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ISOMORPHISMS BETWEEN CURVE GRAPHS OF INFINITE-TYPE SURFACES ARE GEOMETRIC

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This work is dedicated to Marion and Anika Yunuén

ABSTRACT. Let $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ be a simplicial isomorphism between curve graphs of infinite-type surfaces. In this paper, we show that, in this situation, S and S' are homeomorphic and ϕ is induced by a homeomorphism $h : S \to S'$.

1. Introduction. This is the last of three papers on the study of the natural action of the extended mapping class group $\mathrm{Mod}^*(S)$ of an infinite-type (its fundamental group is not finitely generated) surface S on the curve graph $\mathcal{C}(S)$ and whether any isomorphism between curve graphs actually comes from a homeomorphism, see [**5, 6**]. Our main result is the following.

Theorem 1.1. *Let S and S' be infinite-type connected orientable surfaces with empty boundary and $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ a simplicial isomorphism. Then, S is homeomorphic to S', and ϕ is induced by a homeomorphism $h : S \to S'$.*

As an immediate consequence of this result and [**5,** Corollary 1.2], we obtain an analog for infinite-type surfaces of a foundational well-known result by Ivanov (see [**8,** Theorem 1]):

Theorem 1.2. *Let S be an infinite-type connected orientable surface with empty boundary. Then, every automorphism of the curve graph*

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$\mathcal{C}(S)$ is induced by a homeomorphism. More precisely, the natural map:

$$\Psi : \mathrm{Mod}^*(S) \longrightarrow \mathrm{Aut}(\mathcal{C}(S))$$

is an isomorphism.

It is important to remark that both of these results were known to be true for infinite-type surfaces for which all topological ends carry (infinite) genus [**6**]. With this in mind, we highlight the *main contribution* of this text: a new and very simple proof of the fact that every automorphism of the curve graph of an infinite-type surface is geometric. The technology we present is based on *principal exhaustions*, introduced in [**5**]. Roughly speaking, these are nested sequences of finite-type subsurfaces whose union is S, which allow us to "pull" classical results about simplicial actions of mapping class groups to the realm of infinite-type surfaces. In particular, the proof that we present of Theorem 1.1 makes no use of Dehn-Thurston coordinates and is, hence, completely independent from (and simpler than) the proofs presented in [**6**].

Remark 1.3. While this paper was under revision, we learned that Bavard, Dowdall and Rafi proved Theorem 1.1 using different methods and that they used this result to obtain algebraic rigidity of big mapping class groups. For more details, see [**3**].

Applications 1.4. Given that the curve graph of an infinite-type surface has diameter 2, the natural action of $\mathrm{Mod}^*(S)$ on $\mathcal{C}(S)$ gives no large-scale information. However, the results that we present in this text have found the following non-trivial applications.

(1) Let S be an infinite-type surface whose genus is finite and at least 4, e.g., a closed surface of genus 5 to which we have removed a Cantor set, and denote by $\mathrm{PMod}(S)$ the subgroup of $\mathrm{Mod}(S)$ consisting of all orientation preserving mapping classes acting trivially on the topological ends of S. This group is called the *pure mapping class group* of S. There is a natural monomorphism from $\mathrm{Mod}^*(S)$ to $\mathrm{Aut}(\mathrm{PMod}(S))$ given, by conjugation, by the action of $\mathrm{Mod}^*(S)$ on the pure mapping class group. The following result, obtained by Patel and Vlamis [**9**], is a generalization of a famous result of Ivanov [**7**] for mapping class groups of surfaces of finite type.

Theorem 1.5. *If S is an infinite-type surface of genus $4 \leq g < \infty$, then the natural monomorphism from* $\mathrm{Mod}^*(S)$ *modulo its center to* $\mathrm{Aut}(\mathrm{PMod}(S))$ *is an isomorphism.*

Indeed, Patel and Vlamis show that any automorphism of $\mathrm{PMod}(S)$ preserves Dehn twists. As they noted, if every automorphism of $\mathcal{C}(S)$ is induced by a homeomorphism of the surface, then Theorem 1.5 follows by a standard argument that can be found in Ivanov's original paper [**7**].

(2) Let Σ_g be the surface that results from removing a Cantor set from a closed, orientable surface of genus $g \geq 0$. In [**1**], Aramayona and Funar studied $\mathcal{B}_g$, the *asymptotically rigid mapping class group* of Σ_g. This is a finitely presented subgroup of $\mathrm{Mod}^*(\Sigma_g)$ which contains the mapping class group of every surface of genus g with nonempty boundary. Using Theorem 1.2, they proved that $\mathcal{B}_g$ is *rigid*, that is:

Theorem 1.6. *For every $g < \infty$,* $\mathrm{Aut}(\mathcal{B}_g)$ *coincides with the normalizer of $\mathcal{B}_g$ within* $\mathrm{Mod}^*(\Sigma_g)$*.*

More precisely, their result used the next lemma, proven using Theorem 1.2.

