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WELSCHINGER INVARIANTS OF BLOW-UPS OF SYMPLECTIC 4-MANIFOLDS

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ABSTRACT. Using the degeneration technique, we study the behavior of Welschinger invariants under the blow-up and obtain some blow-up formulae of Welschinger invariants. To analyze the variation of Welschinger invariants when replacing a pair of real points in the real configuration by a pair of conjugated points, Welschinger introduced the θ-invariant. In this paper, we also verify that the θ-invariant is the Welschinger invariant of the blow-up of the symplectic 4-manifold.

1. Introduction. Traditional enumerative geometry asks certain questions to which the expected answer is a number: for example, the number of lines incident with two points in the plane, or the number of twisted cubic curves on a quintic 3-fold. For the last two decades, the complex enumerative geometry of curves in algebraic varieties has taken a new direction with the appearance of Gromov-Witten invariants and quantum cohomology. The core of Gromov-Witten invariants is so-called *counting the numbers of rational curves.* On the real enumerative geometry side, a real version of Gromov-Witten invariants has been expected for a long time. In 2005, Welschinger **[36, 37]** first discovered such an invariant in dimensions 4 and 6, which was called the *Welschinger invariant* and revolutionized real enumerative geometry. Recently, it was partially extended to higher dimensions, higher genera and descendant type, see **[8, 31, 32]** and the references therein for details. Itenberg, Kharlamov and Shustin **[23]** also extended the algebraic definition of Welschinger invariants to all del Pezzo surfaces and proved the invariance under deformation in the algebraic setting.

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After the Welschinger invariants were well defined for real symplectic 4-manifolds, the focus of the research on Welschinger invariants turned toward its computation for some manifolds and the understanding of its global structure. Itenberg, Kharlamov and Shustin [**15, 16, 18, 19, 20, 21**] systematically studied the Welschinger invariants of del Pezzo surfaces, including the lower bounds of invariants, the logarithmic equivalence of Welschinger and Gromov-Witten invariants, positivity, and the Caporaso-Harris type formula for Welschinger invariants. Brugallé and Mikhalkin [**4, 5**] provided a method for computing Welschinger invariants via a floor diagram. Using their method, Arroyo, Brugallé and de Medrano [**1**] computed the Welschinger invariants in the projective plane. Based on the open analogues of Kontsevich and Manin axioms and the WDVV equation, Horev and Solomon [**10**] gave a recursive formula of Welschinger invariants of real blow up of the projective plane.

Using the degeneration technique, Itenberg, Kharlamov and Shustin [**24**] studied the positivity and asymptotics of Welschinger invariants of real del Pezzo surfaces of degree ≥ 2 and obtained some new real Caporaso-Harris type formulae as well as real analogues of the Abramovich-Bertram-Vakil formula. In [**6, 7**], Brugallé and Puignau applied the real version of the symplectic sum formula to obtain a real version of the Abramovich-Bertram-Vakil formula in the symplectic setting. Combining their formula with a degeneration formula and the technique of floor diagrams relative to a conic, Brugallé [**2**] computed the Gromov-Witten invariants and Welschinger invariants of some del Pezzo surfaces.

Other important issues in the study of Welschinger invariants are understanding the behavior of Welschinger invariants under geometric transformations and applying Welschinger invariants to investigate the geometry and topology of the underlying manifolds. In [**2, 3, 7, 24**], the authors used the degeneration technique to study the properties of Welschinger invariants. In particular, by locally modifying the real structure, Brugallé [**3**] proved very simple relations among Welschinger invariants of real symplectic 4-manifolds differing by a real surgery along a real Lagrangian sphere. In fact, his real surgery is a kind of real symplectic blow-up along a real Lagrangian sphere, see [**3,** Section 5] for the details.

From the research of algebraic geometry and Gromov-Witten theory [**26, 27**], we know that the invariants obtained from the moduli spaces always have a close relationship with the birational transformation. As is well known, blow-up is the basic birational transformation. The moduli space of genus zero curves is well behaved under blow-up. The absolute value of the Welschinger invariant provided a lower bound for the number of real pseudo-holomorphic curves passing through a particular real configuration and representing a degree, whereas an upper bound is given by the corresponding genus zero Gromov-Witten invariant. Inspired by the works on Gromov-Witten invariants **[3, 11, 12, 13]**, we will study the behavior of Welschinger invariants under real symplectic blow-ups in this paper.

A *real symplectic* 4*-manifold* (X, ω, τ), denoted by $X_{\mathbb{R}}$, is a symplectic 4-manifold (X, ω) with an involution τ on X such that $\tau^*\omega = -\omega$. The fixed point set of τ, denoted by $\mathbb{R}X$, is called the *real part* of X. $\mathbb{R}X$ is either empty or a smooth Lagrangian submanifold of (X, ω). An ω-tamed almost complex structure J is called τ-compatible if τ is J-antiholomorphic. The space of all τ-compatible almost complex structures on X is denoted by $\mathbb{R}\mathcal{J}_\omega$. Let $c_1(X)$ be the first Chern class of the symplectic manifold (X, ω). Let $d \in H_2(X; \mathbb{Z})$ be a homology class satisfying $c_1(X) \cdot d > 0$ and $\tau_* d = -d$. Let L be a connected component of $\mathbb{R}X$. Assume that $\underline{x} \subset X$ is a real configuration consisting of r real points in L and s pairs of τ-conjugated points in $X \backslash \mathbb{R}X$, where $r + 2s = c_1(X) \cdot d - 1$. Fix a τ-invariant class $F \in H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$. Denote by $W_{X_{\mathbb{R}}, L, F}(d, s)$ the Welschinger invariants. For simplicity of notation, we assume that $\mathbb{R}X$ is connected and $F = 0$. In this situation, we denote $W_{X_{\mathbb{R}}}(d, s)$ instead of $W_{X_{\mathbb{R}}, L, F}(d, s)$.

Let $p : X_{a,b} \to X$ be the real symplectic blow-up of X at a real points and b pairs of τ-conjugated points. Denote $p^! d = PDp^*PD(d)$, where PD stands for the Poincaré duality.

From the geometric point of view, an intuitive observation is that we will obtain the same number when we try to count the real rational pseudo-holomorphic curves in X and its blow-up using Welschinger's method if the blown-up points are away from the real configuration. This implies the following theorems.

Theorem 1.1. *Let $X_{\mathbb{R}}$ be a compact real symplectic* 4*-manifold, $d \in H_2(X; \mathbb{Z})$, such that $c_1(X) \cdot d > 0$ and $\tau_* d = -d$. Denote by $p : X_{1,0} \to X_{\mathbb{R}}$ the projection of the real symplectic blow-up of $X_{\mathbb{R}}$ at $x \in \mathbb{R}X$. Then*

$$W_{X_{\mathbb{R}}}(d,s) = W_{X_{1,0}}(p^! d, s), \tag{1.1}$$

$$W_{X_{\mathbb{R}}}(d,s) = W_{X_{1,0}}(p^! d - [E], s) \quad \text{if } c_1(X)\cdot d - 2s \geqslant 2, \tag{1.2}$$

where E denotes the exceptional divisor and $p^! d = PDp^*PD(d)$.

Theorem 1.2. *Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, $d \in H_2(X;\mathbb{Z})$ such that $c_1(X)\cdot d > 0$ and $\tau_* d = -d$. Suppose that y_1, $y_2 \in X \setminus \mathbb{R}X$ is a τ-conjugated pair, i.e., $\tau(y_1) = y_2$. Denote by $p : X_{0,1} \to X_{\mathbb{R}}$ the projection of the real symplectic blow-up of $X_{\mathbb{R}}$ at y_1, y_2. Then,*

$$W_{X_{\mathbb{R}}}(d,s) = W_{X_{0,1}}(p^! d, s), \tag{1.3}$$

$$W_{X_{\mathbb{R}}}(d,s) = W_{X_{0,1}}(p^! d - [E_1] - [E_2], s-1) \quad \text{if } s \geqslant 1, \tag{1.4}$$

where E_1 and E_2 denote the exceptional divisors at y_1, y_2, respectively.

From Theorems 1.1 and 1.2, it is easy to obtain the following.

Corollary 1.3. *Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, $d \in H_2(X;\mathbb{Z})$ such that $c_1(X)\cdot d > 0$ and $\tau_* d = -d$. Suppose that $\underline{x}' \subset X$ is a real set consisting of r' points in $\mathbb{R}X$ and s' pairs of τ-conjugated points in $X\backslash\mathbb{R}X$ with $r' \leqslant r$, $s' \leqslant s$. Denote by $p : X_{r',s'} \to X$ the projection of the real symplectic blow-up of X at $\underline{x}'$. Then:*

$$W_{X_{\mathbb{R}}}(d,s) = W_{X_{r',s'}}(p^! d, s), \tag{1.5}$$

$$W_{X_{\mathbb{R}}}(d,s) = W_{X_{r',s'}}\left(p^! d - \sum_{i=1}^{r'} [E_i] - \sum_{j=1}^{s'} ([E'_j] + [E''_j]), s - s'\right), \tag{1.6}$$

where E_i, E'_j, E''_j denote the exceptional divisors corresponding to the real set $\underline{x}'$, respectively.

Welschinger [**36**] introduced a new θ-invariant to describe the dependence of Welschinger invariants on the number of real points in the real configurations and obtained a *wall-crossing formula*, see [**36,** Theorem 3.2]. More precisely, when replacing a pair of real fixed points in the same component of $\mathbb{R}X$ by a pair of imaginary conjugated points, twice the θ-invariant is the difference of the respective invariants. In this paper, using the degeneration method, we reprove Welschinger's wall-crossing formula and verify that Welschinger's θ-invariants of $X_{\mathbb{R}}$

are the Welschinger invariants of the real blow-up $X_{1,0}$ of the real symplectic manifold at one real point.

Theorem 1.4. *Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, $d \in H_2(X;\mathbb{Z})$ such that $c_1(X)\cdot d \geqslant 4$ and $\tau_* d = -d$. Denote by $p : X_{1,0} \to X$ the projection of the real symplectic blow-up of $X_{\mathbb{R}}$ at $x \in \mathbb{R}X$. If $s \geq 1$, then*

$$W_{X_{\mathbb{R}}}(d, s-1) = W_{X_{\mathbb{R}}}(d, s) + 2W_{X_{1,0}}(p^! d - 2[E], s-1), \tag{1.7}$$

where E denotes the exceptional divisor and $p^! d = PDp^* PD(d)$.

Remark 1.5. The same argument as in the proofs of the previous theorems generalizes formulae (1.5) and (1.6) to the general case where $\mathbb{R}X$ is disconnected. More precisely, assume that $X_{\mathbb{R}}$ is a compact real symplectic 4-manifold and $\mathbb{R}X$ is disconnected. Suppose that $\underline{x}' \subset X$ is a real set comprised of r' points in L and s' pairs of τ-conjugated points in X with $r' \leqslant r$, $s' \leqslant s$. Denote by $\widetilde{L}$ the connected component of $\mathbb{R}X_{r',s'}$ corresponding to L . If only one of the blown-up real points belongs to L, $\widetilde{L} = L \sharp \mathbb{R}P^2$, we assume that the τ-invariant class F has a τ-invariant compact representative $\mathcal{F} \subset X \setminus \underline{x}'$ and denote $\widetilde{F} = p^! F$. Denote by $p : X_{r',s'} \to X$ the projection of the real symplectic blow-up of X at $\underline{x}'$. Then

$$W_{X_{\mathbb{R}},L,F}(d, s) = W_{X_{r',s'},\widetilde{L},\widetilde{F}}(p^! d, s), \tag{1.8}$$

(1.9)

$$W_{X_{\mathbb{R}},L,F}(d, s) = W_{X_{r',s'},\widetilde{L},\widetilde{F}}\left(p^! d - \sum_{i=1}^{r'}[E_i] - \sum_{j=1}^{s'}([E'_j] + [E''_j]), s - s'\right),$$

$$W_{X_{\mathbb{R}},L,F}(d, s-1) = W_{X_{\mathbb{R}},L,F}(d, s) + 2W_{X_{1,0},\widetilde{L},\widetilde{F}}(p^! d - 2[E], s-1), \tag{1.10}$$

where E_i, E'_j, E''_j denote the exceptional divisors corresponding to the real set $\underline{x}'$, respectively.

2. Preliminaries.

2.1. Real blow-ups of the projective plane. In this subsection, we consider how the standard real structure, i.e., the conjugation on $\mathbb{C}P^2$, induces a real structure on $\widetilde{\mathbb{C}P}^2$. For this purpose, we must distinguish the real points from other points in $\mathbb{C}P^2 \setminus \mathbb{R}P^2$.

First, we review the blow-up of $\mathbb{C}P^2$ at a point x. Let U be a neighborhood of x with local coordinate (z_1, z_2). Denote

$$\pi : V := \{((z_1, z_2), [w_1 : w_2]) \in U \times \mathbb{C}P^1 \mid z_i w_j = z_j w_i\} \longrightarrow U$$

the projection to U via the first factor. There is a natural identification map gl between $V \setminus E$ and $U \setminus \{x\}$, where $E = \pi^{-1}(0)$ is the exceptional divisor. We obtain the blow-up

$$\widetilde{\mathbb{C}P}^2 = \mathbb{C}P^2 \setminus \{x\} \bigcup_{gl} V.$$

For a real point $x \in \mathbb{R}P^2 \subset \mathbb{C}P^2$, denote by $\mathbb{C}P^2_{1,0}$ the blow-up of $\mathbb{C}P^2$ at x. We may choose a conjugation invariant neighborhood U of x in $\mathbb{C}P^2$ and the local coordinate (z_1, z_2).

Define an involution $\tau : V \to V$ as

$$\tau((z_1, z_2), [w_1 : w_2]) := ((\overline{z}_1, \overline{z}_2), [\overline{w}_1 : \overline{w}_2]).$$

It is easy to verify that this involution coincides with that induced on $V \setminus E$ by identification with $U \setminus \{x\}$. This implies that the standard real structure on $\mathbb{C}P^2$ naturally induces a real structure on $\mathbb{C}P^2_{1,0}$ at a real point $x \in \mathbb{R}P^2$.

Since there is no conjugation-invariant neighborhood for the points in $\mathbb{C}P^2 \setminus \mathbb{R}P^2$, to obtain a real structure on the blow-up, we need to simultaneously blow up a pair of conjugated points. Denote by $\mathbb{C}P^2_{0,1}$ the blow-up of $\mathbb{C}P^2$ at a pair of conjugated points. The real structure on it can be similarly constructed.

2.2. Symplectic cut. Lerman's symplectic cutting **[25]** is a simple and versatile operation on Hamiltonian S^1-manifolds. Suppose that $X_0 \subset X$ is an open codimension zero connected submanifold with a Hamiltonian S^1-action. Let $H : X_0 \longrightarrow \mathbb{R}$ be a Hamiltonian function with 0 as a regular value. If $H^{-1}(0)$ is a separating hypersurface of X, then we obtain two connected manifolds $X_0^\pm$ with boundary $\partial X_0^\pm = H^{-1}(0)$, where the $+$ side corresponds to $H < 0$. Suppose further that S^1 acts freely on $H^{-1}(0)$. Then, the symplectic reduction $Z = H^{-1}(0)/S^1$ is canonically a symplectic manifold of dimension 2 or less. Collapsing the S^1-action on $\partial X^\pm = H^{-1}(0)$, we obtain smooth closed manifolds $\overline{X}^{\pm}$ containing, respectively, real codimension 2 submanifolds $Z^{\pm} = Z$ with opposite normal bundles. Furthermore, $\overline{X}^{\pm}$ admits a symplectic structure $\overline{\omega}^{\pm}$ which agrees with the restriction of ω away from Z, and whose restriction to $Z^{\pm}$ agrees with the canonical symplectic structure ω_Z on Z from symplectic reduction. The pair of symplectic manifolds $(\overline{X}^{\pm}, \overline{\omega}^{\pm})$ is called the *symplectic cut* of X along $H^{-1}(0)$.

