ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 6, Number 4, Fall 1976

SINGULAR PERTURBATION OF NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. Under certain assumptions on $f(x, y', y, \epsilon)$, this paper discusses the existence and asymptotic behavior of the solution of $\epsilon y'' + f(x, y', y, \epsilon) = 0$ with $y(0) = A$ and $y(1) = B$.

Consider the equations

(1) $y'' + f(x, y, y', \epsilon) = 0$, with $y(0) = A$ and $y(1) = B$, where $x \in [0, 1]$,

(2) $f(x, y_0', y_0) = 0$, with $y_0(1) = B$.

Subtract (2) from (1) and obtain

(3) $\epsilon y'' + f(x, y', y, \epsilon) - f(x, y_0', y_0, 0) = 0$.

Using the mean value theorem for several variables, we obtain

$$\epsilon y'' + \alpha(x, \epsilon)(y' - y_0') + \beta(x, \epsilon)(y - y_0) = 0, \tag{4}$$

where α and β are continuous functions in both x and ϵ, and $\alpha(x, \epsilon)$ is positive (or negative) on $[0, 1]$.

Now we wish to impose certain conditions on $f(x, y', y, \epsilon)$ so that the above conditions on α and β are satisfied. Such assumptions are: f is C^∞ in all variables, and f_y and $f_{y'}$ are positive.

Subtract $(\epsilon y_0'')$ from both sides of (4), and let $y - y_0 = u$; we obtain

$$\epsilon u'' + \alpha(x, \epsilon)\, u' + \beta(x, \epsilon) u = -\epsilon y_0'', \tag{5}$$

with the conditions $u(0) = A - y_0(0)$ and $u(1) = 0$.

Since the function y_0'' is independent of ϵ, then obviously it is sufficient to consider the equation

$$\begin{gathered} \epsilon u'' + \alpha(x, \epsilon) u' + \beta(x, \epsilon) u = 0, \\ \text{with } u(0) = A - y_0(0) \text{ and } u(1) = 0, \end{gathered} \tag{6}$$

i.e., if (6) is stable, so is (5).

If we let $u^{[k]} = \epsilon^k (d^k u / dx^k)$, then (6) becomes

$$u^{[2]} + \alpha(x, \epsilon) u^{[1]} + \epsilon \beta(x, \epsilon) u = 0. \tag{7}$$

Let $\omega_1(x)$ $(i = 1, 2)$ be the roots of $\omega^2 + \alpha(x, 0)\omega = 0$.

LEMMA 1. *For each i $(i = 1, 2)$ there exists an infinite number of functions $u_{i0}, u_{i1}, \cdots$ continuous and with continuous derivatives of all orders such that $u_{i0}(x)$ does not vanish at any point of $[0, 1]$, and* *if the functions* $u_i(x, \epsilon) = \exp[(1/\epsilon) \int_0^x \omega_i(s)\, ds] \sum_{j=0}^{m-1} u_{ij}(x)\epsilon^j$ *are substituted in* (7) *for* u, *then the coefficients of* $\exp[(1/\epsilon) \int_0^x \omega_i(s)\, ds]\epsilon^h$, $i = 1, 2$; $h = 0, 1, \cdots, m$, *vanish identically.*

LEMMA 2. *The* D.E. (7) *has two linearly independent solutions*:

$$y_i(x, \epsilon) = u_i(x, \epsilon) + \epsilon^m E_0$$

for any positive integer m *where* E_0 *is a bounded function.*

The proofs of Lemmas 1 and 2 are in [1].

Now let $Y = c_1 y_1 + c_2 y_2$ and apply the boundary conditions. We find

$$c_1 = \frac{[A - y_0(0)]\, y_2(1)}{-[u_{20}(0) u_{10}(1) + O(\epsilon)]} \to 0, \quad \text{as } \epsilon \to 0,$$

and

$$c_2 = \frac{y_1(1)(A - y_0(0))}{u_{20}(0) u_{10}(1;) + O(\epsilon)}$$

which is bounded as $\epsilon \to 0$.

Since $y_2(x, \epsilon) \to 0$ as $\epsilon \to 0$, we find that $Y(x, \epsilon) \to 0$ as $\epsilon \to 0$. Consequently the solution $u(x, \epsilon)$ of (5) goes to zero as $\epsilon \to 0$ which in turn shows that $y(x, \epsilon)$, the solution of (1), goes to the solution $y_0(x)$ of (2).

REMARKS. Notice that the boundary condition $y_\epsilon(1)$ coincides with $y_0(1)$. However, if we assume $y_\epsilon(0) = y_0(0)$ instead we get $y(x, \epsilon) \to y_0(x) + h(x)$. So it is necessary that the two solutions agree at 1. Another point needs to be mentioned here. Since $y'(x, \epsilon)$ may not be bounded in ϵ at $x = 0$, this necessitates that we impose an extra condition on $f(x, y', y, \epsilon)$ or we would have some exceptions. But if it happens that $y(0, \epsilon) = y_0(0)$ and $y(1, \epsilon) = y_0(1)$, then $y'(x, \epsilon)$ is bounded in $[0, 1]$.

Now we shall apply this technique to the equation

$$\text{(8)} \qquad \begin{aligned} &\epsilon y'' + yy' - y = 0, \text{ with the conditions} \\ &y(0) = A > 0, \text{ and } y(1) = B > A + 1, \end{aligned}$$

whose unperturbed equation is

$$\text{(9)} \qquad yy' - y = 0,$$

which has the solution $y_0(x) = x + B - 1$.

So, subtracting (9) from (8), assuming $y_\epsilon - y_0 = u$ and noticing that $y_0'' = 0$, we obtain by using the mean-value theorem the following equation,

$$\epsilon u'' + y_0 u' + (y' - 1)u = 0, \tag{10}$$
$$\text{with } u(0) = A + 1 - B \text{ and } u(1) = 0.$$

According to Lemma 2, this equation has the two linearly independent solutions

$$y_1 = u_1(x,\epsilon) + \epsilon^m E_0,$$
$$y_2 = u_2(x,\epsilon) + \epsilon^m E_0,$$

where

$$u_1 = \sum_0^{m-1} u_{1j}(x)\epsilon^j, \text{ and}$$

$$u_2 = \left[\exp(-(1/\epsilon) \int_0^x y_0(s)\, ds) \right] \left[\sum_0^{m-1} u_{2j}(x)\epsilon^j \right].$$

Thus, the general solution is, $Y = c_1 y_1 + c_1 y_2$, and with the boundary conditions we obtain

$$c_1 = \frac{[A - B + 1] y_2(1)}{[u_{20}(0) u_{10}(1) + O(\epsilon)]} \to 0 \quad \text{as} \quad \epsilon \to 0$$

and

$$c_2 = \frac{y_1(1)(A - B + 1)}{u_{20}(0) u_{10}(1) + O(\epsilon)} \text{ which is bounded as } \epsilon \to 0.$$

Hence we find $Y(x, \epsilon) \to 0$ as $\epsilon \to 0$ which gives $y_\epsilon(x) \to y_0(x)$ as $\epsilon \to 0$.

Reference

1. H. S. Nur, *Singular Perturbation of Linear P.D.E.*, J.D.E. **6** (1969).

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