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PERIODIC SOLUTION FOR SECOND ORDER DAMPED DIFFERENTIAL EQUATIONS WITH ATTRACTIVE-REPULSIVE SINGULARITIES

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ABSTRACT. In this paper, we investigate a kind of second-order nonlinear differential equation with attractive-repulsive singularities. By applications of Green's function and Schauder's fixed point theorem, we establish the existence of a positive periodic solution for this equation.

1. Introduction. In 1987, Lazer and Solimini [**9**] investigated the model equations with singularity

$$x'' = -\frac{h(t)}{x^{\lambda}} + f(t) \tag{1.1}$$

and

$$x'' = \frac{g(t)}{x^{\lambda}} + f(t), \tag{1.2}$$

where $\lambda > 0$, g, h and f were periodic functions with period ω. Equation (1.1) was determined to have an attractive singularity, whereas equation (1.2) had a repulsive singularity. The authors provided necessary and sufficient conditions for the existence of periodic solutions of equations (1.1) and (1.2) with continuous positive functions h, g and a continuous forcing term f.

Lazer and Solimini's work has attracted the attention of many scholars in singular differential equations [**2**]–[**7**], [**11, 13, 14, 16, 17**]. Most of their work concentrated on repulsive singularity [**3, 4, 5, 11,**

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13, 14, 16] or attractive singularity [**2, 6, 7, 17**]. In 2010, Hakl and Torres [**8**] investigated a kind of second-order differential equation with attractive-repulsive singularities

$$x'' = \frac{g(t)}{x^{\kappa_1}} - \frac{h(t)}{x^{\kappa_2}} + f(t). \tag{1.3}$$

By the method of lower and upper solutions, the authors obtained the existence of a periodic solution for equation (1.3).

In the aforementioned, the authors investigated repulsive singularity or attractive singularity. However, there has been little research on attractive-repulsive singularities. Motivated by [**8**], in this paper, we discuss the existence of a positive periodic solution for the following differential equation with attractive-repulsive singularities

$$x'' + p(t)x' + q(t)x = f(t, x) + e(t), \tag{1.4}$$

where $e \in L^1(\mathbb{R})$ is an ω-periodic function, and $p, q \in C(\mathbb{R}, \mathbb{R})$ are ω-periodic functions; $f \in \mathrm{Car}(\mathbb{R}\times\mathbb{R}^+, \mathbb{R})$ is an L^2-Carathéodory function, which means that it is measurable in the first variable and continuous in the second variable; for every $0 < r < s$, there exists an $h_{r,s} \in L^2[0, \omega]$ such that $|f(t, x)| \le h_{r,s}(t)$ for all $x \in [r, s]$ and almost every $t \in [0, \omega]$; and f is an ω-periodic function on t. The nonlinear term f of equation (1.4) may have an attractive-repulsive singularity at origin.

Attractive-repulsive singularities can be regarded as a generalized Lennard-Jones force [**1, 10**], and they are widely used in molecular dynamics to model the interaction between atomic particles [**12**]. Thus, it is worthwhile and interesting to explore this topic.

The paper is organized as follows. In Section 2, Green's function is given and some useful properties for Green's function are obtained. In Section 3, we find the periodic solution for equation (1.4) with attractive-repulsive singularities. We prove that a weak singularity enables the achievement of new existence criteria through a basic application of Schauder's fixed point theorem.

To conclude this introduction, some notation is presented as follows. We write $d(t) \succ 0$ if $d(t) \ge 0$ for almost every $t \in [0, \omega]$, and it is positive in a set of positive measure. For a given function $e \in L^1[0, \omega]$, we denote the essential supremum and infimum by e^* and e_*, if they exist. Let $X = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t + \omega) = \phi(t)\}$ with the maximum norm $\|\phi\| = \max_{0 \le t \le \omega} |\phi(t)|$. Obviously, X is a Banach space.