This is neatly shown by considering $X_0 \times \mathbb{C}$ equipped with appropriate product symplectic structures and the product S^1-action on $X_0 \times \mathbb{C}$ where S^1 acts on $\mathbb{C}$ by complex multiplication. The extended action is Hamiltonian if we use the standard symplectic structure $\sqrt{-1}\, dw \wedge d\overline{w}$ or its negative on the $\mathbb{C}$ factor, see [**25**]. Denote by $\mu : X_0 \to \mathbb{R}$ the moment map of the S^1-action on X_0. Then, $\overline{X}^{+} = \overline{X}_{\mu \le \epsilon}$, $\overline{X}^{-} = \overline{X}_{\mu \ge \epsilon}$.

The normal connected sum operation [**9, 28**] or the fiber sum operation, is the inverse operation of the symplectic cut. Given two symplectic manifolds containing symplectomorphic codimension 2 symplectic submanifolds with opposite normal bundles, the normal connected sum operation produces a new symplectic manifold by identifying the tubular neighborhoods.

Note that we can apply the normal connected sum operation to the pairs $(\overline{X}^{+}, \overline{\omega}^{+}, Z^{+})$ and $(\overline{X}^{-}, \overline{\omega}^{-}, Z^{-})$ to recover (X, ω).

According to McDuff [**29**], the blow-up operation in symplectic geometry amounts to removal of an open symplectic ball followed by collapse of some boundary directions. In fact, we may apply the symplectic cut to construct the blow-up of the symplectic manifold X at a point p. By the symplectic neighborhood theorem, take X_0 to be a symplectic ball of radius ϵ_0 centered at p with complex coordinates $(z_1, \ldots, z_n)$, where $\dim X = 2n$. Consider the Hamiltonian S^1-action on X_0 by complex multiplication. Fix ϵ with $0 < \epsilon < \epsilon_0$, and consider the moment map

$$H(u) = \sum_{i=1}^{n} |z_i|^2 - \epsilon, \quad u \in X_0.$$

Write the hypersurface $P = H^{-1}(0)$ in X corresponding to the sphere with radius ϵ. We cut X along P to obtain two closed symplectic manifolds $\overline{X}^{+}$ and $\overline{X}^{-}$, one of which is $\mathbb{C}P^n$. Following the notation of [**11, 26**], we denote $\overline{X}^{+} = \mathbb{C}P^n$, where $\overline{X}^{-} = \widetilde{X}$ is the *symplectic blow-up* of X.

2.3. Real symplectic cut. Let (X, ω_X, τ_X) and (Y, ω_Y, τ_Y) be two real compact symplectic manifolds containing a common real symplectic hypersurface V with $e(N_{V|X}) + e(N_{V|Y}) = 0$, where $N_{V|X}$, $N_{V|Y}$ are the normal bundles of V in X and Y, respectively. Denote by ω_V the symplectic form $\omega_X|_V = \omega_Y|_V$ on V and by τ_V the real structure $\tau_X|_V = \tau_Y|_V$. Denote the normal connected sum of X and Y along V by $X\sharp_V Y$. There is a real structure $\tau_\sharp$ on $X\sharp_V Y$ induced by the real structures τ_X, τ_Y. In actuality, the symplectic sum operation will produce a family of symplectic manifolds $(\mathcal{Z}_\lambda, \omega_\lambda)$ parametrized by a small complex number $\lambda \in \Delta$. Suppose that $\pi : \mathcal{Z} \to \Delta$ is the symplectic sum of X and Y along V, cf., **[9, 14, 28]**. Equip the disc Δ with the complex conjugation. The real structures τ_X and τ_Y will induce a real structure $\tau_{\mathcal{Z}}$ on $\mathcal{Z}$ such that the map $\pi : \mathcal{Z} \to \Delta$ is real, see **[3, 7]** for more details regarding the real symplectic sum in dimension 4.

Let (X, ω, τ) be a real symplectic manifold. Assume that $H : X \to \mathbb{R}$ is a τ-invariant smooth Hamiltonian, i.e., $H \circ \tau = H$. Then, we call H a *real Hamiltonian*, cf., **[34]**. A Hamiltonian circle action on (X, ω, τ) is a 1-parameter subgroup $\mathbb{R} \to \mathrm{Symp}(X) : t \mapsto \psi_t$ of symplectomorphisms of X, which is 2π-periodic, i.e., $\psi_{2\pi} = \mathrm{id}$, and which is the integral of a Hamiltonian vector field X_H. The Hamiltonian function $H : X \to \mathbb{R}$ in this case is called the *moment map of the action*. If the Hamiltonian circle action on (X, ω, τ) satisfies

$$\psi_{2\pi - t} \circ \tau = \tau \circ \psi_t \tag{2.1}$$

for all $t \in [0, 2\pi]$, we call it a *real Hamiltonian circle action*. The moment map of a real Hamiltonian circle action is a real Hamiltonian.

Let (X, ω, τ) be a real symplectic manifold with a real Hamiltonian circle action. Suppose that $\mu : X \to \mathbb{R}$ is a real moment map. Let $(\mu^{-1}(0)/S^1, \omega_\mu)$ be the symplectic reduction. There is a natural real structure τ_μ on $\mu^{-1}(0)/S^1$ induced by τ on X. Define

$$\begin{aligned} \tau_\mu : \mu^{-1}(0)/S^1 &\longrightarrow \mu^{-1}(0)/S^1 \\ [x] &\longmapsto [\tau(x)]. \end{aligned}$$

Suppose that x, $y \in \mu^{-1}(0)$ such that $[x] = [y]$. Then, there is a $t \in [0, 2\pi]$ such that $\psi_t(x) = y$. By equation (2.1), $\psi_{2\pi - t}(\tau(x)) = \tau \circ \psi_t(x) = \tau(y)$. Therefore, $[\tau(x)] = [\tau(y)]$, and τ_μ is well defined. Obviously, the reduced space $(\mu^{-1}(0)/S^1, \omega_\mu, \tau_\mu)$ is a real symplectic manifold.

Then, we can find that the real symplectic manifold $(\mu^{-1}(\varepsilon)/S^1, \omega_\mu, \tau_\mu)$ is embedded in both $\overline{X}_{\mu\geqslant\varepsilon}$ and $\overline{X}_{\mu\leqslant\varepsilon}$ as a codimension 2 real symplectic submanifold but with opposite normal bundles. The pair of real symplectic manifolds $\overline{X}_{\mu\geqslant\varepsilon}$, $\overline{X}_{\mu\leqslant\varepsilon}$ is called the *real symplectic cut* of X along $\mu = \varepsilon$.

Remark 2.1. Let $X_0 \subset X$ be an open codimension zero connected real submanifold equipped with real Hamiltonian S^1 action and a real proper momentum map $\mu : X_0 \to \mathbb{R}$. Suppose that μ achieves its maximal value c at a single point $p \in \mathbb{R}X$. For a sufficiently small ε, p is the only critical point in the set $X_{\mu>c-\varepsilon} = \{x \in X_0 | c - \mu(x) < \varepsilon\}$. For all $0 < \delta < \varepsilon$, the real symplectic manifold $\overline{X}_{\mu\leqslant c-\delta}$ is the real blow-up of X_0 at p by a δ amount. We define $\overline{X}^+ := \overline{X}_{\mu\geqslant c-\delta}$ and $\overline{X}^- := (X - X_0) \cup \overline{X}_{\mu\leqslant c-\delta}$. We have $\overline{X}^+ = \mathbb{C}P^n$ and $\overline{X}^- = X_{1,0}$, where $X_{1,0}$ is the real blow-up of X at a real point. The procedure for obtaining $\overline{X}^+$ and $\overline{X}^-$ is to *perform a real symplectic cut* on X at $p \in \mathbb{R}X$. Suppose that μ achieves its maximal value c at an exceptional divisor E. Let ε be small enough. For all $0 < \delta < \varepsilon$, define $\overline{X}^+ := \overline{X}_{\mu\geqslant c-\delta}$ and $\overline{X}^- := (X - X_0) \cup \overline{X}_{\mu\leqslant c-\delta}$. Then, $\overline{X}^+ = \mathbb{C}P^n_{1,0}$ and $\overline{X}^- \cong X$, where $\mathbb{C}P^n_{1,0}$ is the real blow-up of $\mathbb{C}P^n$ at a real point. This procedure is called *performing a real symplectic cut* on X along the exceptional divisor E. One can similarly define performing a real symplectic cut at a pair of τ-conjugated points or conjugated exceptional divisors.

2.4. Welschinger invariants. Let (X, ω, τ) be a compact real symplectic 4-manifold. Denote by $\mathcal{J}_\omega$ the space of almost complex structures of X tamed by ω and which are of class C^l where $l \gg 1$ is a fixed integer which is large enough. Assume that the first Chern class $c_1(X)$ of the symplectic manifold (X, ω) is not a torsion element, and let $d \in H_2(X; \mathbb{Z})$ be a homology class satisfying $c_1(X) \cdot d > 0$ and $\tau_* d = -d$. Let $\underline{x} = (x_1, \ldots, x_m)$ be an ordered set of distinct points of X such that $\underline{x}$ is globally invariant under τ. Such a set is called a *real configuration* of points. Let $\sigma(\tau)$ be the order 2 permutation of $\{1, \ldots, m\}$ induced by τ. Let S be an oriented 2-sphere, $\mathcal{J}_S$ the space of complex structures of class C^l of S compatible with its orientation, and let $\underline{z} = (z_1, \ldots, z_m)$ be m distinct points on S. Denote by

$$\mathcal{S}^d(\underline{x}) = \{u \in W^{k,p}(S, X) | u_*[S] = d \text{ and } u(\underline{z}) = \underline{x}\},$$

where $1 \ll k \ll l$ and $p > 2$. $W^{k,p}(S,X)$ means the space of the continuous maps from S to X which are in local coordinate charts represented by functions in $W^{k,p}(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is an open set and $W^{k,p}(\Omega)$ is the standard Sobolev space. Let

$$\mathcal{P}^d(\underline{x}) = \{(u, J_S, J) \in \mathcal{S}^d(\underline{x}) \times \mathcal{J}_S \times \mathcal{J}_\omega \mid du + J \circ du \circ J_S = 0\}.$$

$\mathcal{P}^d(\underline{x})$ is the space of pseudo-holomorphic maps from S to X which pass through $\underline{x}$ and represent class d. The triple (u, J_S, J) is called a simple map if u cannot be written as $u' \circ \phi$, where $\phi : S \to S'$ is a holomorphic branched covering with $\deg(\phi) > 1$ and $u' : S' \to X$ is a pseudo-holomorphic map. Let $\mathcal{P}^*(\underline{x})$ be the subspace of $\mathcal{P}^d(\underline{x})$ consisting of simple maps.

Denote by $\mathcal{M}^d(\underline{x})$ the quotient of $\mathcal{P}^*(\underline{x})$ by the action of $\mathrm{Diff}^+(S, \underline{z})$. Let $\pi : \mathcal{M}^d(\underline{x}) \to \mathcal{J}_\omega$ be the projection.

Proposition 2.2 ([**36,** Proposition 1.8])**.** *The space $\mathcal{M}^d(\underline{x})$ is a separable Banach manifold of class C^{l-k}. The projection $\pi : \mathcal{M}^d(\underline{x}) \to \mathcal{J}_\omega$ is Fredholm of index* $\mathrm{Ind}_{\mathbb{R}}(\pi) = 2(c_1(X) \cdot d - 1 - m)$.

The manifold $\mathcal{M}^d(\underline{x})$ is equipped with a $\mathbb{Z}/2\mathbb{Z}$ action. Let $\mathbb{R}\mathcal{M}^d(\underline{x})$ denote the fixed point set of this action. $\pi_{\mathbb{R}} : \mathbb{R}\mathcal{M}^d(\underline{x}) \to \mathbb{R}\mathcal{J}_\omega$ is the projection induced by π.

Proposition 2.3. ([**36,** Proposition 1.9])**.** *The projection*

$$\pi_{\mathbb{R}} : \mathbb{R}\mathcal{M}^d(\underline{x}) \longrightarrow \mathbb{R}\mathcal{J}_\omega$$

is Fredholm of index $\mathrm{Ind}_{\mathbb{R}}(\pi_{\mathbb{R}}) = c_1(X) \cdot d - 1 - m$.

Suppose that $\mathbb{R}X$ is connected. Let $c_1(X) \cdot d - 1 = r + 2s$, $\underline{x} \subset X$ be a real configuration consisting of r real points in $\mathbb{R}X$ and s pairs of τ-conjugated points in $X \backslash \mathbb{R}X$. In this case, we denote $\mathcal{C}(d, \underline{x}, J) = \pi^{-1}(J) \subset \mathcal{M}^d(\underline{x})$, $\mathbb{R}\mathcal{C}(d, \underline{x}, J) = \pi_{\mathbb{R}}^{-1}(J) \subset \mathbb{R}\mathcal{M}^d(\underline{x})$.

Proposition 2.4 ([**36**])**.** *If $J \in \mathbb{R}\mathcal{J}_\omega$ is generic enough, the set $\mathcal{C}(d, \underline{x}, J)$ is finite. Moreover, the curves which represent elements of $\mathcal{C}(d, \underline{x}, J)$ are all irreducible and have only transversal double points as singularities. The total number of double points of curve C in $\mathcal{C}(d, \underline{x}, J)$ is equal to*

$$\delta = \frac{1}{2}(d^2 - c_1(X) \cdot d + 2).$$

Assume that $C \in \mathbb{R}\mathcal{C}(d, \underline{x}, J)$. The real double points of C are of two different kinds: either non-isolated or isolated. A real double point is called *non-isolated* if it is the local intersection of two real branches. The real nodal point, which is the local intersection of two complex conjugated branches, is called *isolated*. The *mass* $m_X(C)$ is defined to be the number of its isolated real nodal points which satisfies $0 \leqslant m_X(C) \leqslant \delta$. The integer

$$W_{X_{\mathbb{R}}}(d, s) = \sum_{C \in \mathbb{R}\mathcal{C}(d, \underline{x}, J)} (-1)^{m_X(C)}$$

neither depends upon the choice of J, $\underline{x}$, nor upon the deformation class of $X_{\mathbb{R}}$, cf., **[35, 36]**. These numbers are called *Welschinger invariants* of $X_{\mathbb{R}}$.

When the real part $\mathbb{R}X$ is disconnected, let L be a connected component of $\mathbb{R}X$. Suppose that $f : S \to X$ is an immersed real rational J-holomorphic curve in X such that $f(\mathbb{R}S) \subset L$, for a $J \in \mathbb{R}\mathcal{J}_\omega$. Denoting by S^+ a half of $S \setminus \mathbb{R}S$, $f(S^+)$ defines a class $[f(S^+)]$ in $H_2(X, L; \mathbb{Z}/2\mathbb{Z})$. There exists a well defined pairing

$$H_2(X, L; \mathbb{Z}/2\mathbb{Z}) \times H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

given by the intersection product modulo 2. Fix a τ-invariant class $F \in H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$. Define the (L, F)-mass of f as

$$m_{L,F}(f) = m_L(f) + [f(S^+)] \cdot F,$$

where $m_L(f)$ is the number of real isolated nodes of f in L. $m_{L,F}(f)$ does not depend upon the chosen half of $S \setminus \mathbb{R}S$.

