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SINGULAR PERTURBATION OF SIMPLE EIGENVALUES

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0. **Introduction.** Perturbation theory of linear operators contains many diverse results on behaviour of spectral properties when operators undergo a small change. The subject has occupied an important place in applied mathematics since its beginnings in the works of Rayleigh and Schrödinger. In the last forty years, the regular perturbation method has been developed to a comprehensive theory in the studies of Rellich, Friedrichs, Sz-Nagy, Kato, and others (cf. the bibliography of Kato [**12**]). Not only are criteria known for convergence of perturbation series, but also for asymptotic validity of retention of the first few terms when higher order terms may not exist (cf. Kato [**9**, **10**, **11**, **12**]).

On the other hand, the extensive development of singular perturbation expansions for eigenvalues of differential operators, also dating back to Rayleigh, is not included in the operator theoretic studies. Traditionally, "singular perturbation" means that the "small change" in a differential operator involves derivatives of higher order than those otherwise present. Such a change is of course pathological from the topological point of view, and the regular perturbation method usually fails, even at the first order. Mathematically valid expansion theorems for singular perturbation of eigenvalues have been given by Moser [**16**], Višik and Lyusternik [**19**], and Handelman, Keller, and O'Malley [**6**] (cf. also O'Malley [**17**]). To date the only corresponding operator theoretic results consist of convergence theorems (cf. [**12**], Huet [**8**], and Stummel [**18**]) and rate of convergence theorems (cf. Greenlee [**3**, **4**]).

In this work we prove two operator theoretic theorems which generalize those of asymptotic regular perturbation theory (as in [**10**]), and which apply to the above mentioned singular perturbation problems. Application of these theorems to concrete problems is involved, but the perturbation expansions for eigenvalues and eigenvectors are developed in terms of solutions of linear operator equations. Hence the method of correctors, Lions [**14**], as well as traditional boundary layer techniques, [**17**], can be used to apply these theorems.

Our intent is, however, not merely to derive known singular perturbation results from a more abstract point of view. The current formulation should also be applicable to highly singular "hard core" potential perturbations of the radial equation of quantum mechanics. In the present work we apply our theorems to a comparatively simple model problem whose analysis is basic to that of the quantum mechanical problem. The primary results of this paper, and additional related results, have been announced in Greenlee [5].

1. **Preliminaries.** Let H be a complex Hilbert space with inner product (v, w) and norm $|v|$. Let $b(v, w)$ be a Hermitian symmetric bilinear form defined on a linear manifold $D(b)$ which is dense in H. Further, let the corresponding quadratic form $b(v) \equiv b(v, v)$ be closed and have a positive lower bound. Since $b(v)$ is closed, $D(b)$, with the inner product $b(v, w)$ is a Hilbert space. The corresponding norm will be denoted by $|v|_b$. The Hilbert space $D(b)$ is continuously contained in H, written

$$D(b) \underset{c}{\subset} H;$$

that is $D(b)$ is a vector subspace of H and the injection of $D(b)$ into H is continuous.

Let $a(v, w)$ be a Hermitian symmetric bilinear form defined on a linear manifold $D(a)$ which is dense in $D(b)$ (and therefore in H). Further, let the corresponding quadratic form $a(v) \equiv a(v, v)$ be non-negative and closed in $D(b)$. Then for each $\epsilon > 0$, $D(a)$, with the inner product $\epsilon a(v, w) + b(v, w)$, is a Hilbert space which is continuously contained in $D(b)$ (and hence in H).

Now let B be the operator in H associated with $b(v, w)$, that is B is defined on

$$D(B) = \{v \in D(b) : w \to b(v, w) \text{ is continuous on } D(b) \text{ in the topology induced by } H\},$$

by

$$(Bv, w) = b(v, w) \quad \text{for all } w \in D(b).$$

B is a positive definite self adjoint operator in H (cf. Lions [13]). With the inner product (Bv, Bw), $D(B)$ is a Hilbert space. Similarly for $\epsilon > 0$, let A_ϵ be the operator in H associated with $\epsilon a(v, w) + b(v, w)$, that is

$$(A_\epsilon v, w) = \epsilon a(v, w) + b(v, w).$$

A_ϵ is the singular perturbation of B to be studied in the sequel, and is a positive definite self adjoint operator in H.

