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ENTIRE FUNCTIONS WITH PRESCRIBED ASYMPTOTIC BEHAVIOR

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ABSTRACT. A sufficient condition for a canonical product to be of bounded index is given, from which most of the well known results can be obtained as easy corollaries. Let f be an entire function of exponential type with order ρ and lower order λ. If $\rho - \lambda < 1$ then there exists an entire function g of bounded index such that $\log M(r, f) \sim \log M(r, g)$. This solves a conjecture of S. M. Shah except for the extremal case of $\rho = 1$ and $\lambda = 0$.

1. **Introduction.** An entire function $f(z)$ is said to be of bounded index if there exists a non-negative integer N such that

$$\max_{0 \leqq i \leqq N} \left\{ \frac{|f^{(i)}(z)|}{i!} \right\} \geqq \frac{|f^{(n)}(z)|}{n!} \text{ for all } n \text{ and all } z.$$

The least such integer N is called the index of f, (see [4]).

It is well known that a canonical product having geometrically increasing zeros is of bounded index. We now prove a strong generalization of this result.

THEOREM 1. *Let* $f(z) = \prod_{j=1}^{\infty} (1 + \{z/t_j\}^{q_j})$ *be an entire function with* $t_j \in \mathbb{C} \setminus \{0\}$, $q_j \in N$ *and* $\sum_{j=1}^{\infty} (q_j/|t_j|) < \infty$. *If* $\sum_{j \neq n} 1/|t_n - t_j| = 0(1)$ *as* $n \to \infty$, *then* f *is of bounded index.*

The condition $\sum_{j=1}^{\infty} (q_j/|t_j|) < \infty$ can be replaced by $\limsup_{j\to\infty}(Q_j/|t_j|) < \infty$, where $Q_j = \sum_{i=1}^{j} q_i$, provided f is entire, i.e., the infinite product converges uniformly on every bounded region. However let us remark that the conditions in Theorem 1 are only sufficient and not necessary, (see [2], Theorem 3).

As a direct consequence we obtain the following result of B. S. Lee and S. M. Shah [3].

COROLLARY 2. *Let* $f(z) = \prod_{n=1}^{\infty} (1 - z/a_n)$, *where* $a_n \in R^+$ *and* $(a_{n+1}/a_n) \geqq \alpha > 1$, *then* f *is an entire function of bounded index.*

In 1970 W. J. Pugh and S. M. Shah [5] showed that for any transcendental entire function f of finite order it is always possible to

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find an entire function g of unbounded index such that

$$\log M(r, f) \sim \log M(r, g) \quad (r \to \infty).$$

In [6] S. M. Shah conjectured: If f is an entire function of exponential type then there exists an entire function g of bounded index such that $\log M(r, f) \sim \log M(r, g)$. We now prove this conjecture for functions of exponential type with non-extremal asymptotic behavior.

THEOREM 3. *Let f be an entire function of exponential type with order ρ and lower order λ. If $\rho - \lambda < 1$, then there exists an entire function g of bounded index such that*

$$N\left(r, \frac{1}{g}\right) \sim \log M(r, g) \sim \log M(r, f) \quad (r \to \infty).$$

THEOREM 4. *Let $\phi(t)$ be an increasing, positive function of $t \geqq 1$ with $\limsup_{t\to\infty}(\phi(t)/t) < \infty$. If there exists an integer $n > 0$ such that $\phi(t+1) - \phi(t) \leqq \phi(t)^{(n-1)/n}$ for t sufficiently large, then there exists an entire function f of bounded index such that*

$$N\left(r, \frac{1}{f}\right) \sim \log M(r, f) \sim \int_1^r \frac{\phi(t)}{t}\,dt \quad (r \to \infty).$$

As a straightforward consequence we have

COROLLARY 5. *Let $0 \leqq \lambda \leqq \rho \leqq 1$ be given. Then there exists an entire function f of bounded index and order ρ with lower order λ.*

2. **Proof of Theorem 1.** Let q be a positive integer and let t be a complex number with $|t| > q$.

(i) Let $z \in \mathbb{C}$ with $|z| = |t| + a$, $a > 0$.