Given $J \in \mathbb{R}\mathcal{J}_\omega$, the set $\mathbb{R}\mathcal{C}(d, \underline{x}, J)$ consists of real rational J-holomorphic curves $f : S \to X$ in X realizing class d, passing through $\underline{x}$ and such that $f(\mathbb{R}S) \subset L$. Note that, if $r \geqslant 1$, the condition $f(\mathbb{R}S) \subset L$ is always satisfied. Itenberg, Kharlamov and Shustin **[20]** observed that the integer

$$W_{X_{\mathbb{R}}, L, F}(d, s) = \sum_{C \in \mathbb{R}\mathcal{C}(d, \underline{x}, J)} (-1)^{m_{L,F}(C)}$$

neither depends upon the choice of J, $\underline{x}$, nor upon the deformation class of $X_{\mathbb{R}}$. Note that, if $F = [\mathbb{R}X \setminus L]$, $W_{X_{\mathbb{R}}, L, F}(d, s)$ is the original Welschinger invariant. For simplicity of notation, we assume that $\mathbb{R}X$

is connected and $F = 0$. In this situation, we denote $W_{X_{\mathbb{R}}}(d, s)$ instead of $W_{X_{\mathbb{R}},L,F}(d, s)$.

2.5. Curves with tangency conditions. When we use the degeneration technique to study the behavior of curves under blow-up, we need to deal with curves with tangency conditions. In this subsection, we review some basics on curves with tangency conditions. The applicable reference is [**7,** subsection 2.1].

Two J-holomorphic maps

$$f_1 : C_1 \longrightarrow X \quad \text{and} \quad f_2 : C_2 \longrightarrow X$$

are said to be isomorphic if there is a biholomorphism $\phi : C_1 \to C_2$ such that $f_1 = f_2 \circ \phi$. In the following, maps are always considered up to isomorphism. Given a vector $\alpha = (\alpha_i)_{1 \leqslant i < \infty} \in \mathbb{Z}^{\infty}_{\geqslant 0}$, we use the notation:

$$|\alpha| = \sum_{i=1}^{+\infty} \alpha_i, \qquad I\alpha = \sum_{i=1}^{+\infty} i\alpha_i.$$

For $k \in \mathbb{Z}_{\geq 0}$ and $\alpha = (\alpha_i)_{1 \leq i < \infty}$, denote $k\alpha := (k\alpha_i)_{1 \leq i < \infty}$. Let δ_i denote the vector in $\mathbb{Z}^{\infty}_{\geqslant 0}$, all of whose coordinates are equal to 0 except the ith one which is equal to 1.

Let (X, ω) be a compact and connected four-dimensional symplectic manifold, and let $V \subset X$ be an embedded symplectic curve in X. Let $d \in H_2(X; \mathbb{Z})$ and α, $\beta \in \mathbb{Z}^{\infty}_{\geqslant 0}$ be such that

$$I\alpha + I\beta = d \cdot [V].$$

Choose a configuration $\underline{x} = \underline{x}^{\circ} \sqcup \underline{x}_V$ of points in X, with $\underline{x}^{\circ}$ a configuration of $c_1(X) \cdot d - 1 - d \cdot [V] + |\beta|$ points in $X \setminus V$, and $\underline{x}_V = \{p_{i,j}\}_{0 < j \leqslant \alpha_i, i \geqslant 1}$ a configuration of $|\alpha|$ points in V. Given an ω-tamed almost complex structure J on X such that V is J-holomorphic, denote by $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ the set of rational J-holomorphic maps $f : \mathbb{C}P^1 \to X$ such that

- $f_*[\mathbb{C}P^1] = d$;
- $\underline{x} \subset f(\mathbb{C}P^1)$;
- V does not contain $f(\mathbb{C}P^1)$;
- $f(\mathbb{C}P^1)$ has an order of contact i with V at each of the points $p_{i,j}$;

• $f(\mathbb{C}P^1)$ has an order of contact i with V at exactly β_i distinct points on $V \setminus \underline{x}_V$.

The set of simple maps in $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ is zero-dimensional if the almost complex structure J is chosen to be generic. However, $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ might contain components of positive dimension corresponding to non-simple maps.

Lemma 2.5 ([**7,** Lemma 11])**.** *Suppose that $\beta = (d \cdot [V], 0, \ldots)$ and $\alpha = 0$, or $\beta = (d \cdot [V] - 1, 0, \ldots)$ and $\alpha = (1, 0, \ldots)$. Then, for a generic choice of J, the set $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ contains only simple maps.*

Proposition 2.6 ([**7,** Proposition 13])**.** *Suppose that V is an embedded symplectic sphere with $[V]^2 = -1$ and that $|\beta| \geqslant d \cdot [V] - 1$. Then, for a generic choice of J, the set $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ contains finitely many simple maps. As a consequence, the set*

$$\mathcal{C}^{\alpha,\beta}_*(d, \underline{x}, J) = \{f(\mathbb{C}P^1) \mid (f : \mathbb{C}P^1 \to X) \in \mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)\}$$

is also finite.

In particular, suppose that $X = \mathbb{C}P^2$, $V = H \subset \mathbb{C}P^2$ is the hyperplane in $\mathbb{C}P^2$, and $|\underline{x}^0| = 1$. The set $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ is always finite and comprised of simple maps.

Lemma 2.7. *Suppose that $X = \mathbb{C}P^2$ and $V = H \subset \mathbb{C}P^2$ is the hyperplane in $\mathbb{C}P^2$. Then, the set $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ with $|\underline{x}^0| = 1$ is empty for a generic choice of J, except $\mathcal{C}^{\delta_1,0}([H], \{p\} \cup \underline{x}_V, J)$ which contains a unique element. Moreover, this unique element is an embedding.*

Proof. Suppose that $d = a[H], a \geq 0$. Since $c_1(X) = 3[H]$, we have

$$c_1(X) \cdot (a[H]) - 1 - (a[H]) \cdot [H] + |\beta| = 2a - 1 + |\beta|.$$

Suppose that $2a - 1 + |\beta| = 1$ and $\mathcal{C}^{\alpha,\beta}(a[H], \underline{x}, J) \neq \emptyset$, where $|\underline{x}^0| = 1$. From $I\alpha + I\beta = d \cdot [V] = a[H] \cdot [V] = a$, we can obtain $a[H] \cdot [H] = a \geqslant |\beta|$. The intersection number a of the J-holomorphic curve $f \in \mathcal{C}^{\alpha,\beta}(a[H], \underline{x}, J)$ with V must satisfy $a \geqslant 0$. Thus, this yields $a = 1$, $|\beta| = 0$.

If $\mathbb{C}P^2$ is equipped with the symplectic form ω_{FS} and its standard complex structure J_{st}, it is well known that $\mathcal{C}^{\delta_1,0}([H], \{p\} \cup \underline{x}_V, J_{st})$ consists of a unique element. When ω and J are both varied, the corresponding set still contains at least one element. If there are two distinct curves C_1 and C_2 in $\mathcal{C}^{\delta_1,0}([H],\{p\}\cup \underline{x}_V, J)$, then both C_1 and C_2 pass through $\{p\}\bigcup \underline{x}_V$, which contains at least two points. Therefore, by the positivity of intersections, $C_1 \cdot C_2 = 2$. This is impossible since

$$C_1 \cdot C_2 = [H]\cdot[H] = 1.$$

This contradiction implies that $\mathcal{C}^{\delta_1,0}(H,\{p\}\cup \underline{x}_V, J)$ also consists of a unique element. Due to the adjunction formula, this J-holomorphic curve is an embedding curve. $\square$

Let $\widetilde{\mathbb{C}P}^2$ be the blow-up of $\mathbb{C}P^2$ at a point and E the exceptional divisor. It is easy to see that $\widetilde{\mathbb{C}P}^2 \cong \mathbb{P}_E(\mathcal{O}(-1)\oplus\mathcal{O})$. Let $E_0 := \mathbb{P}_E(0\oplus \mathcal{O})$ and $E_\infty := \mathbb{P}_E(\mathcal{O}(-1)\oplus 0)$. E_0 and E_∞ are two distinguished non-intersecting sections of $\mathbb{P}_E(\mathcal{O}(-1)\oplus\mathcal{O})$. The following may easily be computed:

$$[E_\infty]^2 = -[E_0]^2 = 1.$$

The group $H_2(\widetilde{\mathbb{C}P}^2,\mathbb{Z})$ is the free abelian group generated by $[E_\infty]$ and $[F]$, where F is a fiber of $\mathbb{P}_E(\mathcal{O}(-1)\oplus\mathcal{O})\to E$. The first Chern class of $\widetilde{\mathbb{C}P}^2$ is given by

$$c_1(\widetilde{\mathbb{C}P}^2) = 3[E_\infty]-[E_0] = 2[E_\infty]+[F].$$

In $X = \widetilde{\mathbb{C}P}^2 \cong \mathbb{P}_E(\mathcal{O}(-1)\oplus\mathcal{O})$, if $V = E_\infty$ and $|\underline{x}^0| = 0$, the set $\mathcal{C}^{\alpha,\beta}(d,\underline{x},J)$ is always finite and comprised of simple maps.

Lemma 2.8. *Suppose that $X = \widetilde{\mathbb{C}P}^2$ and $V = E_\infty$. Then, the set $\mathcal{C}^{\alpha,\beta}(d,\underline{x},J)$ with $|\underline{x}^0| = 0$ is empty for a generic choice of J, except $\mathcal{C}^{\delta_1,0}([F],\underline{x}_V,J)$ which contains a unique element. Moreover, the unique element is an embedding.*

Proof. Suppose that $d = a[E_\infty]+b[F]$. Since $c_1(\widetilde{\mathbb{C}P}^2) = 2[E_\infty]+[F]$, we obtain

$$c_1(X)\cdot d - 1 - d\cdot[E_\infty] + |\beta| = 2a+b-1+|\beta|.$$

Suppose that $2a+b-1+|\beta| = 0$ and $\mathcal{C}^{\alpha,\beta}(a[E_\infty]+b[F],\underline{x}_V,J)\neq\emptyset$. Since $|\beta|\leqslant a+b$ and $|\beta|\geqslant 0$, we have $a+b\geqslant 0$. By the positivity of intersection, we obtain

$$\begin{aligned} d\cdot[E_0] &= (a[E_\infty]+b[F])\cdot([E_0]) = b\geqslant 0\\ d\cdot[F] &= (a[E_\infty]+b[F])\cdot[F] \qquad = a\geqslant 0.\end{aligned}$$

We may deduce that $a = |\beta| = 0$, $b = |\alpha| = 1$.

The proof of the remainder of this lemma is similar to that in the proof of Lemma 2.7. □

3. Blow-up formula of Welschinger invariants.

3.1. Blow-up formula at a real point. In this subsection, we consider the behavior of Welschinger invariants under the blow-up of the symplectic 4-manifold at a real point.

Let X be a compact real symplectic 4-manifold. Perform a real symplectic cut on X at the real point $x \in \mathbb{R}X$ (see Remark 2.1). We can obtain two real symplectic 4-manifolds $\overline{X}^+ \cong \mathbb{P}^2$ and $\overline{X}^- \cong X_{1,0}$ which contain a common real symplectic submanifold V. In $\overline{X}^+$, $V \cong H$ is the hyperplane in $\mathbb{P}^2$. In $\overline{X}^-$, $V \cong E$ is the exceptional divisor in $X_{1,0}$.

Let $\pi : \mathcal{Z} \to \Delta$ be the real symplectic sum of $\overline{X}^+$ and $\overline{X}^-$ along V (see subsection 2.3), $d \in H_2(\mathcal{Z}_\lambda; \mathbb{Z})$. Choose $\underline{x}(\lambda)$ as a set of $c_1(X) \cdot d - 1$ real symplectic sections $\Delta \to \mathcal{Z}$ such that $\underline{x}(0) \cap V = \emptyset$. Choose an almost complex structure J on $\mathcal{Z}$ tamed by $\omega_{\mathcal{Z}}$, which restricts to an almost structure J_λ tamed by ω_λ on each fiber $\mathcal{Z}_\lambda$, and is generic with respect to all choices made.

Let $X_\sharp = \overline{X}^+ \cup_V \overline{X}^-$. Denote $\mathcal{C}(d, \underline{x}(0), J_0)$ to be the set $\{\overline{f} : \overline{C} \to X_\sharp\}$ of limits, stable maps, of maps in $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ as λ goes to 0, where $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ is the set of all irreducible rational J-holomorphic curves in $(\mathcal{Z}_\lambda, \omega_\lambda, J_\lambda)$ passing through all points in $\underline{x}(\lambda)$ and realizing the class d. From [**14,** Section 3], we know $\overline{C}$ is a connected nodal rational curve such that:

- $\underline{x}(0) \subset \overline{f}(\overline{C})$;
- any point $p \in \overline{f}^{-1}(V)$ is a node of $\overline{C}$ which is the intersection of two irreducible components $\overline{C}'$ and $\overline{C}''$ of $\overline{C}$, with $\overline{f}(\overline{C}') \subset \overline{X}^+$ and $\overline{f}(\overline{C}'') \subset \overline{X}^-$;
- if, in addition, neither $\overline{f}(\overline{C}')$ nor $\overline{f}(\overline{C}'')$ is entirely mapped into V, then the multiplicities of intersection of both $\overline{f}(\overline{C}')$ and $\overline{f}(\overline{C}'')$ with V are equal.

Given an element

$$\overline{f} : \overline{C} \longrightarrow X_\sharp$$

of $\mathcal{C}(d, \underline{x}(0), J_0)$, denote by C_*, $* = +, -$, the union of the irreducible components of $\overline{C}$ mapped into $\overline{X}^*$.

Proposition 3.1. *Assume that $\underline{x}(0)\cap\overline{X}^+$ contains at most one point, $\underline{x}(0)\cap\overline{X}^-\neq\emptyset$ if $\underline{x}(0)\cap\overline{X}^+\neq\emptyset$. Then, for a generic J_0, the set $\mathcal{C}(d, \underline{x}(0), J_0)$ is finite and only depends upon $\underline{x}(0)$ and J_0. Given an element $\overline{f}:\overline{C}\to X_\sharp$ of $\mathcal{C}(d,\underline{x}(0),J_0)$, the restriction of $\overline{f}$ to any component of $\overline{C}$ is a simple map, and no irreducible component of $\overline{C}$ is entirely mapped into V. Moreover, the following are true*:

(1) *if $\underline{x}(0)\cap\overline{X}^+=\emptyset$, then C_+ is empty. The curve C_- is irreducible, and $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{0,0}(p^!d,\underline{x}(0)\cap\overline{X}^-,J_0)$. The map $\overline{f}$ is the limit of a unique element of $\mathcal{C}(d,\underline{x}(\lambda),J_\lambda)$ as λ goes to* 0.

(2) *If $\underline{x}(0)\cap\overline{X}^+=\{p\}$, then C_+ is irreducible, and $\overline{f}(C_+)$ realizes class $[H]$. The curve C_- is irreducible, and $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{0,\delta_1}(p^!d-[E],\underline{x}(0)\cap\overline{X}^-,J_0)$. The map $\overline{f}$ is the limit of a unique element of $\mathcal{C}(d,\underline{x}(\lambda),J_\lambda)$ as λ goes to* 0.

Proof. From **[14,** Example 11.4, Lemma 14.6], we know that no component of $\overline{C}$ is entirely mapped into V, also see [**7**].