2. Green's function. Firstly, we consider

$$\begin{cases} x'' + p(t)x' + q(t)x = h(t), \\ x(0) = x(\omega), \ x'(0) = x'(\omega), \end{cases} \tag{2.1}$$

where $h \in C(\mathbb{R}, \mathbb{R}^+)$ is an ω-periodic function. The solution of equation (2.1) is written as

$$x(t) = \int_0^\omega G(t,s)h(s)\,ds.$$

In 2005, Wang, Lian and Ge **[15]** discussed the sign of $G(t,s)$; they obtained $G(t,s) > 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$ when the following conditions are satisfied:

(A_1) there are continuous ω-periodic functions a and b such that

$$\int_0^\omega b(s)\,ds > 0, \qquad \int_0^\omega a(s)\,ds > 0$$

and

$$a(t) + b(t) = p(t), \qquad a'(t) + a(t)b(t) = q(t) \quad \text{for } t \in \mathbb{R}.$$

(A_2) $$\left(\int_0^\omega p(s)\,ds \right)^2 \geq 4\omega^2 \exp\left(\frac{1}{\omega} \int_0^\omega \ln q(s)\,ds \right).$$

Obviously, condition (A_2) is hard restrictive for $G(t,s) > 0$. In the following, we consider $G(t,s) > 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$ if only condition (A_1) is satisfied. From (A_1), equation (2.1) is transformed into

$$\begin{cases} y' + b(t)y = h(t), \\ y(0) = y(\omega), \end{cases} \tag{2.2}$$

and

$$\begin{cases} x' + a(t)x = y(t), \\ x(0) = x(\omega). \end{cases} \tag{2.3}$$

The solution of equation (2.2) is written as

$$y(t) = \int_0^\omega G_1(t,s)h(s)\,ds. \tag{2.4}$$

The solution of equation (2.3) is written as

$$x(t) = \int_0^\omega G_2(t,s)y(s)\,ds.$$

In the following, we consider $G_1(t,s)$.

Lemma 2.1. *The periodic boundary problem* (2.2) *is equivalent to the integral equation*

$$y(t) = \int_0^\omega G_1(t,s)h(s)\,ds,$$

where

$$G_1(t,s) = \begin{cases} \dfrac{e^{-\int_s^t b(\tau)\,d\tau}}{1-e^{-\int_0^\omega b(\tau)\,d\tau}} & 0 \le s \le t \le \omega, \\ \dfrac{e^{-\int_s^{\omega+t} b(\tau)\,d\tau}}{1-e^{-\int_0^\omega b(\tau)\,d\tau}} & 0 \le t < s \le \omega. \end{cases} \tag{2.5}$$

Moreover, $G_1(t,s) > 0$ *for all* $(t,s) \in [0,\omega] \times [0,\omega]$ *if* $\int_0^\omega b(t)\,dt > 0$.

Proof. Firstly, it is easy to see that the associate homogeneous equation of equation (2.2) has the solution

$$y(t) = Ce^{-\int_0^t b(\tau)\,d\tau}.$$

Applying the method of variable parameters, we obtain

$$C'(t) = e^{\int_0^t b(\tau)\,d\tau}h(t).$$

Since $y(0) = y(\omega)$, we have

$$C(t) = \frac{\int_0^\omega e^{-\int_s^\omega b(\tau)\,d\tau}h(s)\,ds}{1-e^{-\int_0^\omega b(\tau)\,d\tau}} + \int_0^t e^{\int_0^s b(\tau)\,d\tau}h(s)\,ds.$$

Therefore, we have

$$y(t) = \int_0^\omega G_1(t,s)h(s)\,ds,$$

where $G_1(t,s)$ is as given in equation (2.5). Moreover, from equation (2.5) and $\int_0^\omega b(t)\,dt > 0$, we can get $G_1(t,s) > 0$, for all $(t,s) \in [0,\omega] \times [0,\omega]$. $\square$

