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BOUNDEDNESS FOR SPACES OF CONTINUOUS FUNCTIONS

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1. **Introduction.** Let T be an arbitrary completely regular and Hausdorff space, let $C_c(T)$ denote the space of all real-valued continuous functions on T, endowed with the compact-open topology. One of the two so-called Nachbin-Shirota Theorems tells us that T is replete if and only if $C_c(T)$ is bornological. Deep and beautiful, its proof utilizes the profound work of Hewitt, and has influenced greatly later study of locally convex spaces. Recently the Nachbin-Shirota Theorem was markedly strengthened by De Wilde and Schmets' Theorem, which shows that T is replete if and only if $C_c(T)$ is ultrabornological. Their theorem has ushered in a new era which has involved a broadening of the ideas surrounding the Nachbin-Shirota Theorem. In this respect, we cite especially the articles by Buchwalter and by Buchwalter and Schmets ([4] and [5] respectively).

In the various theorems mentioned above, two structures on $C_c(T)$ come into play. There is the structure of $C_c(T)$ as a locally convex topological vector space, and there is the natural structure of $C_c(T)$ as an ordered vector space of real-valued functions. In the present paper we analyze $C_c(T)$ with respect to both the locally convex related and the order related concepts, and then utilize such an analysis in order to shed further light on the Nachbin-Shirota Theorem and its relatives.

In Section 2 we define the notion of hyper-null sequences in $Cc(T)$, and then discuss the relationships which exist between the following kinds of bounded subsets of $C_c(T)$: null and hyper-null sequences, equicontinuous, order-bounded, relatively compact, and general bounded subsets. Section 3 is devoted to the boundedness — on the just-mentioned kinds of subsets — of linear forms on $C_c(T)$, with Theorem 19 the main result. Rather unexpectedly, we find that, although a linear form which is bounded on all bounded subsets of $C_c(T)$ is obviously bounded on all order-bounded subsets, the converse is not true in general. We end the paper with a historical perspective.

Before we begin on the paper proper, we describe our notational conventions. Throughout the paper T stands for a completely regular Hausdorff space. If $S \subseteq T$, then S^0 is is the interior of S in T, and χ_S is the characteristic function of S. Next, if f is defined on T and $S \subseteq T$, then $f|_S$ denotes the restriction of f to S. If $n \in N$ (with N the set of

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positive integers), then the nth truncating function θ_n is defined on the reals by

$$\theta_n(r) = \begin{cases} r, & |r| \leqq n, \\ nr/|r|, & |r| > n. \end{cases}$$

The collection R^T of all real-valued functions on T carries the product topology. Let $\mathcal{C}(T)$ consist of all real-valued continuous functions on T — without topology — and let $\mathcal{C}^\infty(T)$ be the collection of all bounded functions in $\mathcal{C}(T)$. When $\mathcal{C}^\infty(T)$ has the uniform norm $\| \cdot \|_\infty$, we denote it by $C^\infty(T)$. The simple topology on $\mathcal{C}(T)$ is the topology of pointwise convergence, and the locally convex space which results when we endow $\mathcal{C}(T)$ with the compact-open topology is written $C_c(T)$. The space $M(T)$ is the collection of all real-valued bounded Radon measures on T, the support of $\mu \in M(T)$ is written supp μ, and the point mass at $t \in T$ is δ_t. As in [8], βT is the Stone-Cech compactification of T, whilc νT is the repletion (or the Hewitt realcompactification) of T.

Finally, we would like to thank H. Buchwalter, G. Choquet, and J. Schmets for their many helpful suggestions.

2. **Canonical subsets of $C_c(T)$.** In the analysis of a given locally convex space, the bounded subsets, relatively compact subsets, and null sequences each play a privileged role. If that space is $C_c(T)$, then these sets are given in terms solely of the topology on $C_c(T)$ as a locally convex space. On the other hand, $C_c(T)$ is an ordered vector space, and as such it contains sets bounded in the sense of the order imposed on $C_c(T)$. We recall the definition here.

DEFINITION 1. For an arbitrary $g \in C_c(T)$, let $B_g = \{f \in C_c(T) : |f| \leqq |g|\}$. A subset B of $C_c(T)$ is *order-bounded* iff there exists a $g \in C_c(T)$ such that $B \subseteq B_g$.

In addition to the sets mentioned above, there are equicontinuous subsets of $C_c(T)$, which derive from the space T itself and from the continuity of the functions, and which have flourished under the pen of Buchwalter (see for instance [2] and [4]). We add now another type of subset which depends on the space T, and which will play a role in the sequel analogous to that of null sequences.

DEFINITION 2. A sequence $(f_n)_{n=1}^\infty \subseteq C_c(T)$ is *hyper-null* iff there exists an increasing sequence $(U_k)_{k=1}^\infty$ of open subsets of T such that $\bigcup_{k=1}^\infty U_k = T$ and such that $f_n \xrightarrow[n]{} 0$ uniformly on U_k, for each $k \in N$.

Hyper-null sequences lie, as is immediate from the definition, somewhere between uniformly null sequences and pointwise null sequences.

The aim of this section is to discuss the relationships between the six kinds of subsets of $C_c(T)$ heretofore mentioned, and we begin with a theorem.