Note that $[E]^2=-1$ in the real blow-up $\overline{X}^-=X_{1,0}$, and $c_1(\overline{X}^-)\cdot[E]=1$. Suppose that $\overline{f}_*[C_+]=a[H]$, $a\geq 0$, $\overline{f}_*[C_-]=p^!d-b[E]$, $b\geq 0$. Then, we have

$$a=\overline{f}_*[C_+]\cdot[H]=(p^!d-b[E])\cdot[E]=b.$$

Since $\underline{x}(0)\cap\overline{X}^+$ contains at most one point, we will consider the two cases separately.

(1) $\underline{x}(0)\cap\overline{X}^+=\emptyset$. In this case, we know that $\overline{f}(C_-)$ passes through all of the $c_1(X)\cdot d-1$ points in $\underline{x}(0)\cap\overline{X}^-$ and realizes the class $p^!d-b[E]$ in $H_2(\overline{X}^-;\mathbb{Z})$. Suppose that C_- consists of irreducible components $\{C_{-i}\}_{i=1}^m$ and there are $0\leqslant k\leqslant m$ irreducible components $\{C_{-i}\}_{i=1}^k$ such that the restriction $\overline{f}|_{C_{-i}}$, $i=1,\ldots,k$, is non-simple, which factors through a non-trivial ramified covering of degree $\delta_i\geqslant 2$ of a simple map $f_i:\mathbb{P}^1\to\overline{X}^-$. Assume that $(f_i)_*[\mathbb{P}^1]=d_i$, $i=1,\ldots,k$, and $\overline{f}_*[C_{-j}]=d_j$, $j=k+1,\ldots,m$. Then, $\sum_{i=1}^k\delta_id_i+\sum_{j=k+1}^m d_j=p^!d-b[E]$.

$$\begin{aligned}c_1(\overline{X}^-)\cdot\left(\sum_{i=1}^k d_i\right)-k+c_1(\overline{X}^-)\cdot\left(\sum_{j=k+1}^m d_j\right)-(m-k)&\geq c_1(X)\cdot d-1\\ &=c_1(\overline{X}^-)\cdot\left(\sum_{i=1}^k\delta_id_i+\sum_{j=k+1}^m d_j\right)+b-1.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{k}(1-\delta_i)c_1(\overline{X}^-)\cdot d_i \geqslant m+b-1. \tag{3.1}$$

Since $c_1(\overline{X}^-)\cdot d_i \geqslant 0$, $b \geqslant 0$, $\delta_i \geqslant 2$, so (3.1) only holds when $m=1$, $k=0$ and $b=0$. This implies that C_- is irreducible and $\overline{f}|_{C_-}$ is simple. $b=0$ also implies $\overline{f}_*[C_+]=0$. Therefore, $C_+=\emptyset$.

The previous argument implies that $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{0,0}(p^!d, \underline{x}(0)\cap\overline{X}^-, J_0)$. Moreover, the finiteness of $\mathcal{C}^{0,0}(p^!d,\underline{x}(0)\cap\overline{X}^-, J_0)$ implies that $\mathcal{C}(d,\underline{x}(0),J_0)$ is finite.

(2) $\underline{x}(0)\cap\overline{X}^+=\{p\}$. In this case, the fact that the image of $\overline{f}(C_+)$ must pass $\{p\}$ implies $a=b\geqslant 1$. $\overline{f}(C_-)$ passes through all of the $c_1(X)\cdot d-2$ points in $\underline{x}(0)\cap X^-$ and realizes the class $p^!d-b[E]$ in $H_2(\overline{X}^-;\mathbb{Z})$. Similar to Case I, we know that C_- is irreducible. Next, we prove that $\overline{f}|_{C_-}$ is simple. For this, we assume that $\overline{f}|_{C_-}$ is non-simple. Then, $\overline{f}|_{C_-}$ factors through a non-trivial ramified covering of degree $\delta\geqslant 2$ of a simple map $f_0:\mathbb{P}^1\to\overline{X}^-$, and $(f_0)_*[\mathbb{P}^1]=(1/\delta)(p^!d-b[E])$. Therefore,

$$\frac{1}{\delta}c_1(\overline{X}^-)\cdot(p^!d-b[E])-1\geqslant c_1(X)\cdot d-2.$$

Thus, we have

$$c_1(X)\cdot d+\delta-\delta c_1(X)\cdot d\geqslant b. \tag{3.2}$$

Since $\delta\geqslant 2$, $c_1(X)\cdot d\geqslant 2$, (3.2) implies $b\leq 0$. This is in contradiction with $b\geq 1$. Therefore, $\overline{f}|_{C_-}$ is simple.

From

$$\begin{aligned} c_1(\overline{X}^-)\cdot(p^!d-b[E])-1 &= c_1(X)\cdot d-1-b \\ &\geqslant c_1(X)\cdot d-2, \end{aligned}$$

we obtain $b\leq 1$. Thus, we have $b=1$, and $\overline{f}_*[C_+]=[H]$.

The previous argument implies that $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{0,\delta_1}(p^!d-[E],\underline{x}(0)\cap\overline{X}^-, J_0)$. The finiteness of $\mathcal{C}^{0,\delta_1}(p^!d-[E],\underline{x}(0)\cap\overline{X}^-, J_0)$ implies that $\mathcal{C}(d,\underline{x}(0),J_0)$ is finite.

The number of elements of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ converging to $\overline{f}$ as λ goes to 0 follows from **[14]**. Now, we review the behavior of the elements $f_\lambda : C_\lambda \to \mathcal{Z}_\lambda$ of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ converging to $\overline{f}$ close to the smoothing of the intersection point p of C_- and C_+. In local coordinates (λ, x, y) at $\overline{f}(p)$, the manifold $\mathcal{Z}$ is given by the equation $xy = \lambda$. Locally,

$$\overline{X}^+ = \{\lambda = 0 \text{ and } y = 0\},$$
$$\overline{X}^- = \{\lambda = 0 \text{ and } x = 0\}.$$

Since the order of intersection of $\overline{f}_{C_+}$ and V at $\overline{f}(p)$ is 1, the maps $\overline{f}_{C_+}$ and $\overline{f}_{C_-}$ have expansions

$$x(z) = mz + o(z) \quad \text{and} \quad y(w) = nw + o(w),$$

where z and w are local coordinates at p of C_+ and C_-, respectively.

For $0 < |\lambda| \ll 1$, there exists a solution $\mu(\lambda) \in \mathbb{C}^*$ of

$$\mu(\lambda) = \frac{\lambda}{mn}$$

such that the smoothing of $\overline{C}$ at p is locally given by $zw = \mu(\lambda)$, and the map f_λ is approximated by the map

$$\{zw = \mu(\lambda)\} \subset \mathbb{C}^2 \longmapsto (\lambda, mz, nw)$$

close to the smoothing of p. Furthermore, such maps $f_\lambda \in \mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ converging to $\overline{f}$ are in one-to-one correspondence with the choice of such a $\mu(\lambda)$ for each point of $C_+ \cap C_-$. $\square$

Applying Proposition 3.1, we can obtain a comparison theorem of the Welschinger invariants. Let $\mathbb{R}\mathcal{C}^{\alpha^r+\alpha^c, \beta^r+\beta^c}(d, \underline{x}, J)$ be the subset of real rational curves in $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$, $\alpha = \alpha^r + \alpha^c$ and $\beta = \beta^r + \beta^c$, such that the α (or β) "point(s)" consists of an α^r (or β^r) real "point(s)" and $(1/2)\alpha^c$ (or $(1/2)\beta^c$) pairs of τ-conjugated "points."

Proposition 3.2. *Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, $d \in H_2(X;\mathbb{Z})$ such that $c_1(X) \cdot d > 0$ and $\tau_* d = -d$. Denote by $p : X_{1,0} \to X$ the projection of the real symplectic blow-up of X at $x \in \mathbb{R}X$. Let $\underline{x}(\lambda)$, $\overline{X}^+$, $\overline{X}^-$ and J_0 be as before. Then:*

- *if* $\underline{x}(0)\cap\overline{X}^{+}=\{p\}$, $\underline{x}(0)\cap\overline{X}^{-}\neq\emptyset$,

$$W_{X_{\mathbb{R}}}(d,s)=\sum_{C_-\in\mathbb{R}\mathcal{C}^{0,\delta_1^{\tau}}(p^!d-[E],\underline{x}(0)\cap\overline{X}^{-},J_0)}(-1)^{m_{X_{1,0}}(C_-)},\tag{3.3}$$

- *if* $\underline{x}(0)\cap\overline{X}^{+}=\emptyset$,

$$W_{X_{\mathbb{R}}}(d,s)=\sum_{C_-\in\mathbb{R}\mathcal{C}^{0,0}(p^!d,\underline{x}(0)\cap\overline{X}^{-},J_0)}(-1)^{m_{X_{1,0}}(C_-)},\tag{3.4}$$

where E is the exceptional divisor.

Proof. Equip the small disc Δ with the standard complex conjugation. From subsection 2.3, we know that the symplectic sum $\pi:\mathcal{Z}\to\Delta$ can be equipped with a real structure $\tau_{\mathcal{Z}}$, which is induced by the real structures τ_-, τ_+ on the real symplectic cuts $\overline{X}^{-}$ and $\overline{X}^{+}$ such that the map $\pi:\mathcal{Z}\to\Delta$ is real. Choose a set of real sections $\underline{x}:\Delta\to\mathcal{Z}$. Let $\overline{f}:\overline{C}\to X_{\sharp}$ be a real element in $\mathbb{R}\mathcal{C}(d,\underline{x}(0),J_0)$.

For the case $\underline{x}(0)\cap\overline{X}^{+}=\{p\}$ and $\underline{x}(0)\cap\overline{X}^{-}\neq\emptyset$, from Proposition 3.1 and Lemma 2.7, we know that $\overline{f}_*[C_+]=[H]$ and $\overline{f}|_{C_+}$ is an embedded simple curve. $\overline{f}(C_+)$ has no self-intersection point; thus, $\overline{f}(C_+)$ has no node. Therefore, there is only one possibility for $\overline{f}|_{C_+}$ to recover a real curve $\overline{f}(\overline{C})$ when $\overline{f}|_{C_-}$ is fixed, in other words, the number of real curves $\overline{f}\in\mathbb{R}\mathcal{C}(d,\underline{x}(0),J_0)$ is equal to the number of the real curves $\overline{f}|_{C_-}\in\mathbb{R}\mathcal{C}^{0,\delta_1^{\tau}}(p^!d-[E],\underline{x}(0)\cap\overline{X}^{-},J_0)$. Hence, we have

$$m_{X_{\sharp}}(\overline{f}(\overline{C}))=m_{\overline{X}^{-}}(\overline{f}|_{\overline{C}_-})+m_{\overline{X}^{+}}(\overline{f}|_{C_+})=m_{\overline{X}^{-}}(\overline{f}|_{\overline{C}_-}).\tag{3.5}$$

By Proposition 3.1, an element $\overline{f}$ of $\mathcal{C}(d,\underline{x}(0),J_0)$ is the limit of a unique element of $\mathcal{C}(d,\underline{x}(\lambda),J_\lambda)$; thus, the latter must be real when $\overline{f}$ is real and $\lambda\in\mathbb{C}^*$ is small. The description at the end of the proof of Proposition 3.1 of the local deformation of $\overline{f}$ shows that no node appears in a neighborhood of $V\cap\overline{f}(\overline{C})$ when deforming $\overline{f}$. Combining with (3.5), this implies (3.3).

For the case $\underline{x}(0)\cap\overline{X}^{+}=\emptyset$, we know that $\overline{f}_*[C_+]=0$ from Proposition 3.1. The real curve $\overline{f}(\overline{C})$ is determined by the part $\overline{f}|_{C_-}$, in other words, the number of real curves $\overline{f}\in\mathbb{R}\mathcal{C}(d,\underline{x}(0),J_0)$ is equal to the number of the real curves $\overline{f}|_{C_-}\in\mathbb{R}\mathcal{C}^{0,0}(p^!d,\underline{x}(0)\cap\overline{X}^{-},J_0)$. Then, we have

$$m_{X_{\sharp}}(\overline{f}(\overline{C}))=m_{X^{-}}(\overline{f}|_{\overline{C}_-})$$

The remainder of Proposition 3.2 may be proven similarly to the previous case. □

Remark 3.3. Proposition 3.2 gives that the sum

$$\sum_{C_- \in \mathbb{R}\mathcal{C}^{0,\delta_1^r}(p^! d - [E], \underline{x}(0)\cap \overline{X}^-, J_0)} (-1)^{m_{X_{1,0}}(C_-)}$$

on the right side of formula (3.3) is not dependent upon $\underline{x}(0) \cap \overline{X}^-$ and J_0. It may be seen as a particular case of the relative Welschinger invariants. See **[3, 22]** for more details regarding relative Welschinger invariants.

Now, we perform a real symplectic cut along the exceptional divisor E, see Remark 2.1, of the real symplectic blow-up manifold $\widetilde{X} = X_{1,0}$. This yields two real symplectic cuts:

$$\overline{X}^+_{1,0} \cong \mathbb{P}(N_{E|X_{1,0}} \oplus \mathcal{O}_E) \cong \mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O})$$

and

$$\overline{X}^-_{1,0} \cong X_{1,0},$$

which contain a common real symplectic submanifold V. In $\overline{X}^+_{1,0}$, $V \cong E_\infty$ is the infinity section of $\mathbb{P}_E(\mathcal{O}(-1)\oplus\mathcal{O}) \to E$. In $\overline{X}^-_{1,0} \cong X_{1,0}$, $V \cong E$ is the exceptional divisor.

Let $\widetilde{\mathcal{Z}}$ be the real symplectic sum of $\overline{X}^+_{1,0}$ and $\overline{X}^-_{1,0}$ along V, see subsection 2.3. Let $p^! d - [E] \in H_2(\widetilde{\mathcal{Z}}_\lambda; \mathbb{Z})$, where $d \in H_2(X;\mathbb{Z})$. Choose $\underline{\widetilde{x}}_1(\lambda)$ as a set of $c_1(X)\cdot d - 1$ real symplectic sections $\Delta \to \widetilde{\mathcal{Z}}$ such that $\underline{\widetilde{x}}_1(0) \cap V = \emptyset$, $\underline{\widetilde{x}}_1(0) \cap \overline{X}^+_{1,0} = \emptyset$. Choose $\underline{\widetilde{x}}_2(\lambda)$ as a set of $c_1(X)\cdot d - 2$ real symplectic sections $\Delta \to \widetilde{\mathcal{Z}}$ such that $\underline{\widetilde{x}}_2(0) \cap V = \emptyset$, $\underline{\widetilde{x}}_2(0) \cap \overline{X}^+_{1,0} = \emptyset$. Choose a generic almost complex structure $\widetilde{J}$ on $\widetilde{\mathcal{Z}}$ as above.

Let $\widetilde{X}_\sharp = \overline{X}^+_{1,0} \cup_V \overline{X}^-_{1,0}$. Define $\mathcal{C}(p^! d, \underline{\widetilde{x}}_1(0), \widetilde{J}_0)$, $\mathcal{C}(p^! d - [E], \underline{\widetilde{x}}_2(0), \widetilde{J}_0)$ to be the set $\{\overline{f} : \overline{C} \to \widetilde{X}_\sharp\}$ of limits, as stable maps, of maps in $\mathcal{C}(p^! d, \underline{\widetilde{x}}_1(\lambda), \widetilde{J}_\lambda)$, $\mathcal{C}(p^! d - [E], \underline{\widetilde{x}}_2(\lambda), \widetilde{J}_\lambda)$ as λ goes to 0, respectively. Given an element $\overline{f} : \overline{C} \to \widetilde{X}_\sharp$ of $\mathcal{C}(p^! d, \underline{\widetilde{x}}_1(0), \widetilde{J}_0)$ or $\mathcal{C}(p^! d - [E], \underline{\widetilde{x}}_2(0), \widetilde{J}_0)$, denote by C_*, $* = +, -$, the union of the irreducible components of $\overline{C}$ mapped to $\overline{X}^*_{1,0}$.

