ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 2, 2018

INTERPOLATION MIXING HYPERBOLIC FUNCTIONS AND POLYNOMIALS

J.M. CARNICER, E. MAINAR AND J.M. PEÑA

ABSTRACT. Exponential polynomials as solutions of differential equations with constant coefficients are widely used for approximation purposes. Recently, mixed spaces containing algebraic, trigonometric and exponential functions have been extensively considered for design purposes. The analysis of these spaces leads to constructions that can be reduced to Hermite interpolation problems. In this paper, we focus on spaces generated by algebraic polynomials, hyperbolic sine and hyperbolic cosine. We present classical interpolation formulae, such as Newton and Aitken-Neville formulae and a suggestion of implementation. We explore another technique, expressing the Hermite interpolant in terms of polynomial interpolants and derive practical error bounds for the hyperbolic interpolant.

1. Introduction. Splines in tension were introduced by Schweikert as piecewise solutions of the differential equation

$$f^{(4)}(x) - \tau^2 f''(x) = 0.$$

The meaning of the parameter τ is related with the tension stress exerted on an elastic beam. The set of solutions of this equation is the space $\langle \cosh \tau x, \sinh \tau x, 1, x\rangle$, where $\tau > 0$ is the tension parameter. The space with $\tau = 1$ can be generalized to the mixed space

$$H_n := \langle \cosh x, \sinh x, 1, x, \dots, x^{n-2}\rangle, \quad n \geq 2.$$

For $n = 1$, $H_1 := \langle \cosh x, \sinh x\rangle$. Since $\cosh x$ belongs to H_n, the space contains functions whose graph is a catenary, a remarkable curve in engineering.

We observe that the space H_n, $n \neq 2$, is not the classical space of hyperbolic polynomials

---

2010 AMS *Mathematics subject classification.* Primary 41A05, 41A30, 65D05.

Keywords and phrases. Hyperbolic functions, Hermite interpolation, Newton and Aitken Neville formulae, total positivity, shape preserving representations.

This research was partially supported by MINECO/FEDER, grant No. MTM2015-65433-P.

Received by the editors on March 23, 2016, and in revised form on April 3, 2017.

DOI:10.1216/RMJ-2018-48-2-443

$$\langle 1, \cosh x, \sinh x, \dots, \cosh nx, \sinh nx \rangle,$$

whose algebraic properties lead to explicit interpolation formulae.

Generalized exponential polynomials are solution spaces of linear differential equations with constant coefficients. Approximation properties of generalized exponential polynomials are analyzed in [**1**]. The space H_n, $n \geq 1$, is the set of solutions of the linear differential equation with constant coefficients $f^{(n+1)} - f^{(n-1)} = 0$, and hence, is a space of generalized exponential polynomials. Approximation in the space H_n has different features in contrast to classical polynomial approximation. Since the above differential equation can be related to tension stress, interpolants in H_n tend to reduce the amplitude of the oscillations. On the other hand, the space H_n is invariant under translations in the sense that, if $h \in H_n$, then $h(\cdot + t) \in H_n$ for any $t \in \mathbf{R}$. It is also invariant under reflections in the sense that, if $h \in H_n$, then $h(-\,\cdot) \in H_n$. These properties are useful for considering different bases obtained by translating or reflecting the functions of a standard basis and, by means of a change of variables, this fact offers the possibility of translating or reflecting the position of the nodes. Finally, since the space contains a large subspace of polynomial functions, we can perform simpler manipulations and computations in the space.

Some properties of general spaces

$$\langle 1, x, x^2, \dots, x^{n-2}, u(x), v(x) \rangle$$

with suitable functions u and v were considered in [**5**]. In the case where $u(x) = \cos x$, $v(x) = \sin x$, we have the cycloidal spaces C_n, whose shape preserving properties for design purposes have been analyzed [**2, 3, 7, 13**]. Interpolation problems in cycloidal spaces have been recently studied in [**4**]. If we take $u(x) = \cosh x$, $v(x) = \sinh x$, we recover the space H_n.

General interpolation formulae for extended complete Chebyshev spaces [**8, 9, 10, 12**] can be applied to the space H_n. In Section 2, we review classical interpolation formulae and analyze particular features of the interpolation formulae in H_n. In Section 3, we take advantage of the fact that the space H_n contains P_{n-2}, the space of polynomials of degree less than or equal to $n-2$, to obtain alternative interpolation formulae and error bounds. In Section 4, we illustrate with examples some properties of interpolants in H_n.

2. The Hermite interpolation problem on H_n. For Hermite interpolation problems, we consider sequences of nodes not necessarily distinct.

Definition 2.1. An $(n+1)$-*dimensional extended Chebyshev space* U on an interval I is a subspace of functions of $C^n(I)$ such that the Hermite interpolation problem on U at any sequence of nodes always has a unique solution, that is, for any $f \in C^n(I)$ and any sequence of nodes $x_0, \dots, x_n \in I$, there exists a unique $u \in U$ such that $\lambda_i u = \lambda_i f$, $i = 0, \dots, n$, where

$$\lambda_i f := f^{(r_i - 1)}(x_i), \qquad r_i = \#\{j \le i \mid x_j = x_i\}. \tag{2.1}$$

The extended collocation matrix of $(u_0, \dots, u_n)$ at nodes $x_0, \dots, x_n$ in I is defined by

$$M^* \begin{pmatrix} u_0, \dots, u_n \\ x_0, \dots, x_n \end{pmatrix} := (\lambda_i u_j)_{i,j \in \{0,1,\dots,n\}},$$

where λ_i are the Hermite functionals given in (2.1). We can regard an extended Chebyshev space as a space such that all extended collocation matrices of any basis are nonsingular.

