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$1\frac{1}{2}$ AND 2 GENERATOR IDEALS IN PRÜFER DOMAINS

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An R-module M is *generated by* $1\frac{1}{2}$ *elements* if M can be generated by two elements, the first of which can be an arbitrarily specified element of $M - (\text{rad } R)M$. This last restriction is natural because, by Nakayama's Lemma, elements of $(\text{rad } R)M$ can always be omitted from finite generating sets. If rad $R = 0$ the $1\frac{1}{2}$ generator property says that the "first" generator can be an arbitrarily specified nonzero element of M, a situation familiar in ideals of Dedekind domains.

If every finitely generated ideal in a Prüfer domain R can be generated by $1\frac{1}{2}$ elements, we will call R a $1\frac{1}{2}$ *generator Prüfer domain.*

In §1, "Nonstandard Prüfer Domains," we show that every finite abelian group can occur as the (ideal) class group of a non-noetherian $1\frac{1}{2}$ generator Prüfer domain with radical 0. As the title indicates, the main tool used in the construction is the ultra-power.

In §2, we observe that R. Gilmer's domains of "type $D + M$" afford examples of non-noetherian $1\frac{1}{2}$ generator Prüfer domains with arbitrary class group and radical $\neq 0$.

Unfortunately, we have nothing to say about whether there is a Prüfer domain with a finitely generated ideal requiring 3 or more generators. However, in §3 we produce a Prüfer domain with an ideal that requires 2 *honest generators,* that is, 2 generators will do, but $1\frac{1}{2}$ will not. Here the Prüfer domains are obtained as intersections of discrete rank 2 valuation rings.

In §4, we extend the Steinitz-Kaplansky structure theorems for direct sums of ideals in $1\frac{1}{2}$ generator Prüfer domains of radical zero to those of arbitrary radical.

In §5, we slightly sharpen a result of W. Vasconcelos and J. Sally by showing that every Prüfer domain of Krull dimension one has the $1\frac{1}{2}$ generator property; and having radical $\neq 0$ forces all finitely generated ideals to be principal.

1. **Radical Zero: Non-standard Prüfer Domains.** By the *class group* $\mathcal{C}(R)$ of a Prüfer domain R with quotient field Q, we mean the group of R-isomorphism classes of *fractional ideals* (i.e., nonzero finitely generated R-submodules of Q), multiplication being given by (class A) $\cdot$ (class B) = class AB. This multiplication is well-defined because any

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R-isomorphism of A onto another R-submodule of Q is multiplication by some element of Q [**6**, p. 145, Lemma 22.1].

NOTATION. Let $\{R_i : i \in I\}$ be an infinite family of integral domains R_i (we allow $R_i = R_j$ for $i \neq j$) with quotient field Q_i; and let $\mathcal{G}$ be a *non-principal ultra-filter* of subsets of I, that is: $\mathcal{G}$ is a collection of subsets of I such that (i) If J and K belong to $\mathcal{G}$, so does $J \cap K$; (ii) If $J \in \mathcal{G}$ and $J \subset K \subseteq I$ then $K \in \mathcal{G}$; (iii) $\mathcal{G}$ contains the complement of every finite subset of I (in particular $I \in \mathcal{G}$); and (iv) For every subset J of I exactly one of J and $I - J$ belongs to $\mathcal{G}$ (in particular, the empty set does not belong to $\mathcal{G}$). Elements of the complete direct product $\prod_{i \in I} R_i$ will be denoted by $x = \{x_{(i)}\}_{i \in I}$ or, when subscripts are necessary, $x_k = \{x_{k,(i)}\}_{i \in I}$. Two elements x and y of $\prod R_i$ will be *equivalent* (notation: $x^* = y^*$) if the set of indices $\{i : x_{(i)} = y_{(i)}\}$ belongs to $\mathcal{G}$. The set of these equivalence classes will be written $R^* = \prod R_i/\mathcal{G}$, the *ultra-product* of $\{R_i\}$ determined by $\mathcal{G}$.

We will use the facts that such an $\mathcal{G}$ exists for every infinite set I [9] and that R^* is an integral domain (with coordinate-wise addition and multiplication) whose quotient field is $Q^* = \prod Q_i/\mathcal{G}$ (That R^* is a ring is proved in [9]; that it is a domain with quotient field Q^* can be easily proved from the lemma below, using methods similar to those of Proposition 1.3).