From the above, we know that the solution of equation (2.1) is written as

$$\begin{aligned} x(t) &= \int_0^\omega G_2(t,\tau) \int_0^\omega G_1(\tau,s)h(s)\,ds\,d\tau \\ &= \int_0^\omega \int_0^\omega G_2(t,\tau)G_1(\tau,s)h(s)\,ds\,d\tau \\ &= \int_0^\omega \left[\int_0^\omega G_2(t,s)G_1(s,\tau)\,ds \right] h(\tau)\,d\tau \\ &= \int_0^\omega \left[\int_0^\omega G_2(t,\tau)G_1(\tau,s)\,d\tau \right] h(s)\,ds. \end{aligned}$$

Thus, by writing

$$G(t,s) = \int_0^\omega G_2(t,\tau)G_1(\tau,s)\,d\tau, \tag{2.6}$$

we obtain

$$x(t) = \int_0^\omega G(t,s)h(s)\,ds. \tag{2.7}$$

Lemma 2.2. *Assume that condition* (A_1) *holds. Then,* $G(t,s) > 0$ *for all* $(t,s) \in [0,\omega] \times [0,\omega]$.

Proof. From Lemma 2.1, we know that $G_1(t,s) > 0$. Therefore, from equation (2.6), it is easy to see that $G(t,s) > 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$. $\square$

3. Periodic solution for equation (1.4) with attractive–repulsive singularities. In this section, we establish the existence of the positive periodic solution for equation (1.4) with attractive-repulsive singularities by using Schauder's fixed point theorem, which can be found in [**18,** page 61].

Lemma 3.1. *The compact operator* $A^* : M^* \to M^*$ *has a fixed point provided* M^* *is a bounded, closed, convex and nonempty subset of a Banach space* X *over* $\mathbb{R}$.

We define the function $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\gamma(t) = \int_0^\omega G(t,s)e(s)\,ds,$$

which is the unique ω-periodic solution of

$$x'' + p(t)x' + q(t)x = e(t).$$

Case (I). $\gamma_* > 0$.

Theorem 3.2. *Suppose that condition* (A_1) *holds. Furthermore, assume that the following conditions hold*:

(H_1) *there exist continuous, non-negative functions* $g(x)$ *and* $\zeta(t)$ *such that*

$$0 \le f(t,x) \le \zeta(t)g(x) \quad \textit{for all } (t,x) \in [0,\omega] \times (0,\infty),$$

and $g(x) > 0$ *is non-increasing in* $x \in (0,\infty)$.

(H_2) *There exists a constant* $R > 0$ *such that*

$$g(\gamma_*)\Lambda^* + \gamma^* \le R,$$

where $\Lambda(t) = \int_0^\omega G(t,s)\zeta(s)\,ds$. *If* $\gamma_* > 0$, *then equation* (1.4) *has at least one positive periodic solution.*

Proof. An ω-periodic solution of equation (1.4) is merely a fixed point of the map $T : X \to X$, defined by

$$(Tx)(t) = \int_0^\omega G(t,s)[f(s,x(s)) + e(s)]\,ds = \int_0^\omega G(t,s)f(s,x(s))\,ds + \gamma(t). \tag{3.1}$$

Letting $r := \gamma_*$, we have $R > r > 0$ since $R > \gamma^*$. Now, we define the set

$$K = \{x \in X : r \le x(t) \le R \text{ for all } t\}. \tag{3.2}$$

Obviously, K is a closed convex set.

Next, we prove $T(K) \subset K$. In fact, for every $x \in K$, by applications of the non-negative signs of $G(t,s)$ and $f(t,x)$, we have

$$(Tx)(t) = \int_0^\omega G(t,s)f(s,x(s))\,ds + \gamma(t) \geq \gamma_* := r > 0.$$

On the other hand, for every $x \in K$, from conditions (H_1) and (H_2), we have

$$\begin{aligned}(Tx)(t) &= \int_0^\omega G(t,s)f(s,x(s))\,ds + \gamma(t)\\ &\leq \int_0^\omega G(t,s)\zeta(s)g(x(s))\,ds + \gamma(t)\\ &\leq g(r)\Lambda^* + \gamma^* \leq R.\end{aligned}$$

In conclusion, $T(K) \subset K$.