THEOREM 3. *Let* $A \subseteq C_c(T)$, *and regard the following statements:*
a. A is a hyper-null sequence.
b. A is equicontinuous and pointwise bounded.
c. A is order-bounded.
d. A is a null sequence.
e. A is relatively compact.
f. A is bounded.
Then the following relations hold:

$$\begin{array}{ccccc} a & \Longrightarrow & b & \Longrightarrow & c \\ \Downarrow & & \Downarrow & & \Downarrow \\ d & \Longrightarrow & e & \Longrightarrow & f. \end{array}$$

PROOF. The implications $(d) \Longrightarrow (e) \Longrightarrow (f)$ are obvious. To start off proving $(a) \Longrightarrow (b)$, let $(f_n)_{n=1}^{\infty}$ be a hyper-null sequence, with $(U_k)_{k=1}^{\infty}$ the required associated sequence of open subsets of T. Let $t_0 \in T$ and $\epsilon > 0$. Then there is a k_0 such that $t_0 \in U_{k_0}$. If $n_{k_0,\epsilon}$ is large enough, then

$$|f_n(t) - f_n(t_0)| < \epsilon, \text{ for all } n \geqq n_{k_0,\epsilon} \text{ and } t \in U_{k_0}.$$

On the other hand, there is a neighborhood U of t_0 such that $U \subseteq U_{k_0}$ and such that $t \in U$ implies that

$$|f_n(t) - f_n(t_0)| < \epsilon, \text{ for } n = 1, 2, \cdots, n_{k_0,\epsilon}.$$

From the above two inequalities it is evident that $(f_n)_{n=1}^{\infty}$ is equicontinuous at t_0, and since it is automatically pointwise bounded, we obtain $(a) \Longrightarrow (b)$. Since the sequence $(U_k)_{k=1}^{\infty}$ associated with a given hyper-null sequence covers any given compact subset of T, it follows that a hyper-null sequence converges uniformly on each compact subset of T, so that $(a) \Longrightarrow (d)$. To show that $(b) \Longrightarrow (c)$, let A be equicontinuous, and let g be defined by $g(t) = \sup_{f \in A} |f(t)|$, $t \in T$. Then g is continuous, and moreover, $A \subseteq B_g$, so A is order-bounded. Next, the fact that $(b) \Longrightarrow (e)$ is well-known. A simple proof of it goes like this. The pointwise closure A' in R^T of A is equicontinuous and pointwise bounded, by Theorem 7.14 of [14]. Then Theorem 7.15 of [14] says that the compact-open topology coincides with the pointwise topology on A', which means that A' is compact in $C_c(T)$ iff A' is compact in R^T. But R^T is a Montel space, so that A' is in fact compact in R^T, concluding the proof that $(b) \Longrightarrow (e)$. Finally, since every continuous function on T is bounded on each compact subset of T, the order-bounded subsets are always bounded in $C_c(T)$, proving that $(c) \Rightarrow (f)$. ▎

J. Schmets has observed that Theorem 3 remains true if instead of the compact-open topology we utilize any locally convex topology lying between the simple topology and the topology of uniform convergence on all bounded subsets of T (see [3] for the notion of boundedness in T).

What interests us now is the fact that none of the reverse implications in Theorem 3 holds true in general. The extent to which they do or do not hold occupies us for the remainder of this section.

In order to discuss the relationship between null and hyper-null sequences we introduce an intermediary type of sequence. We say that a sequence $(f_n)_{n=1}^{\infty} \subseteq C_c(T)$ is *bornologically null* iff there exists an unbounded increasing positive sequence $(b_n)_{n=1}^{\infty}$ of numbers such that $(b_n f_n)_{n=1}^{\infty}$ is null in $C_c(T)$. This definition is equivalent to the definitions in, for instance, [12] and [17], for what is also called "convergence in the sense of Mackey". The following proposition follows nearly immediately from the definitions.

PROPOSITION 4 *a. Bornologically null sequences are always null.* *b. hyper-null sequences are always bornologically null.*

PROOF. Part (a) is obvious, and in fact, a bornologically null sequence is in a sense majorized by a null sequence. To prove part (b), let $(f_n)_{n=1}^{\infty} \subseteq C_c(T)$ be hyper-null, with associated sequence $(U_k)_{k=1}^{\infty}$ in T. Without loss of generality let $\|f_n|_{U_i}\|_{\infty} \leqq 1$, for all n, and let $(m_k)_{k=1}^{\infty}$ be a strictly increasing sequence in N such that $m_1 = 1$ and such that

$$\|f_n|_{U_i}\|_{\infty} \leqq 1/k^2, \text{ all } n \geqq m_k, \text{ all } i = 1, 2, \cdots, k, \text{ and all } k.$$

Finally, let $b_n = k$ for all $n \in [m_k, m_{k+1})$, for each $k \in N$. Then $b_n \xrightarrow[n]{} \infty$, and if $t \in T$, then $t \in U_k$ for some k. Thus $|b_n f_n(t)| \leqq k/k^2 = 1/k$ whenever $n \geqq m_k$, so that $(f_n)_{n=1}^{\infty}$ is bornologically null. ▎

Theorems 13, 15, and 18 of [17] give sufficient conditions (though not necessary conditions) that the null sequences and bornologically null sequences of $C_c(T)$ coincide. In particular, when T is hemicompact, they coincide. Thus the converse to 4a holds when T is hemicompact.

In order to find a condition under which the converse to 4b holds, we recall from Proposition 2.3.7 in [2] that T is a k_R-space iff whenever T_1 is any completely regular Hausdorff space and $f: T \to T_1$ is any function, then f is continuous provided its restriction to each compact subset of T is continuous. The k_R-spaces include locally compact spaces, as well as metrizable spaces and k-spaces (see [2] and [14]).

PROPOSITION 5. *Let T be a k_R-space. Then each bornologically null sequence in $C_c(T)$ is hyper-null.*

PROOF. Let $(f_n)_{n=1}^{\infty}$ be bornologically null, so that for some increasing positive unbounded sequence $(b_n)_{n=1}^{\infty}$ of numbers, the sequence $(b_nf_n)_{n=1}^{\infty}$ is null. If $S \subseteq T$ is compact, then $(b_nf_n)_{n=1}^{\infty}$ is null and thus equicontinuous in $C^{\infty}(S)$. If $f = \sup_{n \in N}|b_nf_n|$, then $f|_S$ is continuous, so that by hypothesis f is continuous on T. Therefore $(f_n)_{n=1}^{\infty}$ is hyper-null, since $|f_n| \leqq f/b_n$, for all n. ∎

From our previous remarks, combined with Proposition 5, we know that if T is a hemicompact k_R-space, then the null sequences in $C_c(T)$ are hyper-null. Actually, we can say more, or so it might seem.