Proposition 3.4. *Under the above assumptions, we have the following*:

(1) *For a generic* $\widetilde{J}_0$, *the set* $\mathcal{C}(p^! d, \underline{\widetilde{x}}_1(0), \widetilde{J}_0)$ *is finite and only depends upon* $\underline{\widetilde{x}}_1(0)$ *and* $\widetilde{J}_0$. *Given an element* $\overline{f} : \overline{C} \to \widetilde{X}_\sharp$ *of* $\mathcal{C}(p^! d, \underline{\widetilde{x}}_1(0), \widetilde{J}_0)$, *the restriction of* $\overline{f}$ *to any component of* $\overline{C}$ *is a simple map, and no irreducible component of* $\overline{C}$ *is entirely mapped into* V. *Moreover, the curve* C_- *is irreducible and* $\overline{f}|_{C_-}$ *is an element of* $\mathcal{C}^{0,0}(p^! d, \underline{\widetilde{x}}_1(0) \cap \overline{X}^-_{1,0}, \widetilde{J}_0)$. *The map* $\overline{f}$ *is the limit of a unique element of* $\mathcal{C}(p^! d, \underline{\widetilde{x}}_1(\lambda), \widetilde{J}_\lambda)$ *as* λ *goes to* 0.

(2) *For a generic* $\widetilde{J}_0$, *the set* $\mathcal{C}(p^! d - [E], \underline{\widetilde{x}}_2(0), \widetilde{J}_0)$ *is finite and only depends upon* $\underline{\widetilde{x}}_2(0)$ *and* $\widetilde{J}_0$. *Given an element* $\overline{f} : \overline{C} \to \widetilde{X}_\sharp$ *of* $\mathcal{C}(p^! d - [E], \underline{\widetilde{x}}_2(0), \widetilde{J}_0)$, *the restriction of* $\overline{f}$ *to any component of* $\overline{C}$ *is a simple map, and no irreducible component of* $\overline{C}$ *is entirely mapped into* V. *Moreover, the curve* C_- *is irreducible, and* $\overline{f}|_{C_-}$ *is an element of* $\mathcal{C}^{0,\delta_1}(p^! d - [E], \underline{\widetilde{x}}_2(0) \cap \overline{X}^-_{1,0}, \widetilde{J}_0)$. *The map* $\overline{f}$ *is the limit of a unique element of* $\mathcal{C}(p^! d - [E], \underline{\widetilde{x}}_2(\lambda), \widetilde{J}_\lambda)$ *as* λ *goes to* 0.

Proof. The fact that no component of $\overline{C}$ is entirely mapped into V follows from [**7**], [**14,** Example 11.4, Lemma 14.6].

(1) Suppose that $\overline{f}_*[C_+] = a[F] + b[E_\infty]$, $\overline{f}_*[C_-] = p^! d - k[E]$, $k \geq 0$, where F is a fiber of $\mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}) \to E$ with $F \cdot [E_0] = 1$ and $F \cdot [E_\infty] = 1$. Then

$$a + b = (a[F] + b[E_\infty]) \cdot [E_\infty] = (p^! d - k[E]) \cdot [E] = k,$$
$$a = (a[F] + b[E_\infty]) \cdot [E_0] = (p^! d) \cdot [E] = 0.$$

In $\overline{X}^-_{1,0}$, we know that $\overline{f}|_{C_-}$ passes through

$$|\underline{\widetilde{x}}_1(0)| = c_1(X) \cdot d - 1 = c_1(\overline{X}^-_{1,0}) \cdot (p^! d) - 1$$

distinct points in $\overline{X}^-_{1,0}$. The same argument as in the proof of Proposition 3.1 shows that C_- is irreducible.

Next, we prove that $\overline{f}|_{C_-}$ is simple. For this, we assume that $\overline{f}|_{C_-}$ is non-simple. Then, $\overline{f}|_{C_-}$ factors through a non-trivial ramified covering of degree $\delta \geqslant 2$ of a simple map $f_0 : \mathbb{P}^1 \to \overline{X}^-_{1,0}$ and $(f_0)_*[\mathbb{P}^1] = (1/\delta)(p^! d - k[E])$. Therefore,

$$
\begin{aligned}
&\frac{1}{\delta}c_1(\overline{X}^-_{1,0})\cdot(p^!d-k[E])-1\geqslant c_1(\overline{X}^-_{1,0})\cdot(p^!d)-1 \\
&\qquad(1-\delta)c_1(\overline{X}^-_{1,0})\cdot(p^!d)\geqslant k.
\end{aligned}
\tag{3.6}
$$

Since $c_1(X)\cdot d\geqslant 1$, $\delta\geqslant 2$, $k\geqslant 0$, so (3.6) is impossible. Therefore, $\overline{f}|_{C_-}$ can only be simple.

On the other hand, we have

$$
\begin{aligned}
c_1(\overline{X}^-_{1,0})\cdot(p^!d-k[E])-1 &= c_1(\overline{X}^-_{1,0})\cdot p^!d-k-1 \\
&\geqslant c_1(\overline{X}^-_{1,0})\cdot(p^!d)-1.
\end{aligned}
$$

This implies $k=0$ and $b=0$. Therefore, $C_+=\emptyset$, and $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{0,0}(p^!d,\underline{\widetilde{x}}_1(0)\cap\overline{X}^-_{1,0},\widetilde{J}_0)$, which also is a simple map.

(2) Suppose that $\overline{f}_*[C_+]=a[F]+b[E_\infty]$, $\overline{f}_*[C_-]=p^!d-k[E]$, $k\geqslant 1$, where F is a fiber of $\mathbb{P}_E(\mathcal{O}(-1)\oplus\mathcal{O})\to E$ with $F\cdot[E_0]=1$ and $F\cdot[E_\infty]=1$. Then

$$
\begin{aligned}
a+b&=(a[F]+b[E_\infty])\cdot[E_\infty]=(p^!d-k[E])\cdot[E]=k,\\
a&=(a[F]+b[E_\infty])\cdot[E_0]=(p^!d-[E])\cdot[E]=1.
\end{aligned}
$$

In $\overline{X}^-_{1,0}$, we know that $\overline{f}|_{C_-}$ passes through

$$
c_1(X)\cdot d-2=c_1(\overline{X}^-_{1,0})\cdot(p^!d-[E])-1
$$

distinct points. The same argument as in the proof of Proposition 3.1 shows that C_- is irreducible.

By a similar analysis of the dimension condition as in the proof of Proposition 3.4 (1), we obtain that $\overline{f}_*[C_+]=[F]$, C_+ must have exactly one component, and the image of it must be realized in the class $[F]$. $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{0,\delta_1}(p^!d-[E],\underline{\widetilde{x}}_2(0)\cap\overline{X}^-_{1,0},\widetilde{J}_0)$, which is a simple map.

The proofs of the other parts are the same as those of Proposition 3.1. We omit them here. □

Proposition 3.5. *Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, $d\in H_2(X;\mathbb{Z})$ such that $c_1(X)\cdot d-1=r+2s>0$ and $\tau_*d=-d$. Denote by $p:X_{1,0}\to X$ the projection of the real symplectic blow-up* *of X at $x \in \mathbb{R}X$. Let $\underline{\widetilde{x}}_1(\lambda)$, $\underline{\widetilde{x}}_2(\lambda)$, $\widetilde{J}_0$, $\overline{X}^+_{1,0}$ and $\overline{X}^-_{1,0}$ be as previously defined. Then*:

$$W_{X_{1,0}}(p^!(d), s) = \sum_{C_- \in \mathbb{R}\mathcal{C}^{0,0}(p^! d, \underline{\widetilde{x}}_1(0) \cap \overline{X}^-_{1,0}, \widetilde{J}_0)} (-1)^{m_{X_{1,0}}(C_-)},$$

$$W_{X_{1,0}}(p^!(d) - [E], s) = \sum_{C_- \in \mathbb{R}\mathcal{C}^{0,\delta_1^r}(p^! d - [E], \underline{\widetilde{x}}_2(0) \cap \overline{X}^-_{1,0}, \widetilde{J}_0)} (-1)^{m_{X_{1,0}}(C_-)},$$

where E is the exceptional divisor.

Remark 3.6. Similar to Proposition 3.2, by Proposition 3.4, one can prove Proposition 3.5. Moreover, Propositions 3.2 and 3.5 imply Theorem 1.1.

3.2. Blow-up formula at a conjugated pair. Let (X, ω) be a compact connected real symplectic 4-manifold, and let V_1,$V_2 \subset X$ be two disjoint embedded symplectic curves in X. Let $d \in H_2(X;\mathbb{Z})$ and α^1, α^2, β^1, $\beta^2 \in \mathbb{Z}^{\infty}_{\geqslant 0}$ be such that

$$I\alpha^1 + I\beta^1 = d \cdot [V_1], \qquad I\alpha^2 + I\beta^2 = d \cdot [V_2].$$

Choose a configuration $\underline{x} = \underline{x}^\circ \sqcup \underline{x}_{V_1} \sqcup \underline{x}_{V_2}$ of points in X, with $\underline{x}^\circ$ a configuration of $c_1(X) \cdot d - 1 - d \cdot ([V_1] + [V_2]) + |\beta_1| + |\beta_2|$ points in $X \setminus (V_1 \cup V_2)$, $\underline{x}_{V_1} = \{p_{i,j}\}_{0<j\leqslant\alpha^1_i, i\geqslant 1}$ a configuration of $|\alpha^1|$ points in V_1, and $\underline{x}_{V_2} = \{q_{i,j}\}_{0<j\leqslant\alpha^2_i, i\geqslant 1}$ a configuration of $|\alpha^2|$ points in V_2. Given an ω-tamed almost-complex structure J on X such that V_1 and V_2 are J-holomorphic, denote by $\mathcal{C}^{\alpha^1,\beta^1,\alpha^2,\beta^2}(d, \underline{x}, J)$ the set of rational J-holomorphic maps $f : \mathbb{C}P^1 \to X$ such that

- $f_*[\mathbb{C}P^1] = d$;
- $\underline{x} \subset f(\mathbb{C}P^1)$;
- $V_1 \cup V_2$ does not contain $f(\mathbb{C}P^1)$;
- $f(\mathbb{C}P^1)$ has order of contact i with V_1 at each of the points $p_{i,j}$ and has order of contact i with V_2 at each of the points $q_{i,j}$;
- $f(\mathbb{C}P^1)$ has order of contact i with V_1 at exactly β^1_i distinct points on $V_1 \setminus \underline{x}_{V_1}$ and has order of contact i with V_2 at exactly β^2_i distinct points on $V_2 \setminus \underline{x}_{V_2}$.

Note that $\mathcal{C}^{\alpha^1,\beta^1,\alpha^2,\beta^2}(d,\underline{x},J)$ may contain components of positive dimension corresponding to non-simple maps. However, for the generic J, the set of simple maps in $\mathcal{C}^{\alpha^1,\beta^1,\alpha^2,\beta^2}(d,\underline{x},J)$ is zero-dimensional.

Lemma 3.7. *Let (X,ω) be a compact connected real symplectic 4-manifold. Suppose that V_1 and V_2 are two embedded symplectic spheres in X with $V_1\cdot V_2=0$, $[V_i]^2=-1$, $i=1,2$, and assume that $|\beta^i|=d\cdot[V_i]$, $i=1,2$. Then, for a generic choice of J, the set $\mathcal{C}^{\alpha^1,\beta^1,\alpha^2,\beta^2}(d,\underline{x},J)$ is finite and contains only simple maps which are all immersions.*

Proof. Suppose that $\mathcal{C}^{\alpha^1,\beta^1,\alpha^2,\beta^2}(d,\underline{x},J)$ contains a non-simple map which factors through a non-trivial ramified covering of degree δ of a simple map $f_0:\mathbb{C}P^1\to X$. Let d_0 denote the homology class $(f_0)_*[\mathbb{C}P^1]$. Since $f_0(\mathbb{C}P^1)$ passes through $c_1(X)\cdot d-1=\delta c_1(X)\cdot d_0-1$ points, we have

$$c_1(X)\cdot d_0-1\geqslant \delta c_1(X)\cdot d_0-1\geqslant 0,$$

which is impossible.

Suppose that $\mathcal{C}^{\alpha^1,\beta^1,\alpha^2,\beta^2}(d,\underline{x},J)$ contains infinitely many simple maps. From the Gromov compactness theorem, there exists a sequence $(f_n)_{n\geqslant 0}$ of simple maps in $\mathcal{C}^{\alpha^1,\beta^1,\alpha^2,\beta^2}(d,\underline{x},J)$ which converges to some J-holomorphic map $\overline{f}:\overline{C}\to X$. By genericity of J, the set of simple maps in $\mathcal{C}^{\alpha^1,\beta^1,\alpha^2,\beta^2}(d,\underline{x},J)$ is discrete; hence, either $\overline{C}$ is reducible, or $\overline{f}$ is non-simple. Let $\overline{C}_1,\dots,\overline{C}_m$, $\overline{C}^1_1,\dots,\overline{C}^1_{m_1}$, $\overline{C}^2_1,\dots,\overline{C}^2_{m_2}$ be the irreducible components of $\overline{C}$, labeled in such a way that

- $\overline{f}(\overline{C}_i)\nsubseteq V_1\cup V_2$;
- $\overline{f}(\overline{C}^j_i)\subset V_j$ and $(\overline{f})_*[\overline{C}^j_i]=k^j_i[V_j]$, $j=1,2$.

Define $k^j=\sum_{i=1}^{m_j}k^j_i$, $j=1,2$. The restriction of $\overline{f}$ to $\cup_{i=1}^m\overline{C}_i$ is subject to $c_1(X)\cdot d-1-d\cdot([V_1]+[V_2])+|\beta^1|+|\beta^2|$ point constraints; thus, we have

$$\begin{aligned}c_1(X)\cdot(d-k^1[V_1]-k^2[V_2])-m&\\ &\geqslant c_1(X)\cdot d-1-d\cdot([V_1]+[V_2])+|\beta^1|+|\beta^2|.\end{aligned}$$

Since both V_1 and V_2 are embedded symplectic spheres, from the adjunction formula, we can obtain $c_1(X)\cdot[V_i]=1$, $i=1,2$. Hence, we have

$$c_1(X)\cdot d - k^1 - k^2 - m \geqslant c_1(X)\cdot d - 1 - d\cdot([V_1]+[V_2]) + |\beta^1| + |\beta^2|.$$

From the assumption of Lemma 3.7, we have

$$0 \geqslant m + k^1 + k^2 - 1.$$

Therefore, this yields $m = 1$, $k^1 = k^2 = 0$. The map $\overline{f}$ must be a non-simple map which factors through a non-trivial ramified covering of a simple map $f_0 : \mathbb{C}P^1 \to X$. However, f_0 is subject to more point constraints, which provides a contradiction. □

Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, and suppose that y_1, $y_2 \in X \setminus \mathbb{R}X$ is a τ-conjugated pair. Denote by $p : X_{0,1} \to X$ the projection of the real symplectic blow-up of X at y_1, y_2. Perform a real symplectic cut of X at the τ-conjugated pair y_1, y_2, see Remark 2.1. We obtain

$$\overline{X}^+ = \overline{X}^{+1} \sqcup \overline{X}^{+2} \cong \mathbb{P}^2 \sqcup \mathbb{P}^2, \quad \overline{X}^- \cong X_{0,1}.$$

Both $\overline{X}^+$ and $\overline{X}^-$ contain a common real symplectic submanifold $V = V_1 \sqcup V_2$ of real codimension 2. In $\overline{X}^+$, $V_1 \cong H_1$, $V_2 \cong H_2$, where H_i is the hyperplane of $\overline{X}^{+i}$, $i = 1, 2$. $V_1 \cong E_1$ and $V_2 \cong E_2$ are the associated exceptional divisors in $\overline{X}^-$ at y_i, $i = 1, 2$, respectively.

