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ASYMPTOTIC BEHAVIOR OF INTEGRAL CLOSURES, QUINTASYMPTOTIC PRIMES AND IDEAL TOPOLOGIES

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ABSTRACT. Let R be a Noetherian ring, N a finitely generated R-module and I an ideal of R. It is shown that the sequences $\mathrm{Ass}_R\, R/(I^n)_a^{(N)}$, $\mathrm{Ass}_R (I^n)_a^{(N)}/(I^{n+1})_a^{(N)}$ and $\mathrm{Ass}_R (I^n)_a^{(N)}/(I^n)_a$, $n = 1, 2, \ldots$, of associated prime ideals, are increasing and ultimately constant for large n. Moreover, it is shown that, if S is a multiplicatively closed subset of R, then the topologies defined by $(I^n)_a^{(N)}$ and $S((I^n)_a^{(N)})$, $n \geq 1$, are equivalent if and only if S is disjoint from the quintasymptotic primes of I. By using this, we also show that, if $(R, \mathfrak{m})$ is local and N is quasi-unmixed, then the local cohomology module $H_I^{\dim N}(N)$ vanishes if and only if there exists a multiplicatively closed subset S of R such that $\mathfrak{m} \cap S \neq \emptyset$ and the topologies induced by $(I^n)_a^{(N)}$ and $S((I^n)_a^{(N)})$, $n \geq 1$, are equivalent.

1. Introduction. The important concept of the integral closure of an ideal of a commutative Noetherian ring (with identity), developed by Northcott and Rees in **[15]**, is fundamental to a considerable body of recent and current research both in commutative algebra and algebraic geometry. Let R be a commutative ring (with identity) and I an ideal of R. In the case when R is Noetherian, we denote by $(I)_a$ the integral closure of I, i.e., $(I)_a$ is the ideal of R consisting of all elements $x \in R$ which satisfy

$$x^n + r_1 x^{n-1} + \cdots + r_n = 0,$$

where $r_i \in I^i$, $i = 1, \ldots, n$.

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In **[16]**, Ratliff, Jr., showed that (when R is Noetherian), the sequence of associated prime ideals

$$\mathrm{Ass}_R\, R/(I^n)_a, \quad n = 1, 2, \dots,$$

is increasing and ultimately constant; we use the notation $A_a^*(I)$ to denote $\mathrm{Ass}_R\, R/(I^n)_a$ for large n.

The notion of integral closures of ideals of R relative to a Noetherian R-module N was initiated by Sharp, et al., in **[22]**. An element $x \in R$ is said to be *integrally dependent* on I relative to N if there exists a positive integer n such that

$$x^n N \subseteq \sum_{i=1}^{n} x^{n-i} I^i N.$$

Then, the set

$$I_a^{(N)} = \{x \in R \mid x \text{ is integrally dependent on } I \text{ relative to } N\}$$

is an ideal of R, called the *integral closure* of I relative to N; in the case $N = R$, $I_a^{(N)}$ is the classical integral closure I_a of I. It is clear that $I \subseteq I_a^{(N)}$. We say that I is *integrally closed* relative to N if $I = I_a^{(N)}$.

In Section 2, we show that, when R is a Noetherian ring and N is a finitely generated R-module, the sequences

$$\mathrm{Ass}_R\, R/(I^n)_a^{(N)},$$

$$\mathrm{Ass}_R (I^n)_a^{(N)}/(I^{n+1})_a^{(N)}$$

and

$$\mathrm{Ass}_R (I^n)_a^{(N)}/((I + \mathrm{Ann}_R\, N)^n)_a, \quad n = 1, 2, \dots,$$

of associated primes are ultimately constant; we let

$$A_a^*(I, N) := \mathrm{Ass}_R\, R/(I^n)_a^{(N)}$$

and

$$C_a^*(I, N) := \mathrm{Ass}_R (I^n)_a^{(N)}/((I + \mathrm{Ann}_R\, N)^n)_a,$$

for large n. Pursuing this point of view further we shall show that

$$A_a^*(I + \mathrm{Ann}_R\, N) \setminus C_a^*(I, N) \subseteq A_a^*(I, N).$$

In [**10**], McAdam studied the following, interesting set of prime ideals of R associated with I,

$$\overline{Q}^*(I) = \{\mathfrak{p} \in \operatorname{Spec} R : \exists\ \mathfrak{q} \in \operatorname{mAss} \widehat{R}_{\mathfrak{p}} \text{ s.t. } \operatorname{Rad}(I\widehat{R}_{\mathfrak{p}} + \mathfrak{q}) = \mathfrak{p}\widehat{R}_{\mathfrak{p}}\},$$

and he called $\overline{Q}^*(I)$ the set of *quintasymptotic prime ideals* of I.

On the other hand, Ahn in [**1**] extended the notion of quintasymptotic prime ideals to a finitely generated module over R. More precisely, if N is a finitely generated R-module, then a prime ideal $\mathfrak{p}$ of R is said to be a *quintasymptotic prime ideal* of I with respect to N whenever there exists a $\mathfrak{q} \in \operatorname{mAss}_{\widehat{R}_{\mathfrak{p}}} \widehat{N}_{\mathfrak{p}}$ such that $\operatorname{Rad}(I\widehat{R}_{\mathfrak{p}} + \mathfrak{q}) = \mathfrak{p}\widehat{R}_{\mathfrak{p}}$. The set of all *quintasymptotic prime ideals* of I with respect to N is denoted by $\overline{Q}^*(I, N)$.

In Section 3, for a multiplicatively closed subset S of R, we examine the equivalence between the topologies defined by the filtrations

$$\{(I^n)_a^{(N)}\}_{n\geq 1}, \qquad \{S((I^n)_a^{(N)})\}_{n\geq 1},$$
$$\{S(((I + \operatorname{Ann}_R N)^n)_a)\}_{n\geq 1}, \qquad \{S((I + \operatorname{Ann}_R N)^n)\}_{n\geq 1}$$

by using the quintasymptotic prime ideals of I with respect to N. Some of these results were established by Schenzel [**17, 18**], McAdam [**10**] and Mehrvarz, et al., [**12**], in the case when $N = R$.

A typical result in this direction is the following.