PROPOSITION 6. *If T is hemicompact and $C_c(T)$ is semi-complete, then each null sequence in $C_c(T)$ is hyper-null.*

PROOF. From our comments above, all we need to show is that T is a k_R-space. To that end, let $f \in R^T$ such that $f|_S$ is continuous for each compact $S \subseteq T$. Let $(S_n)_{n=1}^{\infty}$ be an increasing sequence of compact subsets of T such that if S is any compact subset of T, then $S \subseteq S_n$ for some n. For each n, let $f_n \in \mathcal{C}(T)$ such that $f_n = f$ on S_n. Then $f_m - f_n = 0$ on S_k whenever $m, n \geqq k$, so that $(f_n)_{n=1}^{\infty}$ is Cauchy in $C_c(T)$. Since $C_c(T)$ is semi-complete, there exists a $g \in C_c(T)$ such that $f_n \xrightarrow[n]{} g$. But $f_n = f$ on S_k for all $n \geqq k$ and all $k \in N$, which means that $f = g$ and thus $f \in \mathcal{C}(T)$. ∎

If T is a k_R-space, then $C_c(T)$ is necessarily semi-complete. From the proof of Proposition 6 we infer that if T is hemicompact, then T is a k_R-space iff $C_c(T)$ is semi-complete.

Next we expose a type of hemicompact spaces for which the null sequences of $C_c(T)$ are not always hyper-null.

EXAMPLE 7. *Let T be hemicompact, and assume that $(S_n)_{n=0}^{\infty}$ is a sequence of non-empty compact subsets of T such that*

$$\text{(i)} \quad \bigcup_{n=0}^{\infty} S_n = T \textit{ while } \overline{\bigcup_{n=1}^{\infty} S_n} = T,$$

(ii) $S_n \subseteq S_{n+1}^0$, *for all* $n \in N$,

(iii) $S_n \cap S_0 = \emptyset$, *for all* $n \in N$.

Then there exists a sequence $(f_n)_{n=3}^{\infty} \subseteq C_c(T)$ *which is null but not hyper-null.*

To see this, for $n \geqq 3$ let $f_n \in \mathcal{C}(T)$ be such that

$$f_n(t) = \begin{cases} n, & t \in S_n \setminus S_{n-1}^0 \\ 0, & t \in S_{n-1} \cup S_0. \end{cases}$$

Such an f_n exists because of condition (ii). Furthermore, since each $f_n = 0$ on $S_{n-2} \cup S_0$ we know by (i) and (ii) that $(f_n)_{n=3}^{\infty}$ is null in $C_c(T)$. To show that $(f_n)_{n=3}^{\infty}$ is not hyper-null, let U be a neighborhood of $t_0 \in S_0$, and pick $n \geqq 3$. Because $S_n \cap S_0 = \emptyset$ and because both S_n and S_0 are compact, there exists a neighborhood U_n of t_0 such that $U_n \subseteq U$ and $U_n \cap S_n = \emptyset$. By hypothesis (i) we can find a $t' \in U_n \cap S_{m_n}$ for some minimal $m_n \geqq 1$. Evidently $m_n \geqq n$ and $f_{m_n}(t') = n$, while $t' \in U$. Thus $(f_n)_{n=3}^{\infty}$ does not converge uniformly to 0 on U. Because U was arbitrary, $(f_n)_{n=3}^{\infty}$ is not hyper-null, nor is it order-bounded. ∎

A prototype of spaces described in Example 7 is $T = N \cup \{t_0\}$, where $t_0 \in \beta N \setminus N$ and where T inherits its topology from βN. Then T satisfies the conditions of Example 7, so that $C_c(T)$ admits a null sequence which is not hyper-null. Of course the reals R work instead of N, and also one could add any finite (or countable if he is judicious!) number of elements from $\beta N \setminus N$ or from $\beta R \setminus R$ with the same result. In any case, Example 7 and the succeeding special examples show that the converse to Proposition 4b is in general false.

We now show that the converse to Proposition 4a is generally false.

Proposition 8. *Let T be discrete, and assume the continuum hypothesis. Then the following conditions are equivalent:*

a. The cardinality of T is no larger than $\aleph_0$.

b. Null sequences in R^T are hyper-null.

c. Null sequences in R^T are bornologically null.

Proof. Proposition 6 shows that $(a) \Longrightarrow (b)$, while the implication $(b) \Longrightarrow (c)$ is obvious, and Theorem 18 of [17] yields $(c) \Longrightarrow (a)$. ∎

If one does not use the continuum hypothesis, he can in any case prove that if the cardinality of T is no larger than $\aleph_0$, then the null sequences in R^T are hyper-null, while if the cardinality of T is at least $2^{\aleph_0}$, then there exist null sequences in R^T which are not bornologically null. Any further refinement of Proposition 8 to which the continuum hypothesis is not appealed apparently must depend on some new axiom, like, for instance, Martin's axiom.

In any case, if the cardinality of T is at least $2^{\aleph_0}$, we can easily describe a null sequence in R^T which is not bornologically null (and hence not hyper-null). To that end, we identify T (or a suitable subset of T) with the collection of elements in c_0 whose coefficients are rational in $[0, 1]$. Then we define $f_n \in R^T$ by

$$f_n(t) = t(n), t \in T, n \in N.$$

Since each $t \in c_0$, evidently $(f_n)_{n=1}^{\infty}$ is null. However, if $(b_n)_{n=1}^{\infty}$ is a positive, unbounded sequence of reals, and if $t_0(n) = 1/\sqrt{b_n}$ for all n, then $b_n f_n(t_0) = \sqrt{b_n}$, for all n, rendering $(b_n f_n)_{n=1}^{\infty}$ unbounded. Thus $(f_n)_{n=1}^{\infty}$ is not bornologically null.

There is another broad class of spaces T for which the null sequences in $C_c(T)$ are hyper-null. This is the pseudocompact Warnerian spaces, which by Théorème 3.19 of [4] is precisely the collection of T for which there is a denumerable base of bounded sets in $C_c(T)$, or which is the same thing, a sequence converges to 0 in $C_c(T)$ iff it converges to 0 uniformly. The space T of ordinals less than the first uncountable, with the order topology, affords an example of a pseudocompact Warnerian space which is not compact.