An additional operator is needed for the theoretical development of the next section. This operator, $\mathcal{A}$, which "measures the perturbation in terms of the unperturbed form $b(v, w)$", is the operator in $D(b)$ associated with $a(v, w)$, that is

$$b(\mathcal{A}v, w) = a(v, w).$$

$\mathcal{A}$ is a non-negative self adjoint operator in $D(b)$. We note that for $\epsilon > 0$, $B^{-1}A_\epsilon$ is the restriction to $D(A_\epsilon)$ of $\epsilon\mathcal{A} + I$, written $B^{-1}A_\epsilon \subset \epsilon\mathcal{A} + I$, where I is the identity operator on $D(b)$ (cf. Greenlee [2]).

Now let $f: [0, \infty) \to [1, \infty)$ be a Borel function. Then $f(\mathcal{A})$ and $[f(\mathcal{A})]^{-1}$ are well defined by use of the spectral theorem (cf. Dunford and Schwartz [1]) and $D(f(\mathcal{A}))$, provided with the norm $|f(\mathcal{A})v|_b$, is a Hilbert space. In the special case, $f(\nu) = (\nu^2 + 1)^{\tau/2}$, $\tau \in (0, 1]$, the following lemma was proven in [3].

Lemma 1. *Let $f: [0, \infty) \to [1, \infty)$ be a Borel function. If $D(B) \underset{c}{\subset} D(f(\mathcal{A}))$, then there exists a constant $M > 0$ such that*

$$|v| \leqq M|[f(\mathcal{A})]^{-1}v|_b \quad \text{for all } v \in D(b).$$

Proof. For $h \in H$, let u be the unique element of $D(B)$ for which $Bu = h$. By hypothesis there exists $K > 0$ such that $|f(\mathcal{A})w|_b \leqq K|Bw|$ for all $w \in D(B)$. Thus for any $v \in D(b)$,

$$\begin{aligned}
|v| &= \sup\{|(v, h)| : h \in H \text{ and } |h| \leqq 1\} \\
&= \sup\{|(v, Bu)| : u \in D(B) \text{ and } |Bu| \leqq 1\} \\
&= \sup\{|b(v, u)| : u \in D(B) \text{ and } |Bu| \leqq 1\} \\
&\leqq K^{-1} \sup\{|b(v, u)| : u \in D(B) \text{ and } |f(\mathcal{A})u|_b \leqq 1\} \\
&= K^{-1}\sup\{|b([f(\mathcal{A})]^{-1}v, f(\mathcal{A})u)| : u \in D(B) \\
&\qquad \text{and } |f(\mathcal{A})u|_b \leqq 1\} \\
&\leqq K^{-1}|[f(\mathcal{A})]^{-1}v|_b.
\end{aligned}$$

2. **Singular Perturbation of Simple Eigenvalues.** Throughout this section the following assumptions are made: *λ is a simple eigenvalue of B with corresponding eigenvector u having $|u| = 1$; λ is stable under the perturbation, $B \to A_\epsilon$.* Stability of λ means that for ϵ sufficiently small, the intersection of any isolating interval for λ and the specturm of A_ϵ consists of a simple eigenvalue λ_ϵ of A_ϵ, $\lambda_\epsilon \to \lambda$ as $\epsilon \downarrow 0$ (cf. [12]).

Now, $Bu = \lambda u$, and since by [2], $A_\epsilon^{-1}B \subset (\epsilon\mathcal{A} + I)^{-1}$,

$$(A_\epsilon^{-1}u, u) = \lambda^{-1} - \epsilon\lambda^{-2}\lambda_\epsilon',$$

where $\lambda_\epsilon' = b(\mathcal{A}(\epsilon\mathcal{A} + I)^{-1}u, u)$ (cf. also [3]). Letting E be the resolution of the identity for $\mathcal{A}$,

$$\epsilon\lambda_\epsilon' = \int_0^\infty \epsilon\nu(\epsilon\nu + 1)^{-1} b(E(d\nu)u, u),$$

and it follows from Lebesgue's dominated convergence theorem that $\epsilon\lambda_\epsilon' \to 0$ as $\epsilon \downarrow 0$. Use of the Landau symbols o, O, will always denote a limiting process as $\epsilon \downarrow 0$.