Theorem 1.1. *Let N be a finitely generated module over a Noetherian ring R, and let I be an ideal of R. Let S be a multiplicatively closed subset of R. Then, the topologies defined by $(I^n)_a^{(N)}$, $S((I^n)_a^{(N)})$, $S(((I + \operatorname{Ann}_R N)^n)_a)$ and $S((I + \operatorname{Ann}_R N)^n)$, $n \geq 1$, are equivalent if and only if S is disjoint from each of the quintasymptotic prime ideals of I with respect to N.*

The proof of Theorem 1.1 is given in Theorem 3.11. One of our tools for proving Theorem 1.1 is the following, which is a characterization of the quintasymptotic prime ideals of I with respect to N. We use $I_a^{\langle N\rangle}$ to denote the union $I_a^{(N)} :_R s$, where s varies in $R \setminus \cup\{\mathfrak{p} \in \operatorname{mAss}_R N/IN\}$; in particular, for every integer $k \geq 1$ and every prime ideal $\mathfrak{p}$ of R,

$$(\mathfrak{p}^k)_a^{\langle N\rangle} = \bigcup_{s\in R\setminus \mathfrak{p}} ((\mathfrak{p}^k)_a^{(N)} :_R s).$$

Proposition 1.2. *Let R be a Noetherian ring, and let N be a finitely generated R-module. Let $I \subseteq \mathfrak{p}$ be ideals of R such that $\mathfrak{p} \in \operatorname{Supp}(N)$. Then, $\mathfrak{p} \in \overline{Q}^*(I,N)$ if and only if there exists an integer $k \geq 0$ such that, for all integers $m \geq 0$,*

$$(I^m)_a^{(N)} :_R \langle \mathfrak{p} \rangle \nsubseteq (\mathfrak{p}^k)_a^{\langle N \rangle}.$$

Finally, in this section, we derive the next consequence of Theorem 1.1.

Corollary 1.3. *Let R be a Noetherian ring, N a finitely generated R-module and I an ideal of R. Then the following conditions are equivalent*:

(i) $\overline{Q}^*(I,N) = \operatorname{mAss}_R N/IN$.
(ii) *The topologies defined by $\{(I^n)_a^{(N)}\}_{n \geq 0}$ and $\{(I^n)_a^{\langle N \rangle}\}_{n \geq 0}$ are equivalent.*

For any ideal I of R and any R-module N, the ith *local cohomology module of N with respect to I* is defined by

$$H_I^i(N) := \varinjlim \operatorname{Ext}_R^i(R/I^n, N).$$

The reader is referred to [**2**] for basic properties of local cohomology modules. The purpose of Section 4 is to characterize the equivalence between the topologies defined by $(I^n)_a^{(N)}$ and $S((I^n)_a^{(N)})$, $n \geq 1$, in terms of the top local cohomology module $H_I^{\dim N}(N)$. This will generalize the main result of Marti-Farre [**7**] as an extension of the main results of Call [**3,** Corollary 1.4], Call and Sharp [**4**] and Schenzel [**19,** Corollary 4.3].

Theorem 1.4. *If $(R, \mathfrak{m})$ is a local (Noetherian) ring and N a finitely generated quasi-unmixed R-module of dimension d, then $H_I^d(N) = 0$ if and only if there exists a multiplicatively closed subset S of R such that $\mathfrak{m} \cap S \neq \emptyset$, and the topologies induced by $(I^n)_a^{(N)}$ and $S((I^n)_a^{(N)})$, $n \geq 1$, are equivalent.*

The result in Theorem 1.4 is proven in Theorem 4.1. Pursuing this point of view further we show that the support of the $(d-1)$th local cohomology module of a finitely generated R-module N is always finite

($d = \dim N$), which will then be strengthened into a generalized version of a corresponding one by Marley [**6,** Corollaries 2.4 and 2.5] and by Naghipour and Sedghi [**14,** Corollary 3.3].

Theorem 1.5. *Assume that R is a Noetherian ring. Let N be a finitely generated R-module of dimension d and I an ideal of R. Then,* $\mathrm{Supp}(H_I^d(N)) \subseteq \overline{Q}^*(I, N)$. *Moreover, if $(R, \mathfrak{m})$ is local, then*

$$\mathrm{Supp}(H_I^{d-1}(N)) \subseteq \overline{Q}^*(I, N) \cup \{\mathfrak{m}\}.$$

The proof of Theorem 1.5 is given in Corollaries 4.2 and 4.3.

Throughout the paper, all rings are commutative, with identity, unless otherwise specified. We shall use R to denote such a ring, I an ideal of R and N a non-zero module over R. If $(R, \mathfrak{m})$ is a Noetherian local ring and N a finitely generated R-module, then $\widehat{R}$ (respectively, $\widehat{N}$) denotes the completion of R (respectively, N) with respect to the $\mathfrak{m}$-adic topology. Then, N is said to be *quasi-unmixed* if, for every $\mathfrak{p} \in \mathrm{mAss}_{\widehat{R}}\, \widehat{N}$, the condition $\dim \widehat{R}/\mathfrak{p} = \dim N$ is satisfied. For any ideal J of R, the radical of J, denoted by $\mathrm{Rad}(J)$, is defined to be the set $\{a \in R : a^n \in J \text{ for some } n \in \mathbb{N}\}$. Moreover, we use $V(J)$ to denote the set of prime ideals of R containing J. Finally, for any R-module L, we shall use $\mathrm{mAss}_R\, L$ to denote the set of minimal elements of $\mathrm{Ass}_R\, L$. For any unexplained notation or terminology, we refer the reader to [**8, 13**].

2. Asymptotic behavior of integral closures of ideals. The purpose of this section is to study the asymptotic behavior of the integral closure of ideals with respect to a finitely generated module N over a Noetherian ring R. More precisely, we show that the sequences

$$\{\mathrm{Ass}_R\, R/(I^n)_a^{(N)}\}_{n\geq 1},$$

$$\{\mathrm{Ass}_R (I^n)_a^{(N)}/(I^{n+1})_a^{(N)}\}_{n\geq 1},$$

$$\{\mathrm{Ass}_R (I^n)_a^{(N)}/((I + \mathrm{Ann}_R\, N)^n)_a\}_{n\geq 1},$$

of associated prime ideals are ultimately constant; and, pursuing this point of view further, we show that $A_a^*(I + \mathrm{Ann}_R\, N) \setminus C_a^*(I, N) \subseteq A_a^*(I, N)$.

Lemma 2.1. *Let R be a ring (not necessarily Noetherian) and N a Noetherian R-module. Then, for any ideal I of R, the following statements hold:*

(i) $I_a + \mathrm{Ann}_R N \subseteq I_a^{(N)}$.
(ii) $0_a^{(N)} = \mathrm{Rad}(\mathrm{Ann}_R N)$.
(iii) $I_a^{(N)}/\mathrm{Ann}_R N = (I + \mathrm{Ann}_R N/\mathrm{Ann}_R N)_a$; *so that* $\mathrm{Rad}(I_a^{(N)}) = \mathrm{Rad}(I + \mathrm{Ann}_R N)$.
(iv) *For any multiplicatively closed subset S of R,* $(S^{-1}I)_a^{(S^{-1}N)} = S^{-1}(I_a^{(N)})$.
(v) $\cap_{n\geq 1}(I^n)_a^{(N)} = \cap\{\mathfrak{p} \in \mathrm{mAss}_R N \mid \mathfrak{p} + I \neq R\}$. *In particular, if R is local, then*

$$\bigcap_{n\geq 1}(I^n)_a^{(N)} = \mathrm{Rad}(\mathrm{Ann}_R N).$$

(vi) $(I_a^{(N)} J_a^{(N)})_a^{(N)} = (IJ)_a^{(N)}$, *where J is a second ideal of R.*

Proof. (i) and (ii) follow from the definition. For the proof of (iii), see [**22,** Remark 1.6]. Statement (iv) follows from (iii) and [**21,** Lemma 2.3]. In order to show (v), use (iii) and [**9,** Lemma 3.11]. Finally, in order to prove (vi), use (iii) and the fact that $(KL)_a = (K_aL_a)_a$ for all ideals K and L of R. □

The next corollary extends McAdam's result [**10,** Lemma 1.4].