On the other hand, there exist spaces which are neither k_R-spaces nor pseudocompact, but for which the null sequences in $C_c(T)$ are hyper-null. Witness the following example.

EXAMPLE 9. Let ω_1 denote the first uncountable ordinal, and let T consist of ω_1, together with all non-limit ordinals $< \omega_1$, cloaked in the order topology. Let $(f_n)_{n=1}^{\infty}$ be null in $C_c(T)$. Then surely $\lim_n f_n(\omega_1) = 0$, so that for each positive integer k there exists an n_k and a $t_k < \omega_1$ such that $|f_n(t)| < 1/k$, for all $n \geqq n_k$ and all $t \geqq t_k$. Let $t_\infty = \sup_k t_k$ and let $U = \{t \in T : t \geqq t_\infty\}$. Then $(f_n)_{n=1}^{\infty}$ converges to 0 uniformly on U. Since $T \backslash U$ is countable, say $T \backslash U = (s_i)_{i=1}^{\infty}$, if we let $U_k = U \cup (s_i)_{i=1}^{k}$ for each k, then it is easy to check that $(f_n)_{n=1}^{\infty}$ is hyper-null with associated sequence $(U_k)_{k=1}^{\infty}$. The reason that T is not a k_R-space is that the compact-open topology is (nearly) evidently the topology of pointwise convergence, and not every function defined on T is continuous.

We turn now to the compact subsets of $C_c(T)$ and the relationships which exist between them and the equicontinuous and order-bounded subsets. First of all, J. Schmets has pointed out that any compact subset of $C_c(T)$ is equicontinuous at each $t \in T$ with a countable basis of neighborhoods, as is easy to verify. So for any T which satisfies the first countability axiom, every compact subset of $C_c(T)$ is automatically equicontinuous.

In a quite different vein, with Proposition 4.2.3 of [2] H. Buchwalter has extended the several theorems which generally appear under the heading of Arzela's Theorem. For a variety of reasons we feel it wise at this point to present Buchwalter's elegant proof of his theorem.

PROPOSITION 10 (Buchwalter). *Let T be a k_R-space. Then every compact subset of $C_c(T)$ is equicontinuous.*

PROOF. We may as well assume that the compact subset A of $C_c(T)$ is uniformly bounded on T, since in any case $B = \{|f|/(1 + |f|) : f \in A\}$ is compact in $C_c(T)$, so that if B is proved equicontinuous, then A will automatically be equicontinuous. Let $\varphi : T \to C^\infty(A)$ be given by the equation $\varphi(t) = F_t$, where $F_t(f) = f(t)$, for all $f \in A$. By assumption f is bounded on T for each $f \in A$, so that for each $t \in T$ we know that $F_t \in C^\infty(A)$. Now if S is an arbitrary compact subset of T, then $A|_S$ is compact in $C^\infty(S)$, so by the classical Ascoli Theorem [13, p. 237], the set $A|_S$ is equicontinuous. This means that if $s_\lambda \xrightarrow[\lambda]{} s$ in S, then $F_{s_\lambda}(f) = f(s_\lambda) \xrightarrow[\lambda]{} f(s) = F_s(f)$ uniformly with respect to $f \in A$, so that $\varphi(s_\lambda) = F_{s_\lambda} \xrightarrow[\lambda]{} F_s = \varphi(s)$ in the norm of $C^\infty(A)$. Consequently $\varphi|_S$ is continuous. Since $C^\infty(A)$ is completely regular, T is a k_R-space, and S is arbitrary and compact in T, the result is that φ is continuous. Consequently if $t_\lambda \xrightarrow[\lambda]{} t$ in T, then

$$\sup\{|f(t_\lambda) - f(t)| : f \in A\} = \| F_{t_\lambda} - F_t \|_\infty = \|\varphi_{t_\lambda} - \varphi_t\|_\infty \xrightarrow[\lambda]{} 0,$$

rendering A equicontinuous and concluding the proof. ∎

A propos of the proof to Proposition 10, Buchwalter observes that the same argument shows that if T is a k_R-space, then any precompact subset of $C_c(T)$ is equicontinuous. This relates to a very recent theorem of Haydon [9, Corollary 3.2], which says that the precompact subsets of $C_c(T)$ are equicontinuous precisely when every precompact subset of $C_c(T)$ is relatively compact. That k_R-spaces do not completely exhaust the spaces which have this property is known (see for example [10]).

On the other hand, there exist T for which not every compact subset of $C_c(T)$ is equicontinuous. It is in this connection that we derive our next two examples. In the first, which uses the T of Example 9, we display a convex compact subset of $C_c(T)$ which is neither equicontinuous nor even order-bounded, while in the second example we show that for the given T all compact subsets in $C_c(T)$ are order-bounded, although there exist compact subsets in $C_c(T)$ which are not equicontinuous.

EXAMPLE 11. Let ω_1 be the first uncountable ordinal, and let T consist of ω_1, together with all non-limit ordinals $< \omega_1$, endowed with the order topology. Then the compact-open topology of $C_c(T)$ is the simple topology. For each $t \in T \backslash \{\omega_1\}$, if n_t is the positive number such that $t - n_t$ is a limit ordinal (or is 0), then define the set B by $B = \{n_t \chi_{\{t\}} : t \in T \backslash \{\omega_1\}\}$. Let co B denote the convex hull of B, and let A be the closure in $C_c(T)$ of B. We will show that A is (convex and) compact but neither equicontinuous nor order-bounded. Since any function in $C_c(T)$ is bounded on a tail of T, and since B is bounded on no such tail, evidently A is neither equicontinuous nor order-bounded. To begin showing that A is compect, let $T_m = \{t \in T : n_t = m\}$, for all $m \in N$, so that $T = \{\omega_1\} \cup \bigcup_{m=1}^{\infty} T_m$. Note that if $g \in \overline{\text{co } B}^{R^T}$, then $g(\omega_1) = 0$. Next we show that if $g \in \overline{\text{co } B}^{R^T}$ then $g \neq 0$ on at most a countable number of elements of T. Indeed if $g \neq 0$ on an uncountable subset of T, then there must exist an $\epsilon > 0$ and an m such that $|g(t)| > \epsilon$ for an uncountable number of elements T_m' of T_m. Now let $(t_i)_{i=1}^{\infty} \subseteq T_m'$ and let $k \in N$ such that $k > 2m/\epsilon$, so that $(k\epsilon/2) > m$. Next let