THEOREM 1. *The simple eigenvalue* λ_ϵ *of* A_ϵ *such that* $\lambda_\epsilon \to \lambda$ *as* $\epsilon \downarrow 0$ *satisfies the following:*

(i) $\lambda_\epsilon = \lambda + O(\epsilon\lambda_\epsilon')$;

(ii) *if there exists a Borel function* $f: [0, \infty) \to [1, \infty)$ *such that* $f(\nu) \to \infty$ *as* $\nu \to \infty$ *and* $D(B) \underset{c}{\subset} D(f(\mathcal{A}))$, *then*

$$\lambda_\epsilon = \lambda + \epsilon\lambda_\epsilon' + o(\epsilon\lambda_\epsilon');$$

(iii) *if the hypothesis of* (ii) *is satisfied by* $f(\nu) = (\nu + 1)^\tau$ *where* $0 < \tau < 1/2$, *then*

$$\lambda_\epsilon = \lambda + \epsilon\lambda_\epsilon' + O(\epsilon^{2\tau+1}\lambda_\epsilon').$$

Moreover, if the eigenvector u_ϵ *of* A_ϵ *corresponding to* λ_ϵ *is normalized so that* $|u_\epsilon| = 1$ *and* $(u_\epsilon, u) \geqq 0$, *then in Case* (i),

$$|u_\epsilon - u|_b = O([\epsilon\lambda_\epsilon']^{1/2}),$$

in Case (ii),

$$|u_\epsilon - u|_b = o([\epsilon\lambda_\epsilon']^{1/2}),$$

in Case (iii),

$$|u_\epsilon - u|_b = O(\max\{[\epsilon^{2\tau+1}\lambda_\epsilon']^{1/2}, \epsilon\lambda_\epsilon'\}).$$

Recall ([**11**]) that if $u \in D((\mathcal{A} + I)^{1/2}) = D(a)$, the regular perturbation method applies to first order for λ_ϵ. Also observe that part (iii) of Theorem 1 may give more than the first order correction to λ_ϵ, depending on the number of terms in an asymptotic expansion of $\epsilon\lambda_\epsilon'$ which are of lower order than $\epsilon^{2\tau+1}\lambda_\epsilon'$.

PROOF. We calculate $\eta = (A_\epsilon^{-1}u, u)$ and $\theta = |(A_\epsilon^{-1} - \eta)u|$ in order to apply Lemma 4, p. 437, of [**10**]. Now,

$$\begin{aligned}\theta &= \lambda^{-1}|[(\epsilon\mathcal{A} + I)^{-1} - \eta\lambda]u| \\ &\leqq \lambda^{-1}|[(\epsilon\mathcal{A} + I)^{-1} - I]u| + \epsilon\lambda^{-2}\lambda_\epsilon'.\end{aligned}$$

Thus, if f satisfies the hypotheses of Lemma 1, there exists $M > 0$ such that

$$|[(\epsilon\mathcal{A} + I)^{-1} - I]u|^2 \leqq M|[f(\mathcal{A})]^{-1}[(\epsilon\mathcal{A} + I)^{-1} - I]u|_b^2$$

$$= M \int_0^\infty \epsilon^2\nu^2(\epsilon\nu + 1)^{-2}[f(\nu)]^{-2}b(E(d\nu)u, u)$$

$$\leqq M(\sup_{\nu\in[0,\infty)} \{\epsilon\nu(\epsilon\nu + 1)^{-1}[f(\nu)]^{-2}\})$$

$$\cdot \int_0^\infty \epsilon\nu(\epsilon\nu + 1)^{-1}b(E(d\nu)u, u)$$

$$= M\epsilon\lambda_\epsilon' \cdot \sup_{\nu\in[0,\infty)} \{\epsilon\nu(\epsilon\nu + 1)^{-1}[f(\nu)]^{-2}\}.$$

Since $\epsilon\lambda_\epsilon' \to 0$ as $\epsilon \downarrow 0$ and the supremum is necessarily finite, by Lemma 4, p. 437, of [10], $|\lambda_\epsilon^{-1} - \eta| = O(\theta^2)$ and $|u_\epsilon - u| = O(\theta)$. So (i) follows by taking $f(\nu) \equiv 1$, and since under the hypotheses of (ii) the supremum is o(1) as $\epsilon \downarrow 0$, (ii) is also proven. For $f(\nu) = (\nu + 1)^\tau$, $0 < \tau < 1/2$, elementary calculus shows that the supremum is assumed and is $O(\epsilon^{2\tau})$, yielding (iii).