Corollary 2.2. *Let N and R be as in Lemma 2.1. Let I be an integrally closed ideal with respect to N. Then, I has a primary decomposition, each primary component of which is integrally closed with respect to N.*

Proof. The result follows from Lemma 2.1 and [**10,** Lemma 1.4]. □

Lemma 2.3. *Let R be a ring (not necessarily Noetherian) and N a Noetherian R-module. If I is an ideal of R such that it is not contained in any of the minimal prime ideals of* $\mathrm{Ann}_R N$, *then*

$$((I^n)_a^{(N)} :_R I^m) = ((I^n)_a^{(N)} :_R (I^m)_a) = ((I^n)_a^{(N)} :_R (I^m)_a^{(N)}) = (I^{n-m})_a^{(N)}$$

for all integers $n \geq m \geq 0$.

Proof. In view of Lemma 2.1, it is sufficient to show that

$$((I^n)_a^{(N)} :_R I^m) \subseteq (I^{n-m})_a^{(N)}.$$

Toward this end, let $x \in R$ be such that $I^m x \subseteq (I^n)_a^{(N)}$. In order to simplify notation we denote the Noetherian ring $R/\operatorname{Ann}_R N$ by $\widetilde{R}$; the natural image of x in $\widetilde{R}$ is denoted by $\widetilde{x}$ and, for each ideal J of R, write $\widetilde{J}$ for the ideal $J + \operatorname{Ann}_R N/\operatorname{Ann}_R N$. Then, by Lemma 2.1,

$$\widetilde{x} \in ((\widetilde{I^n})_a :_{\widetilde{R}} (\widetilde{I})^m).$$

Now, it follows from [**9,** Lemma 11.27] that $x \in (I^{n-m})_a^{(N)}$, as desired. $\square$

We are now ready to state and prove the main results of this section, namely, we prove that the sequences of associated primes $\{\operatorname{Ass}_R R/(I^n)_a^{(N)}\}_{n\geq 1}$, $\{\operatorname{Ass}_R (I^n)_a^{(N)}/(I^{n+1})_a^{(N)}\}_{n\geq 1}$ and $\{\operatorname{Ass}_R (I^n)_a^{(N)}/((I + \operatorname{Ann}_R N)^n)_a\}_{n\geq 1}$ are increasing and eventually become constant.

Theorem 2.4. *Let I denote an ideal of a ring R (not necessarily Noetherian), and let N be a Noetherian R-module. Then, the sequence of associated primes*

$$\operatorname{Ass}_R R/(I^n)_a^{(N)}, \quad n = 1, 2, \ldots,$$

is increasing and ultimately constant. Moreover, if I is not contained in any of the minimal prime ideals of $\operatorname{Ann}_R N$*, then the sequence*

$$\operatorname{Ass}_R (I^n)_a^{(N)}/(I^{n+1})_a^{(N)}, \quad n = 1, 2, \ldots,$$

is also increasing and eventually constant.

Proof. First, assume that $n \geq 1$ is an integer and $\mathfrak{p} \in \operatorname{Spec} R$. In order to simplify notation we will use $\widetilde{R}$ to denote the commutative Noetherian ring $R/\operatorname{Ann}_R N$, and, for each ideal J of R, we will write $\widetilde{J}$ for the ideal $J + \operatorname{Ann}_R N/\operatorname{Ann}_R N$ of $\widetilde{R}$. Then, by Lemma 2.1, it is easy to see that $\mathfrak{p} \in \operatorname{Ass}_R R/(I^n)_a^{(N)}$ if and only if $\widetilde{\mathfrak{p}} \in \operatorname{Ass}_{\widetilde{R}} \widetilde{R}/(\widetilde{I^n})_a$. Hence, it follows that

$$\bigcup_{n\geq 1} \operatorname{Ass}_R R/(I^n)_a^{(N)} = \{\mathfrak{q} \cap R \mid \mathfrak{q} \in A_a^*(\widetilde{I})\}.$$

Since, by Ratliff's theorem **[16]**, $A_a^*(\widetilde{I})$ is finite, it follows that the set $\cup_{n\geq 1} \operatorname{Ass}_R R/(I^n)_a^{(N)}$ is finite. Moreover, since the sequence $\{\operatorname{Ass}_{\widetilde{R}} \widetilde{R}/(\widetilde{I^n})_a\}_{n\geq 1}$ is increasing, it turns out that the sequence $\{\operatorname{Ass}_R R/(I^n)_a^{(N)}\}_{n\geq 1}$ is increasing, and therefore, ultimately constant.

In order to prove the second part, we assume that I is not contained in any of the minimal prime ideals of $\operatorname{Ann}_R N$. Suppose that $\mathfrak{p} \in \operatorname{Ass}_R R/(I^{n+1})_a^{(N)}$. Then, there exists an $x \in R \setminus (I^{n+1})_a^{(N)}$ such that $\mathfrak{p} = ((I^{n+1})_a^{(N)} :_R x)$. Since $I \subseteq \mathfrak{p}$, it follows that

$$Ix \subseteq (I^{n+1})_a^{(N)}.$$

Hence, by Lemma 2.3, $x \in (I^n)_a^{(N)}$, whence we obtain $\mathfrak{p} \in \operatorname{Ass}_R (I^n)_a^{(N)}/(I^{n+1})_a^{(N)}$. Therefore, when I is not contained in any of the minimal prime ideals of $\operatorname{Ann}_R N$, it follows that

$$\operatorname{Ass}_R R/(I^{n+1})_a^{(N)} = \operatorname{Ass}_R (I^n)_a^{(N)}/(I^{n+1})_a^{(N)}$$

for all integers $n \geq 0$. This finally completes the proof. □

Theorem 2.5. *Suppose that R is a Noetherian ring. Let N denote a finitely generated R-module, and let I be an ideal of R such that it is not contained in any of the minimal prime ideals of* $\operatorname{Ann}_R N$. *Then, the sequence*

$$\{\operatorname{Ass}_R(I^n)_a^{(N)}/(I^n)_a\}_{n\geq 1}$$

is increasing and eventually constant.

