ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 2, 2018

ON MAXIMAL IDEALS OF $C_c(X)$ AND THE UNIFORMITY OF ITS LOCALIZATIONS

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ABSTRACT. A similar characterization, as the Gelfand-Kolmogoroff theorem for the maximal ideals in $C(X)$, is given for the maximal ideals of $C_c(X)$. It is observed that the z_c-ideals in $C_c(X)$ are contractions of the z-ideals of $C(X)$. Using this, it turns out that maximal ideals (respectively, prime z_c-ideals) of $C_c(X)$ are precisely the contractions of maximal ideals (respectively, prime z-ideals) of $C(X)$, as well. Maximal ideals of $C_c^*(X)$ are also characterized, and two representations are given. We reveal some more useful basic properties of $C_c(X)$. In particular, we observe that, for any space X, $C_c(X)$ and $C_c^*(X)$ are always clean rings. It is also shown that $\beta_0 X$, the Banaschewski compactification of a zero-dimensional space X, is homeomorphic with the structure spaces of $C_c(X)$, $C^F(X)$, $C_c(\beta_0 X)$, as well as with that of $C(\beta_0 X)$. F_c-spaces are characterized, the spaces X for which $C_c(X)_P$, the localization of $C_c(X)$ at prime ideals P, are uniform (or equivalently are integral domain). We observe that X is an F_c-space if and only if $\beta_0 X$ has this property. In the class of strongly zero-dimensional spaces, we show that F_c-spaces and F-spaces coincide. It is observed that, if either $C_c(X)$ or $C_c^*(X)$ is a Bézout ring, then X is an F_c-space. Finally, $C_c(X)$ and $C_c^*(X)$ are contrasted with regards to being an absolutely Bézout ring. Consequently, it is observed that the ideals in $C_c(X)$ are convex if and only if they are absolutely convex if and only if $C_c(X)$ and $C_c^*(X)$ are both unitarily absolute Bézout rings.

1. Introduction. As is standard, all topological spaces in this article are infinite completely regular (i.e., infinite Hausdorff Tychonoff spaces). We denote by $C(X)$ ($C^*(X)$) the ring of all real-valued, continuous (bounded) functions on a space X. For each $f \in C(X)$, the

2010 AMS *Mathematics subject classification.* Primary 54C40.

Keywords and phrases. Bézout ring, z_c-ideal, clean, zero-dimensional, F_c-space, F-space.

Received by the editors on March 2, 2016, and in revised form on February 14, 2017.

DOI:10.1216/RMJ-2018-48-2-345

zero-set of f, denoted by $Z(f)$, is the set of zeros of f and $X \setminus Z(f)$ is the *cozero-set* of f and the set of all zero-sets in X is denoted by $Z(X)$. An ideal I in $C(X)$ is called a *z-ideal* if, whenever $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$, then $g \in I$. The space υX is the *Hewitt realcompactification* of X, βX is the *Stone-Čech compactification* of X and, for any $p \in \beta X$, the maximal ideal M^p (respectively, O^p) is the set of all $f \in C(X)$ for which $p \in \mathrm{cl}_{\beta X} Z(f)$ (respectively, $p \in \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z(f)$). Similarly, maximal ideals of $C^*(X)$ are precisely of the form

$$M^{*p} = \{f \in C^*(X) : p \in Z(f^\beta)\},$$

where f^β is the extension of f on βX. Whenever $C(X)/M^p \cong \mathbb{R}$, then M^p is called *real*, else *hyper-real*, and υX is in fact the set of all $p \in \beta X$ such that M^p is real.

The subring of $C(X)$ consisting of those functions with countable (respectively, finite) image, which is denoted by $C_c(X)$ (respectively, $C^F(X)$) is an $\mathbb{R}$-subalgebra of $C(X)$. The subring $C_c^*(X)$ of $C_c(X)$ consists of bounded elements of $C_c(X)$. The rings $C_c(X)$ and $C^F(X)$ are introduced and studied in [**10, 11**]. It is shown [**10**] that, for any topological space X, there exists a zero-dimensional space Y which is a continuous image of X and $C_c(X) \cong C_c(Y)$. A subset S of a space X is called C_c^*-*embedded* in X if every function in $C_c^*(S)$ can be extended to a function in $C_c^*(X)$. We denote by $Z_c(X)$ the set of all zero-sets $Z(f)$, where $f \in C_c(X)$ and an ideal I in $C_c(X)$ is called a z_c-*ideal* if, whenever $f \in I$, $g \in C_c(X)$ and $Z(f) \subseteq Z(g)$, then $g \in I$. An ideal I in $C_c(X)$ or $C_c^*(X)$ and, more generally, in a lattice ordered ring R, is called *absolutely convex* (note that, in the ℓ-group literature, absolutely convex ideals are often called ℓ-*ideals*) if, whenever $a, b \in R$ with $|a| \leq |b|$ and $b \in I$, then $a \in I$. It is easy to see that every z-ideal in $C(X)$ and every z_c-ideal in $C_c(X)$ is an absolutely convex ideal. An ideal I in $C_c(X)$ ($C_c^*(X)$) or in $C(X)$ ($C^*(X)$) is said to be *fixed* if

$$\bigcap_{f \in I} Z(f) \neq \emptyset;$$

otherwise, it is called *free*. However, it seems $C_c(X)$ and $C^F(X)$ are not algebraically defined; thus, we should emphasize here that, whenever $C(X) \cong C(Y)$, then $C_c(X) \cong C_c(Y)$, $C^F(X) \cong C^F(Y)$, and, if $C(X) \cong C_c(Y)$, then $C(X) = C_c(X)$, see [**11,** comments preceding Theorem 3.6] and [**17,** comments preceding Theorem 5.1]. In particular, we always have $C_c(X) \cong C_c(\upsilon X)$, $C^F(X) \cong C^F(\upsilon X)$. In [**10,** Proposition 4.4], it was also shown that X is zero-dimensional if and only if the closed sets and points not contained in them can be completely separated by elements of $C_c(X)$ (or, equivalently, by elements in $C^F(X)$). This and the previous observations show that, in dealing with $C_c(X)$, we may always assume that X is a zero-dimensional space (i.e., a Hausdorff space with a base consisting of clopen sets). In particular, in this paper, all spaces X are zero-dimensional unless otherwise mentioned (note that we sometimes emphasize the zero-dimensionality of the spaces with which we are dealing). As remarked upon in [**6,** introduction], the subject of $C_c(X)$ is receiving increasing attention in the literature. Moreover, we also believe that the present article, together with [**6, 10, 11, 17**] can perhaps provide some basic and necessary results for the study of the subject in the future. Banaschewski has shown that every zero-dimensional space X has a zero-dimensional compactification, denoted by $\beta_0 X$, such that every continuous map $f : X \to Y$, where Y is a zero-dimensional compact space has the extension map

$$\beta_0 f : \beta_0 X \longrightarrow Y.$$

If βX is zero-dimensional, then $\beta X = \beta_0 X$, see [**19,** subsection 4.7] for more details. We also recall a well-known result due to Rudin, Pelczynski and Semadeni (abbreviated *RPS*-theorem in [**10, 11**]), which states that a compact space X is scattered if and only if every member of $C(X)$ has a countable image, that is, $C(X) = C_c(X)$. Recall that a space X is *scattered* if every nonempty subset of X has an isolated point. The reader is referred to [**13**] for undefined terms and notation.

We now give a brief outline of this article. The main part of this article consists of seven sections. In Section 2, we give some algebraic and topological properties of $C_c(X)$ which were previously uncharacterized. First, we start with a useful characterization of $C^F(X)$ and next some new basic properties for $C_c(X)$ are presented. For example, it is observed that $Z_c(X)$ is closed under a countable intersection. We also show that $C_c(X)$ and $C_c^*(X)$ are always clean.

In Section 3, $\beta_0 X$ is studied via $C_c(X)$. In particular, in a natural manner, we observe that $\beta_0 X$ is homeomorphic with the structure spaces of $C_c(X)$, $C^F(X)$ and that of $C_c(\beta_0 X)$.

In Section 4, we fully characterize the maximal ideals of $C_c(X)$ similarly to the maximal ideals in $C(X)$, i.e., we present the counterpart of the Gelfand-Kolmogoroff theorem in $C_c(X)$. It is shown that the maximal ideals (respectively, the prime z_c-ideals) of $C_c(X)$ are the contraction of the corresponding ideals in $C(X)$. We also show that absolutely convex ideals of $C_c^*(X)$ are the contraction of the corresponding ideals in $C^*(X)$ and, using this, we characterize maximal ideals of $C_c^*(X)$.

Section 5 is devoted to F_c-spaces, (i.e., spaces X for which $C_c(X)_P$, the localization of $C_c(X)$ at any prime ideal P is an integral domain). By using the results of the previous sections, which show the counterpart of F-spaces, we study F_c-spaces. It appears that most of the counterparts of the results concerning F-spaces are also naturally valid for F_c-spaces. For example, X is an F_c-space if and only if $\beta_0 X$ is an F_c-space (note that X is an F-space if and only if βX is an F-space).

Section 6 deals with F_c-spaces versus F-spaces. We observe that, whenever X is strongly zero-dimensional (i.e., any pair of disjoint zero-sets are contained in disjoint clopen sets, or equivalently if βX is zero-dimensional), then X is an F-space if and only if it is an F_c-space. The Lindelöf subspaces of F_c-spaces are observed to be F-spaces, and it is shown that every F_c-space satisfying the countable chain condition is an F-space, too.

Finally, in Section 7, the relation of F_c-spaces with the spaces X, for which $C_c(X)$ is a Bézout ring, is investigated. It is proven that the ideals in $C_c(X)$ are convex if and only if they are absolutely convex if and only if $C_c(X)$ and $C_c^*(X)$ are both unitarily absolute Bézout rings.

2. Some previously uncharacterized properties. Let $I_d X$ denote the set of idempotents in $C(X)$. It is well known that $I_d X$ coincides with the set of the characteristic functions of clopen subsets of X. We begin with the following simple and useful fact, which, incidentally and immediately, yields the well-known fact that $C^F(X)$ is always regular (von Neumann), see also **[4, 5, 8]** and **[11,** comment preceding Proposition 4.2].

Proposition 2.1. *The ring $C^F(X)$ coincides with the $\mathbb{R}$-subalgebra of $C(X)$ generated by $I_d X$, i.e., $C^F(X) = \mathbb{R}[I_d X]$. Moreover, every $f \in C^F(X)$ has a unique representation in $\mathbb{R}[I_d X]$ of the form*

$$f = r_1 e_1 + r_2 e_2 + \cdots + r_n e_n,$$

where $\{r_1, r_2, \ldots, r_n\}$ *is the set of nonzero elements in* $f(X)$ *and* $e_1, e_2, \ldots, e_n$ *are some nonzero orthogonal idempotents in* $I_d X$*.*

Proof. Clearly, $\mathbb{R}[I_d X] \subseteq C^F(X)$. Let $r_1, r_2, \ldots, r_n$ be all of the nonzero elements in $f(X)$, where $f \in C^F(X)$. For each r_i, take the idempotent $e_i \in C(X)$, where $e_i^{-1}(\{1\}) = f^{-1}(\{r_i\})$. Set $X_i = e_i^{-1}(\{1\})$, where $i = 1, 2, \ldots, n$, $X_0 = Z(f)$, and let e_0 be the idempotent with $X_0 = e_0^{-1}(\{1\})$ (note that we may have $X_0 = \emptyset$, in which case, $e_0 = 0$). Since $e_i e_j = 0$ for all $i \neq j$, we infer that, if $x \in X$ and $e_i(x) = 1$ for some $0 \leq i \leq n$, we have $e_j(x) = 0$ for all $j \neq i$. It is clear that

$$X = \bigcup_{i=0}^{n} X_i \quad \text{and} \quad X_i \cap X_j = \emptyset \quad \text{for all } i \neq j.$$

Clearly, if f is of the above form, then $f(X_i) = \{r_i\}$, for all i, where we may put $r_0 = 0$. Hence, the representation in this form, which clearly exists for f, is unique. It is apparent that $f \in \mathbb{R}[I_{dX}]$, and we are done. □

Let $f \in C^F(X)$ have the above form, and set

$$g = \frac{1}{r_1} e_1 + \frac{1}{r_2} e_2 + \cdots + \frac{1}{r_n} e_n.$$

Then, $f = f^2 g$; hence, $C^F(X)$ is a regular ring. We also emphasize that the evident fact that X is zero-dimensional if and only if any two disjoint closed sets, of which one is a singleton, are separated by an element in $I_d X$, can be added to the four equivalent statements in [**10,** Proposition 4.4].

In the next lemma, among some other useful facts, we also observe that $Z_c(X)$, similarly to $Z(X)$, is closed under countable intersection. It should be emphasized here that, in the first two parts of the following result, X need not be zero-dimensional.

Lemma 2.2.