Let W be any bounded subset in K. Then for all $x \in W$, we have from Lemma 2.2 that

$$\begin{aligned}\|Tx\| &= \max_{t\in[0,\omega]}\left|\int_0^\omega G(t,s)f(s,x(s))\,ds + \gamma(t)\right|\\ &\leq \max_{t\in[0,\omega]}\left|\int_0^\omega G(t,s)f(s,x(s))\,ds\right| + \gamma^*\\ &\leq M\sqrt{\omega}\left(\int_0^\omega |f(s,x(s))|^2 ds\right)^{1/2} + \gamma^*\\ &\leq M\sqrt{\omega}\|f_R\|_2 + \gamma^* := N,\end{aligned}$$

where

$$\begin{aligned}M &:= \max_{t\in[0,\omega]}|G(t,s)|,\\ |f_R| &:= \max_{r\leq x(t)\leq R}|f(t,x(t))|,\\ \|f_R\|_2 &:= \left(\int_0^\omega |f_R|^2 dt\right)^{1/2}.\end{aligned}$$

The following holds:

$$\left|\frac{dTx}{dt}\right| = \left|\int_0^\omega \frac{\partial G(t,s)}{\partial t}[f(s,x(s)) + e(s)]\,ds\right|$$

$$\le \int_0^\omega \left|\frac{\partial G(t,s)}{\partial t}\right| |f(s,x(s))+e(s)|\,ds$$
$$\le B'(\sqrt{\omega}\|f_R\|_2+\|e\|_1) := N_1,$$

where

$$B' = \max\left|\frac{\partial G(t,s)}{\partial t}\right| \quad \text{for all } (t,s)\in[0,\omega]\times[0,\omega],$$
$$\|e\|_1 = \int_0^\omega |e(s)|\,ds.$$

Using the Arzela-Ascoli theorem, it is easy to show that T is compact in K. Therefore, the proof is finished by Schauder's fixed point theorem. □

In the following, we investigate equation (1.4) with attractive-repulsive singularities.

Corollary 3.3. *Suppose that condition* (A_1) *holds. Assume that the following condition also holds*:

(F_1) *There exist continuous functions* $d(t)\succ 0$, $\alpha>0$, $\beta>0$ *and* $\mu>0$ *such that*

$$f(t,x)\le \frac{d(t)}{x^\alpha}-\frac{\mu d(t)}{x^\beta}$$

for all $x>0$ *and almost every* t. *If* $\gamma_*>0$, *then there exists a positive constant* μ_1 *such that equation* (1.4) *has at least one positive periodic solution for each* $0\le\mu\le\mu_1$.

Proof. We apply Theorem 3.2. We take

$$g(x)=\frac{1}{x^\alpha}, \qquad \zeta(t)\equiv d(t).$$

Firstly, we consider condition (H_2) to be satisfied. We take $R>0$ with

$$R\ge \frac{\Psi^*}{(\gamma_*)^\alpha}+\gamma^*,$$

where $\Psi(t)=\int_0^\omega G(t,s)\,d(s)\,ds$. Next, we consider condition (H_1) also to be satisfied. In fact, $f(t,x)\ge 0$ if and only if $\mu\le x^{\beta-\alpha}$. If $\beta\le\alpha$, then we have $\mu < R^{\beta-\alpha}$. As a consequence, the result holds for

$$\mu_1' := \left(\frac{\Psi^*}{(\gamma_*)^\alpha} + \gamma^* \right)^{\beta-\alpha}.$$

If $\beta > \alpha$, then we have $\mu < r^{\beta-\alpha}$. As a consequence, the result holds for

$$\mu_2'' := \gamma_*^{\beta-\alpha}.$$

Therefore, take $\mu_1 = \min\{\mu_1', \mu_1''\}$. For each $\mu < \mu_1$, the result holds. □

Case (II). $\gamma_* = 0$.