Proof. Assume first that $\mathfrak{p} \in \operatorname{Ass}_R(I^n)_a^{(N)}/(I^n)_a$ for some integer $n \geq 1$. Without loss of generality, we may assume that $(R, \mathfrak{p})$ is a local ring. There exists an element $x \in (I^n)_a^{(N)}$ such that $\mathfrak{p} = ((I^n)_a :_R x)$. Then, in view of Lemma 2.3, $((I^{n+1})_a :_R Ix)$ is a proper ideal of R and $\mathfrak{p} \subseteq (I^{n+1})_a :_R Ix$. Thus, $\mathfrak{p} = ((I^{n+1})_a :_R Ix)$. Since $Ix \subseteq (I^{n+1})_a^{(N)}$, it follows that $\mathfrak{p} \in \operatorname{Ass}_R(I^{n+1})_a^{(N)}/(I^{n+1})_a$. Therefore, the sequence $\{\operatorname{Ass}_R(I^n)_a^{(N)}/(I^n)_a\}_{n\geq 1}$ is increasing. Due to the fact that

$$\bigcup_{n\geq 1} \operatorname{Ass}_R(I^n)_a^{(N)}/(I^n)_a \subseteq A_a^*(I),$$

and $A_a^*(I)$ is finite, we deduce that it is eventually constant for large n. □

Corollary 2.6. *Let R be a Noetherian ring, N a finitely generated R-module and I an ideal of R such that it is not contained in any of the minimal prime ideals of* $\operatorname{Ann}_R N$. *Then, the sequence*

$$\{\operatorname{Ass}_R (I^n)_a^{(N)}/((I+\operatorname{Ann}_R N)^n)_a\}_{n\geq 1}$$

is increasing and ultimately constant.

Proof. This follows from Theorem 2.5 where I is replaced by $I + \operatorname{Ann}_R N$, since the integral closures with respect to N of their powers are the same. □

Definition 2.7. Suppose that R is a Noetherian ring. Let I be an ideal of R. Let N denote a finitely generated R-module. The *eventual constant values* of the sequences

$$\{\operatorname{Ass}_R R/(I^n)_a^{(N)}\}_{n\geq 1} \quad \text{and} \quad \{\operatorname{Ass}_R (I^n)_a^{(N)}/((I+\operatorname{Ann}_R N)^n)_a\}_{n\geq 1}$$

will be denoted by $A_a^*(I,N)$ and $C_a^*(I,N)$, respectively.

It is easy to see that $A_a^*(I,N)$ and $C_a^*(I,N)$ are stable under localization. Moreover,

$$\operatorname{mAss}_R N/IN \subseteq A_a^*(I,N) \quad \text{and} \quad A_a^*(0,N) = \operatorname{mAss}_R N.$$

Proposition 2.8. *Let R be a Noetherian ring, I an ideal of R and N a finitely generated R-module. Then, $A_a^*(I+\operatorname{Ann}_R N)\setminus C_a^*(I,N) \subseteq A_a^*(I,N)$.*

Proof. Let $\mathfrak{p} \in A_a^*(I+\operatorname{Ann}_R N)\setminus C_a^*(I,N)$. Since $A_a^*(I+\operatorname{Ann}_R N)$, $C_a^*(I,N)$ and $A_a^*(I,N)$ behave well under localization, we may assume that $(R,\mathfrak{p})$ is a local ring. Let $\mathfrak{p} = (((I+\operatorname{Ann}_R N)^n)_a :_R x)$ for some $x \in R$ and for large n. Since $\mathfrak{p} \subseteq ((I^n)_a^{(N)} :_R x)$ and $\mathfrak{p} \notin C_a^*(I,N)$, it follows that $\mathfrak{p} = ((I^n)_a^{(N)} :_R x)$. Hence, $\mathfrak{p} \in A_a^*(I,N)$, as required. □

Remark 2.9. Let R be a Noetherian ring, N a finitely generated R-module and I an ideal of R such that it is not contained in any of the minimal prime ideals of $\operatorname{Ann}_R N$ and $(I^nR_{\mathfrak{p}})_a^{(N_{\mathfrak{p}})} = ((IR_{\mathfrak{p}} + \operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}})^n)_a$, for large n and for all $\mathfrak{p} \in C_a^*(I,N)$. Then, $(I^n)_a^{(N)} = ((I+\operatorname{Ann}_R N)^n)_a$ for large n. For this, let $C_a^*(I,N) = \{\mathfrak{p}_1,\dots,\mathfrak{p}_t\}$.

Choose an integer k such that

$$C^*_a(I,N) = \operatorname{Ass}_R (I^n)^{(N)}_a/((I+\operatorname{Ann}_R N)^n)_a$$

for all $n \geq k$, and let $(I^n R_{\mathfrak{p}_i})^{(N_{\mathfrak{p}_i})}_a = ((IR_{\mathfrak{p}_i} + \operatorname{Ann}_{R_{\mathfrak{p}_i}} N_{\mathfrak{p}_i})^n)_a$ for all $i = 1, \ldots, t$. Now, if $(I^n)^{(N)}_a \neq ((I+\operatorname{Ann}_R N)^n)_a$, then there exists an $x \in (I^n)^{(N)}_a \setminus ((I+\operatorname{Ann}_R N)^n)_a$, and thus, $((I+\operatorname{Ann}_R N)^n)_a :_R x$ is a proper ideal of R. As R is Noetherian, there is an $r \in R$ such that $\mathfrak{p} := ((I+\operatorname{Ann}_R N)^n)_a :_R rx$ is a prime ideal of R, and hence, $\mathfrak{p} \in C^*_a(I,N)$. Thus, $(I^n R_{\mathfrak{p}})^{(N_{\mathfrak{p}})}_a = ((IR_{\mathfrak{p}} + \operatorname{Ann}_{R_{\mathfrak{p}}} N_{\mathfrak{p}})^n)_a$; therefore, $x/1 \in ((I+\operatorname{Ann}_R N)^n)_a R_{\mathfrak{p}}$. Hence, there is an $s \in R \setminus \mathfrak{p}$ such that $sx \in ((I+\operatorname{Ann}_R N)^n)_a$, that is, $s \in ((I+\operatorname{Ann}_R N)^n)_a :_R x \subseteq \mathfrak{p}$, which is a contradiction.

3. Quintasymptotic primes and ideal topologies. In this section, we study the equivalence of the topologies defined by $(I^n)^{(N)}_a$, $S((I^n)^{(N)}_a)$, $S(((I+\operatorname{Ann}_R N)^n)_a)$ and $S((I+\operatorname{Ann}_R N)^n)$, $n \geq 1$, by using the quintasymptotic prime ideals of I with respect to N. The main results are Proposition 3.10 and Theorem 3.11. As a consequence, we show that $\overline{Q}^*(I,N) = \operatorname{mAss}_R N/IN$ if and only if the topologies $(I^n)^{(N)}_a$ and $(I^n)^{\langle N \rangle}_a$, $n \geq 1$, are equivalent. We begin with the following elementary result.