LEMMA 1.1. *Let $x^*, y_1^*, \cdots, y_n^*$ be elements of Q^*. Then*

(i) $x^* \in \sum_{k=1}^{n} R^* y_k^*$ $\Leftrightarrow$ *The set of coordinate indices i such that $x_{(i)} \in \sum_{k=1}^{n} R_i y_{k,(i)}$ belongs to $\mathcal{G}$.*

(ii) $x^* \notin \sum_{k=1}^{n} R^* y_k^*$ $\Leftrightarrow$ *The set of coordinate indices i such that $x_{(i)} \notin \sum_{k=1}^{n} R_i y_{k,(i)}$ belongs to $\mathcal{G}$.*

(iii) *If the left side of* (i) (*respectively* (ii)) *holds then the preimage x of x^* can be chosen so that $x_{(i)} \in$ (respectively $\notin$) $\sum_{k=1}^{n} R_i y_{k,(i)}$ for every coordinate index i. (In particular: nonzero elements can be written with all coordinates nonzero.)*

PROOF. (i) follows from the equivalence relation defining R^*, and (ii) follows from (i) and property (iv) in the definition of $\mathcal{G}$. For (iii) choose any preimage x of x^*. Then by (i) or (ii) there is a $J \in \mathcal{G}$ in which the desired condition holds. If we now change the remaining coordinates of x *arbitrarily*, the set of coordinates where the new x equals the old x will contain J, and hence we will not change x^*. Hence the desired conditions can be satisfied.

LEMMA 1.2. *For an element x^* of R^*,*

$$0 \neq x^* \in \operatorname{rad} R^* \Leftrightarrow \{i : 0 \neq x_i \in \operatorname{rad} R_i\} \in \mathcal{G}.$$

PROOF. First recall that $x^* \in \operatorname{rad} R^*$ if and only if

$$(\forall r^* \in R^*)(\exists y^* \in R^*) y^*(1 - r^*x^*) = 1$$

[**1**, Theorem 1, §6, no. 3]. The proof is now completed by a coordinate argument.

PROPOSITION 1.3. *Let R^* be as above. Then*

(i) *If each R_i is a Prüfer domain, so is R^*.*

(ii) *If each* $\operatorname{rad} R_i = 0$, *then* $\operatorname{rad} R^* = 0$; *but if each* $\operatorname{rad} R_i$ *is nonzero, then* $\operatorname{rad} R^* \neq 0$.

(iii) *If each R_i has the $1\frac{1}{2}$ generator property, so has R^*.*

(iv) *If $\mathcal{F}$ contains a countably infinite subset of I (in particular, if I itself is countable), and no R_i is a field, then R^* is nonnoetherian.*

PROOF. For (i) we want to show that every finitely generated nonzero ideal of R^* is invertible; that is, for every finite set of nonzero elements $y_1^*, \cdots, y_n^*$ of R^*, there exist elements x_k^* of Q^* such that

$$\sum_{k=1}^{n} x_k^* y_k^* = 1 \text{ and every } x_j^* y_k^* \in R^* \tag{1}$$

(for then $(\sum R^* x_k^*)(\sum R^* y_k^*) = R^*$). By (iii) of Lemma 1.1 we can suppose that every $y_{k,(i)}$ is nonzero. Since each R_i is a Prüfer domain, there exist elements $x_{k,(i)}$ of Q_i satisfying $\sum_{k=1}^{n} x_{k,(i)} y_{k,(i)} = 1_i$ and $x_{j,(i)} y_{k,(i)} \in R_i$. The elements $x_k^* = \{x_{k,(i)}\}_{i \in I}$ then satisfy (1).

(ii) follows immediately from Lemma 1.2.

To obtain (iii) let nonzero elements $y_1^*, \cdots, y_n^*$ of R^* be given and choose

$$x^* \in \sum_{k=1}^{n} R^* y_k^* - \sum_{k=1}^{n} (\operatorname{rad} R^*) y_k^*.$$

We want

$$z^* \in \sum R^* y_k^* \text{ such that each } y_k^* \in R^* x^* + R^* z^*. \tag{2}$$

By (iii) of Lemma 1.1 we can suppose that every $y^*_{k,(i)}$ is nonzero. Then note that

$$I_1 = \{i : x_{(i)} \notin \sum (\operatorname{rad} R_i) y_{k,(i)}\}$$

belongs to $\mathcal{F}$. For otherwise $I - I_1 \in \mathcal{F}$; and Lemma 1.2 would then yield the contradiction $x^* \in \sum (\operatorname{rad} R^*) y_k^*$. It now follows that

$$I_2 = I_1 \cap \{i : x_{(i)} \in \sum R_i y_{k,(i)}\}$$

belongs to $\mathcal{F}$. We can now satisfy (2) in every coordinate $i \in I_2$, and hence in R^*.

