have $n = \text{order of } p \pmod{q}$. Consequently, $n \mid q - 1$. This completes the proof of Theorem E.

The proof of Theorem E shows that an $S-I$ subgroup A of $\text{Aut}(P)$ has cyclic Sylow q subgroups for $q \neq p$. Černikov has shown [2; p. 723-724] that the Sylow p subgroups of A are also cyclic. The proof of this fact makes essential use of results in the paper [5].

In view of Theorem E and Corollary 1 of Theorem D, we may now give a somewhat different form of Theorem D.

THEOREM D'. $G \in (\mathcal{IN}(s) - \mathcal{N}(s)) \cap L\mathfrak{S}$ if and only if G has an infinite normal Sylow p subgroup P such that

- (i)' P is a Černikov p group,
- (ii)' $G = P]K$ where K is a finite nilpotent p' group and $K \neq 1$,
- (iii)' $Y = [P, K]$ has finite index in P and is a divisible abelian group with $\min$,
- (iv)' $K/C_C(P)$ is a non-trivial cyclic group of $S-I$ automorphisms of Y .

We note the following special case of this theorem.

COROLLARY 4.1. *Suppose in the notation of Theorem D' that $Y = P$ (i.e., P is divisible abelian of finite rank). Let H be an infinite subgroup*

of G , where G satisfies the conditions of Theorem D' . Then either

(a) $P \leq H$, or

(b) $H = (H \cap P) \times L$ where L is conjugate to a subgroup of K .

Consequently, $H \text{ sn } G$.

If, in addition, $C_K(P) = 1$, then $H \text{ sn}_2 G$ ($H \text{ sn}_2 G$ means $H \triangleleft H^G$).

This corollary can be proved by using Theorem E in conjunction with the method of proof of Theorem D. We do not repeat these arguments here.

V. Construction of Groups Satisfying the Conditions of Theorem D. This task turns out to be rather easy if we make use of the results of 2.3.

THEOREM F. *Let E be an elementary abelian p group of rank n . Further, let A be a cyclic p' subgroup of $GL(n, p)$ such that every non-identity subgroup of A acts irreducibly on E . Viewing E as an $R = ZA$ module, let $\bar{E}$ be an R -injective hull of E . Then $\bar{E}$ is a divisible abelian p -group of rank n and the group $G = \bar{E}A$ satisfies the conditions of Theorem D (or Theorem D'). (Here the action of A on $\bar{E}$ is induced from the action of R on $\bar{E}$).*

PROOF. The existence of R injective hulls is well known [6; p. 389]. We first note that $\bar{E}$, as an abelian group, is divisible. This is a consequence of [6; p. 393, Ex. 3]. Let Y be the Sylow p -subgroup of $\bar{E}$. Then Y is a divisible abelian group and is also an R -submodule of $\bar{E}$. Now 2.3.1 implies that Y is injective as an R -module. Since $E \leq Y$, we deduce that $Y = \bar{E}$; thus $\bar{E}$ is a p -group.

Suppose $\bar{E}(p) > E$, and let $y \in \bar{E}(p) - E$. Then by Maschke's theorem, $D = E + yR = E \oplus F$ for some R submodule F of D (yR is a finite elementary p -group). On the other hand, $F \cap E \neq 0$, since $\bar{E}$ is an essential extension of E . This contradiction shows that $\bar{E}(p) = E$; thus, $\bar{E}$ has rank n .

Let $1 \neq a \in A$ and suppose H is an infinite $\langle a \rangle$ invariant subgroup of G . Then $d(H)$ is an infinite divisible group and $d(H)$ is $\langle a \rangle$ invariant. Certainly $d(H) \cap E \neq 1$; this follows from the fact that $\text{rank}(E) = \text{rank}(\bar{E})$. The $\langle a \rangle$ -irreducibility of E now yields $E \leq d(H)$. Thus, $\text{rank}(d(H)) = \text{rank}(\bar{E}) = n$. The divisibility of $d(H)$ now implies that $d(H) = \bar{E}$ (otherwise $d(H)$ would be a proper direct summand of $\bar{E}$). Thus, $H = \bar{E}$ and A is an $S-I$ group of automorphisms of $\bar{E}$.

It is now a simple matter to show that all the conditions of Theorem D hold. This completes the proof of Theorem F.

We note some additional properties of the group G constructed in Theorem F.

5.1 $[\bar{E}, A] = \bar{E}$.

5.2 If H is an infinite subgroup of G , then either

- (a) $\bar{E} < H$, or
- (b) H is abelian and $H \text{ sn}_2 G$.

5.3 Every non-abelian infinite subgroup is normal.

The property 5.1 follows from Corollary 3.1, while 5.2 and 5.3 are consequences of Corollary 4.1.

VI. Non-periodic $\mathfrak{N}(a)$ Groups. The main result here is that an element of infinite order in an $\mathfrak{N}(a)$ group forces the group to be an $\mathfrak{N}(a)$ group. Similar results hold for the class $\mathfrak{N}(\text{sn})$. However, we have been unable to develop a corresponding theory for either of the classes $\mathfrak{N}(s)$ or $\mathfrak{N}(d)$. The key fact that enables us to prove our main result here is the local nilpotence of groups in the class $\mathfrak{N}(a)$ ([18; p. 61]). Whether or not $\mathfrak{N}(s)$ groups (or even $\mathfrak{N}(d)$) groups are locally nilpotent is an unresolved question.

THEOREM G [3; p. 29]. *If G is not periodic and $G \in \mathfrak{N}(a) (\mathfrak{N}(\text{sn}))$, then $G \in \mathfrak{N}(a) (\mathfrak{N}(\text{sn}))$.*

PROOF. Let x be an element of infinite order in the $\mathfrak{N}(a)$ group G and let F be a finite subgroup of G . Then $\langle x \rangle \text{ asc } \langle F, x \rangle$ so that $x^F \in L\mathfrak{N}$ ([9] or [18; p. 61]). The finiteness of F insures that x^F is finitely generated. Thus, x^F is a finitely generated nilpotent group and consequently D , the center of x^F is a finitely generated infinite abelian group. D contains an infinite characteristic torsion free subgroup E . Let $S = EF$ and p be a prime not dividing $|F|$. The group S/E^{p^n} is a finite $\mathfrak{N}(a)$ (and hence nilpotent) group for each positive integer n . Thus, for each n , $[E, F] \leq E^{p^n}$, and this gives $[E, F] \leq \bigcap E^{p^n} = 1$.

We now have $F \triangleleft EF = S \text{ asc } G$ which completes the proof. The same proof shows that non-periodic $\mathfrak{N}(\text{sn})$ groups are $\mathfrak{N}(\text{sn})$.

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