Lemma 3.1. *Let R be a Noetherian ring and N a finitely generated R-module. Let T be a faithfully flat Noetherian ring extension of R. Then, for any ideal I of R,*

$$(IT)^{(N\otimes_R T)}_a \cap R = I^{(N)}_a.$$

Proof. Let $x \in (IT)^{(N\otimes_R T)}_a \cap R$. Then, in view of [**22,** Corollary 1.5], there is an integer $n \geq 1$ such that

$$(IT+Tx)^{n+1}(N\otimes_R T) = IT(IT+Tx)^n(N\otimes_R T).$$

Hence,

$$(I+Rx)^{n+1}(N\otimes_R T) = I(I+Rx)^n(N\otimes_R T).$$

Therefore,

$$(I+Rx)^{n+1}N\otimes_R T = I(I+Rx)^n N\otimes_R T.$$

Now, by faithful flatness, we deduce that $(I+Rx)^{n+1}N = I(I+Rx)^nN$; hence, $x \in I_a^{(N)}$, by [**22,** Corollary 1.5]. Therefore, the conclusion follows since the opposite inclusion is clear by the faithful flatness of T over R. $\square$

Remark 3.2. Before continuing, let us fix some notation employed by Schenzel [**20**] and McAdam [**10**], respectively, in the case $N = R$.

For any multiplicatively closed subset S of R and for each ideal J of R, we use $S(J)$ to denote the ideal $\cup_{s\in S}(J :_R s)$. Note that

$$\operatorname{Ass}_R R/S(J) = \{\mathfrak{p} \in \operatorname{Ass}_R R/J : \mathfrak{p} \cap S = \emptyset\}.$$

In the case where N is a finitely generated R-module and $S = R\backslash \cup\{\mathfrak{p} \in \operatorname{mAss}_R N/JN\}$, we use $J_a^{\langle N\rangle}$ to denote the ideal $S(J_a^{(N)})$, in particular, for every integer $k \geq 1$ and every prime ideal $\mathfrak{p}$ of R, we have

$$(\mathfrak{p}^k)_a^{\langle N\rangle} = \bigcup_{s\in R\backslash\mathfrak{p}} ((\mathfrak{p}^k)_a^{(N)} :_R s).$$

Proposition 3.3. *Let R be a Noetherian ring, and let N be a finitely generated R-module.*

(i) *If $(R, \mathfrak{m})$ is local and $\mathfrak{p} \in \operatorname{mAss}_R N$, then there exists an element $x \in R$ not in $\mathfrak{p}$ such that, for every ideal J of R with $\mathfrak{m}$ minimal over $J + \mathfrak{p}$, $x \in J + \operatorname{Ann}_R N$ or $\mathfrak{m} \in \operatorname{Ass}_R R/J + \operatorname{Ann}_R N$.*

(ii) *If $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathfrak{q} \in \operatorname{mAss}_R N$ with $\mathfrak{q} \subseteq \mathfrak{p}$, then there is an integer $k \geq 1$ such that $\mathfrak{p} \in \operatorname{Ass}_R R/J + \operatorname{Ann}_R N$ for any ideal J of R with $J \subseteq (\mathfrak{p}^k)_a^{\langle N\rangle}$ and $\mathfrak{p} \in \operatorname{mAss}_R R/J + \mathfrak{q}$.*

Proof. In order to show (i), let

$$\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t = \operatorname{Ann}_R N$$

be an irredundant primary decomposition of the ideal $\operatorname{Ann}_R N$ with $\mathfrak{q}_1$ $\mathfrak{p}$-primary. (Note that $\mathfrak{p} \in \operatorname{Ass}_R R/\operatorname{Ann}_R N$.) It follows from $\mathfrak{p} \in \operatorname{mAss}_R N$ that $\cap_{i=2}^t \mathfrak{q}_i \nsubseteq \mathfrak{p}$. Hence, there exists an element $x \in \cap_{i=2}^t \mathfrak{q}_i$ such that $x \notin \mathfrak{p}$. Now, let J be any ideal of R such that $\operatorname{Rad}(J + \mathfrak{p}) = \mathfrak{m}$ and $\mathfrak{m} \notin \operatorname{Ass}_R R/J + \operatorname{Ann}_R N$. It is sufficient to show that $x \in J + \operatorname{Ann}_R N$. Toward this end, let

$$Q_1 \cap \cdots \cap Q_l = J + \operatorname{Ann}_R N$$

be an irredundant primary decomposition of the ideal $J + \operatorname{Ann}_R N$, with Q_i the $\mathfrak{p}_i$-primary ideal, for all $i = 1, \dots, l$. Then, $\mathfrak{m} \neq \mathfrak{p}_i$, for all $i = 1, \dots, l$, and thus, it follows from $\operatorname{Rad}(J + \mathfrak{p}) = \mathfrak{m}$ that $\mathfrak{p} \nsubseteq \mathfrak{p}_i$. Hence, $\mathfrak{p}^v \nsubseteq \mathfrak{p}_i$, where $v \geq 1$ is an integer such that $\mathfrak{p}^v \subseteq \mathfrak{q}_1$. Therefore, since $x\mathfrak{q}_1 \subseteq \operatorname{Ann}_R N$, it follows that $x\mathfrak{p}^v \subseteq Q_i$, for all $i = 1, \dots, l$. Consequently, $x \in Q_1 \cap \cdots \cap Q_l$ so that $x \in J + \operatorname{Ann}_R N$, as required.

In order to prove (ii), without loss of generality, we may assume that $(R, \mathfrak{p})$ is local. Then, $(\mathfrak{p}^k)_a^{\langle N \rangle} = (\mathfrak{p}^k)_a^{(N)}$. Now, let x be as in (i). Then, in view of Lemma 2.1 (v), there exists an integer $k \geq 1$ such that $x \notin (\mathfrak{p}^k)_a^{(N)}$. Therefore, if J is an ideal of R such that $J \subseteq (\mathfrak{p}^k)_a^{\langle N \rangle}$ and $\mathfrak{p} \in \operatorname{mAss}_R R/J + \mathfrak{q}$, then $x \notin J + \operatorname{Ann}_R N$, and thus, it follows from (i) that $\mathfrak{p} \in \operatorname{Ass}_R R/J + \operatorname{Ann}_R N$. □

Proposition 3.4. *Let I be an ideal of a Noetherian ring R and S a multiplicatively closed subset of R. Then, for any finitely generated R-module N,*

$$\bigcap_{n \geq 1} S((I^n)_a^{(N)}) = \bigcap \{\mathfrak{p} \in \operatorname{mAss}_R N \mid (I + \mathfrak{p}) \cap S = \emptyset\}.$$

Proof. Let $x \in \cap_{n \geq 1} S((I^n)_a^{(N)})$. Then, for all $n \geq 1$, there exists an $s \in S$ such that $sx \in (I^n)_a^{(N)}$. Now, let $\mathfrak{p} \in \operatorname{mAss}_R N$ be such that $(\mathfrak{p} + I) \cap S = \emptyset$. Then, it follows from Lemma 2.1 (v) that $x \in \mathfrak{p}$.

