ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 5, 2018

THE SOLUTION OF A NEW CAPUTO-LIKE FRACTIONAL h-DIFFERENCE EQUATION

BAOGUO JIA, XIANG LIU, FEIFEI DU AND MEI WANG

ABSTRACT. Consider the Caputo fractional h-difference equation

$$ {}_a\Delta_{h,*}^{\nu} x(t) = c(t)x(t+\nu), \quad 0<\nu<1,\ t\in (h\mathbb{N})_{a+(1-\nu)h}, $$

where ${}_a\Delta_{h,*}^{\nu} x(t)$ denotes the Caputo-like delta fractional h-difference of $x(t)$ on sets $(h\mathbb{N})_{a+(1-\nu)h}$. Our main results are found in Theorems A and B in Section 1. In Section 3, we show that the proof of a recent result in **[5]** is incorrect. Finally, four numerical examples are given to illustrate the main results.

1. Introduction. Discrete fractional calculus has generated interest in recent years. Some of the research concerns the forward or delta difference. The reader is referred to **[1, 3, 7]**, for example, and more recently, **[8, 10]**. Possibly more work has been developed for the backward or nabla difference; for this, the reader is also referred to **[6, 9]**. Some research developing relations between the forward and backward fractional operators Δ^ν and ∇^ν may be found in **[4]**, and research on fractional calculus on time scales may be found in **[7]**.

One of the central tasks in the qualitative theory of difference systems or equations is stability of solutions. However, due to the lack of a geometric interpretation of fractional derivatives, there are few results concerning how to directly analyze fractional order systems or equations. Some results on stability may be found in **[2, 12, 13, 15, 17]**.

2010 AMS *Mathematics subject classification.* Primary 39A12, 39A70.

Keywords and phrases. Caputo h-fractional difference, Mittag-Leffler function, power rule.

This work was supported by the National Natural Science Foundation of China, grant No. 11271380, the Guangdong Provincicial Key Laboratory of Computational Science and Sun Yat-Sen University, the International Program for Ph.D. Candidates. The third author is the corresponding author.

Received by the editors on July 17, 2017, and in revised form on October 10, 2017.

DOI:10.1216/RMJ-2018-48-5-1607

Consider the following νth order Caputo-like fractional h-difference equation with an initial condition:

$$ {}_a\Delta_{h,*}^{\nu} x(t) = c(t)x(t+\nu h), \tag{1.1} $$

$$ x(a) = x_a, \quad 0 < \nu < 1, \tag{1.2} $$

where $t \in (h\mathbb{N})_{a+(1-\nu)h} := \{a+(1-\nu)h, a+(2-\nu)h, \ldots\}$. This work is motivated by Baleanu, Wu, et al., [**5**], who obtained monotonicity of the solution for $c(t) = c < 0$ and asymptotic stability of (1.1). In this paper, we will further discuss the solution of (1.1). The following theorems are obtained.

Theorem A. *Assume that* $0 < \nu < 1$ *and* $c(t) = c$, $|c| < h^{-\nu}$ *and* $x(a) = 1$. *Then, the solution of equation* (1.1) *is*

$$ E_{c,\nu}^{h}(t,a) := \sum_{i=0}^{\infty} c^i \frac{(t-a+i\nu h-h)_h^{(i\nu)}}{\Gamma(i\nu+1)}, \quad t \in (h\mathbb{N})_a, $$

where we use the convention

$$ \frac{(t-a+i\nu h-h)_h^{(i\nu)}}{\Gamma(i\nu+1)}\bigg|_{t=a} = \begin{cases} \frac{(-h)_h^{(0)}}{\Gamma(1)} = 1 & i = 0, \\ \frac{h^{i\nu}}{\Gamma(0)i\nu} = 0 & i \geq 1. \end{cases} $$

As applications, we obtain that

Theorem B. *Assume that* $0 < \nu < 1$, $x(a) > 0$.

(1) *If there exists a constant* b_1 *such that* $0 < b_1 \leq c(t) < h^{-\nu}$, *then the solution of equation* (1.1) *satisfies* $\lim_{t\to\infty} x(t) = +\infty$.
(2) *If there exists a constant* b_2 *such that* $c(t) \leq b_2 < 0$, *then the solution of equation* (1.1) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

In Section 3, we show that the proof of a recent result in [**5**] is incorrect. Finally, we provide four numerical examples to illustrate the main results.

2. The solution of the initial value problem (1.1)–(1.2). We will be interested in functions defined on sets of the form $(h\mathbb{N})_{a+(1-\nu)h}$, where $a, h \in \mathbb{R}$, $h > 0$. The next two definitions are from [**5**].

Definition 2.1 (Delta fractional sum [**5**]). Let $x : (h\mathbb{N})_a \to \mathbb{R}$ and $\nu > 0$ be given, with a the initial point. The νth *order h-sum* is given by:

$$
\begin{aligned}
{}_a\Delta_h^{-\nu} x(t) &= \frac{1}{\Gamma(\nu)} \int_a^{t-\nu h+h} (t-\sigma(\tau))_h^{(\nu-1)} x(\tau)\Delta_h \tau \\
&= \frac{h}{\Gamma(\nu)} \sum_{\tau\in[a,t-\nu h+h)} (t-\sigma(\tau))_h^{(\nu-1)} x(\tau)
\end{aligned}
\tag{2.1}
$$

for $t \in (h\mathbb{N})_{a+\nu h}$, where $\sigma(\tau) := \tau + h$, $[a, b) := \{t \in (h\mathbb{N})_a : a \le t < b$, $a, b \in (h\mathbb{N})_a\}$ and the *h-falling factorial function* is defined as

$$
t_h^{(\nu)} := h^\nu \frac{\Gamma((t/h)+1)}{\Gamma((t/h)+1-\nu)}, \quad t, \nu \in \mathbb{R}.
$$

Definition 2.2 (Caputo delta difference [**5**]). For $x(t)$ defined on $(h\mathbb{N})_a$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$, $\lceil \cdot \rceil$ is ceiling of the number. The μth *Caputo like fractional difference* is defined as

$$
\begin{aligned}
{}_a\Delta_{h,*}^{\mu} x(t) &= {}_a\Delta_h^{-(m-\mu)} \Delta_h^m x(t) \\
&= \frac{1}{\Gamma(m-\mu)} \int_a^{t-(m-\mu)h+h} (t-\sigma(\tau))_h^{(m-\mu-1)} \Delta_h^m x(\tau)\Delta_h \tau
\end{aligned}
\tag{2.2}
$$

for $t \in (h\mathbb{N})_{a+(m-\mu)h}$, where $\Delta_h x(t) = (x(t+h) - x(t))/h$.