Finally, to obtain (iv), first take I to be the natural numbers and let $x = (x_{(1)}, x_{(2)}, \cdots)$ where each $x_{(i)}$ is a nonzero nonunit of R_i. Let $y = (x_{(1)}, x_{(2)}^2, x_{(3)}^3, \cdots)$ and note that for every positive integer n, $y^*/(x^*)^n \in R^*$, for the set of coordinate indices i for which $y_{(i)}/x_{(i)}^n \in R_i$ is the complement of a finite subset of I, hence belongs to $\mathcal{F}$. Thus

$$R^*\frac{y^*}{1} \subseteq R^*\frac{y^*}{x^*} \subseteq R^*\frac{y^*}{(x^*)^2} \subseteq \cdots \subseteq R^*. \tag{3}$$

In fact, all the inclusions are strict, for if

$$\frac{y^*}{(x^*)^n} = z^*\frac{y^*}{(x^*)^{n-1}}, \qquad (z^* \in R^*),$$

then $1 = x^*z^*$, and hence the sets of indices i for which $1_{(i)} = x_{(i)}z_{(i)}$ belongs to $\mathcal{F}$. But since there are no such indices and the empty set does not belong to $\mathcal{F}$, we have a contradiction. Thus the inclusions are strict; and hence R^* is not noetherian.

For the general case we may take the natural numbers N to be the given countably infinite subset of I which belongs to $\mathcal{F}$. Define $x_{(i)}$ and $y_{(i)}$ as above when $i \in N$, otherwise let $x_{(i)} = y_{(i)} = 1_i$. Then the set $\{i : y_{(i)}/x_{(i)}^n \in R_i\}$ is still the complement of a finite set and hence belongs to $\mathcal{F}$. Thus the inclusions (3) hold. For strictness of these inequalities, note that the set of indices i for which $x_{(i)}$ is invertible in R_i is $I - N \notin \mathcal{F}$. Thus $1 = x^*z^*$ again yields a contradiction.

If all of the given domains R_i are equal to a single domain R, then R^* becomes the *ultra-power* $R^{\mathcal{F}}$. In this case, the "diagonal map" $\Delta : R \to R^*$ given by $r \to \{r_{(i)} : r_{(i)} = r\}^*_{i \in I}$ is a ring monomorphism (It is clearly a ring homomorphism; and $r \neq 0 \Rightarrow$ no coordinate of $\Delta(r)$ is zero), and the same is true of the corresponding map $\Delta : Q \to Q^*$. With this notation we have:

Proposition 1.4. *Let R be a Prüfer domain. Then in the Prüfer domain $R^* = R^{\mathcal{F}}$,*

(i) *$\mathcal{C}(R)$ is monomorphically imbedded in $\mathcal{C}(R^*)$ via the map induced by Δ.*
(ii) *If $\mathcal{C}(R)$ in finite, then the map in* (i) *is an isomorphism.*

Proof. By Proposition 1.3, R^* is a Prüfer domain. Recall that $\mathcal{C}(R) = \mathcal{F}(R)/\mathcal{P}(R)$ where $\mathcal{F}(R)$ is the group of fractional ideals of R (= nonzero, finitely generated R-submodules of Q), and $\mathcal{P}(R)$ is the subgroup of principal fractional ideals Rq ($0 \neq q \in Q$).

For $A \in \mathcal{I}(R)$ let $A^* = R^* \Delta(A)$, the "extension" of A to a fractional ideal of R^*. Since Δ takes each set of generators of A to a set of generators of A^*, the map $A \to A^*$ is a homomorphism: $\mathcal{I}(R) \to \mathcal{I}(R^*)$ which takes $\mathcal{P}(R) \to \mathcal{P}(R^*)$; and hence induces a homomorphism $\delta : \mathcal{C}(R) \to \mathcal{C}(R^*)$.

To see that δ is a monomorphism, suppose $A^* = R^*q^*$ with $0 \neq q^* \in Q^*$. Choose a set of generators $x_1, x_2, \cdots, x_n$ of A, so that $x_1^*, x_2^*, \cdots, x_n^*$ $(x_k^* = \Delta(x_k))$ generate A^*. Then

$$q^* \in \sum_{k=1}^{n} R^*x_k^*, \text{ and each } x_k^* \in R^*q^*. \tag{4}$$

Since $\mathcal{F}$ is closed under finite intersections, the set of coordinates in which (4) holds is not empty (by Lemma 1.1). Considering such a coordinate i, we see that the fractional ideal generated by $x_1, \cdots, x_n$ (namely A) is principal (generated by $q_{(i)}$), and this shows δ to be a monomorphism.

