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ON THE CONJUGATING REPRESENTATION OF A FINITE GROUP

K. L. FIELDS

Very little is known about how the conjugating representation of a finite group decomposes into irreducible representations. In this note we investigate which sets of multiplicities are possible in such a decomposition. The analogous question for the regular representation is also long unsolved. Let $\nu_G = C \cdot 1_G + \gamma_G$ denote the conjugating representation of G (or its character). If c denotes the number of conjugacy classes of G, then the principal representation 1_G does not appear in the decomposition of γ_G. We show here that if G is not abelian (i.e., if $\gamma_G \neq 0$) then γ_G contains at least two inequivalent irreducible representations; moreover, if the group has trivial center and is not the symmetric group on three letters, then γ_G is not multiplicity-free; and if the group is simple, then the *g.c.d.* of the degrees of the irreducible constituents of γ_G is not divisible by the degree of any irreducible representation of G.

Recall that a primary representation is a direct sum of copies of a single irreducible representation.

LEMMA. *If γ_G is primary or multiplicity free then $\gamma_{G/Z(G)}$ must also be respectively primary or multiplicity free.*

PROOF. Let $C[G]$, $C[G/Z(G)]$ denote the complex group algebras of G and $G/Z(G)$ where $Z(G)$ denotes the center of G. Viewing these group algebras as left $C[G/Z(G)]$ modules with the action induced by conjugation, we see that $C[G/Z(G)]$ is a homomorphic image of $C[G]$ (under the mapping induced by $G \rightarrow G/Z(G)$) and so is in fact a direct summand of $C[G]$ by Maschke's Theorem. Hence if $C[G]$ is a direct sum of copies of the trivial module and copies of a single irreducible module, or a direct sum of copies of the trivial module and a multiplicity free module, then so is $C[G/Z(G)]$.

THEOREM 1. *γ_G is never primary unless $\gamma_G = 0$ and G is abelian.*

PROOF. Assume first that $Z(G) = 1$. If $\nu_G = c \cdot 1_G + a_\chi \chi, a_\chi \geqq 0$ then $1^*_{C(x_i)} = 1_G + m_i \chi$ for some $m_i \geqq 0$. Here $\{x_1, \cdots, x_c\}$ is a complete set of non-conjugate elements of G, $C(x_i)$ is the centralizer of x_i and $1^*_{C(x_i)}$ denotes the representation (or character) induced from the trivial representation of $C(x_i)$. Hence $h_i = [G : C(x_i)] = 1 + m_i \chi(1)$,

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so $(h_i, \chi(1)) = 1$ for $i = 1, \cdots, c$. By Burnside's Theorem either $\chi(x_i) = 0$ or $|\chi(x_i)| = \chi(1)$. Now the kernel of χ is the kernel of ν_G, which is $Z(G) = 1$. Hence $Z(G/\mathrm{Ker}\,\chi) = 1$, so $\chi(x_i) = 0$ for $x_i \neq 1$. So

$$1 = \sum_{x \in G} \chi(x)\chi(x^{-1})/|G| = \chi(1)^2/|G|.$$

But this is impossible unless $|G| = 1$.

Let us now allow $Z(G) \neq 1$. By the lemma, if γ_G is primary, so is $\gamma_{G/Z(G)}$. Hence it follows from above that if γ_G is primary then G is nilpotent. But for $G = S_{p_1} \times \cdots \times S_{p_n}$, we have $\nu_G = \nu_{S_{p_1}} \times \cdots \times \nu_{S_{p_n}}$, so if γ_G is primary then γ_{S_p} must be primary for some p and $\gamma_{S_{p_i}} = 0$ (i.e. S_{p_i} is abelian) for $p_i \neq p$. Moreover, since γ_{S_p} is real, we must have $p = 2$.