Then,

$$\frac{|qz^{q-1}|}{|t^q + z^q|} \leqq \frac{q|z|^{q-1}}{|z|^q - |t|^q} = \frac{q}{|z|(1 - \{|t|/|z|\}^q)}$$

$$\leqq \frac{q}{(|t| + a)\left(1 - \dfrac{|t|^q}{|t|^q + aq|t|^{q-1}}\right)}$$

$$= \frac{|t| + qa}{(|t| + q)a} \leqq \frac{1}{a} + \frac{q}{|t| + a} = \frac{1}{a} + \frac{q}{|z|}.$$

(ii) Let $z \in \mathbb{C}$ with $|t| = |z| + b$, $b > 0$.

Then,

$$\frac{|qz^{q-1}|}{|t^q + z^q|} \leqq \frac{q|z|^{q-1}}{(|z| + b)^q - |z|^q} \leqq \frac{q|z|^{q-1}}{bq|z|^{q-1}} = \frac{1}{b}.$$

(iii) Let $z \in \mathcal{C}$ such that $|t| - 1 < |z| \leqq |t| + 1$. Furthermore let $a_1, \cdots, a_q$ denote the zeros of $z^q + t^q$ then

$$\frac{qz^{q-1}}{z^q + t^q} = \sum_{n=1}^{\infty} \frac{1}{z - a_n}.$$

Clearly there exists z' such that $|z'| + 1 = |t|$ and $|z' - z| \leqq 2$. Thus, by (ii), $|\sum_{n=1}^{q} 1/(z' - a_n)| \leqq 1$.

Let $d > 0$ be given and let $|z - a_n| \geqq d$ for $n = 1, 2, \cdots, q$. Obviously the length of the arc from a_i to a_{i+1} for the circle of radius $|t|$ is exactly $2\pi|t|/q$. The distance of two points on a circle is at least the shortest arc length between those points divided by π. Thus, by renumbering the a_i's (so that a_1 is closest to z), we have,

$$|z - a_n| \geqq |a_n - a_2| \geqq \frac{(n-2)2|t|}{q} \geqq 2(n-2) \text{ for } 2 < n \leqq (q+1)/2$$

and

$$|z - a_n| \geqq |a_n - a_q| \geqq \frac{(q-n)2|t|}{q} \geqq 2(q-n) \text{ for } (q+1)/2 < n < q.$$

The same result holds for z' and therefore,

$$\left|\sum_{n=1}^{q} \frac{1}{z - a_n}\right| \leqq \left|\sum_{n=1}^{q} \frac{z - z'}{(z - a_n)(z' - a_n)}\right| + \left|\sum_{n=1}^{q} \frac{1}{z' - a_n}\right|$$

$$\leqq \sum_{n=1}^{q} \frac{2}{|z - a_n|\,|z' - a_n|} + 1$$

$$\leqq \frac{6}{d} + \sum_{n=3}^{q-1} \frac{2}{|z - a_n|\,|z' - a_n|} + 1$$

$$\leqq \frac{6}{d} + \sum_{j=1}^{(q+1)/2} \frac{4}{j^2} + 1$$

$$\leqq \frac{6}{d} + \sum_{j=1}^{\infty} \frac{4}{j^2} + 1 = M.$$

Obviously, $M = M(d)$ is independent of t and q.

We now proceed with the proof of Theorem 1.

For $f(z) = \prod_{j=1}^{\infty} (1 + \{z/t_j\}^{q_j})$ we have,

$$\left| \frac{f'(z)}{f(z)} \right| = \left| \sum_{j=1}^{\infty} \frac{q_j z^{q_j - 1}}{t_j^{q_j} + z^{q_j}} \right| \leqq \sum_{j=1}^{\infty} \frac{q_j |z|^{q_j - 1}}{|z^{q_j} + t_j^{q_j}|}.$$

Let $d > 0$ be given and let $z \in \mathcal{C}$ such that $|z - b_i| \geqq d$ for $i = 1, 2, \cdots$, where the b_i's denote the zeros of f. There exists an integer $n \geqq 0$ such that $|t_n| \leqq |z| < |t_{n+1}|$. Since $\sum_{j=1}^{\infty} q_j/|t_j| < \infty$, there exists an integer $n_0 > 0$ such that $|t_j| > q_j$ for all $j \geqq n_0$. Let $a_j = | \, |t_j| - |z| \, |$, then, by (i), (ii), and (iii), we have, for $|z|$ sufficiently large,