Theorem 3.4. *Suppose that condition* (A$_1$) *holds and that* $f(t,x)$ *satisfies* (H$_1$). *Furthermore, assume that the following conditions hold*:

(H$_3$) *for each* $L > 0$, *there exists a continuous function* $\phi_L \succ 0$ *such that* $f(t,x) \geq \phi_L(t)$ *for all* $(t,x) \in [0,\omega] \times (0,L]$.

(H$_4$) *There exists a constant* $R > 0$ *such that* $R > (\Phi_R)_*$ *and*

$$g((\Phi_R)_*)\Lambda^* + \gamma^* \leq R,$$

where $\Phi_R(t) = \int_0^\omega G(t,s)\phi_R(s)\,ds$. *If* $\gamma_* = 0$, *then equation* (1.4) *has at least one positive periodic solution.*

Proof. We follow the same strategy and notation as in the proof of Theorem 3.2. Let R be the positive constant satisfying condition (H$_4$), and let $r := (\Phi_R)_*$; then we have $R > r > 0$ since $R > (\Phi_R)_*$.

Next, we prove that $T(K) \subset K$ for all $x \in K$. In fact, for every $x \in K$, using the fact that $G(t,s) > 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$, together with condition (H$_3$),

$$\begin{aligned}(Tx)(t) &= \int_0^\omega G(t,s) f(s,x(s))\,ds + \gamma(t) \\ &\geq \int_0^\omega G(t,s)\phi_R(s)\,ds + \gamma(t) \\ &\geq (\Phi_R)_* := r > 0.\end{aligned}$$

On the other hand, for every $x \in K$, from conditions (H_1) and (H_4), we have

$$\begin{aligned}(Tx)(t) &= \int_0^\omega G(t,s)f(s,x(s))\,ds + \gamma(t)\\ &\le \int_0^\omega G(t,s)\zeta(s)g(x(s))\,ds + \gamma(t)\\ &\le g(r)\Lambda^* + \gamma^* \le R.\end{aligned}$$

In conclusion, $T(K) \subset K$.

Meanwhile, using the Arzela-Ascoli theorem, it is easy to show that T is compact in K. Therefore, by Schauder's fixed point theorem, our result is proven. □

Corollary 3.5. *Suppose that condition* (A_1) *holds. Assume, in addition, that the following condition holds*:

(F_2) *there exist constants* $0 < \beta < \alpha < 1$ *and* $\mu > 0$ *such that* $(\alpha + \beta)\alpha < 1$ *and*

$$f(t,x) \le \frac{1}{x^\alpha} - \frac{\mu}{x^\beta}$$

for all $x > 0$ *and almost every* t. *If* $\gamma_* = 0$, *then there exists a positive constant* μ_2 *such that equation* (1.4) *has at least one positive periodic solution for each* $0 \le \mu \le \mu_2$.

Proof. We apply Theorem 3.4. We take

$$\zeta(t) = 1, \qquad g(x) = \frac{1}{x^\alpha}.$$

Firstly, we consider condition (H_3) to be satisfied. Let

$$Q(x) = \frac{1}{x^\alpha} - \frac{\mu}{x^\beta}, \quad x \in (0, +\infty)$$

and

$$s_1 = \mu^{-1/(\alpha-\beta)}, \qquad s_2 = \left(\frac{\alpha}{\mu\beta}\right)^{1/(\alpha-\beta)}.$$

Since $\alpha > \beta$, it can easily be verified that $s_1 < s_2$ and

$$Q(s_1) = 0, \qquad Q'(s_2) = 0, \qquad Q'(s) < 0, \quad s \in (0, s_2).$$

Therefore, $Q(s)$ is decreasing in $(0, s_1) \subset (0, s_2)$.