$$V = \{h \in R^T : \ |h(t_i)| < \epsilon/2, i = 1, 2, \cdots, k\},$$

so that V is a neighborhood of 0 in R^T. If $h \in V$, then $|(g + h)(t_i)| > \epsilon/2$, for $i = 1, 2, \cdots, k$, so that $\sum_{i=1}^{k} |(g + h)(t_i)| > m$. If $f \in$ co B, then $f = \sum_{j=1}^{p} c_j n_{s_j} \chi_{\{s_j\}}$ for appropriate $(s_j)_{j=1}^{p} \subseteq T$ and appropriate $(c_j)_{j=1}^{p}$, where each $c_j \geqq 0$ and where $\sum_{j=1}^{p} c_j = 1$, and where we assume without loss of generality that $(s_j)_{j=1}^{p} \supseteq (t_i)_{i=1}^{k}$. This means that

$$\sum_{i=1}^{k} |f(t_i)| \leqq \sum_{i=1}^{k} c_i m \leqq \sum_{j=1}^{p} c_j m = m.$$

Consequently $(g + V) \cap \text{co } B = \emptyset$, so that $g \notin \overline{\text{co } B}^{R^T}$, with the result that if $g \in \overline{\text{co } B}^{R^T}$, then $g \neq 0$ on at most a countable number of the elements of T, and consequently g is continuous on T. Since the compact subsets of T are finite point-sets, what we have shown is that

$$\overline{\text{co } B}^{R^T} = \overline{\text{co } B}^{C_c(T)} = A.$$

However, $\overline{\text{co } B}^{R^T}$ is bounded pointwise and is closed in R^T, which itself is a Montel space. Thus $\overline{\text{co } B}^{R^T}$ is compact in R^T, meaning therefore that A is compact in $C_c(T)$, as we wished to prove.

Example 11 is to be contrasted with Proposition 3.5 of [9], which says that in $\mathcal{C}(T)$ fortified with the simple topology, each lattice-closed relatively compact subset D is equicontinuous. The set A of Example 11 reveals that the hypothesis of Proposition 3.5 that D be lattice-closed and not just convex is quite essential. Moreover, if it so happened that precompact subsets were always relatively compact in the $C_c(T)$ of Example 11, then Lemma 3.4 and Proposition 3.5 of [9] together would imply that *each* relatively compact subset would be equicontinuous. Consequently for the T of Example 11 there exist precompact subsets of $C_c(T)$ which are not relatively compact.

In our forthcoming Example 12 we must assume the existence of a measurable cardinal (which within the usual axiom scheme may or may not exist). By Theorem 12.2 of [8] a cardinal is measurable iff for a discrete space S with that cardinality we have $\nu S \neq S$.

EXAMPLE 12. Let S be a discrete space with measurable cardinality. Let $s \in \nu S \backslash S$, and let $T = S \cup \{s\}$, as a subset of νS with the restricted topology. Then the compact-open topology on $C_c(T)$ is naturally the simple topology. In addition, it is routine to verify that every function on T is dominated by a suitable continuous function, so that every subset of $C_c(T)$ which is pointwise bounded is order-bounded. Consequently every compact subset of $C_c(T)$ is order-bounded. If $B = \{\chi_{\{t\}} : t \in S\} \cup \{0\}$, then since the compact-open topology is the simple topology, B is compact. However, B is certainly not equicontinuous at s.

We remark that every function defined on the T of Example 12 is bounded on some neighborhood of the non-isolated point s. We know of no essentially different space T which possesses this property, and indeed we know of no space T without isolated points for which every function on T is bounded on some neighborhood of each element of T.

Our next proposition gives a clearer indication of the relationships between compact, equicontinuous, and order-bounded sets in $C_c(T)$, under the stipulation that T be pseudocompact. We thank H. Buchwalter for part (*c*) — both its statement and its proof — and we remark that by Théorème 3.19 of [4], a pseudocompact space T is Warnerian iff every null sequence in $C_c(T)$ is uniformly null.

PROPOSITION 13. *Let T be pseudocompact. Then the following statements hold:*

a. Every convex compact subset of $C_c(T)$ is order-bounded (or is uniformly bounded).

b. Each compact subset of $C_c(T)$ is order-bounded precisely when T is Warnerian.

c. Each compact subset of $C_c(T)$ is equicontinuous precisely when T is Warnerian and simultaneously each compact subset of $C_c(T)$ is metrizable.

PROOF. By Proposition 3.5.2 of [13] it suffices to prove part (a) for an arbitrary $A \in \mathfrak{A}$, where $\mathfrak{A}$ comprises the *balanced* convex compact subsets of $C_c(T)$. Now $A \in \mathfrak{A}$ means that A is compact in $C_c(\nu T)$, via Proposition 2.2 of [4], so A is in particular bounded in $C_c(\nu T)$. But T is assumed pseudocompact, so that $C_c(\nu T) = C^\infty(\beta T)$, and therefore A is order-bounded (or uniformly bounded) in $C^\infty(\beta T)$, meaning that A satisfies the conclusion of part (a). For part (b), we note that by Théorème 3.14 of [4], T is Warnerian iff the bounded subsets are precisely the order-bounded subsets in $C_c(T)$. After a moment's reflection you will see that this is tantamount to the statement of (b). Part (c) is proved as follows. By hypothesis, if A is compact in $C_c(T)$, then A is equicontinuous, so is equicontinuous in $C^\infty(\beta T)$ by Théorème 4.3.3 of [2]. Consequently A is relatively compact in $C^\infty(\beta T)$. The uniqueness of comparable compact Hausdorff topologies shows that the closure in $C^\infty(\beta T)$ of A has the uniform-norm topology inherited from $C^\infty(\beta T)$, so A is metrizable. That T is Warnerian follows from part (b) and Theorem 3. For the other direction in part (c), we assume that A is compact and metrizable in $C_c(T)$, and let $(f_n)_{n=1}^\infty \subseteq A$. Then there exists a subsequence $(f_{n_k})_{k=1}^\infty$ and an $f \in A$ such that $f_{n_k} \xrightarrow[k]{} f$ in $C_c(T)$. Since T is Warnerian, Théorème 3.19 of [4] implies that $f_{n_k} \xrightarrow[k]{} f$ uniformly, so that A is compact in $C^\infty(\beta T)$. By the classical theorem of Arzela, A is therefore equicontinuous. ∎