Lemma 1.7. *If $g < \infty$, then* $\mathrm{Aut}(\mathrm{PMod_c}(\Sigma_g)) = \mathrm{Mod}^*(\Sigma_g)$*.*

Here, $\mathrm{PMod}_c(\Sigma_g)$ denotes the subgroup of $\mathrm{Mod}^*(\Sigma_g)$ formed by *compactly supported pure* mapping classes. For more details, the reader is referred to [**1**].

(3) Let S be an infinite-type surface and $\mathrm{Homeo}(S; \partial S)$ the group of self-homeomorphisms of S that fix the boundary pointwise. Equipped with the compact-open topology, $\mathrm{Homeo}(S; \partial S)$ becomes a topological group. The topology on $\mathrm{Mod}^*(S)$ induced by the epimorphism $\mathrm{Homeo}(S; \partial S) \to \mathrm{Mod}^*(S)$ is called the *compact-open* topology, which induces a topology on the subgroup $\mathrm{PMod}(S) < \mathrm{Mod}^*(S)$ of mapping classes acting trivially on the space of topological ends of S. Using Theorem 1.2, Aramayona, Patel and Vlamis showed that $\mathrm{PMod}(S)$ is Polish, that is, it is separable and completely metrizable. For more details, see [**2**].

In light of these facts, we conjecture that further applications of Theorems 1.1 and 1.2 to the study of big mapping class groups should exist. In particular, it is natural to wonder which of the classical applications of Ivanov's theorem have an analog in the realm of infinite-type surfaces.

In Section 2, we provide a short discussion on the general aspects of pants decompositions. We also recall the notion of principal exhaustion. In Section 3, we prove several topological properties that are preserved under isomorphisms of curve graphs. Finally, in Section 4, we prove that these isomorphisms are geometric.

2. Preliminaries. The main tools that we use in this text to prove our theorem are pants decompositions and a special kind of exhaustion for infinite-type surfaces called *principal exhaustions*. In what follows, we recall the definitions of these objects and the main properties that are necessary.

Abusing language and notation, we call a *curve* a topological embedding $\mathbb{S}^1 \hookrightarrow S$, the isotopy class of this embedding and its image on S. A curve is said to be *essential* if it is neither homotopic to a point nor to the boundary of a neighborhood of a puncture. Hereafter, all curves considered are essential unless otherwise stated. For every two isotopy classes a, b of simple closed curves, we denote by $i(a, b)$ their (geometric) intersection number, that is, the minimal cardinality of $\alpha \cap \beta$, where $\alpha \in a$ and $\beta \in b$.

A collection of essential curves $\mathcal{L}$ in S is *locally finite* if, for every $x \in S$, there exists a neighborhood U_x of x which intersects finitely many elements of $\mathcal{L}$. A locally finite collection of pairwise disjoint non-isotopic essential curves is called a *multicurve*.

Definition 2.1 (The curve graph)**.** The *curve graph* of S, denoted by $\mathcal{C}(S)$, is the *abstract* simplicial graph whose vertices are isotopy classes of essential curves in S, and two vertices α and β span an edge if $i(\alpha, \beta)$=0. We denote the set of vertices of $\mathcal{C}(S)$ by $\mathcal{C}^0(S)$.

Remark 2.2. The curve graph is the 1-skeleton of the *curve complex*, that is, the abstract simplicial complex whose simplices are multicurves of finite cardinality. The curve complex is a *flag* complex, in particular, it is completely determined by its 1-skeleton and, for this reason, in this text we restrain our discussion to the curve graph.

Definition 2.3 (Pants decomposition)**.** A maximal (with respect to inclusion) multicurve is called a *pants decomposition*.

Herein, we call both a maximal multicurve $P = \{\alpha_k\}_{k\in K}$ and its image $\{[\alpha_k]\}_{k\in K}$ in $\mathcal{C}^0(S)$ a pants decomposition. The following lemma gives a simplicial characterization of pants decompositions.

Remark 2.4. Given that every pants decomposition P is locally finite, the index set K in $P = \{\alpha_k\}_{k\in K}$ is at most countable. Moreover, every connected component of $S \setminus P$ is homeomorphic to the thrice-punctured sphere. In particular, every closed subsurface Σ induced by P is homeomorphic to the compact surface of genus zero and three boundary components. This topological surface is called a *pair* of pants.

Lemma 2.5. *Let S be a surface and $P = \{a_k\}_{k\in\mathbb{N}} \subset \mathcal{C}^0(S)$. Then, $P = \{a_k\}_{k\in\mathbb{N}}$ is a pants decomposition of S if and only if this collection satisfies*:

(1) $i(a_k, a_l) = 0$ *for all* $k, l \in \mathbb{N}$;
(2) *for each* $a \in \mathcal{C}^0(S) \setminus P$ *we have*:
  (a) $i(a, a_k) \neq 0$ *for some* $k \in \mathbb{N}$; *and*
  (b) $|\{k \in \mathbb{N} : i(a, a_k) \neq 0\}| < \infty$.