Some interpolation formulae such as Newton type formulae are required to solve an interpolation problem on the first $k+1$ nodes on a $(k+1)$-dimensional subspace of an extended Chebyshev space U, $k = 0, \dots, \dim U - 1$. Thus, in order to conveniently express a Newton type formula, we search for a basis whose first elements generate a suitable interpolation space. An *extended complete Chebyshev system* of functions $(u_0, \dots, u_n)$ defined on an interval I is a system of functions such that, for any $k = 0, \dots, n$, the subsystem $(u_0, \dots, u_k)$ generates a $(k+1)$-dimensional Chebyshev space on I. Mühlbach introduced **[8, 9]** divided differences, a Newton basis and Newton formulae starting from an extended complete Chebyshev system. He also provided Aitken-Neville formulae. In **[10]**, he showed how to generalize these definitions to Hermite interpolation problems. Throughout this section, we shall recover most of his results adapted to the space H_n, pointing out some special features of interpolation in these spaces and taking into account relevant properties such as invariance under translations. Moreover, in this paper, we shall present practical implementations of the Newton formula and error bounds based on the fact that H_n contains polynomials.

Using Pólya's Property W and the results in [**11**], it can be deduced that the set of solutions of any linear differential equation with constant coefficients whose characteristic roots are all real are extended Chebyshev spaces on the whole real line. Since H_n, $n \geq 1$, is the set of solutions of $f^{(n+1)} - f^{(n-1)} = 0$, it follows that H_n, $n \geq 1$, are extended Chebyshev spaces on the whole real line. Furthermore, the system $(\cosh x, \sinh x, 1, \ldots, x^{n-2})$ is an extended complete Chebyshev system on the whole real line.

The unique solution of the Hermite interpolation problem for a given sequence of nodes $x_0, \ldots, x_n$, is called the *hyperbolic interpolant* of f and will be denoted by $H(f; x_0, \ldots, x_n)$.

The next result shows the invariance of hyperbolic interpolants under translation.

Proposition 2.2. *For any $t \in \mathbf{R}$,*

$$H(f; x_0 + t, \ldots, x_n + t)(x) = H(f(\cdot + t); x_0, \ldots, x_n)(x - t). \tag{2.2}$$

Proof. The function h, given by

$$h(x) = H(f(\cdot + t); x_0, \ldots, x_n)(x), \quad x \in \mathbf{R},$$

belongs to H_n. Since the space H_n can be generated by the basis $(1, x, \ldots, x^{n-2}, e^x, e^{-x})$, it readily follows that H_n is invariant under translations. Hence, $h_t(x) = h(x-t)$, $x \in \mathbf{R}$, defines a function $h_t \in H_n$ that interpolates f at $x_0 + t, \ldots, x_n + t$ since

$$h_t^{(r_i-1)}(x_i + t) = h^{(r_i-1)}(x_i) = f^{(r_i-1)}(x_i + t),$$

where $r_i = \#\{j \leq i \mid x_j = x_i\}$ for $i = 0, \ldots, n$. Thus, (2.2) follows from the uniqueness of the interpolant. $\square$

Proposition 2.2 can be used in practice for the computation of the interpolant. Given a sequence $x_0, \ldots, x_n$ of nodes, we suggest taking

$$a = \min_{i=0,\ldots,n} x_i, \qquad b = \max_{i=0,\ldots,n} x_i, \quad t = -(a+b)/2,$$

so that the origin is the midpoint of the shifted interval $[a+t, b+t]$. The absolute values of the elements of the basis

$$(\cosh x,\ \sinh x,\ 1,\ x,\ x^2, \dots, x^{n-2}) \tag{2.3}$$

grow with the distance of x to the origin. With the proposed translation, the computation of the interpolants with respect to the basis (2.3) might involve less cancelations leading to more stable computations.

An alternative basis to (2.3) for H_n is given by the following recurrence:

$$\begin{aligned}\varphi_0(x) &:= \cosh x,\\ \varphi_i(x) &:= \int_0^x \varphi_{i-1}(y)\,dy, \quad i = 1, \dots, n.\end{aligned}$$

Clearly, $\varphi_i \in H_i$, for all $i = 1, \dots, n$, and linear independence follows from the fact that

$$\varphi_i(0) = \varphi_i'(0) = \cdots = \varphi_i^{(i-1)}(0) = 0, \qquad \varphi_i^{(i)}(0) = 1.$$

The function φ_n is the *fundamental function* of H_n.

In order to express that a point x_i appears k times in a sequence of nodes, we shall write $x_i^{[k]}$. The Taylor interpolation problem corresponds to the sequence of nodes $x_0 = x_1 = \cdots = x_n$, giving rise to the Taylor formula

$$H(f; x_0^{[n+1]})(x) = \sum_{k=0}^{n-2} f^{(k)}(x_0)\frac{(x-x_0)^k}{k!} + \sum_{k=n-1}^{n} f^{(k)}(x_0)\varphi_k(x-x_0). \tag{2.4}$$

According to **[8, 9, 10]**, given an extended complete Chebyshev system $(u_0, \dots, u_n)$ of a space U_n and a sequence of nodes $x_0, \dots, x_n$, we define the divided difference of a function as the last coefficient of the interpolant in U_n at $x_0, \dots, x_n$ with respect to that basis

$$\begin{bmatrix} u_0, \dots, u_n \\ x_0, \dots, x_n \end{bmatrix} f := \frac{\det M^*\begin{pmatrix} u_0, \dots, u_{n-1}, f \\ x_0, \dots, x_n \end{pmatrix}}{\det M^*\begin{pmatrix} u_0, \dots, u_n \\ x_0, \dots, x_n \end{pmatrix}}.$$

A corresponding Newton basis function can be defined as the interpolation error of u_n.