Definition 2.3. For $p \in \mathbb{R}$, $|p| < h^{-\nu}$, $0 < \nu < 1$, the discrete Mittag-Leffler function is defined as

$$
E_{p,\nu}^h(t,a) := \sum_{i=0}^{\infty} p^i \frac{(t-a+i\nu h-h)_h^{(i\nu)}}{\Gamma(i\nu+1)}, \quad t \in (h\mathbb{N})_a. \tag{2.3}
$$

Remark 2.4. Using the ratio test and the following property for the Gamma function [**11,** Example 5.6], [**16,** Proposition 2.1.3]:

$$
\lim_{t\to\infty} \frac{\Gamma(t+\alpha)}{\Gamma(t)t^\alpha} = 1, \quad \alpha \in \mathbb{C},
$$

it is easy to see that $E_{p,\nu}^h(t,a)$ is (absolutely) convergent if $|p| < h^{-\nu}$.

The following power rule formula is from **[14]**.

Lemma 2.5. *Let $a \in \mathbb{R}$, $\mu > 0$, be given. Then*:

$$ {}_{a+ph}\Delta_h^{-\mu}(t-a)_h^{(p)} = \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}(t-a)_h^{(p+\mu)}, \tag{2.4} $$

for $t \in (h\mathbb{N})_{a+ph+\mu h}$.

Lemma 2.6. *Assume that $0 < \nu < 1$, $p \in \mathbb{R}$, $|p| < h^{-\nu}$. Then*:

$$ {}_a\Delta_{h,*}^{\nu} E_{p,\nu}^h(t,a) = pE_{p,\nu}^h(t+h\nu, a) $$

for $t \in (h\mathbb{N})_{a+(1-\nu)h}$.

Proof. For simplicity, let $a = 0$. For $t \in (h\mathbb{N})_0$, we have

(2.5)

$$ \begin{aligned} \Delta_h E_{p,\nu}^h(t,0) &= h^{-1}[E_{p,\nu}^h(t+h,0) - E_{p,\nu}^h(t,0)] \\ &= h^{-1}\sum_{i=0}^{\infty} p^i \frac{(t+h+i\nu h-h)_h^{(i\nu)} - (t+i\nu h-h)_h^{(i\nu)}}{\Gamma(i\nu+1)} \\ &= h^{-1}\sum_{i=0}^{\infty} \frac{p^i h^{i\nu}}{\Gamma(i\nu+1)}\left[\frac{\Gamma((t/h)+i\nu+1)}{\Gamma((t/h)+1)} - \frac{\Gamma((t/h)+i\nu)}{\Gamma(t/h)}\right] \\ &= h^{-1}\sum_{i=1}^{\infty} \frac{p^i h^{i\nu}}{\Gamma(i\nu)} \cdot \frac{\Gamma((t/h)+i\nu)}{\Gamma((t/h)+1)}, \end{aligned} $$

where we use the convention $1/\Gamma(0) = 0$. From (2.1), we have

$$ \begin{aligned} {}_0\Delta_{h,*}^{\nu} E_{p,\nu}^h(t,0) &= {}_0\Delta_h^{-(1-\nu)}\Delta_h E_{p,\nu}^h(t,0) \\ &= \frac{1}{\Gamma(1-\nu)}\int_0^{t+\nu h} (t-\sigma(\tau))_h^{(-\nu)}\Delta_h E_{p,\nu}^h(\tau,0)\Delta_h\tau \\ &= \frac{h^{-1}}{\Gamma(1-\nu)}\int_0^{t+\nu h} (t-\sigma(\tau))_h^{(-\nu)}\sum_{i=1}^{\infty}\frac{p^i h^{i\nu}}{\Gamma(i\nu)}\cdot\frac{\Gamma((\tau/h)+i\nu)}{\Gamma((\tau/h)+1)}\Delta_h\tau. \end{aligned} $$

In the following, we first prove that the infinite series

$$\frac{(t-\sigma(\tau))_h^{(-\nu)}}{\Gamma(1-\nu)} \sum_{i=1}^{\infty} \frac{p^i h^{i\nu}}{\Gamma(i\nu)} \cdot \frac{\Gamma((\tau/h)+i\nu)}{\Gamma((\tau/h)+1)}$$

for each fixed t is uniformly convergent for $\tau \in [0, t+\nu h)$.

Taking $t=(1-\nu)h+kh$, $\tau=jh$, $k \in \mathbb{N}_0$, $j=0,1,\ldots,k$, we will show that

$$\left|\frac{(t-\sigma(\tau))_h^{(-\nu)}}{\Gamma(1-\nu)}\right| \le h^{-\nu}, \quad \tau=jh,\ j=0,1,\ldots,k.$$

For $j=k$, we have that

$$\left|\frac{(t-\sigma(\tau))_h^{(-\nu)}}{\Gamma(1-\nu)}\right| = h^{-\nu}.$$

Now, we assume that $j=0,1,\ldots,k-1$.

$$\begin{aligned}\left|\frac{(t-\sigma(\tau))_h^{(-\nu)}}{\Gamma(1-\nu)}\right| &= \left|h^{-\nu}\frac{\Gamma(1-\nu+k-j)}{\Gamma(1-\nu)\Gamma(k+1-j)}\right| \\ &= \left|h^{-\nu}\frac{(k-j-v)(k-j-v-1)\cdots(1-v)}{(k-j)!}\right| \\ &= h^{-\nu}\left|\frac{k-j-v}{k-j}\right|\left|\frac{k-j-v-1}{k-j-1}\right|\cdots\left|\frac{1-v}{1}\right| < h^{-\nu}.\end{aligned}$$

Next, we will show that, for $i \in \mathbb{N}_0$,

$$\left|\frac{\Gamma((\tau/h)+i\nu)}{\Gamma(i\nu)\Gamma((\tau/h)+1)}\right| \le (i\nu+1)^k, \quad \tau=jh,\ j=0,1,\ldots,k.$$

For $j=0$, we have that

$$\left|\frac{\Gamma((\tau/h)+i\nu)}{\Gamma(i\nu)\Gamma((\tau/h)+1)}\right| = 1 \le (i\nu+1)^k.$$

Now, we assume that $j = 1, 2, \ldots, k$. Then,

$$\begin{aligned}\left|\frac{\Gamma((\tau/h)+i\nu)}{\Gamma(i\nu)\Gamma((\tau/h)+1)}\right| &= \left|\frac{\Gamma(j+i\nu)}{\Gamma(i\nu)\Gamma(j+1)}\right| \\ &= \left|\frac{iv+j-1}{j}\right|\left|\frac{iv+j-2}{j-1}\right|\cdots\left|\frac{iv+1}{2}\right|\left|\frac{iv}{1}\right| \\ &\le \left|\frac{iv}{j}+1\right|\left|\frac{iv}{j-1}+1\right|\cdots\left|\frac{iv}{2}+1\right|\left|\frac{iv}{1}\right| \\ &\le (i\nu+1)^j \le (i\nu+1)^k.\end{aligned}$$