(a) *For any space* X*,* $Z \in Z_c(X)$ *if and only if* Z *is a countable intersection of clopen sets in* X*.*

(b) *For any space* X*, if* $f \in C_c(X)$*, then* $\operatorname{pos} f$ *and* $\operatorname{neg} f$ *are a countable union of clopen sets. Moreover, given two disjoint sets* A

and B in X such that both are a countable union of clopen sets, then there exist $f \in C_c(X)$ with $\operatorname{pos} f = A$ *and* $\operatorname{neg} f = B$.

(c) *If X is a zero-dimensional space, then each zero-set in $Z(X)$ contains a member of $Z_c(X)$.*

Proof.

(a) Let $f \in C_c(X)$. We show that $Z(f)$ is a countable intersection of clopen sets. In order to see this, for each $n \in \mathbb{N}$, take $0 < r_n \leq 1/n$ with $r_n \notin f(X)$ and $-r_n \notin f(X)$. Hence,

$$Z(f) = \bigcap_{n=1}^{\infty} f^{-1}((-r_n, r_n)).$$

However, $f^{-1}((-r_n, r_n)) = f^{-1}([-r_n, r_n])$, and we are done.

Conversely, let A be any countable intersection of clopen sets. We show that $A \in Z_c(X)$. To this end, we first recall that, in [**10,** Proof of Theorem 5.5] it was shown that, whenever

$$B = \bigcap_{i=1}^{\infty} Z(e_i),$$

where each e_i is idempotent and $e_i e_j = 0$ for all $i \neq j$, then $B = Z(g)$ for some $g \in C_c(X)$ (note that we may take $g = \sum_{i=1}^{\infty} e_i/2^i$). It is folklore that any countable union of clopen sets can be written as a countable union of disjoint clopen sets; hence, dually, A can be written as

$$A = \bigcap_{i=1}^{\infty} A_i,$$

where each A_i is clopen with $A_i \cup A_j = X$ for all $i \neq j$. Therefore, if we set $A_i = Z(e_i)$, where e_i is idempotent for each i, we have $e_i e_j = 0$ for all $i \neq j$, and hence, from what is observed above, we are through; see also [**6,** Remark 1.2].

(b) Note that, for each $f \in C_c(X)$, $\operatorname{neg} f = \operatorname{pos}(-f)$ and $\operatorname{pos} f = X \setminus Z(g)$, where $g = f + |f| \in C_c(X)$. Moreover, given two disjoint countable union of clopen sets A and B in X, from the above, it can easily be shown that there are infinitely many elements $f \in C_c(X)$ with $\operatorname{pos} f = A$, $\operatorname{neg} f = B$.

(c) In view of part (a), it can easily be seen that, when X is zero-dimensional, any nonempty G_δ-set in X, in particular, any nonempty element in $Z(X)$, contains a nonempty element in $Z_c(X)$. $\square$

Remark 2.3. It is clear that the converse of the first part of Lemma 2.2 (b) is not true in general. For example, let X be an uncountable discrete space, and take any $f \in C(X) \setminus C_c(X)$. In contrast to Lemma 2.2 (c), if X is zero-dimensional, it is apparent that each element in $Z(X)$ is contained in an element in $Z_c(X)$, as well. It is also clear that, if $f \in C_c(X)$ (respectively, $f \in C(X)$) with $\mathrm{int}_X Z(f) \neq \emptyset$, then there exists an idempotent $1 \neq e \in C_c(X)$ such that $Z(e) \subseteq \mathrm{int}_X Z(f)$, and hence, f is a multiple of e by [**10,** Lemma 2.4] (respectively, by [**13,** 1D(1)]). Consequently, X is an almost P-space, i.e., $\mathrm{int}_X Z(f) \neq \emptyset$ for all non-units $f \in C(X)$, if and only if $\mathrm{int}_X Z(f) \neq \emptyset$ for all non-units $f \in C_c(X)$.

Using Lemma 2.2 (a) and in view of [**19,** Theorem 4.7(j)], we have the following fact, found in [**6,** Theorem 1.1].

Proposition 2.4. *X is strongly zero-dimensional, i.e., βX is zero-dimensional, if and only if $Z(X) = Z_c(X)$.*

Remark 2.5. We recall that, whenever A and B are two subsets of X such that A and B are separated by an element in $C_c(X)$ (i.e., $f(A) = 0$ and $f(B) = 1$, for some $f \in C_c(X)$, e.g., take A, B to be contained in two disjoint elements of $Z_c(X)$, see [**10,** Theorem 2.8]), then A and B are contained in two disjoint clopen sets. For example, take $0 < r < 1$ with $r \notin f(X)$, and put $U = f^{-1}((-\infty, r])$. Thus, $A \subseteq U$ and $B \subseteq X \setminus U$, or equivalently, they can be separated by an idempotent element of $C(X)$. In view of Lemma 2.2, it is also clear that, if A and B are two disjoint closed subsets of a zero-dimensional Lindelöf space X, then A is contained in an element of $Z_c(X)$ which is disjoint from B; hence, A and B are contained in two disjoint elements of $Z_c(X)$, which in turn implies that A and B are contained in two disjoint clopen sets. Consequently, we have proved the well-known fact that every zero-dimensional Lindelöf space is a normal strongly zero-dimensional space, see [**9,** Theorem 3.8.2], [**13,** Theorem 16.16] and [**19,** Lemma 4.7(i)].

Corollary 2.6. *Let each element of a Lindelöf subset A of X be separated from a subset B of X by an element in $C_c(X)$ (X may not be zero-dimensional). Then, there exists an element $Z \in Z_c(X)$ containing B disjoint from A.*

Proof. From the first part of the previous remark, we note that, for each $a \in A$, there exist two disjoint clopen sets U_a and V_a with $a \in U_a$ and $B \subseteq V_a$. Hence, there exists a countable set $\{a_1, a_2, \ldots, a_n, \ldots\}$ of elements of A such that

$$A \subseteq \bigcup_{i=1}^{\infty} U_{a_i}$$

and

$$B \subseteq \bigcap_{i=1}^{\infty} V_{a_i}.$$

Now, by Lemma 2.2, we may put

$$Z(f) = \bigcap_{i=1}^{\infty} V_{a_i},$$

where $f \in C_c(X)$. Clearly, $B \subseteq Z(f)$ and $A \cap Z(f) = \emptyset$; hence, we are done. □

Before presenting our next observation, we recall that an element r in a commutative ring R is called *clean* if $r = \sigma + e$, where $\sigma \in R$ is a unit and e is an idempotent of R, and R is called *clean* if every element in R is clean. In view of [**2,** Theorem 2.5], [**15,** Proposition 3.9], [**18,** Theorem 13] and [**20,** Proposition 3.2], see also [**16**], X is strongly zero-dimensional if and only if $C(X)$ (respectively, $C^*(X)$) is clean. Using a proof similar to those of aforementioned results, we may show that $C_c(X)$ is always clean. However, unfortunately, this method cannot be applied to show that $C_c^*(X)$ is also a clean ring. Fortunately, the next lemma, which is a counterpart of [**2,** Lemma 2.1], immediately shows that, for any topological space X (not necessarily Tychonoff), both $C_c(X)$ and $C_c^*(X)$ are clean.

Lemma 2.7. *Let X be a topological space (not necessarily Tychonoff) and*

$$f : X \longrightarrow \mathbb{R}$$

a continuous function which does not take a real number $0 < r < 1$. Then, f is clean.

Proof. It is easy to check that

$$f^{-1}(\{1\}) \subseteq f^{-1}((r,\infty)) \subseteq X \setminus Z(f),$$

and clearly, $f^{-1}((r,\infty))$ is clopen. Hence, $f^{-1}((r,\infty)) = Z(e)$ for some idempotent e. Define $u(x) = f(x)$ for $x \in Z(e)$ and $u(x) = f(x) - 1$ for $x \notin Z(e)$. Now, whenever $x \in Z(e)$, we have $u(x) = f(x) > r$ and, if $x \notin Z(e)$, we have $u(x) = f(x) - 1 < r - 1$. This means that u is bounded away from zero, i.e., u is unit and $f = u + e$ is clean. □

In the proof of the above lemma, it is clear that, if $f \in C_c(X)$ ($f \in C_c^*(X)$), then u is a unit in $C_c(X)$ ($C_c^*(X)$). Using this fact, the following corollary becomes evident. Before stating the corollary, the reader is reminded that every commutative regular ring is clean, see [**18**], hence $C^F(X)$ is clean, too.

Corollary 2.8. *For every topological space X (not necessarily Tychonoff), the rings $C_c(X)$ and $C_c^*(X)$ are always clean.*

3. Some connections between $C_c(X)$ and $\beta_0 X$. In [**10,** Remark 2.12] we observed that all of the results in [**13,** Chapter 2] are valid if we just replace $C(X)$ and z-filters by $C_c(X)$ and z_c-filters, respectively. We should also emphasize here that, by taking X to be zero-dimensional, every single result in [**13**] concerning the convergence of z-filters can be similarly proved for z_c-filters, namely, [**13,** 3.16(a),(b), Theorem 3.17, 3.18(a),(b),(c),(d)] (in the latter five facts, merely replace A_p by A_{cp}, the family of all elements in $Z_c(X)$ containing p). Moreover, [**13,** Theorems 4.8–4.12] are also trivially true in the context of $C_c(X)$ (in carrying over the proofs of these facts for $C_c(X)$, $C^*(X)$ should be replaced by $C^F(X)$), also see [**11,** Theorems 3.8, 3.9], where these results are recorded too. The following fact is also immediate see [**13,** 6.3(a),(b)].

Proposition 3.1. *Let X be dense in a zero-dimensional space T and $Z \in Z_c(X)$ with $p \in \mathrm{cl}_T Z$. Then, there exists a z_c-ultrafilter on X*

containing Z which converges to p. In particular, every point of T is the limit of a z_c-ultrafilter on X.

In [**10, 11**], it is claimed that $C^F(X)$ in $C_c(X)$, in most cases, plays the same role as $C^*(X)$ does in $C(X)$. Part (2) in the next proposition confirms this claim, too. This proposition, together with Proposition 3.3, are the counterparts of [**13,** Theorems 6.4, 6.7]. In particular, these counterparts are essentially well known; however, they are not recorded in the following useful forms in the literature.

Proposition 3.2. *Let X be dense in a zero-dimensional space T. Then, the following statements are equivalent.*

(1) *Every continuous function τ from X into any zero-dimensional compact space Y has a continuous extension $\overline{\tau}$ from T into Y.*

(2) *X is C^F-embedded in T, i.e., every function $f \in C^F(X)$ has an extension to a function $\overline{f} \in C^F(T)$ (hence, $f(X) = \overline{f}(T)$ and the mapping $f \to \overline{f}$ is an isomorphism of $C^F(X)$ onto $C^F(T)$).*

(3) *Every idempotent $e \in C(X)$ has an extension to an idempotent $\overline{e} \in C(T)$.*

(4) *Any two disjoint clopen sets in X have disjoint clopen closures in T.*

(5) *Any two disjoint zero-sets in $Z_c(X)$ have disjoint closures in T.*

(6) *For any two zero-sets Z_1 and Z_2 in $Z_c(X)$, we have*

$$\mathrm{cl}_T(Z_1 \cap Z_2) = \mathrm{cl}_T\, Z_1 \cap \mathrm{cl}_T\, Z_2.$$

(7) *Every point of T is the limit of a unique z_c-ultrafilter on X.*

Proof.

(1) $\Rightarrow$ (2). Let $f \in C^F(X)$. Then, $Y = f(X)$ is a finite discrete subspace (i.e., Y is a zero-dimensional compact space); hence, by (1), there is an $\overline{f} \in C(T)$. It is clear that $\overline{f}$ is unique and $f(X) = \overline{f}(T)$, and $\varphi : C^F(X) \to C^F(T)$, where $\varphi(f) = \overline{f}$ is an isomorphism.

(2) $\Rightarrow$ (3). Let $e \in C(X)$ be an idempotent; hence, $e \in C^F(X)$ has the extension $\overline{e} \in C^F(T)$. Since $e(X) = \overline{e}(T)$, we infer that $\overline{e}$ is an idempotent.

(3) $\Rightarrow$ (4). Let U and V be two disjoint clopen sets in X. We define the idempotents $e_u, e_v \in C(X)$ by $U = e_u^{-1}(\{1\})$ and $V = e_v^{-1}(\{1\})$.

Since $e_u e_v = 0$, we infer that $\overline{e}_u \overline{e}_v = 0$. This implies that

$$\overline{e}_u^{-1}(\{1\}) \cap \overline{e}_v^{-1}(\{1\}) = \emptyset.$$

Note that $\overline{e}_u^{-1}(\{1\}) = \mathrm{cl}_T\, U$, $\overline{e}_v^{-1}(\{1\}) = \mathrm{cl}_T\, V$; hence, we are done.

(4) $\Rightarrow$ (5). Let Z_1 and Z_2 be two disjoint elements in $Z_c(X)$. Then, by Remark 2.5, Z_1 and Z_2 are contained in two disjoint clopen sets in X. Now, by (4), the latter two clopen sets have disjoint closures in T which, in turn, implies that Z_1 and Z_2 have disjoint closures in T.