Proof. The necessity condition is easily verified; thus, we prove the sufficiency. Since S is of infinite type, it is uniformized by the Poincaré disc, and hence, we choose a complete Riemannian metric on S with negative constant curvature. Since the metric is complete, we choose for each a_k the only geodesic representative α_k in its class. Conditions (1) and (2) (a) assure that $\{\alpha_k\}_{k\in\mathbb{N}}$ is a maximal collection of disjoint non-isotopic curves. In order to see that this collection is locally finite, we proceed as follows. Let $s \in S$ be a point and $N \subset S$ a compact finite-type subsurface containing s such that each connected component of ∂N is a closed geodesic which is essential in S. Suppose that there exist finitely many curves in $\{\alpha_k\}_{k\in\mathbb{N}}$ which intersect ∂N. In this case, we can easily find a neighborhood U_s of s in N which intersects only finitely many elements of $\{\alpha_k\}_{k\in\mathbb{N}}$. On the other hand, if infinitely many elements of $\{\alpha_k\}_{k\in\mathbb{N}}$ intersect ∂N, then we get a contradiction with condition (2) (b). $\square$

Corollary 2.6. *Let $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ be a simplicial isomorphism. Then, ϕ sends pants decompositions in S to pants decompositions in S'.*

Let $\mathcal{L}$ be a multicurve. We say that $\mathcal{L}$ *bounds a subsurface* Σ of S, if the elements of $\mathcal{L}$ are *exactly all* the boundary curves of the closure of Σ on S. In addition, we say that Σ is *induced* by $\mathcal{L}$ if there exists a subset $\mathcal{M} \subset \mathcal{L}$ that bounds Σ and there are no elements of $\mathcal{L} \setminus \mathcal{M}$ in its interior. In this case, $\partial\Sigma \neq \emptyset$, and this boundary is contained in $\mathcal{L}$.

Recall that an essential curve α is called *separating* if $S \setminus \alpha$ is disconnected. A separating curve α is called *outer* if it bounds a twice-punctured disc.

Let P be a pants decomposition, and let $\alpha, \beta \in P$. We say that α and β are adjacent with respect to P if there exists a subsurface Σ induced by P such that α and β are two of its boundary curves.

In all of the proofs of our main theorems, we use the following graph associated to a pants decomposition.

Definition 2.7 (Adjacency graph)**.** Let S be a surface and P a pants decomposition of S. We define the *adjacency graph* of P, denoted by $\mathcal{A}(P)$, as the abstract simplicial graph whose set of vertices is P and where two curves α and β span an edge if they are adjacent with respect to P.

Finally, we recall a particular method for exhausting infinite-type surfaces which is used to prove that every automorphism of the curve graph is geometric.

Definition 2.8 (Principal exhaustion)**.** Let $\{S_i\}_{i\in\mathbb{N}}$ be a (set-theoretical) increasing sequence of open connected subsurfaces of S. We say $\{S_i\}_{i\in\mathbb{N}}$ is a *principal exhaustion* of S if $S = \cup_{i\geq 1} S_i$ and, for all $i \geq 1$, it satisfies the following conditions:

(1) S_i is a surface of finite topological type;
(2) S_i is contained in the interior of S_{i+1};

(3) ∂S_i is the finite union of pairwise disjoint essential separating curves on S;
(4) each connected component of $S_{i+1} \setminus \overline{S_i}$ has complexity at least 4; and
(5) each connected component of $S \setminus \overline{S_i}$ is of infinite topological type.

3. Topological properties. In this section, we prove several topological properties preserved under an isomorphism $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$. All surfaces in this section are of infinite type unless otherwise stated.

The following two propositions and lemma can be deduced from the work of Shackleton [**10**]. More precisely, Propositions 3.1 and 3.4 below are [**10,** Lemmata 8, 12]; on the other hand, Lemma 3.2 below follows from [**10,** Lemmata 9, 10] and the fact that ϕ is an isomorphism.

As a matter of fact, Shackelton does not work in the context of infinite-type surfaces, but the arguments that he uses to prove these results are of a local nature, and hence, can immediately be extrapolated to all infinite-type surfaces. For the sake of completeness, we include a sketch of the proof in each case.

Proposition 3.1. *Let $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ be a simplicial isomorphism between curve graphs of infinite-type surfaces. Then, ϕ induces a graph isomorphism*

$$\widetilde{\phi} : \mathcal{A}(P) \longrightarrow \mathcal{A}(\phi(P))$$

for any pants decomposition P of S.