Thus,

(2.6)

$$\left|\frac{(t-\sigma(\tau))_h^{(-\nu)}}{\Gamma(1-\nu)}\sum_{i=1}^{\infty}\frac{p^i h^{i\nu}}{\Gamma(i\nu)}\cdot\frac{\Gamma((\tau/h)+i\nu)}{\Gamma((\tau/h)+1)}\right| \le h^{-\nu}\sum_{i=1}^{\infty}|p|^i h^{i\nu}(i\nu+1)^k.$$

Using the ratio test, the series $\sum_{i=1}^{\infty}|p|^i h^{i\nu}(i\nu+1)^k$ is convergent for $|p| < h^{-\nu}$. From (2.6), for each fixed t, the infinite function series

$$\frac{(t-\sigma(\tau))_h^{(-\nu)}}{\Gamma(1-\nu)}\sum_{i=1}^{\infty}\frac{p^i h^{i\nu}}{\Gamma(i\nu)}\cdot\frac{\Gamma((\tau/h)+i\nu)}{\Gamma((\tau/h)+1)}$$

is uniformly convergent on $\tau \in [0, t+h\nu)$. Thus, integrating, term-by-term, we obtain

$$\begin{aligned}0\Delta_{h,*}^{\nu}E_{p,\nu}^{h}(t,0) &= \frac{h^{-1}}{\Gamma(1-\nu)}\sum_{i=1}^{\infty}\frac{p^i h^{i\nu}}{\Gamma(i\nu)} \\ &\quad\cdot\int_0^{t+\nu h}(t-\sigma(\tau))_h^{(-\nu)}\cdot\frac{\Gamma((\tau/h)+i\nu)}{\Gamma((\tau/h)+1)}\Delta_h\tau \\ &= \frac{1}{\Gamma(1-\nu)}\sum_{i=1}^{\infty}\frac{p^i}{\Gamma(i\nu)} \\ &\quad\cdot\int_0^{t+\nu h}(t-\sigma(\tau))_h^{(-\nu)}(\tau+i\nu h-h)_h^{(i\nu-1)}\Delta_h\tau \\ &\overset{(2.1)}{=}\sum_{i=1}^{\infty}\frac{p^i}{\Gamma(i\nu)}\cdot{}_0\Delta_h^{-(1-\nu)}(t-(h-ih\nu))_h^{(i\nu-1)}\end{aligned}$$

$$\begin{aligned}
&\overset{(2.4)}{=} \sum_{i=1}^{\infty} \frac{p^i}{\Gamma(i\nu)} \frac{\Gamma(i\nu)}{\Gamma(i\nu+1-\nu)} (t+ih\nu-h)_h^{(i\nu-1+1-\nu)} \\
&= p\sum_{j=0}^{\infty} \frac{p^j}{\Gamma((j+1)\nu+1-\nu)} (t+(j+1)h\nu-h)_h^{(j\nu)} \\
&= p\sum_{j=0}^{\infty} p^j \frac{(t+h\nu+jh\nu-h)_h^{(j\nu)}}{\Gamma(j\nu+1)} \\
&= pE_{p,\nu}^h(t+h\nu,0)
\end{aligned}$$

for

$$t \in (h\mathbb{N})_{a+ph+(1-\nu)h} = (h\mathbb{N})_{(h-ih\nu)+(i\nu-1)h+(1-\nu)h} = (h\mathbb{N})_{(1-\nu)h}.$$

This completes the proof. □

From Lemma 2.6, we can get Theorem A.

Remark 2.7. From (2.5), we obtain that, when $0 < c(t) = c < h^{-\nu}$, we have

$$\Delta_h E_{c,\nu}^h(t,a) > 0.$$

Thus, the solution of (1.1) with $x(a) > 0$ is increasing.

3. Asymptotic behavior, $0 < b_1 \le c(t) < h^{-\nu}$.

Lemma 3.1. *Assume that* $0 < \nu < 1$ *and* $0 < p < h^{-\nu}$. *Then, we have*:

$$\lim_{t\to\infty} E_{p,\nu}^h(t,a) = +\infty.$$

Proof. Taking $t = a + kh$, $k \ge 0$. If $i = 0$, we have

$$\lim_{t\to\infty} \frac{(t-a+i\nu h-h)_h^{(i\nu)}}{\Gamma(i\nu+1)} = 1.$$

If $i \ge 1$, we have

$$\begin{aligned}
(3.1) \qquad \lim_{t\to\infty} \frac{(t-a+i\nu h-h)_h^{(i\nu)}}{\Gamma(i\nu+1)} &= \lim_{k\to\infty} \frac{(kh+(i\nu-1)h)_h^{(i\nu)}}{\Gamma(i\nu+1)} \\
&= \lim_{k\to\infty} h^{iv} \frac{\Gamma(k+i\nu)}{\Gamma(k)\Gamma(i\nu+1)}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to\infty} h^{iv} \frac{\Gamma(k+i\nu)}{\Gamma(k)k^{i\nu}} \cdot \frac{k^{i\nu}}{\Gamma(i\nu+1)} \\
&= +\infty,
\end{aligned}
$$

where we use

$$
\lim_{k\to\infty} \frac{\Gamma(k+i\nu)}{\Gamma(k)k^{i\nu}} = 1.
$$

This completes the proof. □

Letting $x(t) = E^h_{p,\nu}(t,a)$, Figure 1 illustrates the validity of Remark 2.7 and Lemma 3.1.

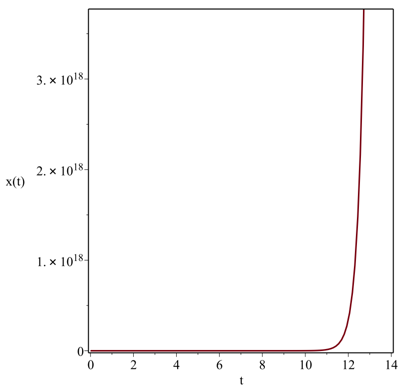

FIGURE 1. Asymptotic behavior of Mittag-Leffler function $E^h_{p,\nu}(t,a)$ for $a = 0$, $\nu = 0.5$, $h = 0.25$, $p = 1.5$.

Next, we will give the delta power rule formulae for the fractional h difference.