(5) $\Rightarrow$ (6). Let Z_1 and Z_2 be two elements in $Z_c(X)$. From (5), we may assume that $Z_1 \cap Z_2 \neq \emptyset$. Hence, it suffices to show that, if there exists a $p \in \mathrm{cl}_T\, Z_1 \cap \mathrm{cl}_T\, Z_2$ with $p \notin \mathrm{cl}_T(Z_1 \cap Z_2)$, we obtain a contradiction. By our assumption, there exists a clopen set $U \subseteq T$ such that $p \in U$ and $U \cap (Z_1 \cap Z_2) = \emptyset$. Clearly, $V = U \cap X$ is clopen in X; hence, $V \in Z_c(X)$ and

$$V \cap (Z_1 \cap Z_2) = (V \cap Z_1) \cap (V \cap Z_2) = \emptyset.$$

Now, by (5), we have

$$\mathrm{cl}_T(V \cap Z_1) \cap \mathrm{cl}_T(V \cap Z_2) = \emptyset.$$

Consequently, p does not belong to one of the latter two closure sets. We suppose that $p \notin \mathrm{cl}_T(V \cap Z_1)$. Hence, there exists a neighborhood W of p with

$$W \cap V \cap Z_1 = W \cap U \cap Z_1 = \emptyset,$$

which is a contradiction ($W \cap U$ is a neighborhood of p and $p \in \mathrm{cl}_T\, Z_1$).

(6) $\Rightarrow$ (7). Evident.

(7) $\Rightarrow$ (1). For the proof of this part, [**13,** Proof of Theorem 6.4, (5) $\Rightarrow$ (1)] can be repeated verbatim without any extra work. $\square$

Proposition 3.3. *Let X be dense in a zero-dimensional space T. Then, the following statements are equivalent.*

(1) *Every continuous function τ from X into any zero-dimensional compact space Y has a continuous extension $\overline{\tau}$ from T into Y.*
(2) *Every $f \in C^F(X)$ has an extension $\overline{f} \in C(T)$.*
(3) *$\beta_0 T = \beta_0 X$.*
(4) *$X \subseteq T \subseteq \beta_0 X$.*

Proof.

(1) $\Rightarrow$ (2). Evident by the previous proposition.

(2) $\Rightarrow$ (3). We first digress for a moment and emphasize that $\overline{f}(T) = f(X)$ (this observation is unnecessary in the proof which follows). Take $f, g \in C_c(X)$ with $Z(f) \cap Z(g) = \emptyset$. Then, by Remark 2.5, there exists an idempotent $e \in C(X)$ with $e(Z(f)) = 0$, $e(Z(g)) = 1$. Hence, by our hypothesis, e has an extension $\overline{e} \in C(T)$. However, $Z(f) \subseteq \overline{e}^{\,-1}(\{0\})$ and $Z(g) \subseteq \overline{e}^{\,-1}(\{1\})$ imply that $Z(f)$ and $Z(g)$ are contained in two disjoint closed subsets of T; thus, they have disjoint closures in T. Consequently, X satisfies part (5) of Proposition 3.2, and therefore, it also satisfies part (1). Now, in order to show that $\beta_0 T = \beta_0 X$, we must prove that X is dense in $\beta_0 T$ and any continuous map $\varphi : X \to Y$, where Y is a zero-dimensional compact space, has a continuous extension from $\beta_0 T$ into Y. It is evident that X is dense in $\beta_0 T$. Since X satisfies part (1), we infer that a continuous map

$$\overline{\varphi} : T \longrightarrow Y$$

exists which is an extension of φ. Consequently, by the definition of $\beta_0 T$, a continuous map

$$\beta_0\overline{\varphi} : \beta_0 T \longrightarrow Y$$

exists which is an extension of $\overline{\varphi}$. It is now evident that $\beta_0\overline{\varphi}$ is an extension of φ, and we are done.

(3) $\Rightarrow$ (4), (4) $\Rightarrow$ (1). Evident. $\square$

If T in the previous proposition is a zero-dimensional compact space, then, in view of [**19,** Corollary 4.7(f)], it is homeomorphic with $\beta_0 X$. In this case, part (2) of Proposition 3.3 raises the question of whether X is C_c^*-embedded in $\beta_0 X$ (it is evident that X is not necessarily C_c-embedded in $\beta_0 X$; for example, take $X = \mathbb{N}$). In view of the next result, the answer to this question is also negative.

Proposition 3.4. *Let X be a strongly zero-dimensional space such that βX is not scattered and $C^*(X) \cong C_c^*(X)$ (e.g., $X = \mathbb{N}$ or $X = \mathbb{Q}$). Then, X is not C_c^*-embedded in $\beta_0 X$. In particular, if we just trade off the strong zero-dimensionality of X with the complete regularity of X, then X is not C_c^*-embedded in βX.*

Proof. Suppose that X is C_c^*-embedded in $\beta_0 X$ in order to seek a contradiction. By our assumption, for every $f \in C_c^*(X)$, there exists an $\overline{f} \in C_c(\beta_0 X)$ with $\overline{f}|_X = f$. Clearly,

$$\varphi : C_c^*(X) \longrightarrow C_c(\beta_0 X);$$

$\varphi(f) = \overline{f}$ is an isomorphism, i.e., $C_c^*(X) \cong C_c(\beta_0 X)$. By our assumption βX is zero-dimensional; hence, $\beta X = \beta_0 X$, and clearly,

$$C(\beta X) \cong C^*(X) \cong C_c^*(X) \cong C_c(\beta_0 X) = C_c(\beta X).$$

Hence, by what we observed in the introduction, $C(\beta X) = C_c(\beta X)$, also see [**11, 17**]. Now, in view of the RPS-theorem, βX must be scattered, which is the desired contradiction. The proof of the last part is exactly similar to the proof of the first part. □

As in βX, for each $p \in X$, where X is zero-dimensional, take A_{cp} to be the unique z_c-ultrafilter whose limit is p and, for $p \in \beta_0 X$, let A_c^p be the unique z_c-ultrafilter with limit p. By convention, we set $A_c^p = A_{cp}$ for $p \in X$. It can be shown that, similarly to the construction of βX, if $Z \in Z_c(X)$ and $\overline{Z} = \{p \in \beta_0 X : Z \in A_c^p\}$, then the topology of $\beta_0 X$ is defined by taking the set

$$\{\overline{Z} : Z \in Z_c(X)\}$$

as a base for the closed sets of $\beta_0 X$. Similarly to the case of βX, it may be shown that $\overline{Z} = \mathrm{cl}_{\beta_0 X} Z$, see [**13,** page 87, part (c)]; hence, $p \in \mathrm{cl}_{\beta_0 X} Z$ if and only if $Z \in A_c^p$.

We conclude this section with the next three remarks. In the remark which follows, we present the form of fixed maximal ideals in $C_c(X)$ and in $C_c^*(X)$. In the second remark, we observe that $\beta_0 X$ is homeomorphic to the structure space of the ring $C_c(X)$.

Remark 3.5. In [**10**], it is observed that, if $p \in X$ and

$$M_{cp} = \{f \in C_c(X) : p \in Z(f)\} = M_p \cap C_c(X),$$

where M_p consists of those elements of $C(X)$ vanishing at p, then M_{cp} is a maximal ideal of $C_c(X)$, which is fixed, see also [**11,** Theorem 3.8 ff.]. Hence, the Jacobson radical of $C_c(X)$ is zero. We also emphasize that M_p (similarly to the case of $C(X)$ and $C^*(X)$, see [**13,** 4.7]) is the only maximal ideal in $C(X)$, fixed or free, whose intersection with $C_c(X)$ is M_{cp}. In order to see this, let $M \neq M_p$ be a maximal ideal in $C(X)$ with $M_{cp} = M \cap C_c(X)$ and seek a contradiction. Take $f \in M$ with

$f(p) \neq 0$. Since X is zero-dimensional, there exists a $g \in C_c(X)$ with $Z(f) \subseteq Z(g)$ and $g(p) \neq 0$. Since M is a z-ideal, we infer that $g \in M$; hence, $g \in M \cap C_c(X) \setminus M_{cp}$, a contradiction. We also emphasize that

$$M^*_{cp} = \{f \in C^*_c(X) : p \in Z(f)\}$$

is a fixed maximal ideal in $C^*_c(X)$. Clearly, $M^*_{cp} = M_{cp} \cap C^*_c(X)$, and M_{cp} is the only maximal ideal in $C_c(X)$ whose intersection with $C^*_c(X)$ is M^*_{cp}. Moreover, if M is an arbitrary maximal ideal in $C_c(X)$, then $M \cap C^*_c(X)$ may not be maximal in $C^*_c(X)$; in addition, free maximal ideals of $C^*_c(X)$ need not be of the latter form. In order to see this, since $C(\mathbb{N}) = C_c(\mathbb{N})$, one can easily see that the entire argument and the example preceding [**13,** Theorem 4.8] applies in this case as well.

Remark 3.6. Let $\mathfrak{M}_c(X) = \mathrm{Max}(C_c(X))$ be the set of all maximal ideals of $C_c(X)$ and, for each $f \in C_c(X)$, define

$$D_f = \{M \in \mathfrak{M}_c(X) : f \notin M\}$$

and

$$V_f = \mathfrak{M}_c(X) \setminus D_f.$$

The topology on $\mathfrak{M}_c(X)$, defined by taking the family of all sets D_f as a base for the open sets, is called the *structure space* of $C_c(X)$, and it is, in fact, the subspace topology of the Zariski topology on $\mathrm{Spec}(C_c(X))$. For a zero-dimensional space X, let $\overline{\mathfrak{M}}_c(X)$ be the subspace of $\mathfrak{M}_c(X)$ consisting of the fixed maximal ideals of $C_c(X)$. It is evident that the correspondence

$$\varphi : p \longrightarrow M_{cp}$$

is a homeomorphism between X and $\overline{\mathfrak{M}}_c(X)$, $\varphi(Z(f)) = V_f \cap \overline{\mathfrak{M}}_c(X) = \overline{\mathfrak{M}}_c(X) \setminus D_f$, where $f \in C_c(X)$. It is also clear that $\mathfrak{M}_c(X)$ is a compact T_1-space and $\overline{\mathfrak{M}}_c(X)$ is dense in $\mathfrak{M}_c(X)$. As in the case of βX and the structure space of $C(X)$, it can be shown that $\beta_0 X$ and $\mathfrak{M}_c(X)$ are homeomorphic. Here, we give a quick proof of this fact. (In [**6**], it is also claimed that a different proof can be modeled after [**6,** reference 6 (Theorem 5.1)].) First, we show that $\mathfrak{M}_c(X)$ is zero-dimensional. Take $M \in \mathfrak{M}_c(X)$ with $M \in G$, where G is open in $\mathfrak{M}_c(X)$. Clearly, $G = D(I) = \{M \in \mathfrak{M}_c(X) : I \not\subseteq M\}$, where I is an ideal of $C_c(X)$. Consequently, $I + M = C_c(X)$. This implies that there exists an $f \in I$ with $1 - f \in M$. Now, in view of Remark 2.5, there exists an idempotent $e \in C_c(X)$ with $Z(f) \subseteq Z(e)$ and $Z(1-f) \subseteq Z(1-e)$. Since $Z(e)$ and $Z(1-e)$ are neighborhoods of $Z(f)$ and $Z(1-f)$, respectively, we infer that $e \in I$ and $1-e \in M$, by [**10,** Lemma 2.4]. Thus, $M \in D_e \subseteq D(I) = G$. Note that D_e is clopen, $D_e = V_{1-e}$; hence, we are done.

Finally, if we show that disjoint clopen sets in $\overline{\mathfrak{M}}_c(X)$ have disjoint closures in $\mathfrak{M}_c(X)$, then, by Proposition 3.2 and the comment following Proposition 3.3, $\mathfrak{M}_c(X)$, $\beta_0(\overline{\mathfrak{M}}_c(X))$ and $\beta_0(X)$ are all homeomorphic. Toward this end, let U and W be two disjoint clopen sets in $\overline{\mathfrak{M}}_c(X)$. Consequently, $A = \varphi^{-1}(U)$ and $B = \varphi^{-1}(W)$ are disjoint clopen sets in X, where φ is the above homeomorphism. Now, take $e \in C_c(X)$ to be the idempotent with $A = Z(e)$ and $B \subseteq Z(1-e)$. This implies that e belongs to every element of U, and $1-e$ belongs to every element of W. Thus, $\mathrm{cl}_{\mathfrak{M}_c(X)} U \subseteq V_e$, $\mathrm{cl}_{\mathfrak{M}_c(X)} W \subseteq V_{1-e}$, and we are done.

Before giving the next remark we remind the reader that, if X is zero-dimensional, then X is compact if and only if very maximal ideal of $C_c(X)$ (respectively, $C^F(X)$) is fixed (note that its proof is identical to [**13,** Proof of Theorem 4.11], see also [**11,** Theorem 3.8]).