Conversely, suppose that $x \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{mAss}_R N$ with $(\mathfrak{p} + I) \cap S = \emptyset$. Then, by virtue of Lemma 2.1 (v), $x/1 \in (S^{-1}I^n)_a^{(S^{-1}N)}$ for all $n \geq 1$. Hence, in view of Lemma 2.1 (iv), $x/1 \in S^{-1}((I^n)_a^{(N)})$, and thus, $sx \in (I^n)_a^{(N)}$ for some $s \in S$. Consequently, we have $x \in S((I^n)_a^{(N)})$, as required. □

Theorem 3.5. *Let R be a Noetherian ring, and let N be a finitely generated R-module. Let I and J be ideals of R. Then*:

$$\bigcap_{n \geq 1} ((I^n)_a^{(N)} :_R \langle J \rangle) = \bigcap \{\mathfrak{p} \in \operatorname{mAss}_R N \mid J \nsubseteq \operatorname{Rad}(I + \mathfrak{p})\}.$$

Proof. In view of Theorem 2.4 the set $A_a^*(I, N) := \cup_{n \geq 1} \operatorname{Ass}_R R/(I^n)_a^{(N)}$ is finite. Let $A_a^*(I, N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$. Let r be an integer such that $0 \leq r \leq t$ and $J \nsubseteq \cup_{i=1}^r \mathfrak{p}_i$; however, $J \subseteq \cap_{i=r+1}^t \mathfrak{p}_j$. Then, there exists an element $s \in J$ such that $s \notin \cup_{i=1}^{r}\mathfrak{p}_i$. Suppose that $S = \{s^i \mid i \geq 0\}$. Then, it is easily seen that

$$((I^n)_a^{(N)} :_R \langle J\rangle) = S((I^n)_a^{(N)})$$

for each integer $n \geq 1$. Now, in view of Proposition 3.4, it is sufficient to show that $J \subseteq \mathrm{Rad}(I+\mathfrak{p})$ if and only if $s \in \mathrm{Rad}(I+\mathfrak{p})$ for each $\mathfrak{p} \in \mathrm{mAss}_R\, N$. In order to do so, since $s \in J$, one direction is clear. For the other direction, let $\mathfrak{q}$ be a minimal prime ideal over $I+\mathfrak{p}$. Then, since $s \in \mathrm{Rad}(I+\mathfrak{p})$ and $I+\mathfrak{p} \subseteq \mathfrak{q}$, we have $s \in \mathfrak{q}$, and hence, in view of the choice of s, it suffices to show that $\mathfrak{q} \in A_a^*(I,N)$. By virtue of Lemma 2.1, we may assume that R is local with maximal ideal $\mathfrak{q}$. Let x be as in Proposition 3.3. Then, by Lemma 2.1, there is an integer $n \geq 1$ such that $x \notin (I^n)_a^{(N)}$. Now, it is easy to see that $\mathfrak{q}$ is minimal over $(I^n)_a^{(N)} + \mathfrak{p}$. Therefore, it follows from Proposition 3.3 that $\mathfrak{q} \in \mathrm{Ass}_R\, R/(I^n)_a^{(N)}$, and thus, $\mathfrak{q} \in A_a^*(I,N)$, as required. □

Corollary 3.6. *Let R be a Noetherian ring and I an ideal of R. Let N be a finitely generated R-module and $\mathfrak{p} \in \mathrm{mAss}_R\, N$. Then,* $\mathrm{mAss}_R\, R/(I+\mathfrak{p}) \subseteq A_a^*(I,N)$.

Proof. The assertion follows from the last argument in the proof of Theorem 3.5. □

Corollary 3.7. *Let $(R,\mathfrak{m})$ be a local (Noetherian) ring, and let N be a finitely generated R-module. Then, for any proper ideal I of R,*

$$\bigcap_{n\geq 1}((I^n)_a^{(N)} :_R \langle\mathfrak{m}\rangle) = \bigcap\{\mathfrak{p} \in \mathrm{mAss}_R\, N \mid \mathrm{Rad}(I+\mathfrak{p}) \subsetneqq \mathfrak{m}\}.$$

Proof. The assertion follows from Theorem 3.5. □

Proposition 3.8. *Let $(R,\mathfrak{m})$ be a local (Noetherian) ring, I a proper ideal of R and N a finitely generated R-module. Then, the following conditions are equivalent*:

(i) *for all* $\mathfrak{p} \in \mathrm{mAss}_R\, N, \mathrm{Rad}(I+\mathfrak{p}) \neq \mathfrak{m}$.
(ii) $\cap_{n\geq 1}(((I+\mathrm{Ann}_R\, N)^n)_a :_R \langle\mathfrak{m}\rangle) \subseteq \mathrm{Rad}(\mathrm{Ann}_R\, N)$.
(iii) $\cap_{n\geq 1}((I+\mathrm{Ann}_R\, N)^n :_R \langle\mathfrak{m}\rangle) \subseteq \mathrm{Rad}(\mathrm{Ann}_R\, N)$.
(iv) $\cap_{n\geq 1}((I^n)_a^{(N)} :_R \langle\mathfrak{m}\rangle) = \mathrm{Rad}(\mathrm{Ann}_R\, N)$.

Proof.

(i) $\Rightarrow$ (ii). In view of Corollary 3.7,

$$\bigcap_{n\geq 1}((I^n)_a^{(N)} :_R \langle \mathfrak{m}\rangle) = \mathrm{Rad}(\mathrm{Ann}_R N).$$

Hence, as

$$\bigcap_{n\geq 1}(((I+\mathrm{Ann}_R N)^n)_a :_R \langle \mathfrak{m}\rangle) \subseteq \bigcap_{n\geq 1}((I^n)_a^{(N)} :_R \langle \mathfrak{m}\rangle),$$

it follows that (ii) holds.

(ii) $\Rightarrow$ (iii). Follows directly.

(iii) $\Rightarrow$ (iv). Suppose the contrary, that is, (iv) is not true. Then,

$$\mathrm{Rad}(\mathrm{Ann}_R N) \subsetneqq \bigcap_{n\geq 1}((I^n)_a^{(N)} :_R \langle \mathfrak{m}\rangle).$$

Hence, according to Corollary 3.7, there exists a $\mathfrak{p} \in \mathrm{mAss}_R N$ such that $\mathrm{Rad}(I+\mathfrak{p}) = \mathfrak{m}$. Moreover, applying the assumption, it is easily seen that $\mathrm{Rad}(I+\mathrm{Ann}_R N) \neq \mathfrak{m}$. Therefore,

$$\mathrm{Rad}((I+\mathrm{Ann}_R N)^n :_R \langle \mathfrak{m}\rangle + \mathfrak{p}) = \mathfrak{m},$$

for each integer $n \geq 1$.

Now, let x be as in Proposition 3.3. Since $\mathfrak{m} \notin \mathrm{Ass}_R R/((I+\mathrm{Ann}_R N)^n :_R \langle \mathfrak{m}\rangle)$, it follows that

$$x \in \bigcap_{n\geq 1}((I+\mathrm{Ann}_R N)^n :_R \langle \mathfrak{m}\rangle).$$

Thus, $x \in \mathrm{Rad}(\mathrm{Ann}_R N)$, i.e., $x \in \mathfrak{p}$, a contradiction.