Now suppose $\mathcal{C}(R)$ is finite; let $A, B, C, \cdots$ be fractional ideals, one from each element of $\mathcal{C}(R)$; and let $X^* = \sum_{k=1}^{n} R^*x_k^*$ be a fractional R^*-ideal in Q^*. We show X^* is isomorphic to one of $A^*, B^*, \cdots$, thereby obtaining (ii). We can assume by Lemma 1.1 that every $x_{k,(i)}$ is nonzero, so that $X_i = \sum_{k=1}^{n} R_i x_{k,(i)}$ is nonzero, and hence, is a fractional R_i-ideal in Q_i.

Let $I_A = \{i \in I : X_i \cong A\}$ and define $I_B, I_C, \cdots$ similarly. Suppose (by way of contradiction) that none of $I_A, I_B, \cdots$ belong to $\mathcal{F}$. Then all of their complements belong to $\mathcal{F}$, and, hence, so does the intersection of these complements ($\mathcal{F}$ is closed under *finite* intersections). In particular, this intersection is nonempty. Thus there is an index i for which X_i is isomorphic to none of $A, B, \cdots$, contrary to our choice of $A, B, \cdots$.

Thus (say), $I_A \in \mathcal{F}$. For each $i \in I_A$, there is, therefore, an element $0 \neq q_i \in Q$ such that $q_iA = X_i$. Define $q^* \in Q$ by $q_{(i)} = q_i$ for $i \in I_A$ and $q_{(i)} = 1$ otherwise. Then $q^*A^* = X^*$ (it is sufficient to check this on a pair of finite generating sets of A^* and X^*), and hence $A^* \cong X^*$ as desired.

A theorem of Luther Claborn [4] states that every abelian group is the class group of some Dedekind domain with radical 0. Thus Proposition 1.3 and 1.4, together with the fact that every Dedekind domain has the $1\frac{1}{2}$ generator property [13, p. 278, Cor. 1] implies our first main result:

THEOREM 1.5. *Every finite abelian group is isomorphic to (and every infinite group is isomorphically contained in) the class group of a non-noetherian $1\frac{1}{2}$ generator Prüfer domain with radical zero.*

2. **Nonzero Radical.** In this section we describe a family of $1\frac{1}{2}$ generator Prüfer domains with radical $\neq 0$. These domains are described in detail in [8, §3]. We summarize just enough to establish the $1\frac{1}{2}$ generator property.

Let V be a valuation ring with maximal ideal M, and suppose V contains a subfield K such that $V = K + M$ (direct sum of additive groups). For example, let V be the power series ring $K[[x]]$. Then let J be an integral domain whose quotient field is K, and set $R = J + M$. Gilmer and Heinzer prove:

THEOREM [8, §3]. *If J is a Prüfer domain, then R is a Prüfer domain in which*

(i) *Each ideal either $\supseteq M$ or $\subseteq M$.*

(ii) *Each ideal A of R which $\supseteq M$ has the form $A = L + M$ where L is an ideal of J (namely, the projection of A in J).*

(iii) *Each finitely generated ideal A of R which is $\subseteq M$ has the form $RH^{-1}\lambda$ for some finitely generated ideal H of J and some element $\lambda \in A$.* (p. 148, item (i)).

(iv) *Each finitely generated ideal $A \neq 0$ of R which is $\subseteq M$ is isomorphic, as an R-module, to one which $\supset M$.* (Keep the notation of (iii) and choose $0 \neq h \in H$. Then $A \cong R(H^{-1}h)$ = an ideal of R which $\not\subseteq M$, hence which $\supset M$).

From the above we obtain our desired examples:

COROLLARY 2.1.

(v) $\operatorname{rad} R = (\operatorname{rad} J) + M \neq 0$; *and*

(vi) *If J is a Dedekind domain (or any other $1\frac{1}{2}$ generator Prüfer domain), R will be a $1\frac{1}{2}$ generator Prüfer domain.*

PROOF. (v) follows immediately from (i) and (ii), and so does

(vii) $R(j + m) = Rj \quad (0 \neq j \in J, m \in M)$.