To prove the eigenvector estimate, first let z_ϵ be the unique solution of

$$\epsilon a(z_\epsilon, v) + b(z_\epsilon, v) = \lambda(u, v) \quad \text{for all} \quad v \in D(a).$$

Since

$$\epsilon a(u_\epsilon, v) + b(u_\epsilon, v) = \lambda_\epsilon(u_\epsilon, v) \quad \text{for all} \quad v \in D(a),$$

by setting $v = z_\epsilon - u_\epsilon$ and subtracting these two equations, we obtain

$$\epsilon a(z_\epsilon - u_\epsilon) + b(z_\epsilon - u_\epsilon) = (\lambda u - \lambda_\epsilon u_\epsilon, z_\epsilon - u_\epsilon).$$

Hence, if c is the reciprocal of the lower bound of $b(v)$, the Cauchy-Schwarz inequality yields

$$b(z_\epsilon - u_\epsilon) \leqq c|\lambda u - \lambda_\epsilon u_\epsilon|\,|z_\epsilon - u_\epsilon|_b.$$

So, since $|u| = 1$,

$$|z_\epsilon - u_\epsilon|_b \leqq c\{|\lambda - \lambda_\epsilon| + \lambda_\epsilon|u - u_\epsilon|\} = O([\epsilon\lambda_\epsilon']^{1/2}),$$

in Case (i). Now, $u_\epsilon - u = (u_\epsilon - z_\epsilon) + (z_\epsilon - u)$ and, since $Bu = \lambda u$, z_ϵ satisfies

$$b((\epsilon\mathcal{A} + I)z_\epsilon, v) = b(u, v) \quad \text{for all} \quad v \in D(b).$$

Hence $z_\epsilon = (\epsilon\mathcal{A} + I)^{-1}u$, so from the first part of the proof with $f(\nu) \equiv 1$ we have that $|z_\epsilon - u|_b = O([\epsilon\lambda_\epsilon']^{1/2})$. Thus the eigenvector estimate for Case (i) follows from the triangle inequality. Cases (ii) and (iii) are obviously similar.

Now one way to investigate the asymptotic behaviour of $\epsilon\lambda_\epsilon'$ is to find $\kappa : (0, \epsilon_0] \to (0, \infty)$, $\kappa(\epsilon) \to 0$ as $\epsilon \downarrow 0$, and $0 \neq u' \in H$ such that

$$|\epsilon\mathcal{A}(\epsilon\mathcal{A} + I)^{-1}u - \kappa(\epsilon)u'| = o(\kappa(\epsilon)). \tag{1}$$

When (1) holds,

$$\begin{aligned}\epsilon\lambda_\epsilon' &= b(\epsilon\mathcal{A}(\epsilon\mathcal{A} + I)^{-1}u, u)\\ &= \lambda(\epsilon\mathcal{A}(\epsilon\mathcal{A} + I)^{-1}u, u)\\ &= \lambda\kappa(\epsilon)(u', u) + o(\kappa(\epsilon)).\end{aligned}$$

THEOREM 2. *If there exist κ, u' for which* (1) *holds, and λ_ϵ is the simple eigenvalue of A_ϵ satisfying $\lambda_\epsilon \to \lambda$ as $\epsilon \downarrow 0$, then*

$$\lambda_\epsilon = \lambda + \epsilon\lambda_\epsilon' - \kappa^2(\epsilon)\lambda(Su', u') + o(\kappa^2(\epsilon)),$$

where S is the bounded self adjoint operator in H defined by $Su = 0$, $S = B(B - \lambda)^{-1}$ on $\{u\}^\perp$. Moreover, letting $v_\epsilon = |u - \kappa(\epsilon)Su'|^{-1}(u - \kappa(\epsilon)Su')$, if the eigenvector u_ϵ of A_ϵ corresponding to λ_ϵ is normalized so that $|u_\epsilon| = 1$ and $(u_\epsilon, v_\epsilon) \geqq 0$, then