Lemma 3.2. *Assume that $h > 0$ and $\alpha \in \mathbb{R}$.*

(i) *The delta h-difference of the h-falling fractional function $(t-\tau)^{(\alpha)}_h$ with respect to t is given by*

$$
{}_t\Delta_h (t-\tau)^{(\alpha)}_h = \alpha (t-\tau)^{(\alpha-1)}_h. \tag{3.2}
$$

(ii) *The delta h-difference of the h-falling fractional function $(t-\tau)_h^{(\alpha)}$ with respect to τ is given by*

$$ {}_{\tau}\Delta_h (t-\tau)_h^{(\alpha)} = -\alpha (t-\sigma(\tau))_h^{(\alpha-1)}. \tag{3.3}$$

Proof. In order to see that (i) holds, consider

$$\begin{aligned} {}_{t}\Delta_h (t-\tau)_h^{(\alpha)} &= \frac{(t+h-\tau)_h^{(\alpha)} - (t-\tau)_h^{(\alpha)}}{h} \\ &= h^{\alpha-1}\left[\frac{\Gamma((t-\tau/h)+2)}{\Gamma((t-\tau/h)+2-\alpha)} - \frac{\Gamma((t-\tau/h)+1)}{\Gamma((t-\tau/h)+1-\alpha)}\right] \\ &= \alpha h^{\alpha-1}\frac{\Gamma((t-\tau/h)+1)}{\Gamma((t-\tau/h)+2-\alpha)} \\ &= \alpha (t-\tau)_h^{(\alpha-1)}. \end{aligned}$$

Hence, (i) holds. In order to see that (ii) holds, consider

$$\begin{aligned} {}_{\tau}\Delta_h (t-\tau)_h^{(\alpha)} &= \frac{(t-\tau-h)_h^{(\alpha)} - (t-\tau)_h^{(\alpha)}}{h} \\ &= h^{\alpha-1}\left[\frac{\Gamma(t-\tau/h)}{\Gamma((t-\tau/h)-\alpha)} - \frac{\Gamma((t-\tau/h)+1)}{\Gamma((t-\tau/h)+1-\alpha)}\right] \\ &= -\alpha h^{\alpha-1}\frac{\Gamma((t-\tau/h))}{\Gamma((t-\tau/h)+1-\alpha)} \\ &= -\alpha (t-\sigma(\tau))_h^{(\alpha-1)}. \end{aligned}$$

This completes the proof. $\square$

In the remainder of the paper, we assume $\Delta_h = {}_t\Delta_h$, for simplicity, when we mention the power rule formula.

The next comparison theorem plays an important role in proving the main results.

Theorem 3.3. *Assume that $c_2(t) \le c_1(t) < h^{-\nu}$, $0<\nu<1$, and $x(t)$, $y(t)$ satisfy*

$$ {}_a\Delta_{h,*}^{\nu} x(t) \ge c_1(t) x(t+\nu h) \tag{3.4}$$

and

$$ {}_a\Delta_{h,*}^{\nu} y(t) \le c_2(t) y(t+\nu h), \tag{3.5}$$

respectively, for $t \in (h\mathbb{N})_{a+(1-\nu)h}$ with the initial condition $x(a) \geq y(a)$. Then, $x(t) \geq y(t)$ for $t \in (h\mathbb{N})_a$.

Proof. Using integration by parts,

$$ {}_{\tau}\Delta_h (t-\tau)_h^{(-\nu)} = \nu (t-\sigma(\tau))_h^{(-\nu-1)} $$

(Lemma 3.2 (ii)),

$$ (-\nu h)_h^{(-\nu)} = h^{-\nu}\Gamma(-\nu+1), $$

we have, for $t \in (h\mathbb{N})_{a+(1-\nu)h}$:

$$ \begin{aligned} {}_a\Delta_{h,*}^{\nu} x(t) &= {}_a\Delta_h^{-(1-\nu)} \Delta_h x(t) \\ &= \frac{1}{\Gamma(1-\nu)} \int_a^{t+\nu h} (t-\sigma(\tau))_h^{(-\nu)} \Delta_h x(\tau) \Delta_h \tau \\ &= \frac{1}{\Gamma(1-\nu)} \Bigg[(t-\tau)_h^{(-\nu)} x(\tau)\big|_{\tau=a}^{t+\nu h} \\ &\qquad\qquad - \nu \int_a^{t+\nu h} (t-\sigma(\tau))_h^{(-\nu-1)} x(\tau) \Delta_h \tau \Bigg] \\ &= h^{-\nu} x(t+\nu h) - \frac{(t-a)_h^{(-\nu)} x(a)}{\Gamma(1-\nu)} \\ &\quad + \frac{h}{\Gamma(-\nu)} \sum_{\tau \in [a, t+\nu h)} (t-\sigma(\tau))_h^{(-\nu-1)} x(\tau). \end{aligned} $$

Take $t = a + (1-\nu)h + kh$, $k \in \mathbb{N}_0$. Then, we have

$$ \begin{aligned} {}_a\Delta_{h,*}^{\nu} x(t) &= h^{-\nu} x(a+kh+h) - \frac{((1-\nu)h+kh)_h^{(-\nu)} x(a)}{\Gamma(1-\nu)} \\ &\quad + \frac{h}{\Gamma(-\nu)} \sum_{\tau \in [a, a+(k+1)h)} (a-\tau-\nu h+kh)_h^{(-\nu-1)} x(\tau) \\ &= h^{-\nu} x(a+kh+h) - \nu h^{-\nu} x(a+kh) \\ &\quad - \frac{\nu(-\nu+1)}{2!} h^{-\nu} x(a+kh-h) - \cdots \end{aligned} $$

$$
\begin{aligned}
&- \frac{\nu(-\nu+1)\cdots(-\nu+k-1)}{k!} h^{-\nu} x(a+h) \\
&- \frac{\Gamma(-\nu+k+1)}{\Gamma(1-\nu)k!} h^{-\nu} x(a).
\end{aligned}
$$

Using (3.4), we obtain

(3.6)

$$
\begin{aligned}
(h^{-\nu} - c_1(t))x(a+kh+h) \geq \nu h^{-\nu} x(a+kh) &\\
+ \frac{\nu(-\nu+1)}{2!} h^{-\nu} x(a+kh-h) + \cdots &\\
+ \frac{\nu(-\nu+1)\cdots(-\nu+k-1)}{k!} h^{-\nu} x(a+h) &\\
+ \frac{\Gamma(-\nu+k+1)}{\Gamma(1-\nu)k!} h^{-\nu} x(a). &
\end{aligned}
$$

Similarly, using (3.5), we obtain

(3.7)

$$
\begin{aligned}
(h^{-\nu} - c_2(t))y(a+kh+h) \leq \nu h^{-\nu} y(a+kh) &\\
+ \frac{\nu(-\nu+1)}{2!} h^{-\nu} y(a+kh-h) + \cdots &\\
+ \frac{\nu(-\nu+1)\cdots(-\nu+k-1)}{k!} h^{-\nu} y(a+h) &\\
+ \frac{\Gamma(-\nu+k+1)}{\Gamma(1-\nu)k!} h^{-\nu} y(a). &
\end{aligned}
$$

Note that the coefficients of $x(a+ih)$, $y(a+ih)$, $i = 0, 1, \ldots, k+1$, are positive. Thus, by using the principle of strong induction and $c_2(t) \leq c_1(t) < h^{-\nu}$, (3.6) and (3.7), it is simple to prove $x(a+kh+h) \geq y(a+kh+h)$ for $k \in \mathbb{N}_0$. This completes the proof. $\square$

Remark 3.4. From the proof of Theorem 3.3, it is easy to see the following result.