In order to write a Newton formula for the spaces H_n, we define, for $n \geq 2$,

$$\omega(x; x_0, \ldots, x_{n-1}) := x^{n-2} - H((\cdot)^{n-2}; x_0, \ldots, x_{n-1})(x). \tag{2.5}$$

Therefore, $\omega(x; x_0, \ldots, x_{n-1})$ is a function in H_n whose coefficient in x^{n-2} with respect to the basis $(\cosh x, \sinh x, 1, x, \ldots, x^{n-2})$ is 1, vanishing on the sequence $x_0, \ldots, x_{n-1}$,

$$\lambda_i \omega(\,\cdot\,; x_0, \ldots, x_{n-1}) = 0, \quad i = 0, \ldots, n-1.$$

We also define the hyperbolic divided difference as:

$$[x_0, \ldots, x_n]_H f := \begin{bmatrix} \cosh x,\ \sinh x,\ 1, \ldots, x^{n-2} \\ x_0, \ldots, x_n \end{bmatrix} f, \quad n \geq 2.$$

Thus, $[x_0, \ldots, x_n]_H f$ is the coefficient in the function x^{n-2} of the hyperbolic interpolant $H(f; x_0, \ldots, x_n)$ when expressed in terms of the basis (2.3).

We observe that $\varphi_2(x - x_0) = \cosh(x - x_0) - 1$, and therefore, $\varphi_2(x - x_0) + 1 \in H_1$. By successive integration, we deduce that

$$\varphi_n(x - x_0) + \frac{x^{n-2}}{(n-2)!} \in H_{n-1}. \tag{2.6}$$

If all nodes coincide, we can use the above formula in the Taylor expansion (2.4) to obtain

$$[x_0^{[n+1]}]_H f = \frac{f^{(n-2)}(x_0) - f^{(n)}(x_0)}{(n-2)!}. \tag{2.7}$$

We deduce a Newton formula for the hyperbolic interpolant.

Proposition 2.3. *Given $x_0, \ldots, x_n$, $n \geq 2$, we have*

$$H(f; x_0, \ldots, x_n)(x) = H(f; x_0, x_1)(x) + \sum_{k=2}^{n} [x_0, \ldots, x_k]_H f \, \omega(x; x_0, \ldots, x_{k-1}),$$

where

$$H(f; x_0, x_0)(x) = f(x_0) \cosh(x - x_0) + f'(x_0) \sinh(x - x_0)$$

and

$$H(f;x_0,x_1)(x) = \frac{f(x_1)\sinh(x-x_0) - f(x_0)\sinh(x-x_1)}{\sinh(x_1-x_0)}$$

if $x_1 \neq x_0$.

Proof. It is sufficient to show

$$\begin{aligned} H(f;x_0,\dots,x_n)(x) = H(f;x_0,\dots,x_{n-1})(x) \\ + [x_0,\dots,x_n]_H f\,\omega(x;x_0,\dots,x_{n-1}), \end{aligned} \tag{2.8}$$

and the result follows by induction. The function defined by

$$d(x) := H(f;x_0,\dots,x_n)(x) - H(f;x_0,\dots,x_{n-1})(x) \in H_n$$

satisfies $\lambda_i d = 0$, $i = 0,\dots,n-1$. Since the interpolation problem at $x_0,\dots,x_n$ has a unique solution,

$$\dim\{h \in H_n \mid \lambda_i h = 0, i = 0,\dots,n-1\} = 1.$$

Since $\omega(\cdot;x_0,\dots,x_{n-1})$ is a nonzero function in the space

$$\{h \in H_n \mid \lambda_i h = 0, i = 0,\dots,n-1\},$$

we conclude that $d(x) = A\omega(x;x_0,\dots,x_{n-1})$ for some real constant A, and taking coefficients in x^{n-2} we deduce that $A = [x_0,\dots,x_n]_H f$. $\square$

We define an interpolation error

$$e(f;x_0,\dots,x_n)(x) := f(x) - H(f;x_0,\dots,x_n)(x).$$

Using Proposition 2.3, a formula for the error in terms of hyperbolic divided differences can be deduced [**12,** Theorem 9.9]

$$e(f;x_0,\dots,x_n)(x) = [x_0,\dots,x_n,x]_H f\,\omega(x;x_0,\dots,x_n).$$

Now, we shall deduce recurrence relations for divided differences. We need the following auxiliary result.

Lemma 2.4. *Let* $x_0,\dots,x_n$, $n \geq 3$, *be a sequence of nodes, and denote*

$$D := [x_1,\dots,x_n]_H(\cdot)^{n-2} - [x_0,\dots,x_{n-1}]_H(\cdot)^{n-2}.$$

If $x_0 \neq x_n$, *then* $D \neq 0$ *and*

$$\omega(x;x_0,\dots,x_{n-1}) - \omega(x;x_1,\dots,x_n) = D\omega(x;x_1,\dots,x_{n-1}). \tag{2.9}$$

Proof. By (2.5),

$$\omega(x;x_0,\dots,x_{n-1}) = x^{n-2} - H((\cdot)^{n-2};x_0,\dots,x_{n-1})(x)$$

and

$$\omega(x;x_1,\dots,x_n) = x^{n-2} - H((\cdot)^{n-2};x_1,\dots,x_n)(x).$$

Subtracting both relations, we obtain

$$\begin{aligned}\omega(x;x_0,\dots,x_{n-1}) &- \omega(x;x_1,\dots,x_n)\\ &= H((\cdot)^{n-2};x_1,\dots,x_n)(x) - H((\cdot)^{n-2};x_0,\dots,x_{n-1})(x).\end{aligned} \tag{2.10}$$