Part (a) of Proposition 13 asserts that if T is pseudocompact, then the convex compact subsets in $C_c(T)$ are relatively tame, in contrast to their possible behavior for other T (e.g., Example 11).

Example 2 after Corollaire 3.12 of [4] describes a pseudocompact, non-Warnerian space. On the other hand, a non-trivial (i.e., noncompact) pseudocompact Warnerian space is afforded by the ordinals T less than the first uncountable, endowed with the natural order topology. This T satisfies the first countability axiom, so we are assured by the comments preceding Proposition 10 that each compact subset of $C_c(T)$ is equicontinuous. Yet we do not know if when T is pseudocompact and Warnerian, then every compact (or even every convex compact) subset of $C_c(T)$ must be equicontinuous. Of course, by our earlier remarks, the only points of T in question with respect to equicontinuity are those which lack a countable basis of neighborhoods.

We conclude this section by discussing the order-bounded subsets of $C_c(T)$, first vis-a-vis the compact and equicontinuous subsets. Recall that if $g \in C_c(T)$, then by definition $B_g = \{f \in C_c(T) : |f| \leqq |g|\}$.

LEMMA 14. *Let $g \in C_c(T)$. Then the set B_g is compact (or is equicontinuous) iff $g(t) = 0$ for each non-isolated $t \in T$.*

PROOF. If B_g is compact in $C_c(T)$, then it is compact in R^T. However, for each $t \in T$, the function $g(t)\,\chi_{\{t\}}$ is a limit point in R^T of the set B_g, and $g(t)\chi_{\{t\}}$ is evidently continuous precisely when $g(t) = 0$ or when t is isolated in T. Conversely, assume that $g(t) = 0$ for each non-isolated $t \in T$. Note first that B_g is equicontinuous at every isolated $t \in T$. Moreover, since g is continuous, we know by the hypothesis that g is small in some neighborhood of each non-isolated $t \in T$, rendering B_g equicontinuous at those points as well. ▌

PROPOSITION 15. *Each order-bounded subset of $C_c(T)$ is relatively compact in $C_c(T)$ (or is equicontinuous) precisely when T is discrete.*

PROOF. An immediate application of Lemma 14. ▌

Next we study when the bounded subsets of $C_c(T)$ are order-bounded.

PROPOSITION 16. *Every bounded subset of $C_c(T)$ is order-bounded iff each lower semi-continuous function which is bounded on each compact subset of T is majorized by a function continuous on T.*

PROOF. It is routine to show that a set B in $C_c(T)$ is bounded iff $B \subseteq B_h = \{f \in C_c(T) : |f| \leqq |h|\}$, for an appropriate lower semi-continuous $|h|$ bounded on each compact subset of T, and the rest follows. |

COROLLARY 17. *If T is paracompact and locally compact, then the bounded subsets of $C_c(T)$ are order-bounded.*

PROOF. Any locally compact and paracompact space is the union of pairwise disjoint, hemicompact, closed and open subsets, so it suffices from Proposition 16 to assume that T is hemicompact and locally compact. Let $|h|$ be lower semi-continuous and bounded on the compact subsets of T, and let $T = \bigcup_{n=1}^{\infty} A_n$, where each A_n is compact and $A_{n+1}^0 \supseteq A_n$. If $\|h\,|_{A_n}\|_\infty \leqq c_n \leqq c_{n+1}$ for each n, then let $f_1 = c_2 = f_2$, and for $n \geqq 3$, let $f_n \in C_c(T)$ be positive, such that $f_n = c_n$ on $A_n \backslash A_{n-1}^0$ and $f_n = 0$ on $A_{n-2} \cup (T \backslash A_{n+1}^0)$. Then $f = \sum_{n=1}^{\infty} f_n \in C_c(T)$, and assuredly $f \geqq |h|$, and $B_f \supseteq B_{|h|}$. ▌

As we mentioned earlier, pseudocompact spaces T are Warnerian iff the bounded subsets of $C_c(T)$ are order-bounded. Example 12 shows that there are T which are neither paracompact nor locally compact nor pseudocompact, but for which the bounded subsets of $C_c(T)$ are all order-bounded.

One might think that there should be an intimate relationship between the coincidence of null and hyper-null sequences on the one hand and the coincidence of the bounded and order-bounded subsets on the other hand. The relationship is not too intimate, however, since for the T of Examples 9 and 11 the null sequences in $C_c(T)$ are hyper-null, while there exist bounded subsets of $C_c(T)$ which are not necessarily order-bounded. In the reverse direction, if T is discrete and cardinality of $T \geqq 2^{\aleph_0}$, then $C_c(T)$ satisfies the reverse pattern, by Proposition 8.

3. **Boundedness of linear forms on $C_c(T)$.** In Section 2 we dwelled on certain types of bounded subsets of $C_c(T)$. Now we turn to linear forms on $C_c(T)$ and discuss how they react on those types of bounded subsets of $C_c(T)$. The following result prepares us for Theorem 19, and is essentially Hewitt's Theorem 22 of [**11**] in a disguised form. Our proof carries the flavor of the proofs of Théorème 14 in [**6**] and of Lemme 1.8 in [**4**].