Let A be a finitely generated ideal $\neq 0$ of R. To establish the $1\frac{1}{2}$ generator property for A, (iv) allows us to suppose that $A \supset M$. Hence, by (ii), $A = L + M$ with L a finitely generated ideal $\neq 0$ of J. Moreover,

(viii) $(\operatorname{rad} R)A = (\operatorname{rad} J)L + M$.

When $\operatorname{rad} J \neq 0$, this follows immediately from (v), (i), and (ii). When $\operatorname{rad} J = 0$, so that $\operatorname{rad} R = M$,

$$(\operatorname{rad} R)A = M(L + M) = (MK)(L + M) = MK = M \ ,$$

so (viii) still holds.

Thus, by (vii) and (viii), A (considered as an ideal of R) inherits the $1\frac{1}{2}$ generator property from L (considered as an ideal of J).

3. **Two Honest Generators.** We begin by constructing rank two valuation rings whose value group Γ is precisely the lexicographic product $\mathbf{Z} \times \mathbf{Z}$. Take a discrete rank one valuation v on a field K, and regard the value group $\mathbf{Z}$ as the isolated subgroup $(0, \mathbf{Z})$ of Γ. Let C be the element $(1,0)$, so $\Gamma = \mathbf{Z} + C\mathbf{Z}$ [and $s + Ct > 0 \Leftrightarrow t > 0$, or $t = 0$ and $s > 0$].

Then, by [2, p. 488], there is a unique valuation w definable on the transcendental extension $K(x)$ which extends v and also gives $w(x) = C$. Moreover, for $a_j \in K$, $w(\sum_j a_j x^j) = \inf_j(v(a_j) + jC) = v(a_j) + jC$ for the least j such that $a_j \neq 0$.

Of course the same technique is equally applicable for giving any transcendental generator of $K(x)$ the value of C. In one case we will set $w(x + 1) = C$.

Construction. Suppose D is a Dedekind domain, with quotient field K, and which has an infinite set of primes $\{P_i\}$ with corresponding P_i-adic valuations $\{v_i\}$. Extend v_i to $K(x)$ by the above procedure to obtain

$$w_1(x + 1) = C$$
$$w_i(x) = C \text{ if } i \neq 1.$$

Each valuation w_i induces a valuation ring W_i. We set $R = \bigcap_i W_i$.

It is beneficial to observe, at this point, that any nonzero element $a \in K(x)$ can be expressed as follows:

$$(*) \qquad a = x^m \frac{xf(x) + d_1}{xg(x) + d_2}, \quad \text{where } m \in \mathbf{Z} \text{ and } d_1, d_2 \in D - \{0\},$$

and so

$$(**) \qquad w_i(a) = v_i(d_1) - v_i(d_2) + mC, \text{ for } i \neq 1.$$

Furthermore, $a \in R$ implies $m \geqq 0$.

Proposition 3.1. *R is a Prüfer domain.*

Proof. We let M_i denote the center of the valuation w_i on R. Then it will suffice to show (See [7, p. 254, (18.1)].)

(A) Every ideal is contained in some M_i; and

(B) $R_{M_i} = W_i$.

To see that (A) holds, consider an arbitrary ideal I. If I were not contained in any M_i, then it would also not be contained in the union of two of them. Hence, $I \not\subseteq M_1 \cup M_k$ where $k \neq 1$ is arbitrary. So pick $a \in I - (M_1 \cup M_k)$. Referring to $(*)$ and $(**)$, noting that $w_k(a) = 0$, we see $m = 0$, and so $w_i(a) = v_i(d_1/d_2)$ for $i \neq 1$. This means $w_i(a)$ is an integer for all i and is nonzero only finitely often. Now by the Chinese Remainder Theorem, there is an element y of D satisfying $v_1(y) = v_k(y) = 0$ and $v_i(y) \geqq w_i(a)$ for every i. (There is no special difficulty when $i = 1$, because $w_1(a) = 0$.) Observe, by considering values, $(y/a) \in R$; so $y \in aR \subseteq I$ and $y \in I \cap D$. As $y \notin P_1 \cup P_k$, we conclude $I \cap D \nsubseteq P_1 \cup P_k$. But the choice of k was arbitrary and so $(I \cap D)$ is not contained in any prime of D. So $I \cap D = D$ and $I = R$.

Part (B) will be done in two cases.

Case 1 ($i = 1$). Consider $a \in W_1$, written in the form $(*)$. Checking values shows that $x^{1-m}a \in R$. Also $x^{m-1} \in R_{M_1}$ since $w_1(x) = 0$. Hence $a = (x^{1-m}a)x^{m-1} \in R_{M_1}$.