(iv) $\Rightarrow$ (i). Follows from Corollary 3.7. □

Theorem 3.9. *Let $(R,\mathfrak{m})$ be a local (Noetherian) ring, let N be a finitely generated R-module and let I be an ideal of R. Then, the following conditions are equivalent*:

(i) $\cap_{n\geq 1}((I^n\widehat{R})_a^{(\widehat{N})} :_{\widehat{R}} \langle \mathfrak{m}\widehat{R}\rangle) = \mathrm{Rad}(\mathrm{Ann}_{\widehat{R}} \widehat{N})$.

(ii) *For all integers $n \geq 1$, there exists an integer $k \geq 1$ such that*

$$(I+\mathrm{Ann}_R N)^k :_R \langle \mathfrak{m}\rangle \subseteq (\mathfrak{m}^n)_a^{(N)}.$$

(iii) *For all integers $n \geq 1$, there exists an integer $k \geq 1$ such that*

$$((I + \operatorname{Ann}_R N)^k)_a :_R \langle \mathfrak{m} \rangle \subseteq (\mathfrak{m}^n)_a^{(N)}.$$

(iv) *For all integers $n \geq 1$, there exists an integer $k \geq 1$ such that*

$$(I^k)_a^{(N)} :_R \langle \mathfrak{m} \rangle \subseteq (\mathfrak{m}^n)_a^{(N)}.$$

Proof. Without loss of generality, we may assume that $(R, \mathfrak{m})$ is a complete local ring as follows by virtue of the faithful flatness of $\widehat{R}$. Now, suppose that (i) is satisfied. Then,

$$\bigcap_{n \geq 1} ((I^n)_a^{(N)} :_R \langle \mathfrak{m} \rangle / \operatorname{Rad}(\operatorname{Ann}_A N)) = 0.$$

Since $R/\operatorname{Ann}_R N$ is a complete local ring, Chevalley's theorem [**13,** Theorem 30.1] states that, for all $n \geq 1$, there exists an integer $k \geq 1$ such that

$$((I^k)_a^{(N)} :_R \langle \mathfrak{m} \rangle)/\operatorname{Rad}(\operatorname{Ann}_A N) \subseteq (\mathfrak{m}/\operatorname{Rad}(\operatorname{Ann}_R N))^n.$$

Therefore,

$$((I^k)_a^{(N)} :_R \langle \mathfrak{m} \rangle) \subseteq \mathfrak{m}^n + \operatorname{Rad}(\operatorname{Ann}_R N) \subseteq (\mathfrak{m}^n)_a^{(N)},$$

and thus, statement (iv) is shown to be true.

Conclusions (iv) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) follow directly. Thus, in order to complete the proof, it is enough to show that (ii) $\Rightarrow$ (i). Toward this end, suppose that, for all $n \geq 1$, there exists an integer $k \geq 1$ such that

$$(I + \operatorname{Ann}_R N)^k :_R \langle \mathfrak{m} \rangle \subseteq (\mathfrak{m}^n)_a^{(N)}.$$

Then, in view of Lemma 2.1, we have

$$\bigcap_{k \geq 1} ((I + \operatorname{Ann}_R N)^k :_R \langle \mathfrak{m} \rangle) \subseteq \operatorname{Rad}(\operatorname{Ann}_R N).$$

Now, use Proposition 3.8 to complete the proof. □

We are now ready to prove the first main result of this section. In fact, there is a characterization of the quintasymptotic prime ideals of I with respect to N, which is a generalization of [**10,** Proposition 3.5].

Proposition 3.10. *Let R be a Noetherian ring, and let N be a finitely generated R-module. Let $I \subseteq \mathfrak{p}$ be ideals of R such that $\mathfrak{p} \in \operatorname{Supp}(N)$. Then, the following conditions are equivalent*:

(i) $\mathfrak{p} \in \overline{Q}^*(I, N)$.

(ii) *There exists an integer* $k \geq 0$ *such that* $\mathfrak{p} \in \operatorname{Ass}_R R/J + \operatorname{Ann}_R N$ *for any ideal* J *of* R *with* $I \subseteq \operatorname{Rad}(J)$ *and* $J \subseteq (\mathfrak{p}^k)_a^{\langle N \rangle}$.

(iii) *There exists an integer* $k \geq 0$ *such that, for all integers* $m \geq 0$,

$$(I + \operatorname{Ann}_R N)^m :_R \langle \mathfrak{p} \rangle \nsubseteq (\mathfrak{p}^k)_a^{\langle N \rangle}.$$

(iv) *There exists an integer* $k \geq 0$ *such that, for all integers* $m \geq 0$,

$$((I + \operatorname{Ann}_R N)^m)_a :_R \langle \mathfrak{p} \rangle \nsubseteq (\mathfrak{p}^k)_a^{\langle N \rangle}.$$

(v) *There exists an integer* $k \geq 0$ *such that, for all integers* $m \geq 0$,

$$(I^m)_a^{(N)} :_R \langle \mathfrak{p} \rangle \nsubseteq (\mathfrak{p}^k)_a^{\langle N \rangle}.$$

Proof.

(i) $\Rightarrow$ (ii). Let $\mathfrak{p} \in \overline{Q}^*(I, N)$. Then, there exists a prime ideal $\mathfrak{q} \in \operatorname{mAss}_{\widehat{R}_\mathfrak{p}} \widehat{N}_\mathfrak{p}$ such that $\operatorname{Rad}(I\widehat{R}_\mathfrak{p} + \mathfrak{q}) = \mathfrak{p}\widehat{R}_\mathfrak{p}$. Now, let k be as in Proposition 3.3 (ii), applied to $\mathfrak{q} \in \operatorname{mAss}_{\widehat{R}_\mathfrak{p}} \widehat{N}_\mathfrak{p}$. Let J be any ideal of R such that $I \subseteq \operatorname{Rad}(J)$ and $J \subseteq (\mathfrak{p}^k)_a^{\langle N \rangle}$. Then, $I\widehat{R}_\mathfrak{p} \subseteq \operatorname{Rad}(J\widehat{R}_\mathfrak{p})$ and $J\widehat{R}_\mathfrak{p} \subseteq (\mathfrak{p}^k \widehat{R}_\mathfrak{p})_a^{(\widehat{N}_\mathfrak{p})}$ by virtue of Lemma 3.1. Since $\mathfrak{p}\widehat{R}_\mathfrak{p}$ is the maximal ideal of $\widehat{R}_\mathfrak{p}$, it follows that $(\mathfrak{p}\widehat{R}_\mathfrak{p})_a^{(\widehat{N}_\mathfrak{p})} = \mathfrak{p}\widehat{R}_\mathfrak{p}$, and thus, $J\widehat{R}_\mathfrak{p}$ is a proper ideal of $\widehat{R}_\mathfrak{p}$. Thus, $\operatorname{Rad}(J\widehat{R}_\mathfrak{p} + \mathfrak{q}) = \mathfrak{p}\widehat{R}_\mathfrak{p}$. Hence, Proposition 3.3 shows that $\mathfrak{p}\widehat{R}_\mathfrak{p} \in \operatorname{Ass}_{\widehat{R}_\mathfrak{p}} \widehat{R}_\mathfrak{p}/J\widehat{R}_\mathfrak{p} + \operatorname{Ann}_{\widehat{R}_\mathfrak{p}} \widehat{N}_\mathfrak{p}$, and thus, by [**8,** Theorem 23.2], we have $\mathfrak{p} \in \operatorname{Ass}_R R/J + \operatorname{Ann}_R N$, that is, (ii) holds.