PROPOSITION 18. *Let F be a linear form on $C_c(T)$, and let F be bounded on all hyper-null sequences of $C_c(T)$. Then F corresponds to a $\mu \in M(\beta T)$ such that* supp μ *is a compact subset of νT.*

PROOF. First of all, F must be bounded on $B_1 = \{f \in C_c(T) : |f| \leqq 1\}$, since otherwise F is unbounded on a hyper-null sequence contained in B_1. Thus $F|_{C^\infty(T)}$ corresponds to a bounded linear form F on $C^\infty(\beta T)$, by $\tilde{F}(\tilde{f}) = F(f)$, for all $f \in C^\infty(T)$, where $\tilde{f}$ is the (unique) continuous extension to βT of f. The Riesz-Kakutani Theorem yields a $\mu \in M(\beta T)$ such that $\tilde{F}(\tilde{f}) = \int_{\beta T} \tilde{f}\, d\mu$. Note that supp μ is compact in βT. Now assume that $s \in (\text{supp } \mu)\backslash \nu T$. Then by Theorem 8.4 of [**8**] there exists a positive, continuous function g on T such that $\tilde{g}(s) = \infty$. For each n, let $P_n = \{t \in \beta T : \tilde{g}(t) > n\}$, so that P_n is open in βT. For each n, since $s \in (\text{supp } \mu) \cap P_n$, there exists an $f_n \in C^\infty(T)$ such that supp $f_n \subseteq P_n$ and such that

$$F(f_n) = \tilde{F}(\tilde{f}_n) = \int_{\beta T} \tilde{f}\, d\mu \neq 0.$$

Then for appropriate $(c_n)_{n=1}^\infty$ in the reals, we have $|F(c_n f_n)| \xrightarrow[n]{} \infty$ while $(c_n f_n)_{n=1}^\infty$ is hyper-null in $C_c(T)$ with respect to $(U_k)_{k=1}^\infty$, where $U_k = (T\backslash P_k)^0$, for $k = 1, 2, \cdots$. This contradicts the hypothesis on F. Consequently the support of μ is compact and lies inside νT. To finish the proof we need only show that F corresponds to μ on all of $C_c(T)$. But if $f \in C_c(T)$, then $\{n(f - \theta_n \circ f)\}_{n=1}^\infty$ is hyper-null, so that $F(f - \theta_n \circ f) \xrightarrow[n]{} 0$, which means finally that

$$F(f) = \lim_n F(\theta_n \circ f) = \lim_n \int_{\beta T} (\widetilde{\theta_n \circ f})\, d\mu = \int_{\nu T} f\, d\mu. \quad ∎$$

Amongst other things, Theorem 19 will demonstrate the simple fact that a linear form F on $C_c(T)$ is bounded on all hyper-null sequences iff it is bounded on all order-bounded sets, and thus it is trivial to translate Proposition 18 into precisely Hewitt's Theorem 22 in [**11**]. However, the proof of Proposition 18 yields a slightly but strictly stronger result than that stated. Our proof shows that if F is bounded on B_1 and is also bounded on those hyper-null sequences $(f_n)_{n=1}^{\infty}$ which eventually vanish on each member of the associated sequence $(U_k)_{k=1}^{\infty}$, then F corresponds to the μ advertised in Proposition 18.

We are now ready for Theorem 19, and remark beforehand that the proof that (i) implies (e) is due to H. Buchwalter.

THEOREM 19. *Let F be a linear form on $C_c(T)$, and regard the following statements:*

a. F is continuous on $C_c(T)$.

b. F is bounded on all bounded subsets of $C_c(T)$.

c. F is bounded on all compact subsets of $C_c(T)$.

d. F is bounded on all null sequences of $C_c(T)$.

e. F is bounded on all convex compact subsets of $C_c(T)$.

f. F is bounded on all order-bounded subsets of $C_c(T)$.

g. F is bounded on all equicontinuous, pointwise-bounded subsets of $C_c(T)$.

h. F is bounded on all hyper-null sequences of $C_c(T)$.

i. F corresponds to a measure in $M(\beta T)$ with compact support in νT.

Then the following relations hold:

$$
\begin{array}{ccccccccc}
 & & & & a & & & & \\
 & & & & \Downarrow & & & & \\
 & & b & \Leftrightarrow & c & \Leftrightarrow & d & & \\
 & & & & \Downarrow & & & & \\
e & \Leftrightarrow & f & \Leftrightarrow & g & \Leftrightarrow & h & \Leftrightarrow & i
\end{array}
$$

Moreover, all the conditions (a) through (i) are equivalent (or indeed $(b) \Rightarrow (a)$) precisely when T is replete. Finally, $(e) \not\Rightarrow (d)$ whenever T is simultaneously pseudocompact and non-Warnerian.

PROOF. For arbitrary locally convex spaces, the implications $(a) \Rightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$ and $(c) \Rightarrow (e)$ are always trivially valid. By Theorem 3, $(e) \Rightarrow (g) \Rightarrow (h)$ and $(f) \Rightarrow (h)$. Next, if $(f_n)_{n=1}^{\infty} \subseteq B_f$ and if $0 < |F(f_n)| = c_n \underset{n}{\rightarrow} \infty$, then F is unbounded on the hyper-null sequence $(f_n/\sqrt{c_n})_{n=1}^{\infty}$, showing that $(h \Rightarrow (f)$. Proposition 18 says that $(h) \Rightarrow (i)$. To prove that $(i) \Rightarrow (e)$, let F satisfy the conditions of (i). Then F is a bounded (in fact continuous!) linear form on $C_c(\nu T)$, and thus is bounded on all convex bounded subsets of $C_c(T)$. Théorème