Case 2 ($i \neq 1$). Consider $a \in W_i$. By referring to $(*)$ and $(**)$, we see that, if $m = 0$, $w_k(a) = v_k(d_1/d_2)$ for $k \neq 1$. $w_i(a) \geqq 0$ implies $(d_1/d_2) \in D_{P_i} \subseteq R_{M_i}$. With at most one nonzero value, $a/(d_1/d_2)$ must either be in R or the inverse of an element outside M_i. Either way, it lies in R_{M_i}, and therefore so does a.

Suppose, on the other hand, that $m > 0$, and let $w_1(a) = s + Cn$. Then checking values shows that $(x + 1)^{1-n}a \in R$ and $x + 1 \notin M_i$; consequently $a \in R_{M_i}$.

The reverse inclusions $(R_{M_i} \subseteq W_i)$ are obvious.

LEMMA 3.2. *Let A and B be finitely generated R-submodules $\neq 0$ of $K(x)$.*

(i) *The minimum $w_i(A) = \min\{w_i(a) \mid 0 \neq a \in A\}$ exists and equals the minimum of $\{w_i(a_1), \cdots, w_i(a_n)\}$ taken over any finite generating set $\{a_i\}$ of A.*

(ii) *If $w_i(A) = w_i(B)$ for all i, then $A = B$.*

(iii) *Suppose P_1 is principal, say $P_1 = Dp$. If $0 \neq a \in K(x)$ and $w_i(a) \in \mathbf{Z}$ for all i, then $(\exists k \in K)(\forall i) w_i(a) = v_i(k)$.*

PROOF. (i) is clear. Since $R = \bigcap W_i$, the inverse of an ideal $A = \{c \in K(x) \mid w_i(c) \geqq -w_i(A)\}$; so the hypothesis of (ii) means $A^{-1} = B^{-1}$. Since R is Prüfer, we get $A = B$.

To obtain (iii), write a in the form $(*)$. Then $w_i(a) \in \mathbf{Z}$ implies $m = 0$. Since $w_1(a) \in \mathbf{Z}$, there will be an integer s such that $w_i(a) = v_i(d_1p^s/d_2)$ for all i, as desired. This uses the fact that $v_i(p) = 0$ for all $i > 1$.

THEOREM 3.3. *If we choose D so that P_1 is a principal ideal and P_2 is not, then R will not satisfy the $1\frac{1}{2}$ generator property. Specifically, the finitely generated ideal M_2 cannot be generated by two elements if one of them is x, although $x \notin (\mathrm{rad}\, R)M_2$.*

PROOF. As $M_2 = RP_2$ (by (ii) of the Lemma), M_2 is finitely generated. Since $x \notin M_1$, it is not in (rad $R)M_2$. Finally, suppose M_2 were generated by x and one more element $a \neq 0$. By (i) of the Lemma, applied to the generating set $\{x, a\}$, we would have

$$w_i(a) = \begin{cases} 1, & \text{if } i = 2\ , \\ 0, & \text{if } i > 2. \end{cases} \tag{1}$$

By hypothesis $P_1 = Dp$ for some p. Let $w_1(a) = s + Ct$. Then $a/p^s(x + 1)^t$ attains all the minimum values of M_2 and so generates it. By (iii) of the Lemma, we find a $k \in K$ with the same values. So $P_2 = Dk$, contradicting the hypothesis.

REMARKS 3.4. (i) In keeping with the spirit of the definition of 1½ generators, we observe that x can be a useful member of *some* finite generating set of M_2, that is $x \notin \text{rad}\, M_2 =$ the intersection of all maximal submodules of M_2. In fact, x doesn't belong to the maximal submodule $M_1 \cap M_2$. Alternatively, let α and β generate the ideal P_2 of D. Then

$$M_2 = RP_2 = R\alpha + R\beta.$$

By checking values, as in (3.3), one can verify that M_2 is generated by $\alpha(x + 1), \beta(x + 1)$, and x; and that x cannot be omitted.

(ii) The ring R in Theorem 3.3 *can have any nonzero abelian group for its class group.* This follows from results of L. Claborn that there exist Dedekind domains with arbitrary class group [4, Theorem 7] and a prime ideal in every class [5, combine 2-1, 2-3, and 2-5], and from the following result which we will need again in §4.