Assume that $c_2(t) \leq c_1(t) < h^{-\nu}$, $0 < \nu < 1$, and $x(t), y(t)$ satisfy

$$
{}_a\Delta_{h,*}^{\nu} x(t) \geq c_1(t) x(t+\nu h) \tag{3.8}
$$

and

$$(3.9) \qquad {}_a\Delta_{h,*}^{\nu} y(t) = c_2(t) y(t+\nu h),$$

respectively, for $t \in (h\mathbb{N})_{a+(1-\nu)h}$ with the initial condition $x(a) \geq y(a) > 0$. Then, $x(t) \geq y(t) > 0$, for $t \in (h\mathbb{N})_a$.

Remark 3.5. Theorem 3.3 can be regarded as an extension of [**5,** Lemma 2.10].

It should be noted that [**5,** Lemma 2.10] is correct, but the proof is incorrect. In the following equations [**5,** page 523], the authors obtained:

(3.10)

$$u(a+k+1) = \frac{u(a)}{1-\lambda h^{\nu}} + \frac{\lambda h^{\nu}}{(1-\lambda h^{\nu})\Gamma(\nu)} \sum_{j=0}^{k-1} \frac{\Gamma(k-j+\nu)}{\Gamma(k+1-j)} u(a+j+1)$$

and

(3.11)

$$g(a+k+1) \leq \frac{g(a)}{1-\lambda h^{\nu}} + \frac{\lambda h^{\nu}}{(1-\lambda h^{\nu})\Gamma(\nu)} \sum_{j=0}^{k-1} \frac{\Gamma(k-j+\nu)}{\Gamma(k+1-j)} g(a+j+1).$$

Due to the fact that $g(a) = u(a)$, we can obtain $g(a+1) \leq u(a+1)$. However, *since* $\lambda < 0$, *using* (3.10) *and* (3.11), *we do not obtain*:

$$g(a+2) \leq u(a+2), \ldots, g(a+k) \leq u(a+k), \; g(a+k+1) \leq u(a+k+1).$$

Theorem 3.6. *Assume* $0 < \nu < 1$, *that there exists a constant* b_1 *such that* $0 < b_1 \leq c(t) < h^{-\nu}$ *and* $x(t)$ *is the solution of the Caputo delta fractional equation*

$$(3.12) \qquad {}_a\Delta_{h,*}^{\nu} x(t) = c(t) x(t+\nu h), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}, \quad x(a) > 0.$$

Then, $x(t) \geq (x(a)/2) E_{b_1,\nu}^{h}(t,a)$ *for* $t \in (h\mathbb{N})_a$.

Proof. From Lemma 2.6, we have

$$ {}_a\Delta_{h,*}^{\nu} E_{b_1,\nu}^{h}(t,a) = b_1 E_{b_1,\nu}^{h}(t+h\nu, a)$$

and $E_{b_1,\nu}^{h}(a,a) = 1$. From Theorem 3.3, take $c_2(t) = b_1$. Then, $x(t)$ and

$$y(t) = \frac{x(a)}{2} E_{b_1,\nu}^{h}(t,a)$$

satisfy

$$ {}_a\Delta_{h,*}^{\nu} x(t) = c(t)x(t+\nu h) \tag{3.13} $$

and

$$ {}_a\Delta_{h,*}^{\nu} y(t) = b_1 y(t+\nu h), \tag{3.14} $$

respectively, and

$$ x(a) > \frac{x(a)}{2} E_{b_1,\nu}^{h}(a,a) = y(a). $$

From Theorem 3.3, we obtain that

$$ x(t) \geq \frac{x(a)}{2} E_{b_1,\nu}^{h}(t,a), $$

for $t \in (h\mathbb{N})_a$. This completes the proof. □

From Lemma 3.1 and Theorem 3.6, we can get the following theorem.

Theorem B$_1$. *Assume that* $0 < \nu < 1$, $x(a) > 0$, *and there exists a constant* b_1 *such that* $0 < b_1 \leq c(t) < h^{-\nu}$. *Then, the solution of equation* (1.1) *satisfies*

$$ \lim_{t\to\infty} x(t) = +\infty. $$

Remark 3.7 (see Example 6.2)**.** If $c(t)$ in Theorem B$_1$ is not a constant, then the solution of equation (1.1) may not be monotonically increasing.

4. Asymptotic behavior, $c(t) \leq b_2 < 0$.

Definition 4.1 (Riemann-Liouville delta difference)**.** For $x(t)$ defined on $(hN)_a$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$, $\lceil \cdot \rceil$ is the *ceiling* of a number. The μth *Riemann-Liouville fractional difference* is defined as

$$ \begin{aligned} {}_a\Delta_h^{\mu} x(t) &= \Delta_h^m \, {}_a\Delta_h^{-(m-\mu)} x(t) \\ &= \frac{1}{\Gamma(m-\mu)} \Delta_h^m \int_a^{t-(m-\mu)h+h} (t-\sigma(\tau))_h^{(m-\mu-1)} x(\tau)\Delta_h \tau, \end{aligned} \tag{4.1} $$

for $t \in (h\mathbb{N})_{a+(m-\mu)h}$, where $\Delta_h x(t) = (x(t+h) - x(t))/h$.

Lemma 4.2 (Leibniz rule). *Assume that* $x : (h\mathbb{N})_{a+\nu h} \times (h\mathbb{N})_a \to \mathbb{R}$. *Then*

(4.2)
$$\Delta_h\left[\int_a^{t-\nu h+h} x(t,\tau)\Delta_h\tau\right] = \int_a^{t-\nu h+2h} \Delta_h x(t,\tau)\Delta_h\tau + x(t, t-\nu h+h),$$

for $t \in (h\mathbb{N})_{a+\nu h}$, *where the* $\Delta_h x(t,\tau)$ *inside the integral means the* (*partial*) *h-difference of* x *with respect to* t.

Proof. In order to see that (4.2) holds for $t \in (h\mathbb{N})_{a+\nu h}$,

$$\begin{aligned}
&\Delta_h\left[\int_a^{t-\nu h+h} x(t,\tau)\Delta_h\tau\right]\\
&= \frac{\int_a^{t-\nu h+2h} x(t+h,\tau)\Delta_h\tau - \int_a^{t-\nu h+h} x(t,\tau)\Delta_h\tau}{h}\\
&= \frac{\int_a^{t-\nu h+2h} x(t+h,\tau)\Delta_h\tau - \int_a^{t-\nu h+2h} x(t,\tau)\Delta_h\tau}{h}\\
&\quad + \frac{\int_{t-\nu h+h}^{t-\nu h+2h} x(t,\tau)\Delta_h\tau}{h}\\
&= \int_a^{t-\nu h+2h} \Delta_h x(t,\tau)\Delta_h\tau + x(t, t-\nu h+h),
\end{aligned}$$

which is the desired result. □

Consider the fractional difference equation

$$ {}_a\Delta_h^{\nu} x(t) = c(t)x(t+\nu h), \quad 0 < \nu < 1, \tag{4.3}$$

$t \in (h\mathbb{N})_{a+(1-\nu)h}$.