On the other hand, we can choose $\mu > 0$ small enough such that $R \in (0, s_1)$. Thus,

$$\min_{s\in(0,R)} Q(s) = Q(R) > Q(s_1) = 0.$$

This implies that condition (H_3) is satisfied if we take

$$\phi_R(t) \equiv Q(R).$$

The existence condition (H_4) becomes

$$\left(\frac{R^{\alpha+\beta}}{R^\beta - \mu R^\alpha}\right)^\alpha \Upsilon^* + \gamma^* \le R, \tag{3.3}$$

where $\Upsilon(t) = \int_0^\omega G(t,s)\,ds$. Since $(\alpha+\beta)\alpha < 1$, we can choose $R > 0$ large enough so that equation (3.3) is satisfied.

Finally, we consider condition (H_1) to be satisfied. In fact, $f(t,x) \ge 0$ if and only if $\mu < x^{\beta-\alpha}$, (H_1) verified for any $\mu < R^{\beta-\alpha}$ since $\beta < \alpha$. As a consequence, the result holds for $\mu_2 := R^{\beta-\alpha}$. $\square$

Case (III). $\gamma^* \le 0$.

Theorem 3.6. *Suppose that condition* (A_1) *holds. In addition, suppose that* $f(t,x)$ *satisfies conditions* (H_1) *and* (H_3). *Furthermore, assume that the following condition holds*:

(H_5) *there exists an* $R > 0$ *such that* $R > (\Phi_R)_* + \gamma_* > 0$, *and*

$$g((\Phi_R)_* + \gamma_*)\Lambda^* \le R.$$

If $\gamma* \le 0$, *then equation* (1.4) *has at least one positive periodic solution.*

Proof. We follow the same strategy and notation as in the proof of Theorem 3.2. Let R be the positive constant satisfying condition (H_4), and let $r := (\Phi_R)_* + \gamma_*$; then, we have $R > r > 0$ since $R > (\Phi_R)_* + \gamma_*$.

Next, we prove that $T(K) \subset K$ for all $x \in K$. In fact, for every $x \in K$, using the fact that $G(t,s) > 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$, together with condition (H_3),

$$(Tx)(t) = \int_0^\omega G(t,s) f(s,x(s))\,ds + \gamma(t)$$

$$
\begin{aligned}
&\geq \int_0^\omega G(t,s)\phi_R(s)\,ds + \gamma(t) \\
&\geq (\Phi_R)_* + \gamma_* := r > 0.
\end{aligned}
$$

On the other hand, for every $x \in K$, from conditions (H_1) and (H_5), we have

$$
\begin{aligned}
(Tx)(t) &= \int_0^\omega G(t,s)f(s,x(s))\,ds + \gamma(t) \\
&\leq \int_0^\omega G(t,s)\zeta(s)g(x(s))\,ds + \gamma(t) \\
&\leq g(r)\Lambda^* \leq R,
\end{aligned}
$$

since $\gamma^* \leq 0$. In conclusion, $T(K) \subset K$.

Meanwhile, using the Arzela-Ascoli theorem, it is easy to show that T is compact in K. Therefore, by Schauder's fixed point theorem, our result is proven. □

Corollary 3.7. *Assume that condition* (A_1) *holds. Assume, in addition, that the following condition holds*:

(F_3) *there exist continuous functions* $c(t), d(t) \succ 0$, $0 < \beta < \alpha < 1$, $\rho > 0$ *and* $\mu > 0$ *such that* $\rho\alpha < 1$ *and*

$$
\frac{c(t)}{x^\rho} \leq f(t,x) \leq \frac{d(t)}{x^\alpha} - \frac{\mu d(t)}{x^\beta}
$$

for all $x > 0$ *and almost every* t. *If* $\gamma^* \leq 0$ *and*

$$
\gamma_* \geq \left(\alpha\rho \frac{C_*}{(\Psi^*)^\rho}\right)^{1/(1-\alpha\rho)} \left(1 - \frac{1}{\alpha\rho}\right),
$$

here $C(t) := \int_0^\omega G(t,s)c(s)\,ds$. *Then, there exists a positive constant* μ_3 *such that equation* (1.4) *has at least one positive periodic solution for each* $0 \leq \mu \leq \mu_3$.