PROPOSITION 3.5. (i) *Every finitely generated ideal $\neq 0$ of R has the form*

$$A = Rx^m(x + 1)^n H\ , \tag{1}$$

for some unique fractional ideal H of D and integers m and $n \geqq 0$.
(ii) *If P_1 is a principal ideal of D, then $\mathcal{C}(R) \cong \mathcal{C}(D)$.*

PROOF. (i) Choose a finite set of generators $a_1, \cdots, a_n$ of A, and write each a_j in the form ($*$). Letting m be the smallest m_j which occurs, we see from ($**$) and (i) of Lemma 3.2, that there exist integers s_i such that

$$w_i(A) = s_i + Cm, \qquad \text{for each } i \neq 1, \tag{2}$$

and that all but finitely many of the integers s_i are zero. (But some of them might be negative if m is strictly positive.) Let $w_1(A) = s_1 + Cn$, and recall that $w_1(x) = 0 = w_i(x + 1)$ for all $i \neq 1$. Therefore,

$$w_i(A) = w_i\left(Rx^m(x+1)^n \prod_{k=1}^{\infty} P_k^{s_k} \right), \quad \text{for all } i,$$

so (1) holds with $H = \prod P_k^{s_k}$.

To obtain (ii), let $P_1 = Dp$. We can define a homomorphism φ of $\mathcal{C}(D)$ into $\mathcal{C}(R)$ by φ (class H) = class RH, and (i) shows that φ is onto. To see that φ is 1-1, it suffices to show that if an *integral* ideal A of R is principal, then so is the fractional D-ideal H in (i). Since A is principal, so is RH, say $RH = Ra$ for some $0 \neq a \in K(x)$. Then $w_i(a) = w_i(H) \in \mathbf{Z}$ for every i (by (i) of the Lemma), so there is a $k \in K$ such that $w_i(k) = w_i(H)$ for all i (by (iii) of the Lemma). Therefore, $H = Dk$ = principal, as desired.

4. **Direct Sums of Ideals.** If $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$ are ideals in any integral domain R with quotient field F, then [11, Lemma 1],

$$(*) \quad A_1 \oplus \cdots \oplus A_n \cong B_1 \oplus \cdots \oplus B_n \Rightarrow (\exists \alpha \neq 0 \text{ in } F)\, \alpha A_1 A_2 \cdots A_n = B_1 B_2 \cdots B_n$$

(isomorphism of R-modules). It is also proved in [11, Theorem 2] that the converse of $(*)$ holds for finitely generated ideals A_i and B_i in a $1\frac{1}{2}$ generator Prüfer domain with radical zero. To remove this restriction on the radical it obviously suffices to prove

THEOREM 4.1. *For finitely generated nonzero ideals A and B in a $1\frac{1}{2}$ generator Prüfer domain R,*

$$A \oplus B \cong R \oplus AB.$$

NOTATION. In what follows, A and B will denote finitely generated nonzero ideals in a Prüfer domain R with quotient field F.

LEMMA 4.2. *The following are equivalent for A and B.*
(i) $A \oplus B \cong R \oplus AB$.
(ii) $\exists A' \cong A$ *and* $B' \cong B$ *such that* $A' + B' = R$.

NOTE. The sum $A' + B'$ is the sum in R, while $A \oplus B$ is an external direct sum.

PROOF. (i) $\Rightarrow$ (ii). Let A' and B' be the images of A and B respectively in R under $A \oplus B \twoheadrightarrow R \oplus AB \xrightarrow{\text{proj}} R$. Then $A' + B' = R$. Since every R-module homomorphism: $A \to R$ equals multiplication by some element of F [3, p. 132], we are done if both A' and B' are nonzero. If, say, $B' = 0$, then $A' = R$; so $A \cong R$. Now observe that $R + B = R$.

(ii) $\Rightarrow$ (i). First recall that in any Prüfer domain,

$$(*) \qquad A \oplus B \cong (A + B) \oplus (A \cap B),$$

which can be seen by mapping $A \oplus B$ onto $A + B$ by $f(a, b) = a - b$. Since R is Prüfer, $A + B$ is a projective R-module, so $A \oplus B \cong (A + B) \oplus \ker f$; and clearly $\ker f \cong A \cap B$.

By (ii) we can suppose $A + B = R$, whence $A \cap B = AB$. Thus, $(*)$ implies $A \oplus B \cong R \oplus AB$.