Sketch of proof. Since pants decompositions are maximal multicurves, $\widetilde{\phi}$ is a bijective correpondence between the set of vertices of $\mathcal{A}(P)$ and the vertices of $\mathcal{A}(\phi(P))$. Hence, we only need verify that $\widetilde{\phi}$ and $\widetilde{\phi}^{-1}$ preserve edges; however, this follows from the fact that any two vertices α and β are adjacent in $\mathcal{A}(P)$ if and only if there exists a curve γ in S that intersects α and β but does not intersect any other element in $P \setminus \{\alpha, \beta\}$. $\square$

Lemma 3.2. *Let $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ be a simplicial isomorphism between curve graphs of infinite-type surfaces. Then, ϕ maps non-outer separating curves to non-outer separating curves, non-separating curves to non-separating curves, and hence, outer curves to outer curves.*

Proof. We observe that, for any pants decomposition P of S, a vertex in $\mathcal{A}(P)$ is a cut vertex if and only if it is non-outer and separating. Since $\widetilde{\phi}$ is an isomorphism, it preserves cut vertices, and thus, ϕ sends non-outer separating curves to non-outer separating curves.

Now, if α is a non-separating curve in S, we can find a pants decomposition P such that α has degree four in $\mathcal{A}(P)$. The result follows from noting that a vertex in $\mathcal{A}(P)$ associated to an outer curve has degree at most two for any pants decomposition P. $\square$

Let α be a non-outer separating curve contained in a pants decomposition P of S. Then, by the definition of adjacency with respect to P, there exist pants decompositions P_1 and P_2 of the connected components of $S \setminus \alpha$ that correspond to the connected components of $\mathcal{A}(P) \setminus \alpha$. By Proposition 3.1 and Lemma 3.2, this decomposition is preserved by ϕ, that is, $\phi(\alpha)$ is a non-outer separating curve in S', and $\phi(P_1)$ and $\phi(P_2)$ are pants decompositions of the connected components of $S' \setminus \phi(\alpha)$. In the particular case where α bounds a finite type subsurface of complexity κ, we have that $\phi(\alpha)$ also bounds a finite type subsurface of complexity κ. These rather obvious facts will be implicit in the proof of several lemmata and propositions in what follows.

Lemma 3.3. *Let $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ be a simplicial isomorphism between curve graphs of infinite-type surfaces and α a non-outer separating curve that bounds a once-punctured torus. Then, $\phi(\alpha)$ also bounds a once-punctured torus.*

Proof. Let $S = \Sigma_1 \cup \alpha \cup \Sigma_2$, where Σ_1 is an once-punctured torus and Σ_2 is a connected infinite-type surface. Let P be a pants decomposition of S that contains α. Then, $P = P_1 \sqcup \alpha \sqcup P_2$ with P_1 and P_2 pants decompositions for Σ_1 and Σ_2, respectively. Moreover, P_1 contains only one non-separating curve β, i.e., Σ_1 has complexity one. Observe that α is a cut vertex in $\mathcal{A}(P)$. By Lemma 3.2, $\phi(\alpha)$ is also a cut vertex in $\mathcal{A}(\phi(P))$. Then, $S' = \Sigma_1' \cup \phi(\alpha) \cup \Sigma_2'$, where Σ_1' and Σ_2' are connected surfaces, and $\phi(P) = \phi(P_1) \sqcup \phi(\alpha) \sqcup \phi(P_2)$, where $\phi(P_1)$ and $\phi(P_2)$ are pants decompositions for Σ_1' and Σ_2', respectively. Therefore, Σ_1' has complexity one and P_1' contains only $\phi(\beta)$. Again, by Lemma 3.2, $\phi(\beta)$ is a non-separating curve in S', and we conclude that $\phi(\alpha)$ bounds a once-punctured torus. $\square$

Proposition 3.4. *Let $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ be a simplicial isomorphism between curve graphs of infinite-type surfaces. Then, S and S' have the same genus.*

Proof. Let $\mathcal{L}$ be a multicurve in S such that each curve in $\mathcal{L}$ bounds a once-punctured torus in S and $S \setminus \mathcal{L}$ has only one connected component of infinite type and genus zero, in other words, $\mathcal{L}$ is the multicurve that "captures" all genera in S, see Figure 1. Hence, the genus of S is equal to the cardinality of $\mathcal{L}$. From Lemma 3.3, for each $\alpha \in \mathcal{L}$, the curve $\phi(\alpha)$ bounds a once-punctured torus in S' induced by $\phi(\mathcal{L})$; hence, genus$(S) \leq$ genus(S'). As ϕ is an isomorphism, we obtain the equality. □

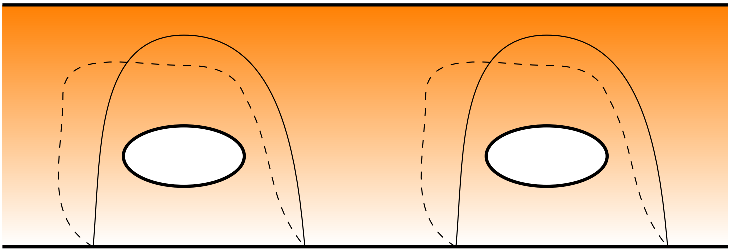

FIGURE 1. Curves capturing the genus of the surface.

Recall that two curves $\{\alpha, \beta\}$ form a *peripheral pair* if they bound a once-punctured annulus.