$$|u_\epsilon - v_\epsilon| = o(\kappa(\epsilon)).$$

This provides a generalization of the second order regular perturbation formula; but observe that a classical asymptotic expansion of λ_ϵ through order $\kappa^2(\epsilon)$ is obtained only if $\epsilon\lambda_\epsilon'$ can be expanded through this order. If $u' \, 2 \, D(b)$, then $Su' \in D(b)$ and, by the method used in the proof of Theorem 1, it can be shown that the eigenvector estimate of Theorem 2 holds with $|u_\epsilon - v_\epsilon|_b$ in place of $|u_\epsilon - v_\epsilon|$.

PROOF. Noting that $|v_\epsilon| = 1$, we calculate $\eta = (A_\epsilon^{-1}v_\epsilon, v_\epsilon)$ and $\theta = |(A_\epsilon^{-1} - \eta)v_\epsilon|$ in order to apply Lemma 4, p. 437, of [10]. Since Su' is orthogonal to u in H,

$$\eta = (1 + \kappa^2(\epsilon)|Su'|^2)^{-1}(A_\epsilon^{-1}(u - \kappa(\epsilon)Su'), (u - \kappa(\epsilon)Su'))$$

and, since A_ϵ^{-1} is self adjoint in H,

$$\begin{aligned}\eta = (1 + \kappa^2(\epsilon)|Su'|^2)^{-1}\{&(A_\epsilon^{-1}u, u) - \kappa(\epsilon)(A_\epsilon^{-1}u, Su')\\ &- \kappa(\epsilon)(Su', A_\epsilon^{-1}u) + \kappa^2(\epsilon)(A_\epsilon^{-1}Su', Su')\}.\end{aligned}$$

Moreover, A_ϵ^{-1} converges strongly to B^{-1} in H as $\epsilon \downarrow 0$ (cf. [12] or [2]), and $|A_\epsilon^{-1}u - \lambda^{-1}(u - \kappa(\epsilon)u')| = o(\kappa(\epsilon))$. So, since S is also self adjoint in H,

$$\eta = (1 + \kappa^2(\epsilon)|Su'|^2)^{-1}\{(A_\epsilon^{-1}u, u) + 2\lambda^{-1}\kappa^2(\epsilon)(Su', u') + \kappa^2(\epsilon)(B^{-1}Su', Su')\} + o(\kappa^2(\epsilon)).$$

Now, letting P be the orthogonal projection in H onto the one dimensional subspace spanned by u,

$$B^{-1}S = \lambda^{-1}[\lambda - B + B]\,S = \lambda^{-1}[P - I + S],$$

where I is the identity operator on H, and

$$((P - I)u', Su') = -(Su', u').$$

Hence,

$$\begin{aligned}\eta &= (1 + \kappa^2(\epsilon)|Su'|^2)^{-1}\{(A_\epsilon^{-1}u, u) + \lambda^{-1}\kappa^2(\epsilon)[(Su', u') + |Su'|^2]\} + o(\kappa^2(\epsilon)),\\ &= (A_\epsilon^{-1}u, u) + \lambda^{-1}\kappa^2(\epsilon)(Su', u') + o(\kappa^2(\epsilon))\\ &= \lambda^{-1} - \epsilon\lambda^{-2}\lambda_\epsilon' + \lambda^{-1}\kappa^2(\epsilon)(Su', u') + o(\kappa^2(\epsilon)).\end{aligned}$$

Now, since $\epsilon\lambda_\epsilon' = \lambda\kappa(\epsilon)(u', u) + o(\kappa(\epsilon))$,

$$\theta = |u - \kappa(\epsilon)Su'|^{-1}|(A_\epsilon^{-1} - \lambda^{-1})u + \lambda^{-1}\kappa(\epsilon)(u', u) - \kappa(\epsilon)(A_\epsilon^{-1} - \lambda^{-1})Su'| + o(\kappa(\epsilon)).$$