$$\begin{aligned}
\left| \frac{f'(z)}{f(z)} \right| &\leqq \sum_{j=1}^{\infty} \frac{q_j |z|^{q_j - 1}}{|z^{q_j} + t_j^{q_j}|} \\
&\leqq \sum_{j=1}^{n_0} \frac{q_j |z|^{q_j - 1}}{|z^{q_j} + t_j^{q_j}|} + \sum_{j=n_0+1}^{n-1} \frac{q_j |z|^{q_j - 1}}{|z^{q_j} + t_j^{q_j}|} \\
&\quad + 2(M + 1) + \sum_{j=n+2}^{\infty} \frac{q_j |z|^{q_j - 1}}{|z^{q_j} + t_j^{q_j}|} \\
&\leqq K + \sum_{j=n_0+1}^{n-1} \left(\frac{1}{a_j} + \frac{q_j}{|t_j| + a_j} \right) + 2(M + 1) + \sum_{j=n+2}^{\infty} \frac{1}{a_j} \\
&\leqq K + \sum_{j \neq n} \left| \frac{1}{|t_n| - |t_j|} \right| + \sum_{j=1}^{\infty} \frac{q_j}{|t_j|} \\
&\quad + 2(M + 1) + \sum_{j \neq n+1} \left| \frac{1}{|t_n| - |t_j|} \right| \\
&\leqq T.
\end{aligned} \tag{2.1}$$

Clearly $f'(z)/f(z)$ is bounded for bounded z with $|z - b_i| \geqq d$ for all i.

Therefore, for each $d > 0$ there exists a constant $L > 0$ such that $|f'(z)| \leqq L|f(z)|$, whenever $|z - b_j| \geqq d$ for all j. Hence, by [2, Theorem 2], f is of bounded index.

Suppose we replace the condition $\sum_{j=1}^{\infty} (q_j/|t_j|) < \infty$ by demanding the infinite product f to be entire and by $\limsup_{j \to \infty} Q_j/|t_j| < \infty$, where $Q_j = \sum_{n=1}^{j} q_n$. It is well known that $f(z)$ of bounded index implies $f(\alpha z)$ is also of bounded index for any $\alpha \in \mathcal{C}$. Thus, without loss of generality we may assume $\limsup_{j \to \infty} Q_j/|t_j| < 1$. Hence, $a_n \leqq Q_n < |t_n|$ for n sufficiently large and therefore we can use the same argument to obtain the inequality (2.1). Since,

$$\sum_{j=n_0+1}^{n-1} \frac{a_j}{|t_j| + a_j} \leqq \sum_{j=1}^{n} \frac{a_j}{|z|} = \frac{Q_n}{|z|} \leqq \frac{Q_n}{|t_n|} < 1,$$

we similarly obtain, f is of bounded index. q.e.d.

3. In this section we assume familiarity with the most elementary results and notations of Nevanlinna's theory of meromorphic functions.

For a transcendental entire function f we have,

$$\log M(r,f) = \log M(r_0,f) + \int_{r_0}^{r} \frac{\Psi(t)}{t} dt \quad (r \geqq r_0), \tag{3.1}$$

where $r_0 > 0$ and $\Psi(t)$ is a non-negative, non-decreasing function of t.

A. Edrei and W. H. T. Fuchs [1] proved that given a positive, non-decreasing function $\Phi(t)$ with $\int_1^r \Phi(t)/t\, dt \leqq r^K$ (for some $K > 0$ and r sufficiently large), then there exists an entire function g of finite order such that

$$N\left(r, \frac{1}{g}\right) \sim \log M(r, g) \sim \int_1^r \frac{\Phi(t)}{t} dt \quad (r \to \infty).$$

We will rely heavily on their construction of the function $g(z)$.

Proof of Theorem 3. Let f be an entire function satisfying the hypothesis of Theorem 3. It is easy to see that the function $\Psi(t)$ we obtain from f according to (3.1) can be replaced by the function $\Phi(t)$ satisfying,

(i) $\Phi(t)$ is continuous,

(ii) $\Phi(1) = 0$ and $\Phi(t)$ is strictly increasing and unbounded,

(iii) $\log M(r, f) \sim \Lambda(r) = \int_1^r \Phi(t)/t\, dt \; (r \to \infty)$.