Proposition 3.5. *Let $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ be a simplicial isomorphism between curve graphs of infinite-type surfaces. Then, ϕ maps peripheral pairs to peripheral pairs.*

Proof. We remark that, if $\{\alpha, \beta\}$ is a peripheral pair, then both curves which form it must either be separating or non-separating. Therefore, we only consider the following three cases:

(1) both α and β are separating curves with α an outer curve;

(2) both α and β are non-outer separating curves; and

(3) both α and β are non-separating curves.

Case (1) Let P be a pants decomposition of S containing both α and β. Then, α is a vertex of degree 1 in $\mathcal{A}(P)$, which is adjacent only to β. These properties are preserved by simplicial isomorphism. Therefore, by Proposition 3.1 and Lemma 3.2, $\phi(\alpha)$ is an outer curve adjacent only to $\phi(\beta)$, and hence, these curves must form a peripheral pair.

Case (2) This is an immediate result of the fact that ϕ is an isomorphism and the following technical lemma, which gives a simplicial characterization of peripheral pairs formed by non-outer separating curves. Recall that the link of a vertex $\alpha \in \mathcal{C}^0(S)$ is the full subgraph of $\mathcal{C}(S)$ induced by all vertices adjacent to α in $\mathcal{C}(S)$. We denote it by $L(\alpha)$. We remark that $L(\alpha)$ is naturally isomorphic to $\mathcal{C}(S \setminus \alpha)$. For any subgraph $\Gamma \subset \mathcal{C}(S)$, we denote by Γ^* the graph whose vertices are $V(\Gamma)$, and two vertices span an edge if they *do not* span an edge in Γ.

Lemma 3.6. *Let α and β be disjoint non-outer separating curves. Then, $(L(\alpha)\cap L(\beta))^*$ has two connected components if and only if $\{\alpha, \beta\}$ forms a peripheral pair.*

Proof. The necessity of the statement is evident. In order to prove the sufficiency, note that $S \setminus \{\alpha, \beta\} = S_1 \sqcup S_2 \sqcup S_3$. Since $(L(\alpha)\cap L(\beta))^*$ has two connected components, there exists a j such that S_j has nonpositive complexity. Moreover, given that α and β are non-outer separating curves, we have that $\alpha \cup \beta = \partial S_j$. A straightforward calculation of the possible topological types for S_j yields the desired result. $\square$

Case (3) Up to homeomorphism, we can find a separating curve γ such that $\{\alpha, \beta, \gamma\}$ bounds a pair of pants as in Figure 2. Let P be a pants decomposition containing $\{\alpha, \beta, \gamma\}$. From Lemma 3.2, $\phi(\gamma)$ is a separating curve; thus, we have that $S' \setminus \phi(\gamma) = S_1 \sqcup S_2$. Additionally, the corresponding vertex in $\mathcal{A}(\phi(P))$ is a cut vertex that separates $\mathcal{A}(\phi(P))$ into a finite graph Γ_1 (whose vertices are $\phi(\alpha)$ and $\phi(\beta)$) and an infinite graph Γ_2. The definition of adjacency with respect to $\mathcal{A}(\phi(P))$ then implies that Γ_1 and Γ_2 are the adjacency graphs of pants decompositions of S_1 and S_2 (we may assume that, up to relabeling, the indices coincide). Thus, S_1 has complexity equal to 2 and contains $\phi(\alpha) \cup \phi(\beta)$. Since both $\phi(\alpha)$ and $\phi(\beta)$ are non-separating curves, S_1 has positive genus. Therefore, S_1 is homeomorphic to a torus with one boundary component (the boundary curve $\phi(\gamma)$) and one puncture, and the result follows. $\square$

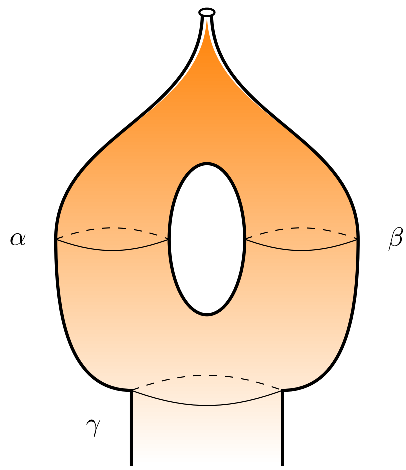

FIGURE 2. Non-separating curves α and β forming a peripheral pair.

Proposition 3.7. *Let $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ be a simplicial isomorphism between curve graphs of infinite-type surfaces. Suppose that α is a non-outer separating curve that bounds a finite-type subsurface $\Sigma \subset S$. Then, $\phi(\alpha)$ bounds a finite-type subsurface $\Sigma' \subset S'$ homeomorphic to Σ.*

Proof. Let $S = \Sigma \cup \alpha \cup \Sigma_1$, where Σ is a connected finite-type surface of genus g with n punctures and one boundary component given by α. Then, by Lemma 3.2 and the paragraph that follows it, $\phi(\alpha)$ bounds a finite-type surface Σ' of the same complexity as S, and we can, thus, write $S' = \Sigma' \cup \phi(\alpha) \cup \Sigma'_1$. Let P be a pants decomposition of Σ, satisfying:

- If $g = 0$, P is composed solely of separating curves.
- If $g > 0$, P contains g curves that bound a once-holed torus, and $P \cup \{\alpha\}$ contains n peripheral pairs.