Theorem 4.3. *Assume that* $c(t) \le 0$. *Then, the solution of equation* (4.3) *with* $x(a) > 0$ *is positive.*

Proof. Using Leibniz formula (4.2), $\Delta_h(t-a)_h^{(-\nu)} = -\nu(t-a)_h^{(-\nu-1)}$ and the convention $1/\Gamma(0) = 0$, we have

$$ {}_a\Delta_h^{\nu} x(t) = \frac{1}{\Gamma(1-\nu)}\Delta_h \int_a^{t-(1-\nu)h+h} (t-\sigma(\tau))_h^{(-\nu)} x(\tau)\Delta_h\tau$$

$$= -\frac{\nu}{\Gamma(1-\nu)}\int_a^{t+\nu h+h} (t-\sigma(\tau))_h^{(-\nu-1)} x(\tau)\Delta_h \tau$$
$$= \frac{h}{\Gamma(-\nu)} \sum_{\tau\in[a,t+\nu h+h)} (t-\sigma(\tau))_h^{(-\nu-1)} x(\tau).$$

Take $t = a + (1-\nu)h + kh$, $\tau = a + jh$, $k \in \mathbb{N}_0$, $j = 0, 1, \ldots, k+1$. Using (4.3), we have

$$\begin{aligned}
{}_a\Delta_h^\nu x(t) &= \frac{h}{\Gamma(-\nu)} \sum_{j=0}^{k+1} ((k-j)h - \nu h)_h^{(-\nu-1)} x(a+jh) \\
&= h^{-\nu}\Bigg[x(a+kh+h) - \nu x(a+kh) \\
&\qquad + \frac{-\nu(-\nu+1)}{2!} x\left(\frac{a+(k-1)}{h}\right) \\
&\qquad\qquad + \cdots + \frac{-\nu(-\nu+1)\cdots(-\nu+k)}{(k+1)!} x(a)\Bigg] \\
&= c(t)x(a+kh+h).
\end{aligned}$$

Thus,

$$\begin{aligned}
(1 - c(t)h^\nu)x(a+kh+h) &= \nu x(a+kh) + \frac{\nu(-\nu+1)}{2!} x\left(\frac{a+(k-1)}{h}\right) \\
&\qquad + \cdots + \frac{\nu(-\nu+1)\cdots(-\nu+k)}{(k+1)!} x(a).
\end{aligned}$$

Due to $c(t) \le 0$, $x(a) > 0$ and $0 < \nu < 1$, using the principle of strong induction, we obtain $x(a+ih) > 0$ for $i \in \mathbb{N}_0$. This completes the proof. $\square$

Theorem 4.4. *Let*

$$F_{p,\nu}^h(t,a) = \sum_{i=1}^{\infty} p^{i-1} \frac{(t-a+i\nu h-h)_h^{(i\nu-1)}}{\Gamma(i\nu)},$$

$t \in (h\mathbb{N})_a$, $0 < \nu < 1$, $|p| < h^{-\nu}$. *Then,* $F_{p,\nu}^h(t,a)$ *is the solution of the initial value problem*:

$$ {}_a\Delta_h^\nu x(t) = p x(t+h\nu), \quad t \in (h\mathbb{N})_{a+(1-\nu)h},$$
$$x(a) = \frac{h^{\nu-1}}{1-ph^\nu}.$$

Proof. For simplicity, let $a = 0$. It is easy to obtain that

$$F^h_{p,\nu}(0,0) = \frac{h^{\nu-1}}{1-ph^{\nu}} > 0$$

and, similar to the proof of Lemma 2.6, we can interchange the sum order and get

$$\begin{aligned}
&{}_0\Delta_h^{\nu} F^h_{p,\nu}(t,0)\\
&\quad = \Delta_h {}_0\Delta_h^{-(1-\nu)} F^h_{p,\nu}(t,0)\\
&\quad = \Delta_h {}_0\Delta_h^{-(1-\nu)} \sum_{i=1}^{\infty} p^{i-1} \frac{(t+ih\nu-h)_h^{(i\nu-1)}}{\Gamma(i\nu)}\\
&\quad = \Delta_h \left[\frac{1}{\Gamma(1-\nu)} \int_0^{t+\nu h} (t-\sigma(\tau))_h^{(-\nu)} \sum_{i=1}^{\infty} p^{i-1} \frac{(\tau+ih\nu-h)_h^{(i\nu-1)}}{\Gamma(i\nu)} \Delta_h\tau\right]\\
&\quad = \Delta_h \left[\frac{1}{\Gamma(1-\nu)} \sum_{i=1}^{\infty} \int_0^{t+\nu h} (t-\sigma(\tau))_h^{(-\nu)} p^{i-1} \frac{(\tau+ih\nu-h)_h^{(i\nu-1)}}{\Gamma(i\nu)} \Delta_h\tau\right]\\
&\quad = \Delta_h \sum_{i=1}^{\infty} p^{i-1} \frac{{}_0\Delta_h^{-(1-\nu)}(t+ih\nu-h)_h^{(i\nu-1)}}{\Gamma(i\nu)}.
\end{aligned}$$

Using (2.4) and $\Delta_h (t-a)_h^{(\nu)} = \nu(t-a)_h^{(\nu-1)}$, we have

$$\begin{aligned}
{}_0\Delta_h^{\nu} F^h_{p,\nu}(t,0) &= \Delta_h \sum_{i=1}^{\infty} \left[p^{i-1} \frac{(t+ih\nu-h)_h^{(i\nu-1+1-\nu)}}{\Gamma(i\nu)} \right.\\
&\qquad\qquad \left. \cdot \frac{\Gamma(i\nu)}{\Gamma(i\nu-1+1+1-\nu)}\right]\\
&= \Delta_h \sum_{i=1}^{\infty} p^{i-1} \frac{(t+ih\nu-h)_h^{(i\nu-\nu)}}{\Gamma(i\nu-\nu+1)}\\
&= \sum_{i=1}^{\infty} p^{i-1} \frac{(i\nu-\nu)(t+ih\nu-h)_h^{(i\nu-\nu-1)}}{\Gamma(i\nu-\nu+1)}\\
&= p \sum_{i=1}^{\infty} p^{i-2} \frac{(t+ih\nu-h)_h^{(i\nu-\nu-1)}}{\Gamma(i\nu-\nu)}
\end{aligned}$$

$$
\begin{aligned}
&= p\sum_{j=0}^{\infty} p^{j-1}\frac{(t+h\nu+jh\nu-h)_h^{(j\nu-1)}}{\Gamma(j\nu)} \\
&= p\sum_{j=1}^{\infty} p^{j-1}\frac{(t+h\nu+jh\nu-h)_h^{(j\nu-1)}}{\Gamma(j\nu)} \\
&= pF_{p,\nu}^h(t+h\nu,0),
\end{aligned}
$$

for $t \in (h\mathbb{N})_{(1-\nu)h}$, where we use the convention $1/\Gamma(0) = 0$. This completes the proof. $\square$

From Theorem 4.3 and Theorem 4.4, we obtain the following.