Let $\mathcal{Z}$ be the real symplectic sum of the two real symplectic manifolds $\overline{X}^+$ and $\overline{X}^-$ along V, see subsection 2.3, and let $d \in H_2(\mathcal{Z}_\lambda;\mathbb{Z})$. Denote $\underline{x}(\lambda)$, J, $X_\sharp = \overline{X}^+ \cup_V \overline{X}^-$, $\mathcal{C}(d, \underline{x}(0), J_0)$, C_{+i}, $i = 1, 2$, as in subsection 3.1. Similar to the proof of Proposition 3.1, we can prove the following.

Proposition 3.8. *Assume that $\underline{x}(0) \cap \overline{X}^{+i}$ contains at most one point, $\underline{x}(0) \cap \overline{X}^- \neq \emptyset$ if $\underline{x}(0) \cap \overline{X}^+ \neq \emptyset$. Let $\underline{x}(0)$, d and J be given as above. Then, for a generic J_0, the set $\mathcal{C}(d, \underline{x}(0), J_0)$ is finite and only depends upon $\underline{x}(0)$ and J_0. Given an element $\overline{f} : \overline{C} \to X_\sharp$ of $\mathcal{C}(d, \underline{x}(0), J_0)$, the restriction of $\overline{f}$ to any component of $\overline{C}$ is a simple map, and no irreducible component of $\overline{C}$ is entirely mapped into V. Moreover, the following are true*:

(1) *if $\underline{x}(0) \cap \overline{X}^{+i} = \{p_i\}$, $i = 1, 2$, the curve C_- is irreducible and $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{0,\delta_1,0,\delta_1}(p^! d - [E_1] - [E_2], \underline{x}(0) \cap \overline{X}^-, J_0)$. The curves C_{+i}, $i = 1, 2$, are irreducible, and the image of C_{+i} represses $[H_i]$ and passes $\{p_i\}$, respectively. The map $\overline{f}$ is the limit of a unique element of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ as λ goes to 0.*

(2) *If* $\underline{x}(0) \cap \overline{X}^{+i} = \emptyset$, $i = 1, 2$, *then* $C_{+i} = \emptyset$, *the curve* C_- *is irreducible and* $\overline{f}|_{C_-}$ *is an element of* $\mathcal{C}^{0,0,0,0}(p^! d, \underline{x}(0) \cap \overline{X}^-, J_0)$. *The map* $\overline{f}$ *is the limit of a unique element of* $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ *as* λ *goes to* 0.

Based on Proposition 3.8, similar to the proof of Proposition 3.2, we can prove the following.

Proposition 3.9. *Let* $X_{\mathbb{R}}$ *be a compact real symplectic* 4*-manifold,* $d \in H_2(X; \mathbb{Z})$ *such that* $c_1(X) \cdot d > 0$ *and* $\tau_* d = -d$. *Suppose that* y_1, $y_2 \in (X \setminus \mathbb{R}X)$ *is a* τ*-conjugated pair. Denote by* $p : X_{0,1} \to X$ *the projection of the real symplectic blow-up of* X *at* y_1, y_2. *Then*:

(1) *If* $\underline{x}(0) \cap \overline{X}^{+i} = \emptyset$, $i = 1, 2$,

$$W_{X_{\mathbb{R}}}(d, s) = \sum_{C_- \in \mathbb{R}\mathcal{C}^{0,0,0,0}(p^! d, \underline{x}(0) \cap \overline{X}^-, J_0)} (-1)^{m_{X_{0,1}}(C_-)}.$$

(2) *If* $\underline{x}(0) \cap \overline{X}^{+i} = \{p_i\}$, $i = 1, 2$, $\underline{x}(0) \cap \overline{X}^- \neq \emptyset$,

$$W_{X_{\mathbb{R}}}(d, s) = \sum_{C_- \in \mathbb{R}\mathcal{C}^{0,\delta_1^c,0,\delta_1^c}(p^! d - [E_1] - [E_2], \underline{x}(0) \cap \overline{X}^-, J_0)} (-1)^{m_{X_{0,1}}(C_-)},$$

where E_1 *and* E_2 *are the exceptional divisors.*

Next, perform the real symplectic cut of $X_{0,1}$ along E_1, E_2, where E_i is the exceptional divisor of the blow-up at y_i, $i = 1, 2$, respectively, see Remark 2.1. We obtain two real symplectic manifolds

$$\overline{\widetilde{X}}^+ = \overline{\widetilde{X}}^{+1} \sqcup \overline{\widetilde{X}}^{+2}$$

and $\overline{\widetilde{X}}^-$ as follows:

$$\overline{\widetilde{X}}^+ \cong \mathbb{P}_{E_1}(\mathcal{O}(-1) \oplus \mathcal{O}) \sqcup \mathbb{P}_{E_2}(\mathcal{O}(-1) \oplus \mathcal{O}), \quad \overline{\widetilde{X}}^- \cong X_{0,1}.$$

Both $\overline{\widetilde{X}}^+$ and $\overline{\widetilde{X}}^-$ contain a common real symplectic submanifold $V = V_1 \sqcup V_2$ of real codimension 2, respectively. In $\overline{\widetilde{X}}^+$, $V_1 \cong E^1_\infty$ and $V_2 \cong E^2_\infty$, where E^i_∞ is the infinity section of $\mathbb{P}_{E_i}(\mathcal{O}(-1) \oplus \mathcal{O}) \to E_i$, respectively. $V_1 \cong E_1$ and $V_2 \cong E_2$ are the exceptional divisors in $\overline{\widetilde{X}}^-$.

Let $\widetilde{\mathcal{Z}}$ be the symplectic sum of the two real symplectic manifolds $\overline{\widetilde{X}}^+$ and $\overline{\widetilde{X}}^-$ along V, see subsection 2.3. Let $p^! d - [E_1] - [E_2] \in H_2(\widetilde{\mathcal{Z}}_\lambda; \mathbb{Z})$, where $d \in H_2(X; \mathbb{Z})$. Choose $\widetilde{\underline{x}}_1(\lambda)$ as a set of $c_1(X) \cdot d - 3$ real symplectic sections $\Delta \to \widetilde{\mathcal{Z}}$ such that $\underline{\widetilde{x}}_1(0) \cap V = \emptyset$ and $\underline{\widetilde{x}}_1(0) \cap \overline{\widetilde{X}}^+ = \emptyset$. Choose $\underline{\widetilde{x}}_2(\lambda)$ as a set of $c_1(X) \cdot d - 1$ real symplectic sections $\Delta \to \widetilde{\mathcal{Z}}$ such that $\underline{\widetilde{x}}_2(0) \cap V = \emptyset$ and $\underline{\widetilde{x}}_2(0) \cap \overline{\widetilde{X}}^+ = \emptyset$. Choose a generic almost complex structure $\widetilde{J}$ on $\widetilde{\mathcal{Z}}$ as above.

Denote $\widetilde{X}_\sharp = \overline{\widetilde{X}}^+ \cup_V \overline{\widetilde{X}}^-$, $\mathcal{C}(p^! d - [E_1] - [E_2], \underline{\widetilde{x}}_1(0), \widetilde{J}_0)$, $\mathcal{C}(p^! d, \underline{\widetilde{x}}_2(0), \widetilde{J}_0)$, C_{+i}, $i = 1, 2$ and C_- as in subsection 3.1. The same argument as in the proofs of Propositions 3.1 and 3.2 shows that Propositions 3.10 and 3.11 hold.

Proposition 3.10. *Let $\underline{\widetilde{x}}(0)$, $p^! d - [E_1] - [E_2]$ and $\widetilde{J}$ be given as above. Then, we have*:

(1) *For a generic $\widetilde{J}_0$, the set $\mathcal{C}(p^! d - [E_1] - [E_2], \underline{\widetilde{x}}_1(0), \widetilde{J}_0)$ is finite and only depends upon $\underline{\widetilde{x}}_1(0)$ and $\widetilde{J}_0$. Given an element $\overline{f} : \overline{C} \to \widetilde{X}_\sharp$ of $\mathcal{C}(p^! d - [E_1] - [E_2], \underline{\widetilde{x}}_1(0), \widetilde{J}_0)$, the restriction of $\overline{f}$ to any component of $\overline{C}$ is a simple map, and no irreducible component of $\overline{C}$ is entirely mapped into V. Moreover, the curve $\overline{f}|_{C_-}$ is irreducible, and $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{0,\delta_1,0,\delta_1}(p^! d - [E_1] - [E_2], \underline{\widetilde{x}}_1(0) \cap \widetilde{X}^-, \widetilde{J}_0)$. C_{+i}, $i = 1, 2$, are irreducible, and the image of C_{+i} under $\overline{f}$ represents the fiber class $[F_i]$ of $\mathbb{P}_{E_i}(\mathcal{O}(-1) \oplus \mathcal{O}) \to E_i$, respectively. The map $\overline{f}$ is the limit of a unique element of $\mathcal{C}(p^! d - [E_1] - [E_2], \underline{\widetilde{x}}_1(\lambda), \widetilde{J}_\lambda)$ as λ goes to* 0.

(2) *For a generic $\widetilde{J}_0$, the set $\mathcal{C}(p^! d, \underline{\widetilde{x}}_2(0), \widetilde{J}_0)$ is finite and only depends upon $\underline{\widetilde{x}}_2(0)$ and $\widetilde{J}_0$. Given an element $\overline{f} : \overline{C} \to \widetilde{X}_\sharp$ of $\mathcal{C}(p^! d, \underline{\widetilde{x}}_2(0), \widetilde{J}_0)$, the restriction of $\overline{f}$ to any component of $\overline{C}$ is a simple map, and no irreducible component of $\overline{C}$ is entirely mapped into V. Moreover, the curve $\overline{f}|_{C_-}$ is irreducible, and $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{0,0,0,0}(p^! d, \underline{\widetilde{x}}_2(0) \cap \overline{\widetilde{X}}^-, \widetilde{J}_0)$. The map $\overline{f}$ is the limit of a unique element of $\mathcal{C}(p^! d, \underline{\widetilde{x}}_2(\lambda), \widetilde{J}_\lambda)$ as λ goes to* 0.

Proposition 3.11. *Let $X_\mathbb{R}$ be a compact real symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ such that $c_1(X) \cdot d > 0$ and $\tau_* d = -d$. Suppose that y_1, $y_2 \in (X \setminus \mathbb{R}X)$ is a τ-conjugated pair. Denote by $p : X_{0,1} \to X$ the projection of the real symplectic blow-up of X at y_1, y_2. Then*:

$$W_{X_{0,1}}(p^! d, s) = \sum_{C_- \in \mathbb{R}\mathcal{C}^{0,0,0,0}(p^! d, \underline{\widetilde{x}}_2(0) \cap \widetilde{X}^-, \widetilde{J}_0)} (-1)^{m_{X_{0,1}}(C_-)}.$$

Moreover, if $s \geq 1$, then

$$W_{X_{0,1}}(p^! d - [E_1] - [E_2], s-1) = \sum_{C_- \in \mathbb{R}\mathcal{C}^{0,\delta_1^c,0,\delta_1^c}(p^! d - [E_1] - [E_2], \underline{\widetilde{x}}_1(0) \cap \widetilde{X}^-, \widetilde{J}_0)} (-1)^{m_{X_{0,1}}(C_-)},$$

where E_1 and E_2 are the exceptional divisors.

Remark 3.12. Propositions 3.9 and 3.11 imply Theorem 1.2.

4. Wall-crossing formula of Welschinger invariants.

4.1. Wall-crossing formula. Welschinger [**36**] introduced a new invariant $\theta_{X_{\mathbb{R}}}(d,s)$ to describe the variation of Welschinger invariants when replacing a pair of real fixed points in the same component of $\mathbb{R}X$ by a pair of τ-conjugated points. Welschinger proved the following wall-crossing formula, [**36,** Theorem 3.2].

Theorem 4.1 ([**36**])**.** *Let (X, ω, τ) be a compact real symplectic 4-manifold such that $\mathbb{R}X$ is connected, $d \in H_2(X;\mathbb{Z})$ such that $c_1(X) \cdot d - 1 > 0$ and $\tau_* d = -d$, and let s be an integer between 1 and $[(c_1(X) \cdot d - 1)/2]$. Then*:

$$W_{X_{\mathbb{R}}}(d, s-1) = W_{X_{\mathbb{R}}}(d,s) + 2\theta_{X_{\mathbb{R}}}(d, s-1).$$

In algebraic geometry, Itenberg, Kharlamov and Shustin [**16**] observed that the invariant $\theta_{X_{\mathbb{R}}}(d,s)$ may be considered as the Welschinger invariants on the blow-up at the fixed real point. In the following, we will use the degeneration technique to verify this observation for symplectic 4-manifolds.

Perform the real symplectic cut on X at the real point $x \in \mathbb{R}X$ (see Remark 2.1). This yields

$$\overline{X}^+ \cong \mathbb{P}^2 \quad \text{and} \quad \overline{X}^- \cong X_{1,0}.$$

In this section, we assume that $d \in H_2(X, \mathbb{Z})$ such that $c_1(X) \cdot d \geq 4$ and $\tau_* d = -d$. Denote $\pi : \mathcal{Z} \to \Delta$, $\underline{x}(\lambda)$, J, $X_\sharp$, $\mathcal{C}(d, \underline{x}(0), J_0)$, C_*, $* = +, -$, as in subsection 3.1. First, we have the following.

Proposition 4.2. *Assume that $\underline{x}(0) \cap \overline{X}^+ = \{p_1, p_2\}$, and $\underline{x}(0) \cap \overline{X}^- \neq \emptyset$. Then, for a generic J_0, the set $\mathcal{C}(d, \underline{x}(0), J_0)$ is finite and only de-*

pends upon $\underline{x}(0)$ *and* J_0. *Given an element* $\overline{f}:\overline{C}\to X_\sharp$ *of* $\mathcal{C}(d,\underline{x}(0), J_0)$, *the restriction of* $\overline{f}$ *to any component of* $\overline{C}$ *is a simple map, and no irreducible component of* $\overline{C}$ *is entirely mapped into* V. *Moreover*:

(1) C_+ *is irreducible, and* $\overline{f}(C_+)$ *realizes the class* $[H]$ *passing through* $\{p_1,p_2\}$. *The curve* C_- *is irreducible, and* $\overline{f}|_{C_-}$ *is an element of* $\mathcal{C}^{\delta_1,0}(p^! d-[E],\underline{x}(0)\cap\overline{X}^-\sqcup\{q\},J_0)$ *for some* $q\in V$. *The map* $\overline{f}$ *is the limit of a unique element of* $\mathcal{C}(d,\underline{x}(\lambda),J_\lambda)$ *as* λ *goes to* 0.

(2) C_+ *has exactly two irreducible components, and the image of each component realizes the class* $[H]$ *and passes through one point of* $\{p_1,p_2\}$. *The curve* C_- *is irreducible, and* $\overline{f}|_{C_-}$ *is an element of* $\mathcal{C}^{0,2\delta_1}(p^! d-2[E],\underline{x}(0)\cap\overline{X}^-,J_0)$. *The map* $\overline{f}$ *is the limit of a unique element of* $\mathcal{C}(d,\underline{x}(\lambda),J_\lambda)$ *as* λ *goes to* 0.