But, $(A_\epsilon^{-1} - \lambda^{-1})u = -\lambda^{-1}\epsilon\mathcal{A}(\epsilon\mathcal{A} + I)^{-1}u$, $|\epsilon\mathcal{A}(\epsilon\mathcal{A} + I)^{-1}u - \kappa(\epsilon)u'| = o(\kappa(\epsilon))$, and $A_\epsilon^{-1} - \lambda^{-1}$ converges strongly to $B^{-1} - \lambda^{-1}$ in H as $\epsilon \downarrow 0$. So,

$$\theta = |u - \kappa(\epsilon)Su'|^{-1}|-\lambda^{-1}\kappa(\epsilon)u' + \lambda^{-1}\kappa(\epsilon)(u', u)u - \kappa(\epsilon)(B^{-1} - \lambda^{-1})Su'| + o(\kappa(\epsilon)).$$

Thus since $(B^{-1} - \lambda^{-1})S = \lambda^{-1}[P - I + S] - \lambda^{-1}S = \lambda^{-1}(P - I)$, and $Pu' = (u', u)u$,

$$\begin{aligned}\theta &= |u - \kappa(\epsilon)Su'|^{-1}|\kappa(\epsilon)\lambda^{-1}[-u' + (u', u)u - (P - I)u']| + o(\kappa(\epsilon))\\ &= o(\kappa(\epsilon)).\end{aligned}$$

Lemma 4, p. 437, of [10] implies that $|\lambda_\epsilon^{-1} - \eta| = O(\theta^2)$ and $|u_\epsilon - v_\epsilon| = O(\theta)$, from which the theorem follows.

3. **An Example.** As mentioned in the Introduction, Theorems 1 and 2 are intended for application to perturbation of the radial equation of quantum mechanics by highly singular "hard core" potentials. We confine attention here to a particular model problem for which theory

and computation are comparatively simple. The techniques used for the present case are however fundamental to the analysis of problems of greater physical interest. More general results are announced in [5].

Consider the eigenvalue problem,

$$-u_\epsilon'' + \epsilon x^{-2\alpha} u_\epsilon = \lambda_\epsilon u_\epsilon,\ u_\epsilon(0) = u_\epsilon(1) = 0, \tag{2}$$

where $' = d/dx$, $\alpha > 1$, and $\epsilon \in (0, \epsilon_0]$ for some $\epsilon_0 > 0$. Letting $b(v, w) = \int_0^1 v'\bar{w}'\, dx$ on $H_0{}^1(0, 1)$ (cf. Lions and Magenes [15], or [1]) and $H = L^2(0, 1)$, the unperturbed problem, $Bu = \lambda u$, is

$$-u'' = \lambda u,\ \ u(0) = u(1) = 0, \tag{3}$$

whose simple eigenvalues are $\lambda = n^2\pi^2$, $n = 1, 2, \cdots$, with corresponding normalized eigenfunctions $u = 2^{1/2} \sin n\pi x$, $n = 1, 2, \cdots$. The quadratic form $\int_0^1 x^{-2\alpha}|v|^2\, dx$ with domain $\{v \in L^2(0, 1) : x^{-\alpha}v \in L^2(0, 1)\}$ is closed in $L^2(0, 1)$. Hence, defining $a(v, w) = \int_0^1 x^{-2\alpha} v\bar{w}\, dx$ on $D(a) = \{v \in H_0{}^1(0, 1) : x^{-\alpha}v \in L^2(0, 1)\}$, and noting that each eigenvalue of B is stable (cf. [12]), the singular perturbation problem (2), (3) lies within the framework of § 2. Noting that if m denotes the greatest integer less than 2α, $D(A_\epsilon)$ with norm $|A_\epsilon v|$ is continuously contained in $H_0{}^{m+1}(0, 1)$, (cf. [15] or [2]), it follows from [2] that the hypotheses of part (iii) of Theorem 1 are satisfied for sufficiently small $\tau > 0$.