Remark 3.7. In Remark 3.6, we have already shown that $\beta_0 X$ and $\mathfrak{M}_c(X)$ are homeomorphic; hence, $C_c(\beta_0 X)$ has a maximal ideal space homeomorphic to $\beta_0 X$. If X is a zero-dimensional space and $x \in X$, $M_x^F = M_x \cap C^F(X)$ is a maximal ideal of $C^F(X)$. Note that

$$f + M_x^F \longrightarrow f(x)$$

is the unique isomorphism of $C^F(X)/M_x^F$ onto $\mathbb{R}$. It is also evident that, in view of Proposition 3.2 (2), $C^F(X) \cong C^F(\beta_0 X)$. Using these comments, we infer that, to consider the maximal ideal space of $C^F(X)$, we may assume that X is compact. Now, it is evident that, if X is a zero-dimensional compact space, the correspondence $x \rightarrow M_x^F$ is a homeomorphism from X onto $\mathrm{Max}(C^F(X))$, the maximal ideal space of $C^F(X)$, also see Theorem 4.1, below. As a consequence, we observe that $C_c(X)$, $C^F(X)$, $C_c(\beta_0 X)$ and $C(\beta_0 X)$ all have the same maximal ideal space homeomorphic to $\beta_0 X$.

4. Characterization of maximal ideals of $C_c(X)$ and $C_c^*(X)$.

In [**11,** Theorem 3.8], it is observed that X is compact if and only if every ideal, or every prime (maximal) ideal, in $C_c(X)$ or in $C^F(X)$ is fixed. If, in Proposition 3.2, we take $T = \beta_0 X$ and, for each $f \in C^F(X)$, put $\overline{f} = f^{\beta_0}$, then we have the following characterization of maximal ideals in $C^F(X)$, the counterpart of [**13,** Theorem 7.2].

Theorem 4.1. *Each maximal ideal M in $C^F(X)$ is of a unique form*

$$M = M^{Fp} = \{f \in C^F(X) : f^{\beta_0}(p) = 0\}, \quad p \in \beta_0 X.$$

Proof. Merely apply [**13,** Proof of Theorem 7.2]. We should also emphasize that M^{Fp} is fixed or free according to whether $p \in X$ (in which case, we put $M^{Fp} = M_p^F$) or $p \notin X$. □

Note that, for each $p \in \beta_0 X$, there is a unique maximal ideal M_c^p in $C_c(X)$, where $M_c^p = Z^{-1}[A_c^p]$. The following fact, the proof of which is exactly the same as the proof of the Gelfand-Kolmogoroff theorem, shows that the counterpart of this theorem is also valid in $C_c(X)$, see [**13,** Theorem 7.3].

Theorem 4.2. *The maximal ideals in $C_c(X)$ are of the form*

$$M_c^p = \{f \in C_c(X) : p \in \mathrm{cl}_{\beta_0 X} Z(f)\}, \quad p \in \beta_0 X.$$

Moreover, M_c^p is fixed if and only if $p \in X$ (in which case, we set $M_c^p = M_{cp}$).

We recall that, if R is a subring of a commutative ring S and I is an ideal in S, then the ideal $I \cap R$ is called the *contraction* of I in R, and it is denoted by I^c. It is well known and easy to prove that every minimal prime ideal P in R is a contraction of a minimal prime ideal, Q say, in S (consider $T = R \setminus P$ to be a multiplicatively closed set in S; hence, there is a minimal prime ideal, say Q, in S with $T \cap Q = \emptyset$), i.e., $P = Q^c$, also see [**10,** comment preceding Corollary 3.4]. Noting that every minimal prime ideal in $C(X)$ (respectively, in $C_c(X)$) is a z-ideal in $C(X)$ (respectively, a z_c-ideal in $C_c(X)$), see [**13,** Theorem 14.7] and [**10,** Corollary 3.4], respectively, the following fact can be considered as an extension of the latter corollary.

Proposition 4.3.

(a) *An ideal J in $C_c(X)$ is a z_c-ideal if and only if it is a contraction of a z-ideal of $C(X)$.*

(b) *An ideal J in $C_c^*(X)$ is an absolutely convex ideal if and only if it is a contraction of an absolutely convex ideal of $C^*(X)$.*

Proof.

(a) Clearly, if $J = I^c$, where I is a z-ideal in $C(X)$, then J is evidently a z_c-ideal in $C_c(X)$.

Conversely, suppose that J is a z_c-ideal of $C_c(X)$, and set

$$I = \{f \in C(X) : Z(g) \subseteq Z(f) \text{ for some } g \in J\}.$$

Clearly, I is a z-ideal in $C(X)$ and $J \subseteq I^c$. On the other hand, if $f \in I^c$, then there exists a $g \in J$ with $Z(g) \subseteq Z(f)$. Inasmuch as J is a z_c-ideal, we infer that $f \in J$; hence, we are done.

(b) Take

$$I = \{f \in C^*(X) : |f| \leq |g| \text{ for some } g \in J\}.$$

Clearly, I is an ideal in $C^*(X)$. In fact, whenever $f \in I$ and $h \in C^*(X)$, then there exist $K \in \mathbb{N}$ and $g \in J$ such that $|h| \leq K$ and $|f| \leq |g|$. Thus, $|fh| \leq K|g| = |Kg|$ implies that $fh \in I$, for $Kg \in J$. Whenever J is proper, then I is too, and it is easily seen that I is absolutely convex containing J and $J = I \cap C_c^*(X)$. □

Remark 4.4. The socle of $C_c(X)$, denoted $\mathrm{Soc}(C_c(X))$, which is the sum of minimal ideals of $C_c(X)$, is fully studied and topologically characterized in [**11**]. It is observed that $\mathrm{Soc}(C_c(X))$ is a z_c-ideal. Consequently, by Proposition 4.3 and in view of [**11,** Proposition 5.3], whenever $\mathrm{Soc}(C_c(X)) \neq 0$, there is a z-ideal I in $C(X)$ with $0 \neq f \in I$ such that $\mathrm{coz} f = X \setminus Z(f)$ is a subset of a finite union of mutually disjoint clopen connected subsets of X, also see [**17,** Proposition 5.2].

Corollary 4.5. *An ideal P in $C_c(X)$ is a prime z_c-ideal if and only if it is a contraction of a prime z-ideal in $C(X)$.*

Proof. Let P be a prime z_c-ideal. Consider $S = C_c(X) \setminus P$ as a multiplicatively closed set in $C(X)$. From Proposition 4.3, P is a contraction of a z-ideal in $C(X)$, I say. Clearly, $I \cap S = \emptyset$; hence, there exists a prime ideal Q in $C(X)$ minimal over I with $Q \cap S = \emptyset$. We recall that, in view of [**13,** Theorem 14.7], Q is a z-ideal. It is clear that $P = I^c \subseteq Q^c \subseteq P$; hence, $P = Q^c$, and we are done. The converse is evident. □

Corollary 4.6. *Every maximal ideal N of $C_c(X)$ is a contraction of a maximal ideal in $C(X)$. Moreover, if $N = M^c$, where M is a maximal ideal in $C(X)$, then N is fixed if and only if M is fixed and, if M is real, then so, too, is N.*

Proof. Let N be a maximal ideal in $C_c(X)$. Since N is a z_c-ideal, see [**10,** Remark 2.12], we infer that $N = I^c$, where I is a z-ideal in $C(X)$, Proposition 4.3 (a). However, there is a maximal ideal M in $C(X)$ containing I. Hence, $N = I^c \subseteq M^c$ implies that $N = M^c$, and we are done. For the last part, it may easily be noted that N is fixed if and only M is fixed as well, by Remark 3.5. Finally, if M is real, then the monomorphism

$$\varphi : \frac{C_c(X)}{N} \longrightarrow \frac{C(X)}{M},$$

where $\varphi(f + N) = f + M$, completes the proof. □

Next, we give a second representation of the maximal ideals of $C_c(X)$. Before doing so, we record some general facts in the following remark.

Remark 4.7. Let A be an $\mathbb{R}$-subalgebra as well as a sublattice of $C(X)$. Define an ideal I in A to be a z_A-ideal if, whenever $f \in I$ and $Z(f) \subseteq Z(g)$, where $g \in A$, then $g \in I$. It is clear that every z_A-ideal is absolutely convex. It is also trivial to see that every maximal ideal in A is z_A-ideal; hence, it is absolutely convex. Many results concerning some appropriate ideals in $C(X)$ remain valid in A, for example, a basic fact, namely, [**13,** Theorem 2.9], is also true for z_A-ideals in A. Consequently, every prime ideal in A is contained in a unique maximal ideal in A. Or, if P is a prime ideal in A which is minimal over a z_A-ideal I in A, then P is a z_A-ideal, too, see [**13,** Theorem 14.7]. Moreover, if $A \subseteq B$ are two $\mathbb{R}$-subalgebras as well as sublattices of $C(X)$, then an ideal I in A is a z_A-ideal, a prime z_A-ideal, if and only if it is a contraction of a z_B-ideal, a prime z_B-ideal, in B, respectively. It is clear that, with regards to the previous results, the subalgebras A and B can be taken to be any two of the subalgebras $\mathbb{R}$, $C^*(X)$, $C_c(X)$, $C_c^*(X)$, $C^F(X)$ and $L_c(X)$, of $C(X)$, as long as $A \subseteq B$. For the definition and properties of $L_c(X)$, see [**17**].

Every maximal ideal of $C(X)$ is of the form M^p, $p \in \beta X$, and every maximal ideal of $C_c(X)$ is of the form M_c^p, where $p \in \beta_0 X$, by Theorem 4.2. Now, using Corollary 4.6, for each $p \in \beta_0 X$, there exists a point $\pi_p \in \beta X$ such that $M_c^p = M^{\pi_p} \cap C_c(X)$. This means that, whenever $f \in C_c(X)$, then $p \in \mathrm{cl}_{\beta_0 X} Z(f)$ if and only if $\pi_p \in \mathrm{cl}_{\beta X} Z(f)$. We note that if, for some $p \in \beta_0 X$, the corresponding point $\pi_p \in \beta X$ is not unique, we may always choose $\pi_p \in \beta X$ to be a unique point in βX corresponding to each point $p \in \beta_0 X$. Hence, we have the next representation for the maximal ideals of $C_c(X)$ as well.

Theorem 4.8. *Every maximal ideal of $C_c(X)$ is precisely of the form*

$$M_c^p = \{f \in C_c(X) : \pi_p \in \mathrm{cl}_{\beta X} Z(f)\}, \quad p \in \beta_0 X.$$

In view of Proposition 4.3 (b) and the fact that maximal ideals in $C_c^*(X)$ are absolutely convex, we have the next immediate result.

Proposition 4.9. *Every maximal ideal of $C_c^*(X)$ is a contraction of a maximal ideal in $C^*(X)$.*

For each $p \in \beta X$, set

$$T_p = \{q \in \beta X : M^{*p} \cap C_c^*(X) = M^{*q} \cap C_c^*(X)\}.$$

Now, for each $p \in \beta X$, take a fixed $q_p \in T_p$, and put

$$T = \{q_p \in T_p : p \in \beta X\}.$$

Therefore, the set of all maximal ideals of $C_c^*(X)$ exactly coincides with $\{M^{*q_p} \cap C_c^*(X) : q_p \in T\}$. Using these facts, we obtain a representation for maximal ideals of $C_c^*(X)$ as follows:

Corollary 4.10. *Maximal ideals of $C_c^*(X)$ are precisely of the form*

$$M_c^{*q} = \{f \in C_c^*(X) : f^\beta(q) = 0\}, \quad q \in T.$$

As in $C(X)$, and similar to the ideals O^p, $p \in \beta X$, for each $p \in \beta_0 X$, we define

$$O_c^p = \{f \in C_c(X) : p \in \mathrm{int}_{\beta_0 X} \mathrm{cl}_{\beta_0 X} Z(f)\}.$$

In the next lemma, we cite some facts concerning maximal ideals and the ideals O_c^p, $p \in \beta_0 X$, in $C_c(X)$ as counterparts of [**13,** 7.12(a),(b), Theorems 7.13, 7.15]. The corresponding proofs, which may be exactly applied for the proofs of these facts, are also left to the reader.

Lemma 4.11. *Let X be a zero-dimensional space. The following statements hold.*

(1) *For $p \in \beta_0 X$, $f \in O_c^p$ if and only if there is a clopen subset V of $\beta_0 X$ containing p such that $V \cap X \subseteq Z(f)$.*
(2) *For $p \in \beta_0 X$, $f \in O_c^p$ if and only if $fg = 0$ for some $g \notin M_c^p$.*
(3) *An ideal I in $C_c(X)$ is contained in a unique maximal ideal M_c^p for some $p \in \beta_0 X$ if and only if $O_c^p \subseteq I$.*
(4) *Every prime ideal P in $C_c(X)$ contains O_c^p for a unique $p \in \beta_0 X$, and M_c^p is the unique maximal ideal containing P.*
(5) *In* (4)*, the prime ideal P can be replaced by a primary ideal Q, also see* [**1,** Remark 2.9].