Let us now define the function $B(r)$ by the condition $B(r) = \Lambda(r)/\log r$ for $r > 1$ and $B(1) = 0$. Since $B'(r) = \Phi(r) \log r - \Lambda(r)/r(\log r)^2 > 0$ we have $B(r)$ is continuous and strictly increasing. Furthermore, f transcendental implies $B(r)$ is unbounded.

Let η be a fixed constant with $0 < \eta < 1/2$ and define a sequence of positive numbers $\{r_n\}_{n=1}^{\infty}$ by

$$n = B^{2\eta}(r_n) \log r_n \quad \text{for } n = 1, 2, \cdots.$$

Since $B^{2\eta}(x) \log x$ is continuous, strictly increasing, and unbounded and since $B^{2\eta}(1) \log 1 = 0$, the sequence $\{r_n\}_1^{\infty}$ is uniquely determined, strictly increasing, and unbounded.

Now set

$$k_j^j = \exp\left(\frac{j}{B^{\eta}(r_j)}\right) = \exp(B^{\eta}(r_j) \log r_j),$$

and notice that the sequence $\{k_j^j\}_{j=1}^{\infty}$ is increasing and unbounded, whereas the sequence $\{k_j\}_{j=1}^{\infty}$ is decreasing and

$$\lim_{j\to\infty} k_j = 1.$$

Denoting by $[y]$ the greatest integer not exceeding y, we define the sequence $\{q_j\}_{j=1}^{\infty}$ by

$$q_j = [2jk_1k_2k_3 \cdots k_j] + 1, \quad \text{for } j = 1, 2, \cdots.$$

It is easily shown that the q_j's satisfy the four following relations:

$$q_j > k_j^j \geqq \exp(\sqrt{j}) \quad (j \geqq 1), \tag{3.2}$$

$$q_{j+1} > q_j \quad (j \geqq 1), \tag{3.3}$$

$$\lim_{j\to\infty} \frac{q_{j+1}}{q_j} = 1, \tag{3.4}$$

$$\lim_{j\to\infty} \frac{q_j}{Q_j} = 0, \text{ where } Q_j = \sum_{i=1}^{j} q_i. \tag{3.5}$$

Define the sequence $\{t_j\}_{j=1}^{\infty}$ of positive, strictly increasing numbers by

$$\Phi(t_j) = Q_j = \sum_{i=1}^{j} q_j \quad (j = 1, 2, \cdots).$$

The existence and uniqueness of $\{t_j\}_1^{\infty}$ is assured by (ii).

Set $s_j = t_j + j(j-1)/2$ and define $n_1(t)$ and $n(t)$ by

$$n_1(t) = \begin{cases} 0 & \text{for } 0 \leqq t < t_1 \\ Q_j & \text{for } t_j \leqq t < t_{j+1}, \quad j = 1, 2, \cdots, \end{cases}$$

$$n(t) = \begin{cases} 0 & \text{for } 0 \leqq t < s_1 \\ Q_j & \text{for } s_j \leqq t < s_{j+1}, j = 1, 2, \cdots. \end{cases}$$

Clearly,

$$1 \leqq \frac{\Phi(t)}{n_1(t)} < 1 + \frac{q_{j+1}}{Q_j} \text{for } t_j \leqq t < t_{j+1}, j \geqq 1,$$

and therefore, by (3.4) and (3.5),

$$\lim_{t\to\infty} \frac{\Phi(t)}{n_1(t)} = 1.$$

Hence,

$$\int_1^r \frac{n_1(t)}{t}\,dt \sim \Lambda(r) \quad (r \to \infty).$$

We will now show that under the hypothesis of Theorem 3 we also have,

$$\int_1^r \frac{n(t)}{t}\,dt \sim \Lambda(r) \quad (r \to \infty).$$

Since f is of exponential type we have, for some $A > 0$,

$$n(t) \leqq n_1(t) \leqq \Phi(t) < At \text{ for } t \geqq 1. \tag{3.6}$$

Thus, $t_j > 1/A\, q_j \geqq 1/A \exp(\sqrt{j})$ and for $\gamma > 0$,

$$j^2 = o(t_j^{\gamma}) \quad (j \to \infty). \tag{3.7}$$

Therefore,

$$\int_{t_j}^{t_{j+1}} \frac{n_1(t)}{t}\,dt = \int_{s_j}^{s_{j+1}-j} \frac{n(t)}{t - j(j-1)/2}$$

$$dt = \{1 + o(1)\} \int_{s_j}^{s_{j+1}-j} \frac{n(t)}{t}\,dt \quad (j \to \infty).$$