Proof. Example 11.4 and Lemma 14.6 in **[14]** imply that no component of $\overline{C}$ is entirely mapped into V. In the real blow-up $\overline{X}^-\cong X_{1,0}$, $[E]^2=-1$. The adjunction formula implies that $c_1(\overline{X}^-)\cdot[E]=1$. Suppose that $\overline{f}_*[C_+]=a[H]$, $\overline{f}_*[C_-]=p^! d-b[E]$. Thus, by considering a representative of V in $\overline{X}^+$ and another in $\overline{X}^-$, respectively, we have

$$a=\overline{f}_*[C_+]\cdot[H]=(p^! d-b[E])\cdot[E]=p^! d\cdot[E]+b=b.$$

Since $\underline{x}(0)\cap\overline{X}^+=\{p_1,p_2\}$, then $\overline{f}(C_+)$ passes through the two points p_1 and p_2. Next, $\underline{x}(0)\cap\overline{X}^-\neq\emptyset$ implies that $a=b\geqslant 1$ and $c_1(X)\cdot d\geq 4$. Therefore, $\overline{f}(C_-)$ passes through all of the $c_1(X)\cdot d-3$ points in $\underline{x}(0)\cap\overline{X}^-$ and realizes the class $p^! d-b[E]$ in $H_2(\overline{X}^-;\mathbb{Z})$.

Suppose that C_- consists of irreducible components $\{C_{-i}\}_{i=1}^m$ with $0\leqslant k\leqslant m$ irreducible components $\{C_{-i}\}_{i=1}^k$ such that the restriction $\overline{f}|_{C_{-i}}$, $i=1,\dots,k$, is non-simple, which factors through a non-trivial ramified covering of degree $\delta_i\geqslant 2$ of a simple map $f_i:\mathbb{P}^1\to\overline{X}^-$. Assume that $(f_i)_*[\mathbb{P}^1]=d_i$, $i=1,\dots,k$, and $\overline{f}_*[C_{-j}]=d_j$, $j=k+1,\dots,m$. Then, $\sum_{i=1}^k\delta_i d_i+\sum_{j=k+1}^m d_j=p^! d-b[E]$.

$$\begin{aligned} c_1(\overline{X}^-)\cdot\left(\sum_{i=1}^k d_i\right)-k+c_1(\overline{X}^-)\cdot\left(\sum_{j=k+1}^m d_j\right)-(m-k)&\geq c_1(X)\cdot d-3\\ &=c_1(\overline{X}^-)\cdot\left(\sum_{i=1}^k\delta_i d_i+\sum_{j=k+1}^m d_j\right)+b-3.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{k}(1-\delta_i)c_1(\overline{X}^-)\cdot d_i \geqslant m+b-3. \tag{4.1}$$

Since $c_1(\overline{X}^-)\cdot d_i \geqslant 0$, we have

$$m+b\leq 3.$$

First, assume that $k\geq 1$. Then (4.1) implies $m+b<3$. Thus, we have $m=b=1$. This implies that C_- has one component. Furthermore, assume that $\overline{f}|_{C_-}$ factors through a non-trivial ramified covering of degree $\delta\geq 2$ of a simple map $\overline{f}_-:\mathbb{P}^1\longrightarrow\overline{X}^-$. Then, we have

$$\frac{1}{\delta}c_1(\overline{X}^-)\cdot(p^!d-b[E])-1\geqslant c_1(X)\cdot d-3.$$

Therefore,

$$c_1(X)\cdot d+2\delta-\delta c_1(X)\cdot d\geqslant b,$$

since $\delta\geqslant 2$ and $c_1(X)\cdot d\geqslant 4$, $b\leqslant 0$. This is in contradiction with $b=1$, which implies that $k=0$.

Next, assume that $k=0$. Equation (4.1) implies that we need only consider the following two cases.

Case I. $m=2$, $b=1$. In this case, $\overline{f}|_{C_+}$ is constrained by $\{p_1,p_2\}$ and $\overline{f}|_{C_+}\in\mathcal{C}^{0,\delta_1}([H],\{p_1,p_2\},J_0)$. Therefore, $\overline{f}|_{C_-}$ must pass the point of intersection of $\overline{f}|_{C_+}$ and V, which is distinct from $\underline{x}(0)\cap\overline{X}^-$. $\overline{f}_{C_-}$ will pass $c_1(X)\cdot d-2=c_1(\overline{X})\cdot(p^!d-[E])-1$ distinct points, which implies that $\overline{f}_{C_-}$ is irreducible. This is in contradiction with the previous statement that C_- has two components, implying that this case is impossible.

Case II. $m=1$, $b=1$ or 2. If $b=a=1$, C_+ must have exactly one component, and its image under $\overline{f}$ realizes the class $[H]$. $\overline{f}|_{C_+}$ is simple due to Lemma 2.5:

$$\overline{f}|_{C_+}\in\mathcal{C}^{0,\delta_1}([H],\{p_1,p_2\},J_0).$$

By the positivity of intersection, only one curve exists in $\mathcal{C}^{0,\delta_1}([H],\{p_1,p_2\},J_0)$, which is an embedded simple curve. Denote by q the point of the intersection of $\overline{f}|_{C_+}$ and V. The point q depends only upon

$\mathcal{C}^{0,\delta_1}([H],\{p_1,p_2\},J_0)$. Therefore, $\overline{f}|_{C_-}$ must pass $\underline{x}(0)\cap\overline{X}^-\sqcup\{q\}$ and $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{\delta_1,0}(p^!d-[E],\underline{x}(0)\cap\overline{X}^-\sqcup\{q\},J_0)$.

If $b=a=2$, $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{0,2\delta_1}(p^!d-2[E],\underline{x}(0)\cap\overline{X}^-,J_0)$. $\overline{f}|_{C_-}$ intersects E transversely at two distinct points. Note that the curve $\overline{C}$ is rational, and any component of $\overline{f}(C_+)$ intersects E in $\overline{X}^+$; thus, C_+ has exactly two irreducible components. Furthermore, each component of $\overline{f}(C_+)$ realizes $[H]$ and passes through one point of $\{p_1,p_2\}$.

The remainder of Proposition 4.2 can be proved similarly to Proposition 3.1. We omit it here. □

Proposition 4.3. *Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, and let $d\in H_2(X;\mathbb{Z})$ be such that $c_1(X)\cdot d\geqslant 4$ and $\tau_*d=-d$. Denote by $p:X_{1,0}\to X$ the projection of the real symplectic blow-up of X at $x\in\mathbb{R}X$. Then, if $s\geqslant 1$, we have*

$$\begin{aligned}W_{X_{\mathbb{R}}}(d,s-1)=&\sum_{C_1\in\mathbb{R}\mathcal{C}^{\delta_1^r,0}(p^!d-[E],\underline{x}(0)\cap\overline{X}^-\sqcup\{q\},J_0)}(-1)^{m_{X_{1,0}}(C_1)}\\&+2\sum_{C_2\in\mathbb{R}\mathcal{C}^{0,2\delta_1^r}(p^!d-2[E],\underline{x}(0)\cap\overline{X}^-,J_0)}(-1)^{m_{X_{1,0}}(C_2)},\end{aligned}\tag{4.2}$$

$$\begin{aligned}W_{X_{\mathbb{R}}}(d,s)=&\sum_{C_1\in\mathbb{R}\mathcal{C}^{\delta_1^r,0}(p^!d-[E],\underline{x}(0)\cap\overline{X}^-\sqcup\{q\},J_0)}(-1)^{m_{X_{1,0}}(C_1)}\\&-2\sum_{C_2\in\mathbb{R}\mathcal{C}^{0,2\delta_1^c}(p^!d-2[E],\underline{x}(0)\cap\overline{X}^-,J_0)}(-1)^{m_{X_{1,0}}(C_2)},\end{aligned}\tag{4.3}$$

where E is the exceptional divisor and q is some particular point in V.

Proof. Equip the small disc Δ with the standard complex conjugation. From subsection 2.3, we know that the symplectic sum $\pi:\mathcal{Z}\to\Delta$ can be equipped with a real structure $\tau_{\mathcal{Z}}$ which is induced by the real structures τ_-, τ_+ on the real symplectic cuts $\overline{X}^-$ and $\overline{X}^+$ such that the map $\pi:\mathcal{Z}\to\Delta$ is real. Choose a set of real sections $\underline{x}:\Delta\to\mathcal{Z}$ such that $\underline{x}(0)\cap\overline{X}^+_{1,0}$ contains two points. Let $\overline{f}:\overline{C}\to X_\sharp$ be a real element in $\mathbb{R}\mathcal{C}(d,\underline{x}(0),J_0)$.

Next, we will divide the proof into two cases according to the type of real configuration points.

Case I. $\underline{x}(0) \cap \overline{X}^+ = \{p_1, p_2\}$ and p_1, $p_2 \in \mathbb{R}X$. From Proposition 4.2, there are two types of the limited curve $\overline{f}$ as follows.

Type I-1. C_+ only has one component. $\overline{f}_*[C_+] = [H]$ and $\overline{f}|_{C_+} \in \mathcal{C}^{0,\delta_1}([H], \{p_1, p_2\}, J_0)$ is an embedded simple curve. The intersection point q of $\overline{f}(C_+)$ with V, determined by $\mathcal{C}^{0,\delta_1}([H], \{p_1, p_2\}, J_0)$, must be real. In this case, since $\overline{f}(C_+)$ has no self-intersection point; thus, $\overline{f}(C_+)$ has no node. Therefore, there is only one possibility to recover a real curve $\overline{f}(\overline{C})$ from $\overline{f}|_{C_+}$ when $\overline{f}|_{C_-}$ is fixed. Thus, we have

$$\begin{aligned} m_{X_\sharp}(\overline{f}(\overline{C})) &= m_{\overline{X}^-}(\overline{f}|_{\overline{C}_-}) + m_{\overline{X}^+}(\overline{f}|_{C_+}) \\ &= m_{\overline{X}^-}(\overline{f}|_{\overline{C}_-}). \end{aligned} \tag{4.4}$$

Type I-2. C_+ has exactly two irreducible components C_{+i}, $i = 1, 2$. In this case, $\overline{f}_*[C_{+i}] = [H]$, and $\overline{f}|_{C_{+i}}$ is an embedded simple curve. By the positivity of intersections, $\overline{f}(C_{+1})$ intersects $\overline{f}(C_{+2})$ at one point. This point must be a real node of $\overline{f}(C_+)$ due to the fact that $\overline{f}(C_+)$ is real. Since $\overline{f}(C_{+i})$ passes $p_i \in \mathbb{R}X$, $\overline{f}(C_{+1})$ and $\overline{f}(C_{+2})$ cannot be two τ-conjugated components. Therefore, the real nodal point of $\overline{f}(C_+)$ must be non-isolated. Moreover, each $\overline{f}(C_{+i})$ intersects V at a real point q_i transversally and $\overline{f}_{C_{+i}} \in \mathcal{C}^{\delta_1^r,0}([H], \{p_i\} \sqcup \{q_i\}, J_0)$. From Proposition 4.2 (2), we obtain that the curve C_- is irreducible, and $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{0,2\delta_1}(p^!d - 2[E], \underline{x}(0) \cap \overline{X}^-, J_0)$. $\overline{f}|_{C_+}$ and $\overline{f}|_{C_-}$ form the limited curve $\overline{f}$. We know that $\overline{f}(C_-)$ intersects V at two real non-prescribed points transversally. Therefore, $\overline{f}|_{C_-} \in \mathcal{C}^{0,2\delta_1^r}(p^!d - 2[E], \underline{x}(0) \cap \overline{X}^-, J_0)$. Furthermore, there are two possibilities for recovering a real curve $\overline{f}(\overline{C})$ from $\overline{f}|_{C_+}$ when $\overline{f}|_{C_-}$ is fixed. We have

$$\begin{aligned} m_{X_\sharp}(\overline{f}(\overline{C})) &= m_{\overline{X}^-}(\overline{f}|_{\overline{C}_-}) + m_{\overline{X}^+}(\overline{f}|_{C_+}) \\ &= m_{\overline{X}^-}(\overline{f}|_{\overline{C}_-}). \end{aligned} \tag{4.5}$$

By Proposition 4.2, an element $\overline{f}$ of $\mathcal{C}(d, \underline{x}(0), J_0)$ is the limit of a unique element of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ as λ goes to 0. Thus, the latter must be real when $\overline{f}$ is real and $\lambda \in \mathbb{C}^*$ is small. When deforming $\overline{f}$, no node appears in a neighborhood of $V \cap \overline{f}(\overline{C})$. From the analysis of Case I, we know that the elements of $\mathcal{C}(d, \underline{x}(0), J_0)$ have two different types. Therefore, the elements of $\mathbb{R}\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ will degenerate into two types: in Type I-1, an element of $\mathbb{R}\mathcal{C}^{\delta_1^r,0}(p^!d - [E], \underline{x}(0) \cap \overline{X}^- \sqcup \{q\}, J_0)$

corresponds to a unique element of the limited curve; in Type I-2, an element of $\mathbb{R}\mathcal{C}^{0,2\delta_1^r}(p^! d - 2[E], \underline{x}(0) \cap \overline{X}^-, J_0)$ corresponds to two limited curves with the same mass. We may easily obtain formula (4.2) from (4.4) and (4.5).

Case II. $\underline{x}(0) \cap X^+ = \{p, p'\}$ with $p,\ p' \in X \setminus \mathbb{R}X$ and $\tau(p) = p'$. From Proposition 4.2, we also obtain two types of the limited curve $\overline{f}$ as follows.

Type II-1. C_+ has only one component. $\overline{f}_*[C_+] = [H]$, and $\overline{f}|_{C_+} \in \mathcal{C}^{0,\delta_1}([H], \{p, p'\}, J_0)$ is an embedded simple curve. The τ-conjugated pair $p, p' \in X \setminus \mathbb{R}X$ can be chosen such that the intersection point, determined by $\mathcal{C}^{0,\delta_1}([H], \{p, p'\}, J_0)$, is also q. Thus, $\overline{f}|_{C_-}$ belongs to $\mathcal{C}^{\delta_1^r,0}(p^! d - [E], \underline{x}(0) \cap \overline{X}^- \sqcup \{q\}, J_0)$, and the remaining argument is the same as that in Case I. We omit it here.