Remark 4.12. By Theorem 4.2, $p \in \mathrm{cl}_{\beta_0 X} Z(g)$ if and only if $g \in M_c^p$, where $g \in C_c(X)$. Consequently, if $f, g \in C_c(X)$, then $\mathrm{cl}_{\beta_0 X} Z(f)$ is a neighborhood of $\mathrm{cl}_{\beta_0 X} Z(g)$ if and only if $f \in O_c^p$ whenever $g \in M_c^p$, see [**13,** 7.14]. Moreover, $\mathrm{cl}_{\beta_0 X} Z(f)$ is a neighborhood of $\mathrm{cl}_{\beta_0 X} Z(g)$ if and only if there exists an $h \in C_c(X)$ with $Z(g) \subseteq X \setminus Z(h) \subseteq Z(f)$. In order to see this, it suffices to invoke Proposition 3.2 and apply the proof of [**13,** Theorem 7.14].

For $p \in X$, the ideal O_p consists of elements $f \in C(X)$ such that $Z(f)$ is a neighborhood of p, and M_p is the only maximal ideal containing O_p, see [**13,** 4I]. In [**3,** Theorem 2.4], it is shown that X is a zero-dimensional space (respectively, strongly zero-dimensional space) if and only if, for each $p \in X$, the ideal O_p is generated by a set of idempotents (respectively, for each $p \in \beta X$, the ideal O^p is generated by a set of idempotents). Using Lemma 4.11 (1), we obtain the next result, which is the counterpart of the above facts.

Proposition 4.13. *Let X be a zero-dimensional space. Then, for each $p \in \beta_0 X$, the ideal O_c^p is generated by a set of idempotents in $C_c(X)$.*

Proof. For each $p \in \beta_0 X$, let $\mathfrak{B}_p$ be a local base at p consisting entirely of clopen sets. For each $V \in \mathfrak{B}_p$, define the idempotent $e_v \in C_c(X)$ with $e_v = 0$ on $V \cap X$ and $e_v = 1$ on $X \setminus V$. Now, it is clear that O_c^p is generated by the set $\{e_v : V \in \mathfrak{B}_p\}$. In fact, each e_v belongs to O_c^p and, whenever $f \in O_c^p$, then there is a $V \in \mathfrak{B}_p$ such that $Z(e_v) = V \cap X \subseteq Z(f)$, by Lemma 4.11 (1). Thus, $Z(f)$ is a neighborhood of $Z(e_v)$ and, in view of [**10,** Lemma 2.4], we infer that $f = ge_v$ for some $g \in C_c(X)$, and we are through. $\square$

We should emphasize here that the above proof, in fact, proves the next result, which asserts the validity of a stronger property than the fact that O_c^p is merely generated by idempotents.

Corollary 4.14. *Given $p \in \beta_0 X$ and $f \in O_c^p$, where X is zero-dimensional, there exists an idempotent $e \in O_c^p$ and $g \in C_c(X)$ such that $f = eg$.*

5. When are the localization of $C_c(X)$ at its prime ideals uniform? We recall that a completely regular Hausdorff space X is called an *F-space* if each ideal O^p, $p \in \beta X$, is a prime ideal of $C(X)$. In [**1,** Proposition 2.5], it is observed that X is an F-space if and only if $C(X)$ is locally a domain. It is very easy to see that a commutative reduced ring R is a domain if and only if it is a uniform ring (a ring is *uniform* if its nonzero ideals mutually intersect non-trivially), also see [**1,** Proposition 2.5]. Motivated by the latter simple fact and in order to answer the question, it is natural that we study and determine spaces X for which $C_c(X)$ is locally a domain ($C_c(X)_P$ is reduced). We similarly call a space X (not necessarily zero-dimensional) an *F_c-space* if every localization $C_c(X)_P$ of $C_c(X)$ at a prime ideal P is a domain. It is clear that an F_c-space may not be an F-space; for example, take X to be a connected Tychonoff space which is not an F-space (X can be any connected metric space).

Our main aim in this section is to study and characterize F_c-spaces. Similarly to the characterization of F-spaces, we shall first give several algebraic and topological characterizations of F_c-spaces, see [**13,** Theorem 14.25]. Toward this end, we also need the following facts which are the counterparts of [**13,** Lemma 14.21, Corollary 14.22, Lemma 14.23]. We recall that, if I is an ideal in a commutative ring R, then, for each $a \in R$, the element $a + I \in R/I$ is denoted by $\overline{a}$.

Lemma 5.1. *Let $f \in C_c(X)$, $(\overline{f}, |\overline{f}|)$ be a principal ideal in $C_c(X)/I$, where I is an ideal in $C_c(X)$. Then, there exists a zero-set $Z \in Z_c[I]$ such that $Z \cap \operatorname{pos} f$ and $Z \cap \operatorname{neg} f$ are contained in two disjoint clopen sets in X.*

Proof. Using the same proof as [**13,** Proof of Lemma 14.21], two disjoint zero-sets $Z_1, Z_2 \in Z_c[X]$ and $Z \in Z_c[I]$ may be found such that $Z \cap \operatorname{pos} f \subseteq Z_1$ and $Z \cap \operatorname{neg} f \subseteq Z_2$. From Proposition 3.2, $\operatorname{cl}_{\beta_0 X} Z_1 \cap \operatorname{cl}_{\beta_0 X} Z_2 = \emptyset$. Since $\beta_0 X$ is compact, $\operatorname{cl}_{\beta_0 X} Z_1$ and $\operatorname{cl}_{\beta_0 X} Z_2$ are also compact; hence, they are contained in two disjoint clopen subsets of $\beta_0 X$ (any two disjoint compact sets in a zero-dimensional space are contained in two disjoint clopen sets), and we are done. $\square$

From Lemma 5.1, the next corollary is now immediate, also see [**13,** Corollary 14.22].

Corollary 5.2. *The following statements are equivalent for any* $f \in C_c(X)$.

(1) $\operatorname{pos} f$ *and* $\operatorname{neg} f$ *are contained in two disjoint clopen subsets in* X.
(2) *There exists a unit* $u \in C_c(X)$ *such that* $|f| = uf$.
(3) $(f, |f|)$ *is a principal ideal of* $C_c(X)$.

If we consider the "order" defined in [**13,** Theorem 5.2] on $C_c(X)/I$, where I is a z_c-ideal in $C_c(X)$, then, similar to the result [**13,** 5.4(a)], it is easy to see that, in the factor ring $C_c(X)/I$, we have $\overline{f} \geq 0$, where $f \in C_c(X)$, if and only if f is nonnegative on some element of $Z_c[I]$. We apply this fact in the proof of the following result.

Lemma 5.3. *Let* I *be a* z_c*-ideal of* $C_c(X)$ *containing* O_c^p*, where* $p \in \beta_0 X$. *If the ideal* $(\overline{f}, |\overline{f}|)$ *in* $C_c(X)/I$ *is principal for every* $f \in C_c(X)$*, then* I *is prime.*

Proof. We follow the proof of [**13,** Lemma 14.23]. In view of Lemma 5.1, there exists a $Z \in Z_c[I]$ such that $Z \cap \operatorname{pos} f$ and $Z \cap \operatorname{neg} f$ are contained in two disjoint clopen sets, say U and V, respectively; hence, by Proposition 3.2, $\operatorname{cl}_{\beta_0 X} U \cap \operatorname{cl}_{\beta_0 X} V = \emptyset$. Therefore, there exists a $Z' \in Z_c[O_c^p]$ disjoint from $Z \cap \operatorname{neg} f$, say. Evidently, f is nonnegative on $Z \cap Z' \in Z_c[I]$, and this means that $\overline{f} \geq 0$ in $C_c(X)/I$. Hence, $C_c(X)/I$ is a totally ordered ring, and, since I is a z_c-ideal, it is a prime ideal, in view of [**10,** Theorem 2.13]. The fact in [**13,** 5.4(c)] is also valid in the context of $C_c(X)$. $\square$

The following proposition is needed in the sequel.

Proposition 5.4. *Let* I *be an ideal in* $C_c(X)$. *Then*

$$I = \bigcap_{p \in \beta_0 X} (I + O_c^p).$$

Proof. Obviously,

$$I \subseteq \bigcap_{p\in\beta_0 X} (I + O_c^p).$$

For the reverse inclusion, we take

$$f \in \bigcap_{p\in\beta_0 X} (I + O_c^p).$$

For each $p \in \beta_0 X$, there exists a $g_p \in I$ such that $f - g_p \in O_c^p$. If we set $Z_p = Z(f - g_p)$, then $p \in \operatorname{int}_{\beta_0 X} \operatorname{cl}_{\beta_0 X} Z_p$. The collection

$$\mathcal{C} = \{\operatorname{int}_{\beta_0 X} \operatorname{cl}_{\beta_0 X} Z_p : p \in \beta_0 X\}$$

is an open cover of $\beta_0 X$. Therefore, there exists a finite subcover whose union is $\beta_0 X$, say

$$\beta_0 X = \bigcup_{i=1}^{n} \operatorname{int}_{\beta_0 X} \operatorname{cl}_{\beta_0 X} Z_{p_i}.$$

Since $\beta_0 X$ is zero-dimensional and compact, we easily infer that there exists a finite collection of disjoint clopen sets in $\beta_0 X$, say $\{O_1, \ldots, O_m\}$, which covers $\beta_0 X$ and is a refinement of the latter subcover. For each $1 \leq s \leq m$, choose g_{p_s} such that $O_s \subseteq \operatorname{int}_{\beta_0 X} \operatorname{cl}_{\beta_0 X} Z(f - g_{p_s})$. For each $1 \leq s \leq m$, $V_s = O_s \cap X$ is clopen in X and $\{V_1, \ldots, V_m\}$ covers X. Since, for $i \neq j$, $V_i \cap V_j = \emptyset$, it is easy to see that

$$f = \sum_{s=1}^{m} e_s g_{p_s},$$

where e_s is the idempotent in $C_c(X)$ with $e_s^{-1}(\{1\}) = V_s$ ($e_s e_r = 0$ where $r \neq s$). Thus, $f \in I$, and we are finished. $\square$

The next proposition is necessary for the characterization of F_c-spaces.

Proposition 5.5. *Let X be a zero-dimensional space.*

(1) *If the idempotents in $C(X \setminus Z(f))$, where $f \in C_c(X)$, are extendable to the idempotents in $C(X)$, then, for each $g \in C_c(\beta_0 X)$, the idempotents in $C(\beta_0 X \setminus Z(g))$ are also extendable to the idempotents in $C(\beta_0 X)$.*

(2) *If, for each* $g \in C_c(\beta_0 X)$, $\operatorname{pos} g$ *and* $\operatorname{neg} g$ *are contained in two disjoint clopen sets in* $\beta_0 X$*, then, for each* $f \in C_c(X)$, $\operatorname{pos} f$ *and* $\operatorname{neg} f$ *are also contained in two disjoint clopen sets in* X*.*

Proof.

(1) Let $g \in C_c(\beta_0 X)$, and consider the restriction $f = g|_X \in C_c(X)$. Take

$$h : \beta_0 X \setminus Z(g) \longrightarrow \{0, 1\}$$

to be a nontrivial idempotent. Hence, the idempotent

$$h|_{X \setminus Z(f)} : X \setminus Z(f) \longrightarrow \{0, 1\}$$

has an extension to an idempotent in $C(X)$, by our hypothesis, and hence, to an idempotent in $C(\beta_0 X)$, say

$$H : \beta_0 X \longrightarrow \{0, 1\},$$

by Proposition 3.2 ($X \setminus Z(f) = (\beta_0 X \setminus Z(g)) \cap X$ and $X \setminus Z(f)$ is dense in $\beta_0 X \setminus Z(g)$). Evidently, $H|_{\beta_0 X \setminus Z(g)} = h$; hence, we are done.

(2) Let $f \in C_c(X)$. From Lemma 2.2, there exist two sequences of clopen sets $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$ such that

$$\operatorname{pos} f = \bigcup_{n=1}^{\infty} U_n, \qquad \operatorname{neg} f = \bigcup_{n=1}^{\infty} V_n,$$

and clearly, $U_n \cap V_m = \emptyset$, for each $m, n \in \mathbb{N}$. Hence, $\operatorname{cl}_{\beta_0 X} U_n \cap \operatorname{cl}_{\beta_0 X} V_m = \emptyset$, for all $m, n \in \mathbb{N}$. In addition, the collections

$$\{\operatorname{cl}_{\beta_0 X} U_n : n \in \mathbb{N}\} \quad \text{and} \quad \{\operatorname{cl}_{\beta_0 X} V_n : n \in \mathbb{N}\}$$

are sequences of clopen sets in $\beta_0 X$, by Proposition 3.2. Now, in view of Lemma 2.2 (b), there exists a $g \in C_c(\beta_0 X)$ such that

$$\operatorname{pos} g = \bigcup_{n=1}^{\infty} \operatorname{cl}_{\beta_0 X} U_n$$

and

$$\operatorname{neg} g = \bigcup_{n=1}^{\infty} \operatorname{cl}_{\beta_0 X} V_n.$$

Hence, by our hypothesis, there exist clopen sets U and V in $\beta_0 X$ such that $\operatorname{pos} g \subseteq U$ and $\operatorname{neg} g \subseteq V$. Take $U_1 = X \cap U$ and $V_1 = X \cap V$. Clearly, $\operatorname{pos} f \subseteq U_1$ and $\operatorname{neg} f \subseteq V_1$. Hence, we are done. $\square$

Proposition 5.5 (1) and Proposition 2.1 immediately yield the following corollary.