Hence, for $s_j \leqq r < s_{j+1}$ and $j \to \infty$,

$$\int_1^{r-j^2} \frac{n_1(t)}{t}\,dt \leqq \{1 + o(1)\} \int_1^r \frac{n(t)}{t}\,dt \leqq \{1 + o(1)\} \int_1^r \frac{n_1(t)}{t}\,\mathrm{dt}.$$

This leaves to show,

$$\int_{r-j^2}^{r} \frac{n_1(t)}{t}\,dt = o\left(\int_1^r \frac{n_1(t)}{t}\,dt\right). \qquad (j \to \infty).$$

(a) Suppose f is of lower order $\lambda > 0$ then, since $\Lambda(r) \sim \int_1^r n_1(t)/t\,dt$, we have for r sufficiently large,

$$\int_1^r \frac{n_1(t)}{t}\,dt > r^{\lambda/2}.$$

Thus, by (3.6) and (3.7),

$$\int_{r-j^2}^{r} \frac{n_1(t)}{t}\,dt \leqq j^2 A = o(r^{\lambda/2}) = o\left(\int_1^r \frac{n_1(t)}{t}\,dt\right).$$

(b) Suppose f is of order $\rho < 1$, then there exists γ, $0 < \gamma < 1 - \rho$ such that, for t sufficiently large,

$$\frac{n_1(t)}{t} \leqq \frac{\Phi(t)}{t} < t^{-\gamma}.$$

Therefore, by (3.7), we have for $s_j \leqq r < s_{j+1}$

$$\int_{r-j^2}^{r} \frac{n_1(t)}{t} dt = o(1) \quad (j \to \infty).$$

Hence, since f is either of order $\rho < 1$ or of lower order $\lambda > 0$,

$$(3.8) \qquad \int_1^r \frac{n_1(t)}{t}\, dt \sim \int_1^r \frac{n(t)}{t}\, dt \sim \Lambda(r) \quad (r \to \infty).$$

We consider next the infinite product

$$(3.9) \qquad g(z) = \prod_{j=1}^{\infty} \left(1 + \left\{\frac{z}{s_j}\right\}^{q_j}\right).$$

Let $|z| = r$ with $r < R$ and define p by $s_p \leqq R < s_{p+1}$.

By (3.3), $q_m - q_n \geqq m - n$ and therefore,

$$(3.10) \qquad \begin{aligned} \sum_{s_j > R} \left| \frac{z}{s_j} \right|^{q_j} &\leqq \sum_{s_j > R} \left\{\frac{r}{R}\right\}^{q_j} < \sum_{j=p}^{\infty} \left\{\frac{r}{R}\right\}^{q_j} \\ &= \left\{\frac{r}{R}\right\}^{q_p} \sum_{j=p}^{\infty} \left\{\frac{r}{R}\right\}^{q_j - q_p} \leqq \left\{\frac{r}{R}\right\}^{q_p} \frac{R}{R-r}. \end{aligned}$$

This shows that the infinite product in (3.9) converges uniformly in every bounded region. Hence $g(z)$ is an entire function.

Now, $n(r, 1/g) = n(r)$ and therefore by (3.8),

$$N\left(r, \frac{1}{g}\right) = \int_1^r \frac{n(t)}{t}\, dt \sim \Lambda(r) \quad (r \to \infty).$$

For $r < R$ and $s_p \leqq R < s_{p+1}$,

$$\begin{aligned} \log M(r, g) = {} & \sum_{s_j \leqq r} q_j \log \frac{r}{s_j} + \sum_{s_j \leqq r} \log\left(1 + \left\{\frac{s_j}{r}\right\}^{q_j}\right) \\ & + \sum_{r < s_j \leqq R} \log\left(1 + \left\{\frac{r}{s_j}\right\}^{q_j}\right) \\ & + \sum_{s_j > R} \log\left(1 + \left\{\frac{r}{s_j}\right\}^{q_j}\right) \\ \leqq {} & N\left(r, \frac{1}{g}\right) + p \log 2 + \sum_{s_j > R} \left\{\frac{r}{s_j}\right\}^{q_j}. \end{aligned}$$