Proof. We apply Theorem 3.6. Take

$$
\phi_R(t) = \frac{c(t)}{R^\rho}, \qquad \zeta(t) = d(t), \qquad g(x) = \frac{1}{x^\alpha}.
$$

Then, condition (H_1) is satisfied.

Next, we consider condition (H_5) to be satisfied. Taking $R = \Psi^*/r^\alpha$, then $(\Phi_R)_* + \gamma_* > 0$ holds if r verifies

$$\frac{C_*}{(\Psi^*)^\rho}(r)^{\alpha\rho} + \gamma_* \geq r,$$

or equivalently,

$$\gamma_* \geq f(r) := r - \frac{C_*}{(\Psi^*)^\rho}(r)^{\alpha\rho}.$$

The function $f(r)$ possesses a minimum at

$$r_0 := \left(\alpha\rho\frac{C_*}{(\Psi^*)^\rho}\right)^{1/(1-\alpha\rho)}.$$

Let $r = r_0$. Then, $(\Phi_R)_* + \gamma_* > 0$ holds if $\gamma_* \geq f(r_0)$, which is merely the condition

$$\gamma_* \geq \left(\alpha\rho\frac{C_*}{(\Psi^*)^\rho}\right)^{1/(1-\alpha\rho)}\left(1 - \frac{1}{\alpha\rho}\right).$$

Then, condition (H_5) directly holds by the choice of R, and it would remain to prove that $R = \Psi^*/(r_0)^\rho > r_0$. This is easily verified through elementary computations.

Finally, we consider that condition (H_3) is satisfied. In fact, $d(t)/x^\alpha - \mu d(t)/x^\beta \geq c(t)/R^\rho$ if and only if

$$\mu \leq x^{\beta-\alpha} - \frac{c(t)}{d(t)}\frac{x^\beta}{R^\rho}.$$

(H_3) is verified for any $\mu \leq R^{\beta-\alpha} - (c(t)/d(t))R^{\beta-\rho}$ since $\beta < \alpha$. As a consequence, the result holds for

$$\mu_3 := \frac{(\Psi^*)^{\beta-\alpha}}{(\alpha\rho(C_*/((\Psi^*)^\rho)))^{\alpha(\beta-\alpha)/(1-\alpha\rho)}} - \frac{c_*}{d^*}\frac{(\Psi^*)^{\beta-\rho}}{(\alpha\rho(C_*/(\Psi^*)^\rho))^{\alpha(\beta-\rho)/(1-\alpha\rho)}}.$$

□

On the other hand, condition (H_1) implies, in particular, that the nonlinearity $f(t,x)$ is non-negative for all values (t,x), which is quite a difficult restriction. In the following, we show how to avoid this restriction for $\gamma_* > 0$.

Theorem 3.8. *Suppose that condition* (A_1) *holds. Assume that the following conditions hold*:

(H_6) *for each* $L > l > 0$*, there exists a continuous function* $\widehat{\phi}_l \prec 0$ *such that* $f(t,x) \geq \widehat{\phi}_l(t)$ *for all* $(t,x) \in [0,\omega] \times [l,L]$.

(H_7) *There exist continuous, non-negative functions* $g(x)$ *and* $\zeta(t)$ *such that*

$$f(t,x) \leq \zeta(t)g(x) \quad \text{for all } (t,x) \in [0,\omega] \times (0,\infty),$$

and $g(x) > 0$ *is non-increasing in* $x \in (0,\infty)$.