Corollary 5.6. *If the idempotents in $C(X \setminus Z(f))$, where $f \in C_c(X)$, are extendable to the idempotents in $C(X)$, then $X \setminus Z(f)$ (respectively, $\beta_0 X \setminus Z(h)$, where $h \in C_c(\beta_0 X)$) is C^F-embedded in X (respectively, C^F-embedded in $\beta_0 X$). Moreover, if $g \in C^F(X \setminus Z(f))$ (or $g \in C^F(\beta_0 X \setminus Z(h))$), then there is an extension $\overline{g} \in C^F(X)$ (or $\overline{g} \in C^F(\beta_0 X \setminus Z(h))$) of g such that g and $\overline{g}$ have the same image.*

Now, we are ready to give some algebraic and topological characterizations of F_c-spaces. We remind the reader that, if P is a prime ideal minimal over a z_c-ideal I, then P is a z_c-ideal, too, see Remark 4.7, [**10,** Corollary 3.4] and [**13,** Theorem 14.7]. We also recall that an ideal I in a commutative ring R is *pseudoprime* if, for each $a, b \in R$ with $ab = 0$, then either $a \in I$ or $b \in I$. Next, we note that, whenever $f \in C_c(X)$ with $|f| \leq 1$, then

$$\sum_{n=1}^{\infty} 2^{-n} |f|^{1/n}$$

belongs to $C_c(X)$. Using the latter fact, it is shown in [**10,** Theorem 3.10] that any ideal and its radical in $C_c(X)$ have the same largest z_c-ideal. Applying this fact and the proof of [**12,** Theorem 4.1], it is easily seen that an ideal I in $C_c(X)$ is pseudoprime if and only if it contains a prime ideal of $C_c(X)$.

For each prime ideal P in $C_c(X)$, put

$$O_P = \{f \in C_c(X) : fg = 0 \text{ for some } g \notin P\}.$$

Thus, in view of Lemma 4.11 (2), we have $O_c^p = O_{M_c^p}$ for all $p \in \beta_0 X$, also see [**1,** comment preceding Theorem 2.12]. Consequently, we immediately have the following result which is the counterpart of [**1,** Lemma 2.1].

Lemma 5.7. *Let X be zero-dimensional. Then, for every $p \in \beta_0 X$, $C_c(X)/O_c^p \cong C_c(X)_{M_c^p}$. In particular, O_c^p is prime if and only if $C_c(X)_{M_c^p}$ is a domain.*

In view of [**1,** Corollary 2.4, Proposition 2.5], [**10,** Theorem 2.13] and Lemma 5.7, the following result is now immediate.

Corollary 5.8. *Let X be a zero-dimensional space. Then, the following statements are equivalent.*

(1) *X is an F_c-space.*
(2) *O_c^p is a prime ideal in $C_c(X)$ for all $p \in \beta_0 X$.*
(3) *Given $p \in \beta_0 X$ and $f \in C_c(X)$, there is a zero-set of O_c^p on which f does not change sign.*
(4) *$C_c(X)_M$ is a domain for all maximal ideals M of $C_c(X)$.*
(5) *Every prime ideal of $C_c(X)$ contains a unique minimal prime ideal.*
(6) *$C_c(X)$ is locally uniform, i.e., for any prime ideal P in $C_c(X)$, any two nonzero ideals in $C_c(X)_P$ intersect non-trivially.*

The following characterization of F_c-spaces is the counterpart of [**13,** Theorem 14.25] and, although its proof is nearly identical, we present one here for the sake of completeness.

Theorem 5.9. *For every zero-dimensional space X, the following statements are equivalent.*

(1) *X is an F_c-space.*
(2) *The prime ideals of $C_c(X)$ contained in any given maximal ideal of $C_c(X)$ form a chain.*
(3) *Each ideal of $C_c(X)$ is an intersection of pseudoprime ideals.*
(4) *Each principal ideal of $C_c(X)$ is an intersection of pseudoprime ideals.*
(5) *For all $f \in C_c(X)$, the ideal $(f, |f|)$ is principal.*
(6) *For each $f \in C_c(X)$, there exists a unit $u \in C_c(X)$ such that $f = u|f|$.*
(7) *For each $f \in C_c(X)$, $\operatorname{pos} f$ and $\operatorname{neg} f$ are contained in two disjoint clopen sets in X.*
(8) *For each $f \in C_c(X)$, $X \setminus Z(f)$ is C^F-embedded in X.*
(9) *$\beta_0 X$ is an F_c-space.*

Proof.

(1) $\Rightarrow$ (2). From Corollary 5.8 (2), O_c^p is a prime ideal. Consequently, the prime ideals in $C_c(X)$ containing O_c^p are comparable, see [**10,** Corollary 3.8]. This implies that the prime ideals of $C_c(X)_{M_c^p}$, where $p \in \beta_0 X$, form a chain, by Lemma 5.7. The latter fact, in turn, implies that the prime ideals in the maximal ideal M_c^p form a chain, too.

(2) $\Rightarrow$ (3). First, we recall that the localization of any commutative reduced ring is reduced. Thus, $C_c(X)_{M_c^p}$ is a reduced ring, which means the intersection of its prime ideals is zero. By our assumption, the prime ideals of $C_c(X)_{M_c^p}$ form a chain; hence, their intersection is a prime ideal. This implies that the zero ideal in $C_c(X)_{M_c^p}$, which is the intersection of the prime ideals of $C_c(X)_{M_c^p}$, must be a prime ideal, i.e., $C_c(X)_{M_c^p}$ is a domain. Thus, from Lemma 5.7, we infer that O_c^p is a prime ideal for all $p \in \beta_0 X$. Now, let I be any ideal of $C_c(X)$, then by the last part of the comment following Corollary 5.6, we note that $I + O_c^p$ is a pseudoprime ideal for each $p \in \beta_0 X$. Thus, by Proposition 5.4, we are done.

(3) $\Rightarrow$ (4). Evident.

(4) $\Rightarrow$ (5). For each $f \in C_c(X)$, $(|f|)$ is an intersection of pseudoprime ideals, by our assumption. Since $(f - |f|)(f + |f|) = 0$, either $f - |f|$ or $f + |f|$ belongs to each pseudoprime ideal containing $|f|$. In any case, f belongs to each pseudoprime ideal containing $(|f|)$, which implies that $f \in (|f|)$. Hence, $(f, |f|) = (|f|)$.

(5) $\Rightarrow$ (6) $\Rightarrow$ (7). Evident, by Corollary 5.2.

(7) $\Rightarrow$ (8). In view of Corollary 5.6, it suffices to show that the idempotents in $C(X \setminus Z(f))$ can be extended to idempotents in $C(X)$. For the proof of this part, we follow the proof of [**13,** (4) $\Rightarrow$ (6), Theorem 14.25]. Let $e \in C(X \setminus Z(f))$ be a nontrivial idempotent. Set $A = e^{-1}(\{1\})$, $B = e^{-1}(\{0\})$ and define $g \in C^F(X \setminus Z(f))$ with $g(A) = 1$ and $g(B) = -1$. Define the real function h as follows:

$$h(x) = \begin{cases} 0 & x \in Z(f), \\ g(x)|f(x)| & x \in X \setminus Z(f). \end{cases}$$

Clearly, $h(X)$ is countable and, since g is bounded on $X \setminus Z(f)$, we infer that h is continuous, also see [**13,** Proof of Theorem 5.5]; hence, $h \in C_c(X)$. It is clear that $A = \operatorname{pos} h$ and $B = \operatorname{neg} h$. By our assumption, A and B are contained in two disjoint clopen sets U, V in X, respectively. Now, we may define the idempotent $e^* \in C(X)$ with $U = e^{*-1}(\{1\})$ and $X \setminus U = e^{*-1}(\{0\})$ which evidently extends e.

(8) $\Rightarrow$ (9). First, in view of Corollary 5.6, note that (8) is still valid if we replace X by $\beta_0 X$. Clearly, for every $f \in C_c(X)$, $X \setminus Z(f) = \operatorname{pos} f \cup \operatorname{neg} f$. Hence, we may define the idempotent $e \in C(X \setminus Z(f))$

with $e^{-1}(\{1\}) = \operatorname{pos} f$ and $e^{-1}(\{0\}) = \operatorname{neg} f$. Thus, by our assumption, the idempotent e can be extended to an idempotent $\overline{e} \in C(X)$. This shows that $\operatorname{pos} f$ and $\operatorname{neg} f$ are contained in two disjoint clopen sets in X. Consequently, in view of Corollary 5.2, Lemma 5.3, we infer that O_c^p is prime, which, in turn, implies that X is an F_c-space, by Corollary 5.8. Incidentally, from our observations at the beginning of this proof, we have already shown that $\beta_0 X$ is also an F_c-space.

(9) $\Rightarrow$ (1). Since $\beta_0 X$ is an F_c-space, part (7) of this theorem is also valid if we replace X by $\beta_0 X$. Now, in view of Proposition 5.5, part (7) still remains valid for X; hence, part (8) is valid for X. Consequently, from what we have shown in the proof of (8) $\Rightarrow$ (9), X is an F_c-space. The proof is finished. □

6. F_c-spaces versus F-spaces. In the next result, we observe that in the class of strongly zero-dimensional spaces, F_c- and F-spaces coincide.

Proposition 6.1. *Let X be a strongly zero-dimensional space. Then, X is an F_c-space if and only if it is an F-space.*

Proof. First suppose that X is an F_c-space and $f \in C(X)$. In order to see that X is an F-space, it suffices to show that $\operatorname{pos} f$ and $\operatorname{neg} f$ are completely separated, by [**13,** Theorem 14.25]. In light of Proposition 2.4, there exist $u, v \in C_c(X)$ such that $Z(|f| + f) = Z(u)$ and $Z(|f| - f) = Z(v)$. Define $h = u^2 - v^2 \in C_c(X)$. Since $Z(u) \cup Z(v) = X$, we have

$$\operatorname{pos} f = X \setminus Z(u) \subseteq \operatorname{pos} h$$

and

$$\operatorname{neg} f = X \setminus Z(v) \subseteq \operatorname{neg} h.$$

However, by Theorem 5.9, $\operatorname{pos} h$ and $\operatorname{neg} h$ are contained in two disjoint clopen sets in X. Consequently, $\operatorname{pos} f$ and $\operatorname{neg} f$ are also contained in these two disjoint clopen sets; hence, they are completely separated. Conversely, let X be an F-space. Since X is strongly zero-dimensional, $\beta X = \beta_0 X$. Thus, $O_c^p = O^p \cap C_c(X)$ for each $p \in \beta X = \beta_0 X$. However, O^p is prime in $C(X)$ for each $p \in \beta X$; hence, O_c^p is also prime in $C_c(X)$ for each $p \in \beta_0 X$, i.e., X is an F_c-space. □

It is well known that a zero-dimensional Lindelöf space is normal and strongly zero-dimensional; Remark 2.5 provides a simple proof. From Proposition 6.1 and Theorem 5.9, the next fact is now immediate.

Corollary 6.2. *Let X be a zero-dimensional Lindelöf F_c-space. Then, X is an F-space and $\beta_0 X = \beta X$ is an F-space as well as an F_c-space.*

Before presenting the next proposition, we observe that Theorem 5.9 and Proposition 6.1 immediately imply that a zero-dimensional space X is an F_c-space if and only if $\beta_0 X$ is both an F-space and an F_c-space. Note that a zero-dimensional compact space (and a Lindelöf space) is strongly zero-dimensional.