Type II-2. C_+ has exactly two irreducible components $C_{+i},\ i = 1, 2$. In this case, $\overline{f}_*[C_{+i}] = [H]$, and $\overline{f}|_{C_{+i}}$ is an embedded simple curve. By the positivity of intersections, $\overline{f}(C_{+1})$ intersects $\overline{f}(C_{+2})$ at one real point which is a real node of $\overline{f}(C_+)$. Since $\overline{f}(C_{+i})$ passes one point of $\{p, p'\} \subset X \setminus \mathbb{R}X$ with $\tau(p) = p'$, $\overline{f}(C_{+1})$ and $\overline{f}(C_{+2})$ are two τ-conjugated components. Therefore, the real nodal point of $\overline{f}(C_+)$ must be isolated. Moreover, each $\overline{f}(C_{+i})$ intersects V at a point q_i transversally with $\tau(q_1) = q_2$. From Proposition 4.2 (2), we can obtain that the curve C_- is irreducible and $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{0,2\delta_1}(p^! d - 2[E], \underline{x}(0) \cap \overline{X}^-, J_0)$. $\overline{f}|_{C_+}$ and $\overline{f}|_{C_-}$ form the limited curve $\overline{f}$. We know that $\overline{f}(C_-)$ intersects V at two τ-conjugated non-prescribed points transversally. Therefore, $\overline{f}|_{C_-} \in \mathcal{C}^{0,2\delta_1^c}(p^! d - 2[E], \underline{x}(0) \cap \overline{X}^-, J_0)$. There are two possibilities for recovering a real curve $\overline{f}(\overline{C})$ from $\overline{f}|_{C_+}$ when $\overline{f}|_{C_-}$ is fixed. We have

$$
\begin{aligned}
m_{X_\sharp}(\overline{f}(\overline{C})) &= m_{\overline{X}^-}(\overline{f}|_{\overline{C}_-}) + m_{\overline{X}^+}(\overline{f}|_{C_+}) \\
&= m_{\overline{X}^-}(\overline{f}|_{\overline{C}_-}) + 1.
\end{aligned}
\tag{4.6}
$$

By Proposition 4.2, an element $\overline{f}$ of $\mathcal{C}(d, \underline{x}(0), J_0)$ is the limit of a unique element of $\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ as λ goes to 0. Thus, the latter must be real when $\overline{f}$ is real and $\lambda \in \mathbb{C}^*$ is small. When deforming $\overline{f}$, no node appears in a neighborhood of $V \cap \overline{f}(\overline{C})$. From the analysis of Case II, we know the elements of $\mathcal{C}(d, \underline{x}(0), J_0)$ have two different types. Therefore, the elements of $\mathbb{R}\mathcal{C}(d, \underline{x}(\lambda), J_\lambda)$ will degenerate into two types: in Type II-1, an element of $\mathbb{R}\mathcal{C}^{\delta_1^r,0}(p^! d - [E], \underline{x}(0) \cap \overline{X}^- \sqcup \{q\}, J_0)$ corresponds to a unique element of the limited curve; in Type II-2, an element of $\mathbb{R}\mathcal{C}^{0,2\delta_1^c}(p^! d - 2[E], \underline{x}(0) \cap \overline{X}^-, J_0)$ corresponds to two limited curves with the same mass. Formula (4.3) is easily obtained from (4.4) and (4.6). $\square$

Next, perform the real symplectic cut along the exceptional divisor E in $X_{1,0}$, see Remark 2.1. We can get $\overline{X}_{1,0}^+ \cong \mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O})$, $\overline{X}_{1,0}^- \cong X_{1,0}$, V as in subsection 3.1.

Let $\widetilde{\mathcal{Z}}$ be the real symplectic sum of $\overline{X}_{1,0}^+$ and $\overline{X}_{1,0}^-$ along V (see subsection 2.3). Let $p^! d - 2[E] \in H_2(\widetilde{\mathcal{Z}}_\lambda; \mathbb{Z})$, where $d \in H_2(X; \mathbb{Z})$. Choose $\underline{\widetilde{x}}(\lambda)$ as a set of $c_1(X) \cdot d - 3$ real sections $\Delta \to \widetilde{\mathcal{Z}}$ such that $\underline{\widetilde{x}}(0) \cap V = \emptyset$ and $\underline{\widetilde{x}}(0) \cap \overline{X}_{1,0}^+ = \emptyset$. Choose an almost complex structure $\widetilde{J}$ on $\widetilde{\mathcal{Z}}$ as before. Denote $\widetilde{X}_\sharp$, $\mathcal{C}(p^! d - 2[E], \underline{\widetilde{x}}(0), \widetilde{J}_0)$, C_*, $* = +, -$, as in subsection 3.1.

Proposition 4.4. *For a generic $\widetilde{J}_0$, the set $\mathcal{C}(p^! d - 2[E], \underline{\widetilde{x}}(0), \widetilde{J}_0)$ is finite and only depends upon $\underline{\widetilde{x}}(0)$ and $\widetilde{J}_0$. Given an element $\overline{f} : \overline{C} \to \widetilde{X}_\sharp$ of $\mathcal{C}(p^! d - 2[E], \underline{\widetilde{x}}(0), \widetilde{J}_0)$, the restriction of $\overline{f}$ to any component of $\overline{C}$ is a simple map, and no irreducible component of $\overline{C}$ is entirely mapped into V. Moreover, the curve C_- is irreducible and $\overline{f}|_{C_-}$ is an element of $\mathcal{C}^{0,2\delta_1}(p^! d - 2[E], \underline{\widetilde{x}}(0) \cap \overline{X}_{1,0}^-, \widetilde{J}_0)$. The curve C_+ has two irreducible components. Each component of $\overline{f}(C_+)$ realizes the fiber class $[F]$ in $\mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}) \to E$. The map $\overline{f}$ is the limit of a unique element of $\mathcal{C}(p^! d - 2[E], \underline{\widetilde{x}}(\lambda), \widetilde{J}_\lambda)$ as λ goes to 0.*

Proof. As before, we know that no component of $\overline{C}$ is entirely mapped into V. Since $\overline{f}_*[\overline{C}] = p^! d - 2[E]$, we may suppose that $\overline{f}_*[C_+] = a[F] + b[E_\infty]$, $a, b \geq 0$, $\overline{f}_*[C_-] = p^! d - k[E]$, $k \geqslant 0$, where F is a fiber of $\mathbb{P}_E(\mathcal{O}(-1) \oplus \mathcal{O}) \to E$ with $F \cdot [E_0] = 1$ and $F \cdot [E_\infty] = 1$. Then

$$a + b = (a[F] + b[E_\infty]) \cdot [E_\infty] = (p^! d - k[E]) \cdot [E] = k,$$

$$a = (a[F] + b[E_\infty]) \cdot [E_0] = (p^! d - 2[E]) \cdot [E] = 2.$$

This implies $k \geq 2$.

In $\overline{X}_{1,0}^-$, we know that $\overline{f}|_{C_-}$ passes through

$$c_1(X) \cdot d - 3 = c_1(\overline{X}_{1,0}^-) \cdot (p^! d - 2[E]) - 1$$

distinct points. The same argument as in the proof of Proposition 3.1 shows that C_- is irreducible.

Assume that $\overline{f}|_{C_-}$ is non-simple. Then $\overline{f}|_{C_-}$ factors through a non-trivial ramified covering of degree $\delta \geq 2$ of a simple map $f_0 : \mathbb{P}^1 \to \widetilde{X}$. Then, $(f_0)_*[\mathbb{P}^1] = (1/\delta)(p^! d - k[E])$. Therefore,

$$\frac{1}{\delta} c_1(\overline{\widetilde{X}}^-) \cdot (p^! d - k[E]) - 1 \geqslant c_1(X) \cdot d - 3.$$

This implies

$$c_1(X) \cdot d - \delta c_1(X) \cdot d + 2\delta \geqslant k.$$

Since $\delta \geqslant 2$, $c_1(X) \cdot d \geqslant 4$, we have $k \leq 0$. This is in contradiction with $k \geq 2$. Thus, $\overline{f}|_{C_-}$ is simple.

On the other hand, we have

$$c_1(\widetilde{X}^-) \cdot (p^! d - k[E]) - 1 = c_1(X) \cdot d - k - 1 \geqslant c_1(X) \cdot d - 3.$$

This implies $k \leq 2$, and we have $k = 2$, $b = 0$. Since the image of C_- under $\overline{f}$ intersects V transversally in two distinct points, C_+ has two irreducible components C_{+i} such that $\overline{f}_*[C_{+i}] = [F]$, $i = 1, 2$.

The remainder of Proposition 4.4 can be obtained by a similar argument as that in the proof of Proposition 3.1. We omit it here. $\square$

Proposition 4.5. *Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold, $d \in H_2(X;\mathbb{Z})$ such that $c_1(X) \cdot d \geqslant 4$ and $\tau_* d = -d$. Denote by $p : X_{1,0} \to X$ the projection of the real symplectic blow-up of X at $x \in \mathbb{R}X$. Then, if $s \geqslant 1$,*

$$\begin{aligned} &W_{X_{1,0}}(p^!(d) - 2[E], s-1) \\ &\quad = \sum_{C_- \in \mathbb{R}\mathcal{C}^{0,2\delta^r_1}(p^! d - 2[E], \underline{\widetilde{x}}(0) \cap \widetilde{X}^-_{1,0}, \widetilde{J}_0)} (-1)^{m_{X_{1,0}}(C_-)} \\ &\quad\quad + \sum_{C_- \in \mathbb{R}\mathcal{C}^{0,2\delta^c_1}(p^! d - 2[E], \underline{\widetilde{x}}(0) \cap \widetilde{X}^-_{1,0}, \widetilde{J}_0)} (-1)^{m_{X_{1,0}}(C_-)}, \end{aligned}$$

where E is the exceptional divisor.

Remark 4.6. The proof of Proposition 4.5 is similar to Proposition 3.2. Propositions 4.3 and 4.5 imply Theorem 1.4.

4.2. Generating function. In this subsection, we restate our formulae in the form of generating functions. Denote by

$$W^d_{X_{\mathbb{R}},L,F}(T) = \sum_{s=0}^{[(c_1(X)\cdot d-1)/2]} W_{X_{\mathbb{R}},L,F}(d,s)T^s \in \mathbb{Z}[T]$$

the generating function of Welschinger invariants which encodes all of the information of the Welschinger invariants.

Let $X_{\mathbb{R}}$ be a compact real symplectic 4-manifold. If $\mathbb{R}X$ is disconnected, the previous formulae are still true, and can be proved in the same method as Theorem 1.4 and Corollary 1.3. Suppose that $\underline{x}' \subset X$ is a real set consisting of r' points in L and s' pairs of τ-conjugated points in X with $r' + 2s' \leqslant c_1(X)\cdot d - 1$. We denote the connected component of $\mathbb{R}X_{r',s'}$ corresponding to L by $\widetilde{L}$. If there is only one blown-up real point in L, $\widetilde{L} = L\sharp\mathbb{R}P^2$. We assume that F has a τ-invariant compact representative $\mathcal{F} \subset X \setminus \underline{x}'$, and denote $\widetilde{F} = p^! F$. Denote by $p : X_{r',s'} \to X$ the projection of the real symplectic blow-up of X at $\underline{x}'$. Then

$$W^d_{X_{\mathbb{R}},L,F}(T) = W^{p^! d}_{X_{r',s'},\widetilde{L},\widetilde{F}}(T),$$

$$\begin{aligned} W^d_{X_{\mathbb{R}},L,F}(T) - W_{X_{\mathbb{R}},L,F}(d,0) - \cdots - W_{X_{\mathbb{R}},L,F}(d,s'-1)T^{s'-1} \\ = W^{p^! d - \sum_{i=1}^{r'}[E_i] - \sum_{j=1}^{s'}([E'_j]+[E''_j])}_{X_{r',s'},\widetilde{L},\widetilde{F}}(T)T^{s'}, \end{aligned}$$

$$W^d_{X_{\mathbb{R}},L,F}(T)T = W^d_{X_{\mathbb{R}},L,F}(T) - W_{X_{\mathbb{R}},L,F}(d,0) + 2W^{p^! d - 2[E]}_{X_{1,0},\widetilde{L},\widetilde{F}}(T)T,$$

where E_i, E'_j, E''_j denote the exceptional divisors corresponding to the real set $\underline{x}'$, respectively.

5. Real enumeration. In this section, we will give some applications of the blow-up formula of Welschinger invariants.

5.1. Blow-up of $\mathbb{C}P^2$. Let $\mathbb{C}P^2_{r,s}$ denote the blow-up of $\mathbb{C}P^2$ at r real points and s pairs of conjugated points. The projective plane with the standard symplectic structure and complex conjugation is a real symplectic manifold. In **[1]**, a recursive formula of Welschinger invariants in the projective plane was proven. Using the results of **[1]**

and the blow-up formula of Welschinger invariants (Corollary 1.3), we can compute the invariants of $\mathbb{C}P^2_{r,s}$, as shown in Tables 1 and 2.

TABLE 1. Welschinger invariants of $\mathbb{C}P^2_{r,s}$ with $r+2s \leqslant 8$.

s	0	1	2	3	4
$W(c_1(X),0)$	8	6	4	2	0
$W(c_1(X),1)$	6	4	2	0	–
$W(c_1(X),2)$	4	2	0	–	–
$W(c_1(X),3)$	2	0	–	–	–

TABLE 2. Welschinger invariants with purely real point constraints.

	$W([H],0)$	$W(2[H],0)$	$W(4[H],0)$
$X=\mathbb{C}P^2_{3,0}$	1	1	240
$X=\mathbb{C}P^2_{1,1}$	1	1	144

Note that the Welschinger invariants of $\mathbb{C}P^2_{r,s}$ with purely real point constraints were computed in [**15, 16, 17, 19, 20, 21, 24**]. In addition, the Welschinger invariants of $\mathbb{C}P^2_{r,s}$ were completely computed in [**10**]. The invariants for $r+2s \leqslant 3$ with arbitrary real and conjugated pairs of point constraints were studied in [**5**]. For $6 \leqslant r+2s \leqslant 8$, $W_{\mathbb{C}P^2_{r,s}}(d,s')$ was considered with point constraints in [**2**].

5.2. Conic bundles and del Pezzo surfaces of degree 2. Recall that there are 12 topological types of degree 2 real del Pezzo surfaces. Itenberg, Kharlamov and Shustin [**24**] computed the Welschinger invariants with purely real point constraints of all degree 2 real del Pezzo surfaces. Brugallé [**2**], computed the Welschinger invariants with arbitrary point constraints of real degree 2 del Pezzo surfaces with a non-orientable real part. Horev and Solomon [**10**] also computed Welschinger invariants with arbitrary point constraints of some degree 2 del Pezzo surfaces with a non-orientable real part. Brugallé [**2, 3**] computed the Welschinger invariants of the entire real degree 1 del Pezzo surface and showed that every degree $2n-1$ del Pezzo surface which is not the minimal del Pezzo surface is the blow-up of a degree $2n$ del Pezzo surface at a real point. We can use the blow-up formula to compute the Welschinger invariants with conjugated point constraints in the remaining five topological types of degree 2 real del Pezzo surfaces with no non-orientable real part.

Let $\mathbb{B}^n$ be the real conic bundle with $2n$ singular fibers and X^1 the minimal real del Pezzo surface of degree 2. Endow $\mathbb{P}^1 \times \mathbb{P}^1$ with the standard real structure. Thus, X^1, $\mathbb{B}^3$, $\mathbb{B}^2_{0,1}$, $\mathbb{B}^1_{0,2}$, $(\mathbb{P}^1 \times \mathbb{P}^1)_{0,3}$ are the five topological types of real del Pezzo surfaces of degree 2 with real parts $\sqcup 4S^2 \sqcup 3S^2$, $\sqcup 2S^2$, S^2 and $S^1 \times S^1$, respectively. The next tables are from [**2, 3, 24**].

Table 3.

	X^1	$\mathbb{B}^3$	$\mathbb{B}^2_{0,1}$	$\mathbb{B}^1_{0,2}$	$(\mathbb{P}^1 \times \mathbb{P}^1)_{0,3}$
$W(2c_1(X), 0)$	0	0	0	8	32

Table 4.

	$X^1_{1,0}$	$\mathbb{B}^3_{1,0}$	$\mathbb{B}^2_{1,1}$	$\mathbb{B}^1_{1,2}$	$(\mathbb{P}^1 \times \mathbb{P}^1)_{1,3}$
$W(2c_1(X), 0)$	18	10	6	6	10

Using Welschinger's wall-crossing formula

$$W_{X_{\mathbb{R}}}(d, s-1) = W_{X_{\mathbb{R}}}(d, s) + 2W_{X_{1,0}}(p^! d - 2[E], s-1),$$

we can obtain the following values.

Table 5.

	X^1	$\mathbb{B}^3$	$\mathbb{B}^2_{0,1}$	$\mathbb{B}^1_{0,2}$	$(\mathbb{P}^1 \times \mathbb{P}^1)_{0,3}$
$W(2c_1(X), 1)$	-36	-20	-12	-4	12

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