Proof (of Theorem 4.1). Suppose first that $A \not\subseteq \operatorname{rad} R$. By the Lemma, it suffices to find $\beta \neq 0$ in F such that $A + \beta B = R$; that is, such that

$$(1) \qquad AB^{-1} + \beta R = B^{-1}.$$

Note that B^{-1} has the 1½ generator property (because, for any $0 \neq b \in B$, $B^{-1} \cong B^{-1}b \subseteq R$). Since $A \not\subseteq \operatorname{rad} R$, we conclude from invertibility of B that $AB^{-1} \not\subseteq (\operatorname{rad} R)B^{-1}$. If, therefore, we take any $\alpha \in AB^{-1} - (\operatorname{rad} R)B^{-1}$, the 1½ generator property for B^{-1} gives us β such that $R\alpha + R\beta = B^{-1}$. Hence (1) also holds.

The problem is thus reduced to finding $A' \cong A$ such that $A' \not\subseteq \operatorname{rad} R$. By invertibility of A, there exist elements β_i in A^{-1} such that $\sum A\beta_i = R$. At least one of the terms $A\beta_i$ must be $\not\subseteq \operatorname{rad} R$, and hence will do for A'.

We close this section with an amusing proof which extends another result from [11, Theorem 2] to the case of arbitrary radical.

Theorem 4.3. *Let R be any integral domain in which $A \oplus B \cong R \oplus AB$ for all invertible A and B (for example any 1½ generator Prüfer domain). The direct sum of infinitely many nonzero invertible ideals of R must be a free R-module.*

Proof. It suffices to consider direct sums of countably many ideals. We employ the hypothesis:

$$\begin{aligned}
& A \oplus B \oplus C \oplus D \oplus E \oplus F \oplus \cdots \\
&\cong (R \oplus AB) \oplus (A^{-1}B^{-1} \oplus ABCD) \oplus (A^{-1}B^{-1}C^{-1}D^{-1} \oplus ABCDEF) \oplus \cdots \\
&\cong R \oplus (\ \oplus\ ) \oplus (\ \oplus\ ) \oplus \cdots \\
&\cong R \oplus (R \oplus R) \oplus (R \oplus R) \oplus \cdots
\end{aligned}$$

Example 4.4. Prüfer domains without the 1½ generator property can also satisfy the conclusions of Theorems 4.1 and 4.3. For example, choose D and R as in Theorem 3.3, and let A and B be finitely generated nonzero ideals of R. By Proposition 3.5, $A \cong RH$ and $B \cong RK$ for fractional ideals H and K of the Dedekind domain D. (In fact we can take H and $K \subseteq D$). Then $H \oplus K \cong D \oplus HK$ as D-modules and therefore $A \oplus B \cong R \oplus AB$ as R-modules.

5. **Krull Dimension One.** Here we obtain the following slight improvement of [**12**, Cor. 4.3].

THEOREM. *In a Prüfer domain R of Krull dimension* 1, *every finitely generated ideal can be generated by* $1\frac{1}{2}$ *elements. If* rad $R \neq 0$, *then every finitely generated ideal is principal.*

PROOF. Let b be any nonzero element of the given finitely generated ideal I. We want to show that the R-module I/Rb is cyclic. (If rad $R \neq 0$ we choose $b \in (\text{rad } R)I$ and conclude, by Nakayama's Lemma, that I is principal.)

Let J/Rb be the Jacobson radical of I/Rb. It will suffice to show that I/J (which has radical zero) is cyclic.

Since I is invertible, the submodules between I and J are in one-to-one inclusion-preserving correspondence with those between R and $I^{-1}J$. Therefore the ring $R/I^{-1}J$ has Jacobson radical zero, and, hence, has nilradical zero, too. But a commutative ring without nilpotent elements and of Krull dimension zero is known to be *von Neumann regular*. (For instance, one can check that every localization at a maximal ideal has Krull dimension zero and no nilpotent elements; hence is a field.)

Now $I/J = I/I(I^{-1}J)$ is a projective module over $R/I^{-1}J$ which is regular, hence semi-hereditary, so I/J is a direct sum of cyclic modules [3, p. 14]. Also, since I invertible, it is projective *of rank* 1 as an R-module [1, p. 117], and therefore I/J is projective of rank 1 as an $R/I^{-1}J$-module.

Finally, over any commutative ring, a projective module of rank 1 which is a direct sum of cyclics is itself cyclic. (If $Rm_1 \oplus \cdots \oplus Rm_t$ is projective of rank 1, then it is easily checked, locally, that it equals $R(m_1 + \cdots + m_t)$.)

ADDED IN PROOF. Heitmann has recently shown that, in Prüfer domains of Krull dimension n, each finitely generated ideal can be generated by $n + 1$ elements, the first of which can be selected arbitrarily (but $\neq 0$). [To appear.]

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