(H_8) *There exist positive constants* $R > r > 0$ *such that* $R > (\widehat{\Phi}_r)_* + \gamma_* > 0$ *and*

$$g((\widehat{\Phi}_r)_* + \gamma_*)\Lambda^* + \gamma^* \leq R,$$

where $\widehat{\Phi}_r = \int_0^\omega G(t,s)\widehat{\phi_r}(s)\,ds$.

If $\gamma_* > 0$*, then equation* (1.4) *has at least one positive periodic solution.*

Proof. We follow the same strategy and notation as in the proof of Theorem 3.2. Let R be the positive constant satisfying condition (H_8), and let $r := (\widehat{\Phi}_r)_* + \gamma_*$; then, we have $R > r > 0$ since $R > (\widehat{\Phi}_r)_* + \gamma_*$.

Next, we prove that $T(K) \subset K$ for all $x \in K$. In fact, for every $x \in K$, using the fact that $G(t,s) > 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$, together with condition (H_6),

$$\begin{aligned}(Tx)(t) &= \int_0^\omega G(t,s)f(s,x(s))\,ds + \gamma(t)\\ &\geq \int_0^\omega G(t,s)\widehat{\phi}_r(s)\,ds + \gamma(t)\\ &\geq (\widehat{\Phi}_r)_* + \gamma_* := r > 0.\end{aligned}$$

On the other hand, for every $x \in K$, from conditions (H_7) and (H_8), we have

$$\begin{aligned}(Tx)(t) &= \int_0^\omega G(t,s)f(s,x(s))\,ds + \gamma(t)\\ &\leq \int_0^\omega G(t,s)\zeta(s)g(x(s))\,ds + \gamma(t) \leq g(r)\Lambda^* + \gamma^* \leq R.\end{aligned}$$

In conclusion, $T(K) \subset K$.

Meanwhile, using the Arzela-Ascoli theorem, it is easy to show that T is compact in K. Therefore, by Schauder's fixed point theorem, our result is proven. $\square$

Corollary 3.9. *Suppose that condition* (A_1) *holds. Assume that the following condition holds*:

(F_4) *there exist continuous functions* $b(t), d(t) \succ 0$ *and constants* α, $\mu > 0$ *such that*

$$-\frac{c(t)}{x^\beta} \le f(t,x) \le \frac{d(t)}{x^\alpha} - \frac{\mu d(t)}{x^\beta}$$

for all $x > 0$ *and almost every* t. *If*

$$\gamma_* > (\beta C^*)^{1/(\beta+1)}\left(1 + \frac{1}{\beta}\right),$$

then equation (1.4) *has at least one positive periodic solution.*

Proof. We apply Theorem 3.6. We take

$$\zeta(t) = d(t), \qquad \widehat{\phi}_l(t) = -\frac{c(t)}{r^\beta}, \qquad g(x) = \frac{1}{x^\alpha}.$$

Then, conditions (H_6) and (H_7) are satisfied, and the existence condition (H_8) is also satisfied if r verifies

$$-\frac{C^*}{r^\beta} + \gamma_* \ge r, \tag{3.4}$$

or equivalently,

$$\gamma_* \ge f_1(r) := r + \frac{C^*}{r^\beta}.$$

The function $f(r)$ possesses a minimum at $r_1 := (\beta C^*)^{1/(1+\beta)}$. Let $r = r_1$. Then, $(\widehat{\Phi}_r)_* + \gamma_* > 0$ holds in equation (3.4) if $\gamma_* \ge f(r_1)$, which is merely the condition

$$\gamma_* \ge (\beta C^*)^{1/(\beta+1)}\left(1 + \frac{1}{\beta}\right).$$

Moreover, we choose R such that $(\Psi^*/r_1^\alpha) + \gamma^* \le R$. $\square$

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