We remark that these properties of P are preserved by ϕ. If $g = 0$, then $\phi(P)$ is also composed solely of separating curves; thus, Σ' has genus zero. If $g > 0$, then $\phi(P)$ is a pants decomposition with g curves that bound a once-holed torus, and $\phi(P) \cup \{\phi(\alpha)\}$ contains n peripheral pairs. Since Σ and Σ' have the same complexity and exactly one boundary component, a straightforward calculation shows that they are homeomorphic. $\square$

4. Proof of Theorem 1.1. In this section, we use the topological results from the previous section to prove Theorem 1.1.

Let $\{S_i\}$ be a fixed principal exhaustion of S. For each $1 \leq i$, we denote by B_i the set of boundary curves of S_i and $B := \cup_{1 \leq i} B_i$.

In the proof of the next theorem, we use the following notation: given a pants decomposition P and a subset $Y \subseteq P$, we denote by $V(Y)$ the set of vertices in $\mathcal{A}(P)$ defined by elements in Y.

Theorem 4.1. *Let S and S' be infinite-type connected, orientable surfaces with empty boundary and $\phi : \mathcal{C}(S) \to \mathcal{C}(S')$ a simplicial isomorphism. Then, S is homeomorphic to S'. Moreover, we can construct a homeomorphism $f : S \to S'$ such that $f(\beta) = \phi(\beta)$ for all $\beta \in B$.*

Proof. Let $S \setminus B = \sqcup_{1 \leq j} \mathrm{int}(\Sigma_j)$, where the collection $\{\Sigma_j\}_{1 \leq j}$ is formed by closed subsurfaces of S of finite type whose complexity is at least 4 and such that, for any $j \neq k$, $\Sigma_j \cap \Sigma_k$ is either empty or formed by boundary curves of S_i for some $i \in \mathbb{N}$. See Figure 3. For each $1 \leq j$, let P_j be a pants decomposition of Σ_j which contains a multicurve such that each curve in it bounds a once-punctured torus in Σ_j; in other words, we choose a pants decomposition P_j that captures the genus of Σ_j, as in Figure 1. Then, $P = (\cup_{1 \leq j} P_j) \cup (\cup_{1 \leq i} B_i)$ is a pants decomposition of S. Let $\mathcal{A}(P)$ be the adjacency graph of P and $\widetilde{\phi} : \mathcal{A}(P) \to \mathcal{A}(\phi(P))$ the corresponding graph isomorphism, see Proposition 3.1. Curves in B are, by definition, non-outer and separating; hence, every element $v \in V(B)$ is a cut vertex. Then, $\mathcal{A}(P) \setminus V(B) = \sqcup_{1 \leq j} \Gamma_j$, where each Γ_j is a finite subgraph whose vertex set $V(\Gamma_j)$ is precisely the pants decomposition P_j of Σ_j. Recalling that ϕ sends non-outer separating curves to non-outer separating curves (see Lemma 3.2) and defining Σ'_j as the closed subsurface of S' bounded by

$\phi(\partial\Sigma_j)$, we have that

$$\widetilde{\phi}(P) = \bigcup_{1\leq j} \widetilde{\phi}(P_j) \cup \bigcup_{1\leq i} \widetilde{\phi}(B_i)$$

is such that $\widetilde{\phi}(P_j)$ is a pants decomposition of $\Sigma'_j \subset S'$. Given that $\widetilde{\phi}(P_j)$ is connected in $\mathcal{A}(\phi(P))$, the definition of adjacency with respect to $\mathcal{A}(P)$ implies that Σ'_j is connected. Since P_j captures the genus of Σ_j, ϕ is an isomorphism and, by construction, both surfaces have the same number of boundary components, and a direct calculation of the complexity of Σ_j and Σ'_j shows that they must be homeomorphic. Recall that, by construction, we have that $\partial\Sigma'_j = \{\phi(\alpha) : \alpha \subset \partial\Sigma_j\}$. Hence, we can find a collection of orientation preserving homeomorphisms $\{f_j : \Sigma_j \to \Sigma'_j\}$ such that each f_j maps a boundary curve $\alpha \subset \partial\Sigma_j$ to $\phi(\alpha)$. These homeomorphisms can be glued together to define a global homeomorphism $f : S \to S'$, which coincides with ϕ on B. $\square$

With this result, we have proved the topological rigidity, and we only need prove that isomorphisms between curve complexes are geometric.

Hereafter, $f : S \to S'$ denotes the homeomorphism obtained from Theorem 4.1. We remark that every homeomorphism h of the form $f \circ g$ with $g \in \mathrm{stab}_{pt}(B)$, where

$$\mathrm{stab}_{pt}(B) := \{g \in \mathrm{Homeo}(S) : g \text{ fixes } B \text{ pointwise}\},$$

also coincides with ϕ on B.