From (2.8), we deduce that

$$\begin{aligned}H((\cdot)^{n-2};x_0,\dots,x_{n-1})(x) &= H((\cdot)^{n-2};x_1,\dots,x_{n-1})(x)\\ &\quad + [x_0,\dots,x_{n-1}]_H(\cdot)^{n-2}\omega(x;x_1,\dots,x_{n-1}),\\ H((\cdot)^{n-2};x_1,\dots,x_n)(x) &= H((\cdot)^{n-2};x_1,\dots,x_{n-1})(x)\\ &\quad + [x_1,\dots,x_n]_H(\cdot)^{n-2}\omega(x;x_1,\dots,x_{n-1}),\end{aligned}$$

and, inserting these relations into (2.10), we obtain (2.9). If $D=0$, then

$$h(x) := \omega(x;x_0,\dots,x_{n-1}) = \omega(x;x_1,\dots,x_n)$$

is a nonzero function in H_n such that $\lambda_i h = 0$, $i=0,\dots,n$, where λ_i, $i=0,\dots,n$, are the functionals given in (2.1), contradicting the fact that H_n is an extended Chebyshev space. $\square$

Using the previous lemma, we obtain the following result.

Theorem 2.5 (Recurrence relations for hyperbolic divided differences)**.** *Let $x_0,\dots,x_n$, $n\ge 3$, be a sequence of nodes with $x_0\ne x_n$. Then,*

$$D := [x_1,\dots,x_n]_H(\cdot)^{n-2} - [x_0,\dots,x_{n-1}]_H(\cdot)^{n-2} \neq 0$$

and

$$[x_0,\dots,x_n]_H f = \frac{[x_1,\dots,x_n]_H f - [x_0,\dots,x_{n-1}]_H f}{D}.$$

Proof. Let $h(x) := H(f; x_0, \ldots, x_n)(x)$. From Proposition 2.3, we have

$$h(x) = H(f; x_0, \ldots, x_{n-1})(x) + [x_0, \ldots, x_n]_H f \omega(x; x_0, \ldots, x_{n-1}), \tag{2.11}$$

$$h(x) = H(f; x_1, \ldots, x_n)(x) + [x_0, \ldots, x_n]_H f \omega(x; x_1, \ldots, x_n). \tag{2.12}$$

Subtracting both relations and using Lemma 2.4, we deduce that

$$\begin{aligned} &H(f; x_1, \ldots, x_n)(x) - H(f; x_0, \ldots, x_{n-1})(x) \\ &\quad = [x_0, \ldots, x_n]_H f(\omega(x; x_0, \ldots, x_{n-1}) - \omega(x; x_1, \ldots, x_n)) \\ &\quad = D[x_0, \ldots, x_n]_H f \omega(x; x_1, \ldots, x_{n-1}), \end{aligned}$$

with $D \neq 0$. Taking coefficients in the basis function x^{n-3} with respect to the basis $(\cosh x, \sinh x, 1, \ldots, x^{n-3})$ of H_{n-1} the result follows. $\square$

Lemma 2.4 can also be used to deduce an Aitken-Neville formula for hyperbolic interpolants.

Theorem 2.6 (Aitken-Neville formula). *Let $x_0, \ldots, x_n$, $n \geq 3$, be a sequence of nodes, and denote*

$$D := [x_1, \ldots, x_n]_H (\cdot)^{n-2} - [x_0, \ldots, x_{n-1}]_H (\cdot)^{n-2}.$$

Then, we have

$$\begin{aligned} &D\omega(x; x_1, \ldots, x_{n-1}) H(f; x_0, \ldots, x_n)(x) \\ &\quad = \omega(x; x_0, \ldots, x_{n-1}) H(f; x_1, \ldots, x_n)(x) \\ &\qquad - \omega(x; x_1, \ldots, x_n) H(f; x_0, \ldots, x_{n-1})(x). \end{aligned}$$

Proof. Multiplying (2.12) by $\omega(x; x_0, \ldots, x_{n-1})$, (2.11) by $\omega(x; x_1, \ldots, x_n)$ and subtracting, we obtain

$$\begin{aligned} &(\omega(x; x_0, \ldots, x_{n-1}) - \omega(x; x_1, \ldots, x_n)) H(f; x_0, \ldots, x_n)(x) \\ &\quad = \omega(x; x_0, \ldots, x_{n-1}) H(f; x_1, \ldots, x_n)(x) \\ &\qquad - \omega(x; x_1, \ldots, x_n) H(f; x_0, \ldots, x_{n-1})(x). \end{aligned}$$

Using Lemma 2.4, the result follows. $\square$

Let us extend the definition of $[x_i,\dots,x_{i+k}]_H f$ and the recurrence relations in Theorem 2.5 to the cases $k=0,1$. We can define

$$[x_0]_H f := f(x_0)/\cosh x_0$$

and $[x_0,x_1]_H f$ as the coefficient in $\sinh x$ of the interpolant in H_1

$$[x_0,x_1]_H f := \begin{cases} \dfrac{f(x_1)\cosh x_0 - f(x_0)\cosh x_1}{\sinh(x_1-x_0)} & \text{if } x_0 < x_1,\\ f'(x_0)\cosh x_0 - f(x_0)\sinh x_0 & \text{if } x_0 = x_1. \end{cases}$$

The recurrence relations can be extended for $k=0,1$, in the following way:

$$[x_0,x_1]_H f = \frac{[x_1]_H f - [x_0]_H f}{[x_1]_H \sinh - [x_0]_H \sinh}$$

and

$$[x_0,x_1,x_2]_H f = \frac{[x_1,x_2]_H f - [x_0,x_1]_H f}{[x_1,x_2]_H 1 - [x_0,x_1]_H 1}.$$

Let us see how to compute hyperbolic divided differences and the corresponding Newton basis functions. For the sake of simplicity, we shall assume that the nodes are in increasing order, that is, $x_0 \le \cdots \le x_n$.