Application of Theorems 1 and 2 requires asymptotic solution of the boundary value problem,

$$-w_\epsilon'' + \epsilon x^{-2\alpha} w_\epsilon = u,\quad w_\epsilon(0) = w_\epsilon(1) = 0. \tag{4}$$

This problem can be solved explicitly in terms of Bessel functions (cf. Hildebrand [7]). The solution is more conveniently analyzed when written with the change of variables $x = (2\beta)^{2\beta} t^{-2\beta}$, where $\beta = 1/2(\alpha - 1)$. Letting $\mu = \epsilon^{1/2}$, and using a standard abuse of notation, $w_\epsilon = A_\epsilon^{-1} u$ is given by

$$\begin{aligned} w_\epsilon(t) = (2\beta)^{2+4\beta} \Bigg\{ &- \frac{I_\beta(2\beta\mu)}{K_\beta(2\beta\mu)} t^{-\beta} K_\beta(\mu t) \\ &\cdot \int_{2\beta}^{\infty} \tau^{-1-3\beta} u\left(\left[\frac{2\beta}{\tau} \right]^{2\beta} \right) K_\beta(\mu\tau)\, d\tau \\ &+ t^{-\beta} I_\beta(\mu t) \int_t^{\infty} \tau^{-1-3\beta} u\left(\left[\frac{2\beta}{\tau} \right]^{2\beta} \right) K_\beta(\mu\tau)\, d\tau \\ &+ t^{-\beta} K_\beta(\mu t) \int_{2\beta}^{t} \tau^{-1-3\beta} u\left(\left[\frac{2\beta}{\tau} \right]^{2\beta} \right) I_\beta(\mu\tau)\, d\tau \Bigg\}. \end{aligned} \tag{5}$$

Herein, I_β and K_β are the modified Bessel functions of the first and second kind, respectively. The underlying Hilbert space, H, is now the space of functions square integrable on $(2\beta, \infty)$ with respect to the weight function $|dx/dt| = (2\beta)^{1+2\beta}t^{-1-2\beta}$, and $A_\epsilon^{-1}u = w_\epsilon \to \lambda^{-1}u$ (observe $u(x) = u((2\beta t^{-1})^{2\beta}))$ in the sense of this space as $\epsilon \downarrow 0$.

Formula (5) is now used to find $\kappa(\epsilon)$, u' for application of Theorem 2 (in [5] an application of Theorem 1 without availability of $\kappa(\epsilon)$, u' is mentioned). A tedious, but relatively straightforward, analysis using the series expansions for I_β and K_β near zero, the asymptotic expansions for I_β and K_β near infinity, the simple zero of u at $x = 0$ ($t = \infty$), and Lebesgue's dominated convergence theorem, yields

$$w_\epsilon \sim \lambda^{-1}(u - \kappa(\epsilon)u'),$$

($\sim$ denoting asymptotic in the preceding L^2 space) where,

$$\kappa(\epsilon) = \begin{cases} \epsilon^\beta & \text{if } 0 < \beta < 1, \\ \epsilon \log \epsilon, & \text{if } \beta = 1, \\ \epsilon, & \text{if } \beta > 1. \end{cases}$$

The inner product (w_ϵ, u) is evaluated similarly, with the result that;

$$\lambda_\epsilon \sim \begin{cases} \lambda + \lambda_1\epsilon^\beta + \lambda_2\epsilon^{2\beta} & \text{if } 0 < \beta \leqq 1/2, \\ \lambda + \lambda_1\epsilon^\beta + \lambda_2\epsilon, & \text{if } 1/2 < \beta < 1, \\ \lambda + \lambda_1\epsilon \log \epsilon + \lambda_2\epsilon, & \text{if } \beta = 1, \\ \lambda + \lambda_1\epsilon + \lambda_2\epsilon^\beta, & \text{if } 1 < \beta < 2, \\ \lambda + \lambda_1\epsilon + \lambda_2\epsilon^2 \log \epsilon, & \text{if } \beta = 2, \\ \lambda + \lambda_1\epsilon + \lambda_2\epsilon^2, & \text{if } \beta > 2, \end{cases}$$

where $\lambda_1 = \lambda_1(\beta)$ and $\lambda_2 = \lambda_2(\beta)$ are independent of ϵ. Observe that for $\beta > 1$ (i.e., $\alpha < 3/2$) the regular perturbation method applies to first order, while for $\beta > 2$, (i.e., $\alpha < 5/4$) the regular perturbation method applies to second order (cf. [12]).

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