Corollary 4.5. *Assume that $-h^{-\nu} < p < 0$. Then,*

$$
F_{p,\nu}^h(t,a) = \sum_{i=1}^{\infty} p^{i-1}\frac{(t-a+i\nu h-h)_h^{(i\nu-1)}}{\Gamma(i\nu)} > 0
$$

for $t \in (h\mathbb{N})_a$.

Theorem 4.6. *Assume that $-h^{-\nu} < p < 0$. Then,*

$$
\lim_{t\to\infty} E_{p,\nu}^h(t,a) = 0.
$$

Proof. For simplicity, let $a = 0$. From Corollary 4.5, we have

$$
\begin{aligned}
\Delta_h E_{p,\nu}^h(t,0) &= h^{-1}\sum_{i=0}^{\infty}\frac{p^i h^{i\nu}}{\Gamma(i\nu)}\cdot\frac{\Gamma((t/h)+i\nu)}{\Gamma((t/h)+1)} \\
&= \sum_{i=0}^{\infty} p^i\frac{(t+hi\nu-h)_h^{(i\nu-1)}}{\Gamma(i\nu)} = pF_{p,\nu}^h(t,0) < 0
\end{aligned}
\tag{4.4}
$$

for $t \in (h\mathbb{N})_a$. From Remark 3.4 and $E_{p,\nu}^h(0,0) = 1$, we obtain that $E_{p,\nu}^h(t,0) > 0$. Set $u(t) := E_{p,\nu}^h(t,0)$. From (4.4), we have that $\lim_{t\to\infty} u(t)$ exists. Arguing by contradiction, we assume that $\lim_{t\to\infty} u(t) = l > 0$ for $t \in (h\mathbb{N})_0$. Using $u(t) - u(0) ={}_{(1-\nu)h}\,\Delta_h^{-\nu}{}_0\Delta_{h,*}^{\nu}u(t)$, we have

$$
\begin{aligned}
u(t) - u(0) &= {}_{(1-\nu)h}\Delta_h^{-\nu} pu(t+\nu h) \\
&= \frac{h}{\Gamma(\nu)}\sum_{\tau\in[(1-\nu)h,\,t-\nu h+h)} (t-\sigma(\tau))_h^{(\nu-1)} pu(\tau+\nu h).
\end{aligned}
$$

Due to $u(\tau+\nu h)\geq u(t)>0$, $\tau=(1-\nu)h,\,(1-\nu)h+h,\dots,t-\nu h$, and $p<0$, using ${}_\tau\Delta_h(t-\tau)_h^{(\nu)}=-\nu(t-\sigma(\tau))_h^{(\nu-1)}$, we obtain:

$$\begin{aligned}
u(t)-u(0) &\leq \frac{phu(t)}{\Gamma(\nu)}\sum_{\tau\in[(1-\nu)h,t-\nu h+h)}(t-\sigma(\tau))_h^{(\nu-1)}\\
&= -\frac{phu(t)}{\nu\Gamma(\nu)}(t-\tau)_h^{(\nu)}\Big|_{\tau=(1-\nu)h}^{t-\nu h+h}\\
&= \frac{phu(t)}{\Gamma(\nu+1)}(t-(1-\nu)h)_h^{(\nu)},
\end{aligned}$$

where we use the convention $1/\Gamma(0)=0$. Taking $t=kh$, $k\in\mathbb{N}_0$, we have

$$\begin{aligned}
u(t)-u(0) &\leq phu(kh)\cdot\frac{h^\nu\Gamma(k+\nu)}{\Gamma(\nu+1)\Gamma(k)}\\
&= ph^{\nu+1}u(kh)\frac{k^\nu}{\Gamma(\nu+1)}\cdot\frac{\Gamma(k+\nu)}{k^\nu\Gamma(k)}\longrightarrow-\infty
\end{aligned}$$

as $k\to\infty$, where we use

$$\lim_{k\to\infty}\frac{\Gamma(k+\nu)}{k^\nu\Gamma(k)}=1.$$

However,

$$u(t)-u(0)=u(kh)-u(0)\longrightarrow l-u(0)<0$$

as $k\to\infty$. This yields a contradiction, and we obtain that $\lim_{t\to\infty}u(t)=0$. $\square$

Remark 4.7. From (4.4), we can get that, when $-h^{-\nu}<c(t)=c<0$, we have

$$\Delta_hE_{c,\nu}^h(t,a)<0.$$

Thus, the solution of (1.1) with $u(a)>0$ is decreasing.

Let $x(t)=E_{p,\nu}^h(t,a)$. Figure 2 illustrates the validity of Theorem 4.6 and Remark 4.7.

Theorem $\mathbf{B}_2$. *Assume that $0<\nu<1$, $x(a)>0$. If there exists a constant b_2 such that $c(t)\leq b_2<0$, then the solution of equation* (1.1)

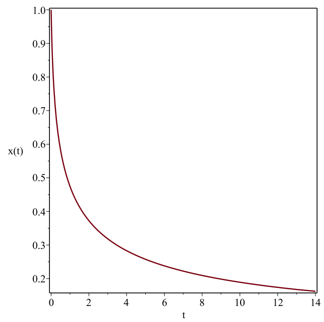

FIGURE 2. Asymptotic behavior of Mittag-Leffler function $E^h_{p,\nu}(t,a)$ for $a=0$, $\nu=0.5$, $h=0.25$, $p=-0.9$.

satisfies

$$\lim_{t\to\infty} x(t)=0.$$

Proof. Assume that $b_2 > -h^{-\nu}$. Otherwise, we can choose $0 > b_2' > -h^{-\nu}$, $b_2' > b_2$ and replace b_2 by b_2'. From Lemma 2.6, we have

$$_a\Delta^{\nu}_{h,*}E^h_{b_2,\nu}(t,a)=b_2E^h_{b_2,\nu}(t+h\nu,a)$$

for $t\in(h\mathbb{N})_{a+(1-\nu)h}$ and $E^h_{b_2,\nu}(a,a)=1$.

In Theorem 3.3, take $c_1(t)=b_2$. Then, $x(t)$ and $y(t)=2x(a)E^h_{b_2,\nu}(t,a)$ satisfy

$$_a\Delta^{\nu}_{h,*}x(t)=c(t)x(t+\nu h)$$

and

$$_a\Delta^{\nu}_{h,*}y(t)=b_2y(t+\nu h),$$

respectively, for $t\in(h\mathbb{N})_{a+(1-\nu)h}$ and

$$x(a)<2x(a)=2x(a)E^h_{b_2,\nu}(a,a)=y(a).$$

From Remark 3.4, we obtain

$$0<x(t)\le 2x(a)E^h_{b_2,\nu}(t,a)$$

for $t\in(h\mathbb{N})_a$. From Theorem 4.6, we get that

$$\lim_{t\to\infty} x(t) = 0.$$

This completes the proof. □

Remark 4.8 (see Example 6.4)**.** If $c(t)$ in Theorem B_2 is not a constant, then the solution of equation (1.1) may not be monotonically decreasing.