We denote

$$d_{i,k} f := [x_i,\dots,x_{i+k}]_H f, \quad i=0,\dots,n-k,\ k=0,\dots,n.$$

We shall start computing hyperbolic divided differences

$$d_{i,0}(\cdot)^j, \quad j=0,\dots,n-2, \quad d_{i,0} f \text{ for } i=0,\dots,n,$$

and

$$d_{i,1}(\cdot)^j, \quad j=0,\dots,n-2, \quad d_{i,1} f \text{ for } i=0,\dots,n-1,$$

using the above formulae. For $k \ge 2$, if all nodes are coincident, we use formula (2.7). Otherwise, the hyperbolic divided differences can be computed by successive applications of Theorem 2.5. Thus, for $k=2,\dots,n$, and for each $i=0,\dots,n-k$, we compute

$$(2.13) \qquad d_{i,k}(\cdot)^j = \begin{cases} \dfrac{d_{i+1,k-1}(\cdot)^j - d_{i,k-1}(\cdot)^j}{d_{i+1,k-1}(\cdot)^{k-2} - d_{i,k-1}(\cdot)^{k-2}} & \text{if } x_i < x_{i+k},\\ x_i^{j-k}\left(\binom{j}{k-2}x_i^2 - k(k-1)\binom{j}{k}\right) & \text{otherwise,} \end{cases}$$

for $j = k-1, \ldots, n-2$, and

$$(2.14) \qquad d_{i,k}f = \begin{cases} \dfrac{d_{i+1,k-1}f - d_{i,k-1}f}{d_{i+1,k-1}(\cdot)^{k-2} - d_{i,k-1}(\cdot)^{k-2}} & \text{if } x_i < x_{i+k}, \\ \dfrac{f^{(k-2)}(x_i) - f^{(k)}(x_i)}{(k-2)!} & \text{otherwise.} \end{cases}$$

We observe that the denominators in (2.14) and (2.13) in the kth step were previously computed in the $(k-1)$th step in (2.13).

Starting from

$$(2.15) \qquad \omega_0(x) := \cosh(x), \qquad \omega_1(x) := \frac{\sinh(x-x_0)}{\cosh(x_0)},$$

we compute

$$\omega_k(x) := \omega(x; x_0, \ldots, x_{k-1}), \quad k = 2, \ldots, n,$$

for each evaluation point x. We apply Newton's formula (2.8) in the definition of ω to obtain

$$(2.16) \qquad \omega_j(x) = x^{j-2} - \sum_{k=0}^{j-1} d_{0,k}(\cdot)^{j-2}\,\omega_k(x), \quad j = 2, \ldots, n.$$

Note that

$$d_{0,k}(\cdot)^{j-2}, \quad k = 0, \ldots, j-1,$$

were previously obtained in (2.13) during the computation of cycloidal divided differences. Finally, the cycloidal interpolant is given by the following extension of Newton's formula (2.8)

$$H(f; x_0, \ldots, x_n)(x) = \sum_{k=0}^{n} d_{0,k}f\,\omega_k(x).$$

3. A formula relating hyperbolic and polynomial interpolants. In this section, $[x_0, \ldots, x_n]f$ denotes the usual polynomial divided difference and $P(f; x_0, \ldots, x_n)$ the polynomial interpolant of f at $x_0, \ldots, x_n$. We introduce the $(k+1) \times (m+1)$ matrix

$$M_{k,n}(f_0,\dots,f_m) := \begin{pmatrix} [x_0,\dots,x_{n-k}]f_0 & \cdots & [x_0,\dots,x_{n-k}]f_m \\ \vdots & \ddots & \vdots \\ [x_0,\dots,x_{n-1}]f_0 & \cdots & [x_0,\dots,x_{n-1}]f_m \\ [x_0,\dots,x_n]f_0 & \cdots & [x_0,\dots,x_n]f_m \end{pmatrix},$$

where $k \le n$.

We begin with the following auxiliary result.

Proposition 3.1. *If the Hermite interpolation problem has a unique solution in the space generated by*

$$(1, x, \dots, x^{n-k-1}, f_0(x), \dots, f_k(x))$$

at the sequence of nodes $x_0,\dots,x_n$, *then* $M_{k,n}(f_0,\dots,f_k)$ *is nonsingular.*

Proof. Let

$$\omega_j(x) := \prod_{i=0}^{j-1}(x - x_i), \quad j = 0,\dots,n,$$

denote the functions of the polynomial Newton basis. Applying the Hermite functional λ_i to Newton's polynomial formula

$$P(f; x_0,\dots,x_i)(x) = \sum_{j=0}^{i} [x_0,\dots,x_j]f\omega_j(x),$$

we obtain

$$\lambda_i f = \sum_{j=0}^{i} t_{ij}[x_0,\dots,x_j]f, \quad t_{ij} = \lambda_i\omega_j.$$

Let $T = (t_{ij})_{i,j\in\{0,1,\dots,n\}}$ be the lower triangular matrix whose (i,j) entry is $\lambda_i\omega_j$. Then, we can write

$$M^*\begin{pmatrix} 1,\dots,x^{n-k-1}, f_0,\dots,f_k \\ x_0,\dots,x_n \end{pmatrix} = TM,$$

where $M = (m_{ij})_{i,j\in\{0,1,\dots,n\}}$ is the matrix whose $(i+1)$th row is given by

$$m_{ij} := [x_0,\dots,x_i](\cdot)^j, \quad j = 0,\dots,n-k-1,$$

and

$$m_{ij} := [x_0, \ldots, x_i] f_{j+k-n}, \quad j = n-k, \ldots, n.$$

Since the collocation matrix is nonsingular, we deduce that M is nonsingular. Taking into account that $m_{jj} = 1$ and $m_{ij} = 0$, $i > j$, $j = 0, \ldots, k-1$, we deduce that $\det M_{k,n}(f_0, \ldots, f_k) = \det M \neq 0$. $\square$

We express the hyperbolic interpolant as a linear combination of polynomial interpolants.