Hence, by (3.10) and elementary inequalities of Nevalinna's theory

$$
(3.11) \qquad \begin{aligned} N\left(r, \frac{1}{g}\right) \leqq \log M(r, g) \leqq N\left(r, \frac{1}{g}\right) \\ + p \log 2 + \left\{\frac{r}{R}\right\}^{q_p} \frac{R}{R-r} \quad (r < R). \end{aligned}
$$

Now, let $R = 2r$ and p defined by $s_p \leqq R < s_{p+1}$. Then,

$$
q_p \leqq Q_p \leqq \Phi(2r) < \int_{2r}^{2er} \frac{\Phi(t)}{t} dt < \Lambda(2er) \text{ and}
$$

thus, $q_p = 0(r)$ $(p = p(2r), r \to \infty)$.

By (3.2),

$$
q_p > k_p{}^p = \exp(B^{\eta}(r_p) \log r_p).
$$

Hence,

$$
B^{\eta}(r_p) \log r_p = 0(\log r) \quad (r \to \infty),
$$

and since $B^{\eta}(x)$ is strictly increasing,

$$
r_p < r \text{ for } r \text{ sufficiently large.}
$$

Thus, since $B^{2\eta}(x) \log x$ is strictly increasing,

$$
p = B^{2\eta}(r_p) \log r_p \leqq B^{2\eta}(r) \log r = \Lambda(r)\, B^{1-2\eta}(r).
$$

Since $\lim_{r \to \infty} B^{1-2\eta}(r) = 0$, we have,

$$
p = o(\Lambda(r)) \quad (r \to \infty, p = p(2r)).
$$

Thus, we obtain for (3.11),

$$
N(r, 1/g) \leqq \log M(r, g) \leqq N(r, 1/g) + o(\Lambda(r)) + o(1) \quad (r \to \infty).
$$

Hence,

$$
\log M(r, g) \sim N(r, 1/g) \sim \Lambda(r) \sim \log M(r, f) \quad (r \to \infty).
$$

Since $s_{j+1} - s_j \geqq j + 1$, we have

$$
\lim_{n \to \infty} \sum_{j \neq n} \left| \frac{1}{s_n - s_j} \right| = 0.
$$

Clearly, $\lim \sup_{j \to \infty} Q_j / s_j < \infty$ and therefore, by Theorem 1, $g(z)$ is of bounded index. q.e.d.

4. **Proof of Theorem 4.** Without loss of generality we may assume $\Phi(1) = 0$, $\Phi(t)$ continuous, strictly increasing and unbounded. The condition $\lim \sup_{t \to \infty} \Phi(t)/t < \infty$ assures us that the function f we are

about to construct is of exponential type. Let $B(r)$, $\{q_j\}_1^\infty$, $\{t_j\}_1^\infty$, and $n_1(t)$ be defined as in the proof of Theorem 3.

Now let $f(z) = \prod_{j=1}^{\infty}(1 + \{z/t_j\}^{q_j})$, then by the same argument as in the proof of Theorem 3, $f(z)$ is an entire function and

$$N\left(r, \frac{1}{g}\right) = \int_1^r \frac{n_1(t)}{t}dt \sim \log M(r,f) \sim \int_1^r \frac{\Phi(t)}{t}dt \quad (r \to \infty).$$

Then, for $t \geqq t_j$ and j sufficiently large,

$$\begin{aligned}\Phi(t+1) - \Phi(t) &\leqq \Phi(t)\{\Phi(t)\}^{-1/n}\\ &\leqq Q_j^{-1/n}\Phi(t)\\ &\leqq e^{-\sqrt{j/n}}\,\Phi(t) \leqq 1/j^3\,\Phi(t).\end{aligned}$$

Hence, for j sufficiently large,

$$\Phi(t_j + j) \leqq (1 + 1/j^3)^j\Phi(t_j) \leqq (1 + 1/j)\,\Phi(t_j).$$

Therefore,

$$\Phi(t_{j+1}) - \Phi(t_j) = q_{j+1} > (1/j)Q_j = (1/j)\,\Phi(t_j), \text{ and thus}$$

$t_{j+1} - t_j > j$ for j sufficiently large.

Now, $\sum_{j\neq n} 1/|t_n - t_j| = o(1)$ and, by Theorem 1, f is of bounded index. q.e.d.

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