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PERIODIC SOLUTIONS TO A WAVE EQUATION

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ABSTRACT. The problem is to establish the existence of small amplitude, time periodic solutions of

$$\text{(WE)} \qquad \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\sigma\left(\frac{\partial u}{\partial x}\right) = 0, \quad 0 < x < 1 \text{ and}$$

$$\text{(BC)} \qquad u(0, t) = u(1, t) = 0,$$

where

$$\sigma(\gamma) = \gamma^3\left(1 + \sum_{n=1}^{\infty} \sigma_n \gamma^{2n}\right) \tag{1}$$

and the series (1) converges in a neighborhood of $\gamma = 0$. The basic result is that the above problem has small amplitude, periodic solutions of the form

$$u(x, t) = A\underline{u}U(x)\mathcal{A}(At/T_1) + A^3 w(x, t; A), \quad 0 < A \ll 1. \tag{2}$$

The numbers T_1 and $\underline{u}$ are given by

$$\begin{aligned} T_1 &= (2^{3/2}/\pi)\int_0^1 \frac{da}{(1 - a^4)^{1/2}}, \text{ and} \\ \underline{u}^{-1/2} &= 2^{3/4}3^{1/4}\int_0^1 \frac{du}{(1 - u^2)^{1/4}}; \end{aligned} \tag{3}$$

the functions $U(\cdot)$ and $\mathcal{A}(\cdot)$ which satisfy

$$\underline{u}^2 \frac{d}{dx}(U_x^3) + U = 0 \quad \text{and} \quad \frac{d^2\mathcal{A}}{d\tau^2} + T_1\mathcal{A}^3 = 0 \tag{4}$$

are given by

$$\begin{cases} \displaystyle\int_0^{U(x)} \frac{du}{(1 - u^2)^{1/4}} = (2/3)^{1/4} x/\underline{u}^{1/2}, & 0 \leqq x \leqq 1/2, \\ U(x) = U(1 - x), & 1/2 \leqq x \leqq 1 \end{cases} \tag{5}$$

and

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(6)\qquad \begin{cases} \int_0^{\mathcal{A}(\tau)} \dfrac{da}{(1-a^4)^{1/2}} = \dfrac{T_1\tau}{2^{1/2}}, \quad 0 \leqq \tau \leqq \dfrac{\pi}{2}, \\ \mathcal{A}(\tau) = \mathcal{A}(\pi - \tau), \pi/2 \leqq \tau \leqq \pi, \\ \mathcal{A}(\tau) = -\mathcal{A}(-\tau), \quad -\pi \leqq \tau \leqq 0, \text{ and} \\ \mathcal{A}(\tau + 2n\pi) = \mathcal{A}(\tau); \end{cases}
$$

and the period of w is $2\pi T_1/A$.

The proof of the above result may be found in [1]. The result hinges on the invertibility of the variational operator

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(7)\qquad \mathcal{L}w = \frac{\partial^2 w}{\partial \tau^2} - 3\underline{u}^2 T_1^{\,2} \mathcal{A}^2(\tau) \frac{\partial}{\partial x}\left(U_x^{\,2} \frac{\partial w}{\partial x} \right)
$$

$$
(8)\qquad w(0,\tau) = \frac{\partial w}{\partial x}(1/2,\tau) = 0, \quad 0 < \tau < \pi/2, \text{ and}
$$

$$
(9)\qquad w(x,0) = \frac{\partial w}{\partial \tau}(x,\pi/2) = 0, \quad 0 < x < 1/2.
$$

Reference

1. J. M. Greenberg, *Smooth and time periodic solutions to the quasilinear wave equation,* The Archive for Rational Mechanics and Analysis **60** (1975), 29-50.

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