Theorem 3.2. *Let $x_0, \ldots, x_n$, $n \geq 1$, be a sequence of nodes. Then,*

$$H(f; x_0, \ldots, x_n)(x) = P(f; x_0, \ldots, x_n)(x) + c_0 e_0(x) + c_1 e_1(x), \tag{3.1}$$

where

$$e_i(x) := \varphi_i(x) - P(\varphi_i; x_0, \ldots, x_n)(x), \quad i = 0, 1, \tag{3.2}$$

are the polynomial interpolation errors of $\varphi_0(x) = \cosh x$ and $\varphi_1(x) = \sinh x$, and $c = (c_0, c_1)^T$ is the solution of the linear system

$$M_{1,n}(\varphi_0, \varphi_1)\, c = M_{1,n}(f). \tag{3.3}$$

Proof. We denote by $h(x) = H(f; x_0, \ldots, x_n)$ the interpolant of f in H_n. Then, there exist $c_0, c_1 \in \mathbf{R}$ such that

$$q(x) := h(x) - c_0 \varphi_0(x) - c_1 \varphi_1(x) \tag{3.4}$$

belongs to P_{n-2}. From the definition of the polynomial interpolants, we have $\lambda_i e_0 = \lambda_i e_1 = 0$ for $i = 0, \ldots, n$, where λ_i are the Hermite functionals defined in (2.1). Therefore,

$$\lambda_i (h - c_0 e_0 - c_1 e_1) = \lambda_i f, \quad i = 0, \ldots, n,$$

and $h - c_0 e_0 - c_1 e_1$ interpolates f at $x_0, \ldots, x_n$. From the equality

$$\begin{aligned} h(x) - c_0 e_0(x) - c_1 e_1(x) \\ &= q(x) + c_0 P(\varphi_0; x_0, \ldots, x_n) + c_1 P(\varphi_0; x_0, \ldots, x_n), \end{aligned}$$

we deduce that $h - c_0 e_0 - c_1 e_1$ is a polynomial in P_n. Therefore,

$$h(x) - c_0 e_0(x) - c_1 e_1(x) = P(f; x_0, \ldots, x_n)(x),$$

and (3.1) holds for some constants $c_0, c_1 \in \mathbf{R}$. In order to find c_0, c_1, we insert the relations (3.1) and (3.2) into (3.4) and deduce that

$$\begin{aligned} q(x) = {} & P(f; x_0, \dots, x_n)(x) \\ & - c_0 P(\varphi_0; x_0, \dots, x_n)(x) - c_1 P(\varphi_1; x_0, \dots, x_n)(x). \end{aligned}$$

According to Newton's polynomial formula, we have

$$q(x) = \sum_{k=0}^{n} [x_0, \dots, x_k](f - c_0\varphi_0 - c_1\varphi_1) \prod_{j=0}^{k-1} (x - x_j).$$

Since q belongs to P_{n-2}, c_0 and c_1 satisfy

$$[x_0, \dots, x_k](f - c_0\varphi_0 - c_1\varphi_1) = 0, \quad k = n-1, n,$$

conditions that are equivalent to (3.3). Observe that $c = (c_0, c_1)^T$ is the unique solution of (3.3) since the coefficient matrix is nonsingular by Proposition 3.1. □

Now, comparing the coefficients in x^{n-2} in (3.1), we deduce a relationship between hyperbolic and polynomial divided differences.

Proposition 3.3. *Let $x_0, \dots, x_n$, $n \geq 2$, be a sequence of nodes. Then*

$$\begin{aligned} [x_0, \dots, x_n]_H f = {} & [x_0, \dots, x_{n-2}] f \\ & - c_0 [x_0, \dots, x_{n-2}]\varphi_0 - c_1 [x_0, \dots, x_{n-2}]\varphi_1, \end{aligned}$$

where $c = (c_0, c_1)$ is the solution of the linear system (3.3).

Now, we find bounds for the solution of the linear system (3.3) and apply it to obtain error bounds.

Lemma 3.4. *Let $x_0, \dots, x_n \in [a, b]$, $n \geq 1$, be a sequence of nodes and c the solution of the linear system* (3.3). *Then,*

$$\|c\|_\infty \leq L \max(K_{n-1}, K_n), \quad i = 0, 1, \tag{3.5}$$

where $L := \exp\max(|a|, |b|)$ and

$$K_j := \max_{x \in [a,b]} |f^{(j)}(x)|, \quad j = n-1, n.$$

Proof. By (3.4), the error can be written as

$$e(x) = f(x) - q(x) - c_0\varphi_0(x) - c_1\varphi_1(x),$$

for some $q \in P_{n-2}$, where $c = (c_0, c_1)^T$ is the solution of the linear system (3.3). Since e vanishes at $x_0, \dots, x_n$, we deduce from Rolle's generalized Rolle's that there exist two points ξ_0, ξ_1 in $[a, b]$ such that the derivatives of order $n-1$ and n of the error vanish on ξ_0, ξ_1, respectively,

$$f^{(n-1)}(\xi_0) - c_0\varphi_0^{(n-1)}(\xi_0) - c_1\varphi_1^{(n-1)}(\xi_0) = 0,$$
$$f^{(n)}(\xi_1) - c_0\varphi_0^{(n)}(\xi_1) - c_1\varphi_1^{(n)}(\xi_1) = 0.$$