We now show that γ_{S_2} is not primary unless $\gamma_{S_2} = 0$ and S_2 is abelian by induction on $|S_2| = 2^m$. If $m = 1, 2$ then $G = S_2$ is abelian. If $m \geqq 3$, G is not abelian, and γ_G is primary, then $\gamma_{G/Z(G)}$ is primary by the lemma and so contradicts our induction hypothesis unless $G/Z(G)$ is abelian. In this case there are at least two distinct subgroups of G of index two which contain centralizers of elements; hence by Roth [1 Thm. 3.2.], γ_G must contain at least two distinct linear characters having these subgroups as kernels, and so γ_G is *not* primary.

The argument of the first part of the proof yields,

Corollary 1. *If G is simple and $\nu_G = C \cdot 1_G + \sum a_\chi \chi$ then g.c.d.$_{a_\chi \neq 0}$ $\chi(1)$ is not divisible by any $\chi(1)$.*

Theorem 2. *If γ_G is multiplicity free, then G contains a nilpotent normal subgroup of index two. Moreover, if the center of G is trivial, then G is in fact the symmetric group on 3 letters, S_3.*

Proof. By the lemma we are reduced to considering the case when $Z(G) = 1$. In this case $\nu_G = \sum 1^*_{C(x_i)} = \sum_i (1_G + \sum_j \chi_{ij})$ where χ_{ij} are certain irreducible characters of G, at least one appearing for each $x_i \neq 1$ since the conjugacy class of $x_i \neq 1$ contains more than one element. Since the number of distinct irreducible characters equals the number of conjugacy classes, we must in fact have exactly one irreducible character appearing for each $x_i \neq 1$, i.e., $1^*_{C(x_i)} = 1_G + \chi_i$, and $\chi_i \neq \chi_j$ for $i \neq j$ in order to prevent multiplicities from occurring. Otherwise put, *G must act doubly transitively and pairwise inequivalently on its conjugacy classes.* We will show that *such a group must be S_3.*

To this end, let us first observe that since $1^*_{C(x_i)} = 1_G + \chi_i$ is rational valued and each irreducible character $\chi \neq 1_G$ appears among the $c - 1$ characters χ_i, all of the irreducible characters of G are rational valued. Hence, by a well known result of Brauer, x is conjugate to x^n whenever n is prime to the order of x; in particular x is conjugate to x^{-1}.

Now let a be an involution in $Z(S_2(G))$, the center of a 2-Sylow subgroup of G ($|G|$ is even since G is doubly transitive). Let x be an element of odd order in $G - C(a)$ (recall that $Z(G) = 1$). Then $G = C(x) \cup C(x)aC(x)$; hence, from the remarks made above, the only distinct conjugate of x in G is $x^a = x^{-1} = x^2$. Thus the conjugacy class of x consists of $\{x, x^2\}$ and so $C(x)$ is normal of index 2 in G. We claim, moreover, that $C(x) = \{1, x, x^2\}$. For let $y \in C(x)$ and assume that y has odd order:

(a) If $y \notin C(a)$, then $G = C(y) \cup C(y)aC(y)$ and, as above, $aya = y^{-1} = y^2$. Then $\{a, y^{-1}ay = ya, x^{-1}ax = xa\}$ are conjugate and so there is an element $h \in C(a)$ such that $xa = h^{-1}yah = h^{-1}yha$; hence $x = h^{-1}yh$ and so either $y = x$ or $y = x^2$.

(b) If $y \in C(a)$ then $yx \notin C(a)$ and also yx has odd order. So from (a) either $yx = x$ or $yx = x^2$, i.e. $y = 1$ or $y = x$. Thus $C(x)$ has no elements of odd order other than $\{x, x^2\}$. Hence $\{1, x, x^2\}$ is a normal subgroup of index 2^n in $C(x)$, and so is normal of index 2^{n+1} in G. Therefore (Schur) G is a semi-direct product, $G = \langle x \rangle \times_{sd} S_2(G)$. If we let $A = S_2(G) \cap C(x)$ then $S_2(G) = A \cup Aa$; but now if G has trivial center then so must A, i.e., $A = 1$, and so $S_2 = \langle a \rangle$ and $G = S_3$, the symmetric group on three letters.

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UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112