2.5 of [4] tells us that $C_c(\nu T)$ is the ultrabornological space associated to $C_c(T)$, whereupon the convex compact subsets of $C_c(T)$ are convex and bounded in $C_c(\nu T)$ by Proposition 2.2 of [4]. Now condition (e) follows directly. Next, the fact that all the implications are equivalent iff T is replete follows immediately from the foregoing and from the fact that each one-point evaluation, for $t \in \nu T$, gives rise to a linear form on $C_c(T)$ which is bounded on all order-bounded subsets of $C_c(T)$, and all such forms are continuous precisely when $T = \nu T$. Finally we prove that if T is pseudocompact and non-Warnerian, then $(e) \not\Rightarrow (d)$. For such a T there exists a bounded subset B of $C_c(T)$ which is not order-bounded, and by the proof of Proposition 16 we may assume that $B = \{f \in C_c(T) : |f| \leqq h\}$, for an appropriate lower semi-continuous function h. Let us find $(f_n)_{n=1}^{\infty} \subseteq B$, $(t_n)_{n=1}^{\infty} \subseteq T$, and $(V_n)_{n=1}^{\infty}$ a sequence of pairwise disjoint open subsets of T, such that for each n the following conditions hold: $t_n \in V_n$, $|f_n(t_n)| > n \cdot 2^n$, and $h > n \cdot 2^n$ on V_n. Then for n there exists a $g_n \in \mathcal{C}(T)$ such that $|g_n| \leqq |f_n| \leqq h$, $g_n(t_n) = n \cdot 2^n$, and $g_n = 0$ on $T \backslash V_n$. Define F by $F(f) = \sum_{n=1}^{\infty} [f(t_n)/2^n]$, for all $f \in C_c(T)$. Since T is pseudocompact, F is linear and bounded on all order-bounded subsets of $C_c(T)$, and F corresponds to $\sum_{n=1}^{\infty} [\delta_{t_n}/2^n] \in M(T) \subseteq M(\nu T) = M(\beta T)$. But $g_n \in B$ and $F(g_n) = n$ for each n, so that F is not bounded on the bounded set B, which means that $(e) \not\Rightarrow (d)$, as we wished to prove. ∎

The example just used in proving that in general $(e) \not\Rightarrow (d)$ shows that even innocuous-looking measures defined in terms of elements of T can yield unbounded linear forms on $C_c(T)$. We mention also that (e) can imply (d) even if neither all the compact subsets nor all the null sequences of $C_c(T)$ are order-bounded. In fact, this is what happens if $T = T_1 \oplus T_2$, where T_1 is the space exhibited in Example 7 and where T_2 is the space given in Examples 9 and 11. It results from our earlier discussions of T_1 and T_2 that neither all the compact subsets nor all the null sequences of $C_c(T)$ are order-bounded. On the other hand, if F is a linear form on $C_c(T)$, then F is bounded on the order-bounded subsets of $C_c(T)$ iff F is bounded on the order-bounded subsets of both $C_c(T_1)$ and $C_c(T_2)$, and a like statement holds for the continuity of F on $C_c(T)$. However, T_1 and T_2 are replete, so all linear forms bounded on the order-bounded subsets of $C_c(T_1)$ and on $C_c(T_2)$ are continuous. Therefore $(i) \Rightarrow (a)$, so $(e) \Rightarrow (d)$ for this T.

We conclude the paper by describing what Hewitt, Nachbin and Shirota, De Wilde and Schmets, and Buchwalter have in turn proved, and by comparing their results in light of the ideas appearing in Sections 2 and 3. First let us recall that a locally convex space E is bornological (resp. ultrabornological) iff E is an inductive limit of normed (resp. Banach) spaces, which happens iff E is a Mackey space and additionally every linear form on E which is bounded on all bounded (resp. convex compact) subsets of E is continuous.

The collection of theorems began with E. Hewitt, who proved in 1950 that T is replete iff every linear form on $C_c(T)$ which is bounded on all order-bounded subsets of $C_c(T)$ is continuous [11, Theorem 22]. In 1954, L. Nachbin and T. Shirota simultaneously solved a problem posed by J. Dieudonné; at the same time they each established that T is replete iff $C_c(T)$ is bornological (see [15] and [16]). (Actually, our Theorem 19 tells us that the proof given by Shirota is a bit stronger than that stated, since Shirota utilized Hewitt's Theorem and proved besides that if T is replete, then $C_c(T)$ is a Mackey space. Nachbin's proof did not rely on Hewitt's Theorem; the proof was direct, and it gave the result as stated.) In 1971, M. De Wilde and J. Schmets proved in [7] that T is replete iff $C_c(T)$ is ultrabornological. (Indeed, their proof too yields a slightly different conclusion, because what they really proved was that T is replete iff $C_c(T)$ is an inductive limit of the Banach spaces $(E_f)_{f\in\mathcal{C}(T)/f\geqq 0}$, where for each such f, E_f is the span of B_f.) Finally, in 1972, H. Buchwalter [4] proved that T is replete iff $C_c(T)$ is the inductive limit of the Banach spaces $(E_H)_{H\in\mathcal{H}}$, where $\mathcal{H}$ is the collection of all balanced, convex, pointwise closed, equicontinuous and pointwise bounded subsets of $\mathcal{C}(T)$, and where E_H is the span of H, for each $H\in\mathcal{H}$.

The half of each of the above-mentioned theorems which is the difficult one to prove is the half in which T is assumed to be replete and $C_c(T)$ is shown to possess an appropriate property. With this in mind, one might conjecture that the Nachbin-Shirota Theorem is stronger than the Hewitt Theorem, that the De Wilde-Schmets Theorem is stronger than the Nachbin-Shirota Theorem, and finally that the Buchwalter Theorem is stronger than the De Wilde-Schmets Theorem.

However, Theorem 19 settles the score — somewhat differently from the conjectures mentioned above. Indeed, because in general we have $(f)\not\Rightarrow(b)$, the Nachbin-Shirota Theorem (as stated) is not exactly stronger than the Hewitt Theorem. On the other hand, since $(b)\Rightarrow(e)$ but in general $(e)\not\Rightarrow(b)$, the De Wilde-Schmets Theorem is genuinely stronger than Hewitt's Theorem and the Nachbin-Shirota Theorem. Finally, the equivalence of (e) and (g), together with the fact that a subset A of $C_c(T)$ is equicontinuous and pointwise bounded iff the balanced, convex, pointwise closed hull of A is also equicontinuous and pointwise bounded, yields the equivalence of the theorems of De Wilde-Schmets and of Buchwalter.

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