For every subsurface Σ of S with complexity at least 2, we have that the natural inclusion $\iota : \Sigma \to S$ induces a simplicial map $\iota_* : \mathcal{C}(\Sigma) \to \mathcal{C}(S)$ which is an isomorphism on its image. Abusing notation, we denote by $\mathcal{C}(\Sigma)$ the image of ι_* on $\mathcal{C}(S)$. Analogously, we do the same for subsurfaces of S'.

Lemma 4.2. *For all $1 \leq i$, and for all curves $\alpha \in \mathcal{C}(S_i) < \mathcal{C}(S)$, we have that $\phi(\alpha) \in \mathcal{C}(f(S_i)) < \mathcal{C}(S')$. In particular, for each $1 \leq i$, the restriction of ϕ to $\mathcal{C}(S_i)$ defines an injective simplicial map $\phi_i : \mathcal{C}(S_i) \to \mathcal{C}(f(S_i))$.*

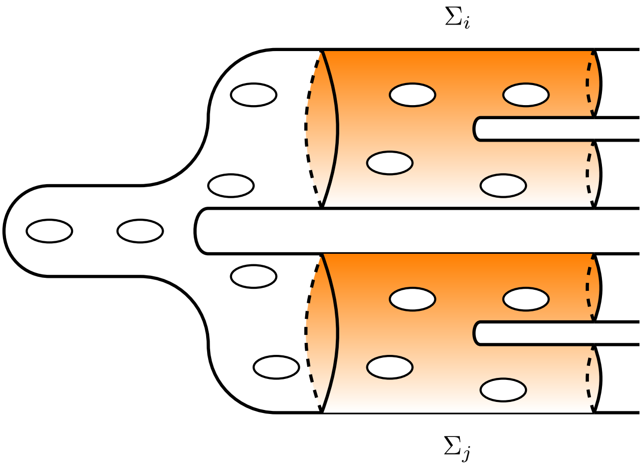

FIGURE 3. The collection of subsurfaces $\{\Sigma_j\}_{j\geq 1}$.

Proof. Let $1 \leq i$ be fixed, P_1 a pants decomposition of S_i, and P_2 a pants decomposition of $S \setminus S_i$. Then, $P = P_1 \cup B_i \cup P_2$ is a pants decomposition of S.

Recall that f coincides with ϕ on B, and that the curves in $\partial(f(S_i))$ are all separating curves. Then, by the same argument as in Theorem 4.1, we have that $\phi(P_1)$ is a pants decomposition of $f(S_i)$ and is contained in the interior of $f(S_i)$. Analogously, the curves in $\phi(P_2)$ are contained in the interior of $f(S \setminus S_i)$.

Now, let α be a curve contained in S_i. If $\alpha \in P_1$, then $\phi(\alpha) \in \mathcal{C}(f(S_i))$, as above. If $\alpha \notin P_1$, then there exists a $\beta \in P_1$ such that $i(\alpha, \beta) \neq 0$. Since we have that:

- $\phi(\alpha)$ is disjoint from every element in $\phi(B_i) = f(B_i)$;
- $\phi(\beta)$ is contained in $f(S_i)$; and
- $i(\phi(\alpha), \phi(\beta)) \neq 0$;

we can conclude that $\phi(\alpha)$ is contained in $f(S_i)$. $\square$

With Lemma 4.2 and Shackleton's result on combinatorial rigidity (see **[10,** Theorem 1]), we obtain for each $1 \leq i$ a homeomorphism

$g_i : S_i \to f(S_i)$ that induces ϕ_i, that is, such that, for all $\alpha \in \mathcal{C}(S_i)$, we have that $\phi(\alpha) = g_i(\alpha)$.

We affirm that each g_i can be extended, up to isotopy, to a homeomorphism $\overline{g_i} : \overline{S_i} \to \overline{f(S_i)}$ between the closure on S and S' of the respective subsurfaces. This is explained in the following paragraphs.

Each S_i is an open subsurface of S, and its punctures can be classified into two categories: those that persist when we take the closure $\overline{S_i}$ of S_i in S (which are precisely those punctures of S_i which are also punctures of S) and those that do not (these "become" curves contained in $B_i = \partial S_i$ when taking the closure of S_i in S). An obstruction to the extension of g_i could be that g_i exchanges a puncture of S_i that persists in $\overline{S_i}$ with one that does not. We suppose this is the case, and we derive a contradiction.

Let α, β, γ bound a pair of pants in S such that $\alpha \subset \partial\overline{S_i}$ and $\beta, \gamma \in \mathcal{C}(S_i)$. Note that this implies that $\{\beta, \gamma\}$ is a peripheral pair in S_i. If g_i exchanges the puncture of S_i, defined by $\overline{S_i} \setminus \alpha$ with a puncture of S, then $\{\phi(\beta), \phi(\gamma)\}$ would be a peripheral pair in S. By Proposition 2, $\{\beta, \gamma\}$ is also a peripheral pair in S. This situation is depicted in Figure 4. It is clear that $S \setminus \alpha$ has one connected component, the complexity of which is *strictly less* than 3. This is a contradiction, for both connected components of $S \setminus \alpha$ have complexity *at least* 4.