These relations can be written in matrix form in the following manner:

$$\begin{pmatrix} \varphi_0^{(n-1)}(\xi_0) & \varphi_1^{(n-1)}(\xi_0) \\ \varphi_0^{(n)}(\xi_1) & \varphi_1^{(n)}(\xi_1) \end{pmatrix} c = \begin{pmatrix} f^{(n-1)}(\xi_0) \\ f^{(n)}(\xi_1) \end{pmatrix}.$$

We observe that the determinant of the coefficient matrix of the above system is $\pm\cosh(\xi_1 - \xi_0)$, depending upon whether n is an odd or even integer. Since the derivatives of $\varphi_0(x) = \cosh x$ and $\varphi_1(x) = \sinh x$ are again one of both functions, we have that the sum of the absolute values of the entries of each column equals $|\sinh \xi_i| + \cosh \xi_{1-i}$ for some $i \in \{0, 1\}$, and we deduce the following bounds:

$$|\sinh \xi_i| + \cosh \xi_{1-i} \leq \sinh \max(|a|, |b|) + \cosh \max(|a|, |b|) = L.$$

Thus, applying Cramer's rule, we deduce that

$$|c_i| = \frac{|f^{(n-1)}(\xi_0)\varphi_{1-i}^{(n)}(\xi_1) - f^{(n)}(\xi_1)\varphi_{1-i}^{(n-1)}(\xi_0)|}{\cosh(\xi_1 - \xi_0)}$$

and, since $\cosh(\xi_1 - \xi_0) \geq 1$, formula (3.5) follows. □

The next proposition gives a formula and a bound for the interpolation error

$$e(x) = f(x) - H(f; x_0, \dots, x_n)(x).$$

Proposition 3.5. *Let $x_0, \dots, x_n \in [a, b]$, $n \geq 1$, be a sequence of nodes. Then,*

$$\begin{aligned} e(x) &:= f(x) - H(f; x_0, \dots, x_n)(x) \\ &= e_2(x) - c_0 e_0(x) - c_1 e_1(x), \end{aligned} \tag{3.6}$$

where e_0 *and* e_1 *are given by* (3.2) *and*

$$e_2(x) := f(x) - P(f; x_0, \dots, x_n)(x).$$

Therefore, a bound for the interpolation error is

$$|e(x)| \le \frac{K_{n+1} + L^2 \max(K_{n-1}, K_n)}{(n+1)!} \prod_{i=0}^{n} |x - x_i|,$$

where $L := \exp \max(|a|, |b|)$ *and*

$$K_j := \max_{x \in [a,b]} |f^{(j)}(x)|, \quad j = n-1, n, n+1.$$

Proof. Using Theorem 3.2,

$$\begin{aligned} e(x) &= f(x) - H(f; x_0, \dots, x_n)(x) \\ &= f(x) - P(f; x_0, \dots, x_n)(x) - c_0 e_0(x) - c_1 e_1(x), \end{aligned}$$

formula (3.6) is confirmed. From the polynomial interpolation error formulae, we have

$$|e_2(x)| \le \frac{K_{n+1}}{(n+1)!} \prod_{i=0}^{n} |x - x_i|$$

and

$$|e_i(x)| \le \frac{K_{n+1}^i}{(n+1)!} \prod_{i=0}^{n} |x - x_i|, \quad i = 0, 1,$$

where

$$K_{n+1}^i := \max_{x \in [a,b]} |\varphi_i^{(n+1)}(x)|, \quad i = 0, 1.$$

We observe that

$$K_{n+1}^0 + K_{n+1}^1 \le L = \exp \max (|a|, |b|),$$

and then we deduce

$$|e(x)| \le \frac{K_{n+1} + L \|c\|_\infty}{(n+1)!} \prod_{i=0}^{n} |x - x_i|.$$

Finally, we can apply Lemma 3.4, and the result follows. □

4. Examples. We have taken equidistant nodes

$$x_i = -5 + 10i/n, \quad i = 0, \dots, n,$$

in the interval $[-5, 5]$ and data $y_i = f(x_i)$, $i = 0, \dots, n$, from a function f. We have computed the hyperbolic and polynomial interpolants and compared their interpolation errors. In order to avoid numerical instabilities, we have used double precision arithmetic and degrees $n \leq 15$. Depending on the function f, the error in hyperbolic interpolation might be lower or greater. Typically, if the function is close to the space H_n, in the sense that $\|f^{(n+1)} - f^{(n-1)}\|_\infty$ is small, then hyperbolic interpolants give better results. Figure 1 illustrates this behavior for

$$f(x) = \exp(1.1x) - 4\exp(0.8x)$$

and $n = 5$. The maximum error is 8.122 for the polynomial interpolant and 3.867 for the hyperbolic interpolant.

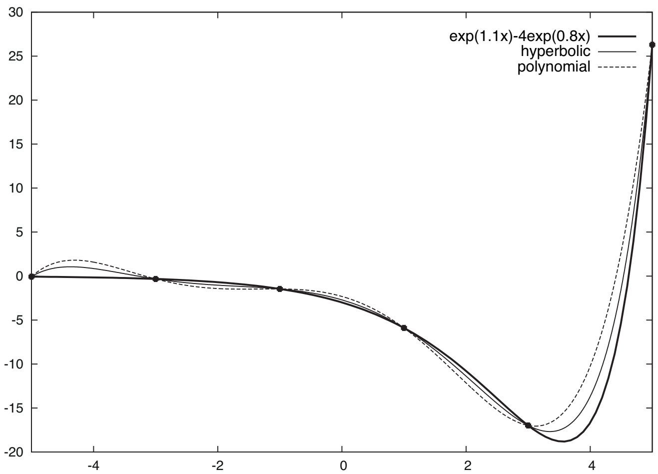

FIGURE 1. Polynomial and hyperbolic interpolants for $n = 5$ to $\exp(1.1x) - \exp(0.8x)$.