Proposition 6.3. *Let Y be a Lindelöf subspace of a zero-dimensional F_c-space X. Then, Y is C^F-embedded in X. In particular, Y is an F_c-space as well as an F-space.*

Proof. With the aid of Proposition 2.1, in order to show that Y is C^F-embedded in X, it suffices to show that every idempotent $e \in C(Y)$ can be extended to an idempotent of $C(X)$. Set $A = e^{-1}(\{1\})$ and $B = e^{-1}(\{0\})$. Clearly, A and B are disjoint clopen subsets of Y. For every $p \in A$, let $U(p)$ be a clopen neighborhood of p in X such that $U(p) \cap B = \emptyset$. Similarly, for every $q \in B$, let $V(q)$ be a clopen neighborhood of q in X with $V(q) \cap A = \emptyset$. Note that A and B are Lindelöf subspaces of Y, and also of X. Hence, there are countable subcovers, say $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$, of the covers $\{U(p) : p \in A\}$ of A and $\{V(q) : q \in B\}$ of B, respectively. Inductively, we may define $\widetilde{U}_1 = U_1$, $\widetilde{V}_1 = V_1$, $\widetilde{U}_n = U_n \setminus (V_1 \cup \cdots \cup V_n)$, $\widetilde{V}_n = V_n \setminus (U_1 \cup \cdots \cup U_n)$ for $n \geq 2$. For each $n \geq 1$, $\widetilde{U}_n$ and $\widetilde{V}_n$ are clopen subsets of X. Finally, define

$$\widetilde{U} = \bigcup_{n=1}^{\infty} \widetilde{U}_n \quad \text{and} \quad \widetilde{V} = \bigcup_{n=1}^{\infty} \widetilde{V}_n.$$

It is easy to see that $\widetilde{U} \cap \widetilde{V} = \emptyset$, $A \subseteq \widetilde{U}$ and $B \subseteq \widetilde{V}$. Now, by Lemma 2.2, there exists an $f \in C_c(X)$ such that $\widetilde{U} = \operatorname{pos} f$ and $\widetilde{V} = \operatorname{neg} f$. Since X is an F_c-space, there exist two disjoint clopen subsets W_1 and W_2 of X with $A \subseteq \operatorname{pos} f \subseteq W_1$ and $B \subseteq \operatorname{neg} f \subseteq W_2$, by Theorem 5.9. Now, define the idempotent $\overline{e} \in C(X)$ with $W_1 = \overline{e}^{\,-1}(\{1\})$ and $W_2 = \overline{e}^{\,-1}(\{0\})$, which extends the idempotent e.

For the proof of the final part, we first note that Y is C^F-embedded in $\beta_0 X$ since we have already shown that Y is C^F-embedded in X and, in turn, X is C^F-embedded in $\beta_0 X$, by Proposition 3.2. This implies that $\mathrm{cl}_{\beta_0 X} Y = \beta_0 Y (\mathrm{cl}_{\beta_0 X} Y$ as a subspace of $\beta_0 X$ is a compact zero-dimensional space and Y is dense in $\mathrm{cl}_{\beta_0 X} Y$, too). Now, we note that $\beta_0 Y$ as a closed subspace of $\beta_0 X$ is C^*-embedded in it. Hence, by [**13,** 14.26], we infer that $\beta_0 Y$ as a C^*-embedded subspace of the F-space $\beta_0 X$ is an F-space (from the comment preceding Proposition 6.3, $\beta_0 X$ is both an F- and an F_c-space). However, in view of the last part of Remark 2.5, Y is strongly zero-dimensional; hence, $\beta Y = \beta_0 Y$ is also an F-space. This means that Y is an F-space, by [**13,** Theorem 14.25], which, in turn, implies that Y is also an F_c-space, by Proposition 6.1. Hence, we are finished. □

Motivated by the above proof, we now record the following three facts.

Proposition 6.4. *If Y is a C^F-embedded subspace of a zero-dimensional space X, then $\mathrm{cl}_{\beta_0 X} Y = \beta_0 Y$. In particular, if X is an F_c-space, then Y is also an F_c-space.*

Proof. We have already shown in the previous proof that $\mathrm{cl}_{\beta_0 X} Y = \beta_0 Y$. Finally, if X is an F_c-space then so too is $\beta_0 X$, by Theorem 5.9 and it is also an F-space by Proposition 6.1 (note that $\beta_0 X$ as a compact zero-dimensional space is strongly zero-dimensional). Since $\mathrm{cl}_{\beta_0 X} Y = \beta_0 Y$ is closed in $\beta_0 X$, we infer that $\beta_0 Y$ is C^*-embedded in the compact space $\beta_0 X$. Hence, by [**13,** 14.26], $\beta_0 Y$ is an F-space and, in view of Proposition 6.1, it is also an F_c space. This implies that Y is also an F_c-space as well, by Theorem 5.9. □

Proposition 6.4 and Theorem 5.9 (8) immediately yield the next fact.

Corollary 6.5. *Let X be a zero-dimensional F_c-space. Then, for any $f \in C_c(X)$, $X \setminus Z(f)$ is also an F_c-space. In particular,* $\mathrm{pos}\, f$ *and* $\mathrm{neg}\, f$ *are F_c-spaces.*

It is well known that, if X is a locally compact F-space, then $\beta X \setminus X$ is also an F-space, see [**13,** 14O(3)]. The next result is its counterpart.

Corollary 6.6. *Let X be a zero-dimensional F_c-space. Then, every closed subset of $\beta_0 X$ (if X is locally compact, $\beta_0 X \setminus X$ is a closed subset of $\beta_0 X$) is both an F- and an F_c-space.*

Proof. First, note that $\beta_0 X$ is an F_c-space, by Theorem 5.9; hence, it is an F-space, by Proposition 6.1. Since every closed subspace of $\beta_0 X$ is C^*-embedded in $\beta_0 X$, we infer that it is an F-space, by [**13,** 14.26], which is also an F_c-space by Proposition 6.1. $\square$

Remark 6.7. Recall that a space X is *basically* (*extremally*) *disconnected* if every cozero-set (open set) has an open closure. It is well known that every basically disconnected space is zero-dimensional, see [**13,** 16O]. Hence, whenever X is basically disconnected, then, for every $f \in C_c(X)$, $\mathrm{cl}_X \operatorname{pos} f$ is clopen. If we define a function u such that $u(\mathrm{cl}_X \operatorname{pos} f) = 1$ and $u(X \setminus \mathrm{cl}_X \operatorname{pos} f) = -1$, then u is a unit of $C_c(X)$ and, clearly, $f = u|f|$ ($\mathrm{cl}_X \operatorname{pos} f \cap \operatorname{neg} f = \emptyset$). Now, using Theorem 5.9 (6), we conclude that every basically disconnected space is an F_c-space. The converse is not true, in general. For example, $\beta\mathbb{N} \setminus \mathbb{N}$ is an F_c-space ($\beta\mathbb{N} \setminus \mathbb{N}$ is, in fact, an F-space, see [**13,** 14O(3)], and, since $\beta\mathbb{N} \setminus \mathbb{N}$ is strongly zero-dimensional, it is also an F_c-space, by Proposition 6.1). However, $\beta\mathbb{N} \setminus \mathbb{N}$ is not basically disconnected, see [**13,** 6W(3)].

In the next result, we show that every F_c-space satisfying the countable chain condition is extremely disconnected, and hence, it is an F-space, see [**19,** 6L(8)]. Recall that a topological space X satisfies the *countable chain condition* if every family of pairwise disjoint open subsets of X is countable.

Proposition 6.8. *Let X be a zero-dimensional F_c-space which satisfies the countable chain condition. Then, it is extremely disconnected.*

Proof. Let U and V be two disjoint open subsets of X. It suffices to show that $\mathrm{cl}_X U \cap \mathrm{cl}_X V = \emptyset$. Since X satisfies the countable chain condition, there exist countable families $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$ of pairwise disjoint clopen sets such that $\bigcup_{i\in\mathbb{N}} U_i$ and $\bigcup_{i\in\mathbb{N}} V_i$ are dense in U and V, respectively (we may consider a maximal collection of mutually disjoint clopen sets in U, V, respectively). Now, by Lemma 2.2, there exist $f, g \in C_c(X)$ such that

$$\bigcup_{i\in\mathbb{N}} U_i = X \setminus Z(f)$$

and

$$\bigcup_{i\in\mathbb{N}} V_i = X \setminus Z(g).$$

Clearly, $Z(f) \cup Z(g) = X$. We define, $h = f^2 - g^2$; hence, $X \setminus Z(f) \subseteq \operatorname{pos} h$ and $X \setminus Z(g) \subseteq \operatorname{neg} h$. Since X is an F_c-space, there exist disjoint clopen sets A, B in X with $\operatorname{pos} h \subseteq A$ and $\operatorname{neg} h \subseteq B$, by Theorem 5.9. It is clear that we must have $U \subseteq A$ (otherwise, $U \cap (X \setminus A) \neq \emptyset$, which contradicts the density of $X \setminus Z(f)$ in U) and $V \subseteq B$. Consequently, $\operatorname{cl}_X U \subseteq A$ and $\operatorname{cl}_X V \subseteq B$; hence, $\operatorname{cl}_X U \cap \operatorname{cl}_X V = \emptyset$, and we are done. □

We digress for a moment and refer the reader to [**21,** Theorem 3.31] for the definition of the DuBois-Reymond separability and its connection to the F-space $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. In particular, in [**21,** Proposition 2.23], it is shown that the Boolean algebra of clopen subsets of a zero-dimensional compact F-space X is a DuBois-Reymond separability. We noted earlier that, for every $f \in C_c(X)$, there exist increasing sequences $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$ of clopen sets such that $\operatorname{pos} f = \bigcup_{n=1}^{\infty} U_n$, $\operatorname{neg} f = \bigcup_{n=1}^{\infty} V_n$, see Lemma 2.2. Using this fact, Theorem 5.9, and [**21,** Proof of Proposition 2.23], we may record the next fact as well.

Proposition 6.9. *Let X be a zero-dimensional space. Then, the Boolean algebra of clopen subsets of X is DuBois-Reymond separable if and only if X is an F_c-space.*

In view of Propositions 6.1 and 6.9, it may be observed that the converse of [**21,** Proposition 2.23], in the following sense, is also true.

Proposition 6.10. *If the Boolean algebra of clopen subsets of a strongly zero-dimensional space X is DuBois-Reymond separable, then X is an F-space.*

In view of Proposition 6.3 and the fact that no point of an F-space is a limit of a sequence of distinct points, see [**13,** 14N], we have the following two facts.

Corollary 6.11. *No point of a zero-dimensional F_c-space is the limit of a sequence with distinct points.*

Corollary 6.12. *In a zero-dimensional F_c-space X, any point with a countable base of neighborhoods is isolated. In particular, any first countable subspace of X, e.g., any metrizable subspace, is discrete.*

We remind the reader that a topological space X (not necessarily zero-dimensional) is called a CP-space in [**10**] if $C_c(X)$ is a regular ring (von Neumann). In [**10,** Theorem 5.5], it is shown that X is a CP-space if and only if $Z(f)$ is open for each $f \in C_c(X)$. In [**10,** Proposition 5.3, Corollary 5.7], it is shown that any P-space is a CP-space, and, when X is zero-dimensional, the converse is also true.

The next fact is the counterpart of [**13,** 14Q], and its proof is almost identical.

Proposition 6.13. *Let X and Y be two zero-dimensional spaces. If $X \times Y$ is an F_c-space, then either X or Y is a P-space.*

Proof. Suppose, on the contrary, that neither X nor Y is a P-space, and seek a contradiction. Since X and Y are both zero-dimensional, neither of them is a CP-space by the above comment. This implies that there are $f \in C_c(X)$, $g \in C_c(Y)$ such that neither $Z(f)$ nor $Z(g)$ is an open set in X and in Y, respectively. We should emphasize here that, without loss of generality, both functions f and g can be taken to be non-negatives. Since $Z(f)$ is not open and $X = Z(f) \cup \mathrm{cl}_X(X \setminus Z(f))$, we infer that there exists a $p \in Z(f) \cap \mathrm{cl}_X(X \setminus Z(f))$. Similarly, there exists a $q \in Z(g) \cap \mathrm{cl}_X(X \setminus Z(g))$. We may now define the continuous function

$$h : X \times Y \longrightarrow \mathbb{R}$$

by $h(x,y) = f(x) - g(y)$, for each $(x,y) \in X \times Y$. Note that $(p,q) \in \mathrm{cl}_{X\times Y} \operatorname{pos} h$ and also $(p,q) \in \mathrm{cl}_{X\times Y} \operatorname{neg} h$. Consequently, $\operatorname{pos} h$ and $\operatorname{neg} h$ cannot be contained in two disjoint clopent sets. Therefore, $X \times Y$ is not an F_c-space, by Theorem 5.9 (7), the desired contradiction. □

7. $C_c(X)$ as a Bézout ring. Finally, we conclude this article with some miscellaneous facts for $C_c(X)$ concerning F_c-spaces. First, we recall the following fact for commutative Bézout rings R, i.e., every finitely generated ideal in R is principal, which is merely a variant of [**13,** 14L].

Lemma 7.1. *Let R be a commutative Bézout ring. Then, the prime ideals inside a proper ideal of R are comparable. In particular, for any prime ideal P in R, the prime ideals of R_P form a chain.*

Corollary 7.2. *Let $C_c(X)$ be a Bézout ring. Then, the following statements hold.*

(1) *$C_c(X)_M$ is a valuation domain, where M is a maximal ideal of $C_c(X)$.*

(2) *The primary ideals of $C_c(X)$ inside any maximal ideal of $C_c(X)$ form a chain.*

(3) *The primary ideals of $C_c(X)$ inside any proper ideal of $C_c(X)$ form a chain.*

(4) *X is an F_c-space.*

Proof.