Remark 4.9. In [**5**], when $c(t) = c < 0$ and $x(a) > 0$, Baleanu, Wu, et al., using qualitative theory, proved that the solution of equation (1.1) satisfies $\lim_{t\to\infty} x(t) = 0$.

5. Asymptotic behavior with initial value, $x(a) < 0$. Consider the following νth order Caputo-like fractional h-difference equation with an initial condition:

$$ {}_a\Delta_{h,*}^{\nu} x(t) = c(t)x(t+\nu h), \tag{5.1}$$

$$ x(a) = x_a < 0, \quad 0 < \nu < 1, \tag{5.2}$$

where $t \in (h\mathbb{N})_{a+(1-\nu)h} = \{a + (1-\nu)h, a + (2-\nu)h, \ldots\}$. By making the transformation $x(t) = -y(t)$ and using Theorem A, Theorem B_1, and Theorem B_2, we obtain the following results.

Theorem C_1. *Assume that $0 < \nu < 1$, $x(a) < 0$ and there exists a constant b_1 such that $0 < b_1 \le c(t) < h^{-\nu}$. Then, the solution of equation* (5.1) *satisfies* $\lim_{t\to\infty} x(t) = -\infty$.

Theorem C_2. *Assume that $0 < \nu < 1$, $x(a) < 0$. If there exists a constant b_2 such that $c(t) \le b_2 < 0$, then the solution of equation* (5.1) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

6. Examples. We now consider the numerical solution of equation (1.1) to show the validity of Theorem B_1, Theorem B_2, Remark 3.7 and Remark 4.8. Let $a = 0$ and $t = (1-\nu)h + kh$, $k \ge 0$. We have the numerical formula for equation (1.1):

$$x((k+1)h) = \frac{h^{-\nu}}{-h^{-\nu} + c((1-\nu)h + kh)} \cdot \left[\sum_{i=1}^{k} \frac{(-\nu)^{\bar{i}}}{i!} x((k+1-i)h) - \frac{(1-\nu)^{\bar{k}}}{k!} x(0) \right],$$

where $x^{\bar{k}} = \Gamma(x+k)/\Gamma(x)$.

Example 6.1. Consider the linear discrete fractional equation

$$\begin{aligned} {}_0\Delta_{0.5,*}^{0.9} x(t) &= 0.3x(t+0.45), \\ x(0) = 0.1, \quad \nu &= 0.9, \quad h = 0.5, \\ t \in (h\mathbb{N})_{(1-\nu)h} &= (0.5\mathbb{N})_{0.05}. \end{aligned} \tag{6.1}$$

Plot the solution $x(t)$ in Figure 3. It is easily seen that $x(t)$ monotonically tends toward ∞ as $t \to \infty$.

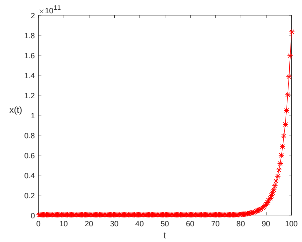

FIGURE 3. Asymptotic behavior of $x(t)$ for $\nu = 0.9$, $h = 0.5$, $x(0) = 0.1$, $c(t) = 0.3$.

Example 6.2. Consider the linear discrete fractional equation

$$\begin{aligned} {}_0\Delta_{0.5,*}^{0.9} x(t) &= \left(\frac{1}{2}\sin^2 t + \frac{1}{200}\right) x(t+0.45), \\ x(0) = 0.1, \quad \nu &= 0.9, \quad h = 0.5, \\ t \in (h\mathbb{N})_{(1-\nu)h} &= (0.5\mathbb{N})_{0.05}. \end{aligned} \tag{6.2}$$

Plot the solution $x(t)$ in Figure 4. It is easily seen that $x(t)$ tends toward ∞ as $t \to \infty$ but is not monotonically increasing.

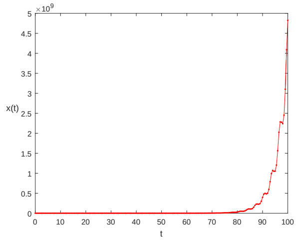

FIGURE 4. Asymptotic behavior of $x(t)$ for $\nu = 0.9$, $h = 0.5$, $x(0) = 0.1$, $c(t) = (1/2)\sin^2 t + 1/200$.

Example 6.3. Consider the linear discrete fractional equation

$$\begin{aligned} {}_0\Delta^{0.5}_{0.5,*}x(t) &= -5x(t+0.25), \\ x(0) = 0.1, \quad \nu &= 0.5, \quad h = 0.5, \\ t \in (h\mathbb{N})_{(1-\nu)h} &= (0.5\mathbb{N})_{0.25}. \end{aligned} \tag{6.3}$$

Plot the solution $x(t)$ in Figure 5. It is easily seen that $x(t)$ monotonically tends toward 0 as $t \to \infty$.

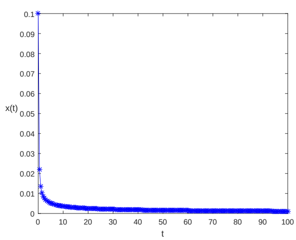

FIGURE 5. Asymptotic behavior of $x(t)$ for $\nu = 0.5$, $h = 0.5$, $x(0) = 0.1$, $c(t) = -5$.

Example 6.4. Consider the linear discrete fractional equation

$$\begin{aligned} {}_0\Delta^{0.5}_{0.5,*}x(t) &= (-2-\sin t)x(t+0.25), \\ x(0) = 0.1, \quad \nu &= 0.5, \quad h = 0.5, \\ t \in (h\mathbb{N})_{(1-\nu)h} &= (0.5\mathbb{N})_{0.25}. \end{aligned} \tag{6.4}$$

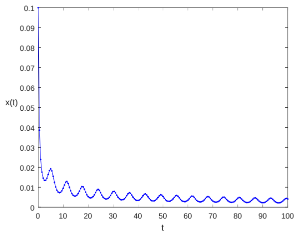

FIGURE 6. Asymptotic behavior of $x(t)$ for $\nu = 0.5$, $h = 0.5$, $x(0) = 0.1$, $c(t) = -2 - \sin t$.

Plot the solution $x(t)$ in Figure 6. It is easily seen that $x(t)$ tends to 0 as $t \to \infty$ but is not monotonically decreasing.

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Sun Yat-Sen University, School of Mathematics, Guangzhou, 510275, China
Email address: **mcsjbg@mail.sysu.edu.cn**

Sun Yat-Sen University, School of Mathematics, Guangzhou, 510275, China
Email address: **1065265114@qq.com**

Sun Yat-Sen University, School of Mathematics, Guangzhou, 510275, China
Email address: **qinjin65@126.com**

Sun Yat-Sen University, School of Mathematics, Guangzhou, 510275, China
Email address: **1196338819@qq.com**