We have also tested several interpolants in the spaces

$$H_n(\tau) := \langle \cosh(\tau x), \sinh(\tau x), 1, x, x^2, \dots, x^{n-2} \rangle$$

corresponding to different values of τ. Observe that a change of variables $\xi = \tau x$ allows us to reduce an interpolation problem in $H_n(\tau)$ to interpolation problems on the space H_n with $\tau = 1$. We remark that, when $\tau \to 0$, the hyperbolic interpolant tends to the polynomial interpolant. The hyperbolic interpolants seem to better reproduce some features of the function when the nodes are close to the boundary.

We have solved Hermite interpolation problems with repeated nodes. In our second example, we have taken the function

$$g(x) = \cos(2\pi x^2),$$

degree $n = 11$, and the node sequence

$$(-1.0, -1.0, -0.80, -0.80, -0.60, -0.60, 0.60, 0.60, 0.80, 0.80, 1.0, 1.0).$$

We remark that each node is repeated twice, and there are no nodes close to the center of the interval $[-1, 1]$. This fact leads to large errors in the polynomial interpolant at the center of the interval.

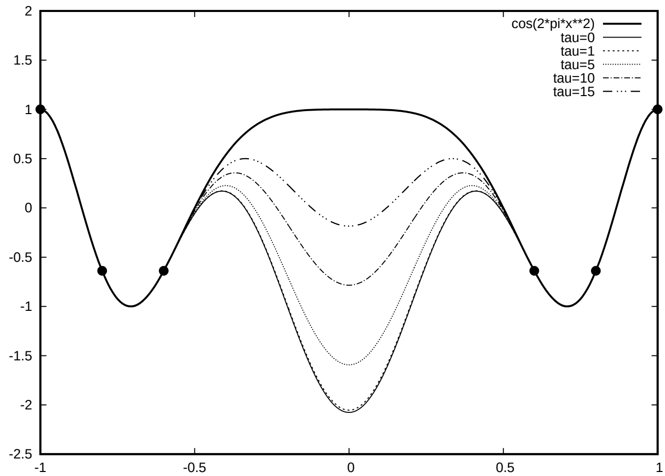

FIGURE 2. Polynomial and hyperbolic interpolants for $n = 11$ to $\cos(2\pi x^2)$.

We have chosen values of $\tau = 1, 5, 10, 15$. In this case, the interpolation error at the center of the interval can be considerably reduced, depending upon the choice of τ (see Figure 2). The maximum error for polynomial interpolant ($\tau = 0$) is 3.0765. The choice of $\tau = 1, 5, 10, 15$, leads to maximum errors $3.053, 2.593, 1.785$ and 1.185, respectively. We note that a convenient choice of the parameter τ can improve the approximation properties of the interpolant.

REFERENCES

1. J.M. Aldaz, O. Kounchev and H. Render, *Bernstein operators for exponential polynomials*, Constr. Approx. **29** (2009), 345–367.

2. J.M. Carnicer, E. Mainar and J.M. Peña, *Critical length for design purposes and extended Chebyshev spaces*, Constr. Approx. **20** (2004), 55–71.

3. ______, *On the critical length of cycloidal spaces*, Constr. Approx. **39** (2014), 573–583.

4. ______, *Interpolation on cycloidal spaces*, J. Approx. Th. **187** (2014), 18–29.

5. P. Costantini, T. Lyche and C. Manni, *On a class of weak Tchebycheff systems*, Numer. Math. **101** (2005), 333–354.

6. E. Mainar and J.M. Peña, *A general class of Bernstein-like bases*, Comp. Math. Appl. **53** (2007), 1686–1703.

7. E. Mainar, J.M. Peña and J. Sánchez-Reyes, *Shape preserving alternatives to the rational Bézier model*, Comp. Aided Geom. Design **18** (2001), 37–60.

8. G. Mühlbach, *A recurrence formula for generalized divided differences and some applications*, J. Approx. Th. **9** (1973), 165–172.

9. ______, *The general recurrence relation for divided differences and the general Newton-interpolation-algorithm with application to trigonometric interpolation*, Numer. Math. **32** (1979), 393–408.

10. ______, *A recurrence relation for generalized divided differences with respect to* ECT-*systems*, Numer. Algorithms **22** (1999), 317–326.

11. G. Pólya, *On the mean value theorem corresponding to a given linear homogeneous differential equation*, Trans. Amer. Math. Soc. **24** (1922), 312–324.

12. L.L. Schumaker, *Spline functions*: *Basic theory*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2007.

13. J. Zhang, C-*curves*: *An extension of cubic curves*, Comp. Aided Geom. Design **13** (1996), 199–217.

Universidad de Zaragoza, Departamento de Matemática Aplicada/IUMA, Pedro Cerbuna, 12 50009 Zaragoza, Spain
Email address: **carnicer@unizar.es**

Universidad de Zaragoza, Departamento de Matemática Aplicada/IUMA, María de Luna, 3 50018 Zaragoza, Spain
Email address: **esmemain@unizar.es**

Universidad de Zaragoza, Departamento de Matemática Aplicada/IUMA, Pedro Cerbuna, 12 50009 Zaragoza, Spain
Email address: **jmpena@unizar.es**