(1) From Lemma 7.1, it is evident that the prime ideals inside any maximal ideal of $C_c(X)$ form a chain. Consequently, X is an F_c-space, by Theorem 5.9 (2). We should also emphasize that, for any maximal ideal M of $C_c(X)$, $C_c(X)_M$ is a domain, by Corollary 5.8 (5). Clearly, $C_c(X)_M$ is a Bézout domain with a unique maximal ideal; hence, it is a valuation domain, see [**1,** comment preceding Corollary 2.8].

(2) First, we recall that, for a prime ideal P of a commutative ring R, there is a one-one correspondence between the primary ideals inside P and the primary ideals of R_P which preserves the inclusion relation. Hence, we are through, by part (1).

(3) Evident by part (2).

(4) We have already shown that X is an F_c-space. □

The next lemma, which is the counterpart of [**13,** 1E(1)] for $C_c(X)$, is necessary for what follows.

Lemma 7.3. *Let $f \in C_c(X)$. Then, there exist an $f^* \in C_c^*(X)$ and a positive unit $u \in C_c(X)$ with $f = uf^*$ and $u^{-1} \in C_c^*(X)$. In particular, for every principal ideal (f) in $C_c(X)$, we may take f to be an element in $C_c^*(X)$ which, in turn, implies that every ideal in $C_c(X)$ is generated by some of its elements in $C_c^*(X)$.*

Proof. Although the same proof as that of [**13,** 1E(1)] works well (only put $f^* = (f \vee -1) \wedge 1$, $u = |f| \vee 1$), we may also simply take $f^* = f/(1+f^2)$ and $u = 1 + f^2$. The last part is now evident. □

It is interesting to note that any of the following equivalent statements implies that $C_c(X)$ (respectively, $C_c^*(X)$) is a Bézout ring, also see the next proposition.

Theorem 7.4. *For a space X, the following statements are equivalent.*

(1) *For every $f \in C_c(X)$, $X \setminus Z(f)$ is C_c^*-embedded in X.*
(2) *Every ideal of $C_c(X)$ is absolutely convex.*
(3) *Every ideal of $C_c(X)$ is convex.*
(4) *Every principal ideal of $C_c(X)$ is convex.*
(5) *For all $f, g \in C_c(X)$, $(f, g) = (|f| + |g|)$.*

Proof.

(1) $\Rightarrow$ (2). Let I be an ideal in $C_c(X)$, $f \in C_c(X)$ and $g \in I$ such that $|f| \leq |g|$. If we define

$$h : X \setminus Z(g) \longrightarrow \mathbb{R},$$

by $h = f/g$, then $h \in C_c^*(X \setminus Z(g))$. Now, by (1), there exists an $\overline{h} \in C_c^*(X)$ such that $\overline{h}|_{X \setminus Z(g)} = h$. Clearly, $f = \overline{h}g$, which means that $f \in I$, i.e., I is absolutely convex.

(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4). Evident.

(4) $\Rightarrow$ (5). The proof of this part is similar to the proof of (7) $\Rightarrow$ (8) in [**13,** Theorem 14.25].

(5) $\Rightarrow$ (1). Let $f \in C_c(X)$ and $h \in C_c^*(X \setminus Z(f))$. First, we assume that $h \geq 0$. Without loss of generality, we may also assume that $f \geq 0$ $(Z(f) = Z(|f|))$. Now, define

$$g(x) = \begin{cases} f(x)h(x) & x \in X \setminus Z(f), \\ 0 & x \in Z(f). \end{cases}$$

Since h is non-negative and bounded, we infer that g is continuous; hence, $0 \leq g \in C_c(X)$. Now, by (5), $(f, g) = (|f| + |g|) = (f + g)$. However, $f \in (f + g)$ implies that $f = s(f + g)$ for some $s \in C_c(X)$. We also note that $f + g = f + fh = f(1 + h)$ on $X \setminus Z(f)$. Hence, $f = s(f+g) = sf(1+h)$, and consequently, we have $f(s(1+h)-1) = 0$ on $X \setminus Z(f)$. Therefore, $s(1 + h) = 1$ on $X \setminus Z(f)$, i.e., $s = 1/(1 + h)$ on $X \setminus Z(f)$. Since h is non-negative and bounded on $X \setminus Z(f)$, there exists a positive real number M such that $1 \leq 1 + h \leq M$. Hence, $0 < 1/M \leq s \leq 1$ on $X \setminus Z(f)$. Take

$$t = \left(\frac{1}{M} \vee s\right) \wedge 1.$$

Clearly, $t \in C_c(X)$. Now, it is enough to define $\overline{h} = 1/t - 1$, and easily observe that $\overline{h} \in C_c^*(X)$ and $\overline{h}|_{X\setminus Z(f)} = h$. Hence, we are done in this case. Now, we assume that $h \in C_c^*(X \setminus Z(f))$ is an arbitrary element. Set $h_1 = h \vee 0$ and $h_2 = -(h \wedge 0)$. Then, $h_1, h_2 \geq 0$, $h = h_1 - h_2$ and $h_1, h_2 \in C_c^*(X \setminus Z(f))$. Consequently, from what has already been proved, there exist $\overline{h}_1$, $\overline{h}_2$ such that $\overline{h}_1|_{X\setminus Z(f)} = h_1$, $\overline{h}_2|_{X\setminus Z(f)} = h_2$. Note that $\overline{h} = \overline{h}_1 - \overline{h}_2 \in C_c^*(X)$ and $\overline{h}|_{X\setminus Z(f)} = h$, which implies that any element $h \in C_c^*(X \setminus Z(f))$ can be extended to an element $\overline{h} \in C_c^*(X)$. This completes the proof. $\square$

We should remind the reader that, whenever R_1 is a subring of a commutative ring R_2 such that for every principal ideal (a) in R_2 we may take a to be an element in R_1 (e.g., $R_1 = C^*(X), R_2 = C(X)$ or $R_1 = C_c^*(X), R_2 = C_c(X)$), then it is clear that, if R_1 is a Bézout ring, so too is R_2. (If $a_1, a_2, \ldots, a_n$ and b are elements in R_1 with $(a_1, a_2, \ldots, a_n) = (b)$ as ideals in R_1, then the latter equality holds as ideals in R_2, too.) In particular, if $C_c^*(X)$ is a Bézout ring, then so too is $C_c(X)$ (it is well-known that $C(X)$ is Bézout if and only $C^*(X)$ is Bézout). Before concluding our article with the next result, let us call $C_c(X)$ (respectively, $C_c^*(X)$) an *absolutely Bézout* ring if, for all $f, g \in C_c(X)$ (respectively, $f, g \in C_c^*(X)$), $(f, g) = (u|f| + v|g|)$ in $C_c(X)$, where u and v are positive elements in $C_c(X)$ (respectively, in $C_c^*(X)$, where u and v are positive elements in $C_c^*(X)$). In particular, if, in the latter definition, $u = v = 1$, then $C_c(X)$ (respectively, $C_c^*(X)$) is called *unitarily absolute Bézout.*

Proposition 7.5. *Let $C_c^*(X)$ be an absolutely Bézout ring. Then, $C_c(X)$ is one as well. Conversely, if $C_c(X)$ is unitarily absolute Bézout, then so, too, is $C_c^*(X)$. In particular, if every principal ideal of $C_c(X)$ is convex, then $C_c(X)$ and $C_c^*(X)$ are both unitarily absolute Bézout.*

Proof. First, assume that $C_c^*(X)$ is an absolutely Bézout ring, and let $f, g \in C_c(X)$. Note that (f, g) and

$$\left(\frac{f}{1+f^2}, \frac{g}{1+g^2}\right)$$

as two ideals in $C_c(X)$ coincide. Now, we consider

$$\left(\frac{f}{1+f^2}, \frac{g}{1+g^2}\right)$$

as the ideal in $C_c^*(X)$ generated by

$$\frac{f}{1+f^2}, \ \frac{g}{1+g^2} \in C_c^*(X).$$

By our assumption,

$$\left(\frac{f}{1+f^2}, \ \frac{g}{1+g^2}\right) = \left(u\left|\frac{f}{1+f^2}\right| + v\left|\frac{g}{1+g^2}\right|\right),$$

where u and v are positive elements of $C_c^*(X)$. However, from the comment preceding Proposition 7.5, the latter equality of ideals also holds in $C_c(X)$ if we consider both sides as ideals in $C_c(X)$. Now, if we put

$$u_f = \frac{u}{1+f^2} \quad \text{and} \quad v_g = \frac{v}{1+g^2},$$

then we have $(f,g) = (u_f|f| + v_g|g|)$ as an equality of two ideals in $C_c(X)$, where u_f, v_g are positive elements of $C_c(X)$ (although unnecessary, u_f and v_g are still elements of $C_c^*(X)$). Consequently, $C_c(X)$ is an absolutely Bézout ring, and we are done.

Conversely, let $(f,g) = (|f| + |g|)$, where $f, g \in C_c(X)$. We must show that the latter equality is true when the two ideals are considered as ideals in $C_c^*(X)$, where $f, g \in C_c^*(X)$. By our assumption, $C_c(X)$ is a Bézout ring; hence, the ideals $(f, |f|)$ and $(g, |g|)$ are principal in $C_c(X)$. Thus, in view of Corollary 5.2, there are units $u, v \in C_c(X)$ such that $f = u|f|$ and $g = v|g|$ (we may assume that u, v are invertible elements in $C_c^*(X)$, see the comment following **[13,** Corollary 14.22]). This shows that, for all $f, g \in C_c^*(X)$ the two ideals (f,g) and $(|f|, |g|)$ of $C_c^*(X)$ coincide. Hence, in order to show that $C_c^*(X)$ has the required property, it suffices to prove that the ideals $(|f|, |g|)$ and $(|f| + |g|)$ of $C_c^*(X)$ coincide for all $f, g \in C_c^*(X)$. Clearly, $(|f| + |g|) \subseteq (|f|, |g|)$. Thus, it remains to be shown that $|f|, |g| \in (|f| + |g|)$ in $C_c^*(X)$.

In what follows, we aim only to show that $|f| \in (|f| + |g|)$ since the proof of $|g| \in (|f| + |g|)$ is similar. However, by our assumption, $(|f| + |g|) = (|f|, |g|)$ as two ideals in $C_c(X)$. Thus, there exists an $h \in C_c(X)$ with $|f| = h(|f| + |g|)$, and it is evident that we may assume that $h \geq 0$. In view of the proof of Lemma 7.3, there exists a $k \in C_c^*(X)$ with

$$kh = (h \vee -1 \wedge 1) \in C_c^*(X).$$

Clearly, $h \vee -1 = h$; hence, $kh = h \wedge 1 \in C_c^*(X)$.

Now, we claim that $|f| = kh(|f| + |g|)$, which completes the proof. Toward this end, take any $x \in X$ and consider two cases. First, let $h(x) < 1$. Then, $kh(x) = (h \wedge 1)(x) = h(x)$. Hence,

$$(kh(|f| + |g|))(x) = h(x)(|f| + |g|)(x) = |f|(x),$$

by our assumption. For the second case, let $h(x) \geq 1$; hence, $kh(x) = (h \wedge 1)(x) = 1$. Consequently,

$$(kh(|f| + |g|))(x) = (|f| + |g|)(x).$$

Since $|f|(x) = h(x)(|f| + |g|)(x)$ and $h(x) \geq 1$, we infer that $|f|(x) \geq (|f| + |g|)(x)$, which in turn, implies that $|g(x)| = 0$. Therefore,

$$kh(|f| + |g|)(x) = |f|(x),$$

which shows that, in any case, $|f| \in (|f| + |g|)$, where the latter ideal is considered to be an ideal of $C_c^*(X)$. Hence, we are finished. The last part is now evident by Theorem 7.4. □

In conclusion, we admit that we know of no example of an F-space which is not an F_c-space. In contrast to the well-known fact that, X is an F-space if and only if $C(X)$ is Bézout, it was shown, in Corollary 7.2, that, if $C_c(X)$ is Bézout, then X is an F_c-space. However, we are undecided about the converse of this result. We acknowledge here that these unsettled questions are also of interest to the referee.

Note added in proofs. In what follows, we present another proof of Proposition 6.1, which seems to be more natural, and it is also "shorter." First, note that, since X is strongly zero-dimensional, $\beta X = \beta_0 X$. Thus, $O_c^p = O^p \cap C_c(X)$ for each $p \in \beta X = \beta_0 X$. Now, suppose that X is an F_c-space. Then, O_c^p is a prime z_c-ideal in $C_c(X)$ for each $p \in \beta_0 X$, and hence, there exists a prime z-ideal P in $C(X)$ such that $O_c^p = P \cap C_c(X)$, by Corollary 4.5. Then, $Z[O^p \cap C_c(X)] = Z[P \cap C_c(X)]$. However, by Proposition 2.4, $Z(X) = Z_c(X)$ and O^P and P are z-ideals. Then, $Z[O^p] = Z[P]$, and hence, $O^p = P$. Consequently, O^p is prime in $C(X)$ for each $p \in \beta X$, i.e., X is an F-space. The proof of the converse remains intact.

Acknowledgments. The authors would like to thank the well-informed referee for reading the article carefully and giving useful suggestions.

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