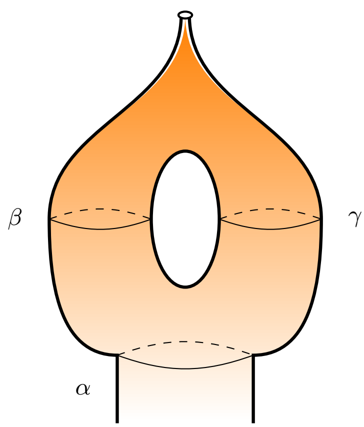

FIGURE 4. $\{\beta, \gamma\}$ is a peripheral pair.

Thus, we can isotope g_i to a homeomorphism $\widetilde{g_i} : S_i \to f(S_i)$, which can be extended to a homeomorphism $\overline{g}_i : \overline{S_i} \to \overline{f(S_i)}$ that still induces ϕ_i. Using the next lemma, we can assert that, for each $1 \leq i$, $\overline{g_i}$ coincides with f on B_i.

Lemma 4.3. *Let α, β and γ be curves on S such that α is a separating curve, and let α, β and γ bound a pair of pants on S. Then, $\phi(\alpha)$, $\phi(\beta)$ and $\phi(\gamma)$ also bound a pair of pants on S'.*

Proof. Let P be a pants decomposition of S with $\alpha, \beta, \gamma \in P$. Then, with respect to P, α is adjacent to β, β is adjacent to γ and α is adjacent to γ. From Proposition 3.1, we know that adjacency is preserved under ϕ.

The only possibility for this to occur and have that $\phi(\alpha)$, $\phi(\beta)$ and $\phi(\gamma)$ do *not* bound a pair of pants on S', is (up to homeomorphism) illustrated in Figure 5. However, if this were to occur, we could find a curve δ on S' that would intersect $\phi(\alpha)$ exactly once, which is impossible, since $\phi(\alpha)$ is a separating curve due to Proposition 3.1. Therefore, $\phi(\alpha)$, $\phi(\beta)$ and $\phi(\gamma)$ bound a pair of pants on S'. $\square$

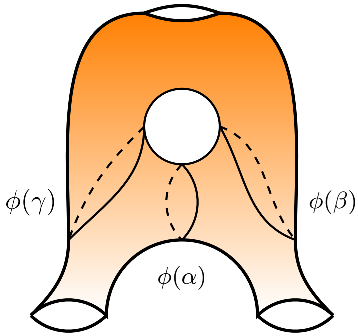

FIGURE 5. The curves $\phi(\alpha)$, $\phi(\beta)$ and $\phi(\gamma)$ do *not* bound a pair of pants.

For each $1 \leq i$, we can choose a homeomorphism $l_i : S \setminus S_i \to S' \setminus f(S_i)$ of the form $f \circ \eta$ with $\eta \in \mathrm{stab}_{pt}(B)$ and define:

$$h_i(x) = \begin{cases} \overline{g_i}(x) & \text{if } x \in S_i, \\ l_i(x) & \text{if } x \in S \setminus S_i. \end{cases}$$

In this manner, we obtain a family of homeomorphisms $h_i : S \to S'$ which, by construction, satisfies that $h_i(\alpha) = \phi(\alpha) = h_j(\alpha)$ for all $i < j$ and $\alpha \in \mathcal{C}(S_i) \subset \mathcal{C}(S_j) \subset \mathcal{C}(S)$. As a consequence of the Alexander method (see [**4,** Chapter 2.3]) we have, for each $i < j$, that $h_{i|S_i} = h_{j|S_i} \circ M_i$, where $M_i \in \mathrm{Homeo}(S_i)$ is a multitwist whose support is contained in a neighborhood in S_i of ∂S_i. In other words, for each $1 \leq i$, there exists a subsurface $\widetilde{S}_i \subset S_i \subset S$, isotopic within S_i to S_i such that the support of the multitwist M_i is contained in $S_i \setminus \widetilde{S}_i = \sqcup_{k=1}^{s} A_k$, where each A_k is an annulus. In particular, $M_{|\widetilde{S_i}} = Id_{\widetilde{S_i}}$ and hence, for each $i < j$, we have that $h_{i|\widetilde{S_i}} = h_{j|\widetilde{S_i}}$. In this manner, we can define the following map:

$$\begin{aligned} h : S &\longrightarrow S', \\ s \in \widetilde{S}_i &\longmapsto h_i(s). \end{aligned} \tag{4.1}$$

Since, for all $1 \leq i < j$, we have that $h_i|_{\widetilde{S}_i} = h_j|_{\widetilde{S}_i}$. This map is well defined. Moreover, it is a homeomorphism, and, by construction, it coincides with ϕ on the entire curve graph $\mathcal{C}(S)$, as desired. □

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