(ii) $\Rightarrow$ (iii). This follows easily from the fact that

$$\mathfrak{p} \notin \operatorname{Ass}_R R/((I + \operatorname{Ann}_R N)^n :_R \langle \mathfrak{p} \rangle),$$

for all integers $n \geq 0$.

Conclusions (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) follow directly.

(v) $\Rightarrow$ (i). Toward this end, suppose that there is an integer $k \geq 0$ such that $((I^m)_a^{(N)} :_R \langle \mathfrak{p} \rangle) \nsubseteq (\mathfrak{p}^k)_a^{\langle N \rangle}$ for all integers $m \geq 0$. Then, by Lemma 3.1,

$$((I^m \widehat{R}_\mathfrak{p})_a^{(\widehat{N}_\mathfrak{p})} :_{\widehat{R}_\mathfrak{p}} \langle \mathfrak{p}\widehat{R}_\mathfrak{p} \rangle) \nsubseteq (\mathfrak{p}^k \widehat{R}_\mathfrak{p})_a^{(\widehat{N}_\mathfrak{p})}.$$

Hence, in view of Proposition 3.8, there is a $\mathfrak{q} \in \operatorname{mAss}_{\widehat{R}_\mathfrak{p}} \widehat{N}_\mathfrak{p}$ such that $\operatorname{Rad}(I\widehat{R}_\mathfrak{p} + \mathfrak{q}) = \mathfrak{p}\widehat{R}_\mathfrak{p}$, and thus, $\mathfrak{p} \in \overline{Q}^*(I, N)$, as required. $\square$

We are now ready to state and prove the second main theorem of this section, which is a characterization of the equivalence between the topologies $\{(I^n)_a^{(N)}\}_{n\geq 1}$, $\{S((I^n)_a^{(N)})\}_{n\geq 1}$, $\{S(((I+\operatorname{Ann}_R N)^n)_a)\}_{n\geq 1}$ and $\{S((I+\operatorname{Ann}_R N)^n)\}_{n\geq 1}$ in terms of the quintasymptotic primes of I with respect to N. This will generalize the main result of McAdam [**10**].

Theorem 3.11. *Let R be a Noetherian ring, N a finitely generated R-module and I an ideal of R. Then, for any multiplicatively closed subset S of R, the following are equivalent*:

(i) $S \subseteq R \setminus \cup\{\mathfrak{p} \in \overline{Q}^*(I,N)\}$.

(ii) *The topologies defined by $\{S((I^n)_a^{(N)})\}_{n\geq 0}$ and $\{(I^n)_a^{(N)}\}_{n\geq 0}$ are equivalent.*

(iii) *The topology defined by $\{S(((I+\operatorname{Ann}_R N)^n)_a)\}_{n\geq 0}$ is finer than the topology defined by $\{(I^n)_a^{(N)}\}_{n\geq 0}$.*

(iv) *The topology defined by $\{S((I+\operatorname{Ann}_R N)^n)\}_{n\geq 0}$ is finer than the topology defined by $\{(I^n)_a^{(N)}\}_{n\geq 0}$.*

(v) *For all $\mathfrak{p} \in \operatorname{Supp}(N) \cap V(I)$, the topology defined by*

$$\{S((I^n)_a^{(N)})\}_{n\geq 0}$$

is finer than the topology defined by $\{(\mathfrak{p}^n)_a^{\langle N\rangle}\}_{n\geq 0}$.

(vi) *For all $\mathfrak{p} \in \operatorname{Supp}(N) \cap V(I)$, the topology defined by*

$$\{S(((I+\operatorname{Ann}_R N)^n)_a)\}_{n\geq 0}$$

is finer than the topology defined by $\{(\mathfrak{p}^n)_a^{\langle N\rangle}\}_{n\geq 0}$.

(vii) *For all $\mathfrak{p} \in \operatorname{Supp}(N) \cap V(I)$, the topology defined by*

$$\{S((I+\operatorname{Ann}_R N)^n)\}_{n\geq 0}$$

is finer than the topology defined by $\{(\mathfrak{p}^n)_a^{\langle N\rangle}\}_{n\geq 0}$.

Proof.

(i) $\Rightarrow$ (ii). Let $\mathfrak{p} \in \operatorname{Spec} R$ with $I+\operatorname{Ann}_R N \subseteq \mathfrak{p}$, and let $l \geq 1$. We first show that there exists an integer $m \geq 1$ such that $S((I^m)_a^{(N)}) \subseteq (\mathfrak{p}^l)_a^{\langle N\rangle}$. In order to do so, let S' be the natural image of S in $R_\mathfrak{p}$. Since $\overline{Q}^*(I,N)$ behaves well under localization, we have $S' \subseteq R_\mathfrak{p} \setminus \cup\{\mathfrak{q} \in \overline{Q}^*(IR_\mathfrak{p}, N_\mathfrak{p})\}$. Moreover, it is easy to see that $S'((I^m R_\mathfrak{p})_a^{(N_\mathfrak{p})}) \subseteq (\mathfrak{p}^l R_\mathfrak{p})_a^{\langle N_\mathfrak{p}\rangle}$ implies $S((I^n)_a^{(N)}) \subseteq (\mathfrak{p}^n)_a^{\langle N\rangle}$. Therefore, we may assume that R is local with maximal ideal $\mathfrak{p}$. Then, $(\mathfrak{p}^n)_a^{\langle N\rangle} = (\mathfrak{p}^n)_a^{(N)}$. From Lemma 3.1 and

[**1,** Proposition 3.8], we may assume, in addition, that R is complete, whence, in view of [**1,** Lemma 3.5], for any $\mathfrak{q} \in \operatorname{mAss}_R N$, S is disjoint from $I + \mathfrak{q}$. Therefore, by Proposition 3.4, we have $\cap_{n\geq 1} S((I^n)_a^{(N)}) = \operatorname{Rad}(\operatorname{Ann}_R N)$. Consequently,

$$\bigcap_{n\geq 1} S((I^n)_a^{(N)})/\operatorname{Rad}(\operatorname{Ann}_R N) = 0.$$

Since the ring $R/\operatorname{Rad}(\operatorname{Ann}_R N)$ is complete, Chevalley's theorem [**13,** Theorem 30.1] implies the existence of an integer $m \geq 1$ such that

$$S((I^m)_a^{(N)})/\operatorname{Rad}(\operatorname{Ann}_R N) \subseteq (\mathfrak{p}/\operatorname{Rad}(\operatorname{Ann}_R N))^l.$$

Hence,

$$S((I^m)_a^{(N)}) \subseteq \mathfrak{p}^l + \operatorname{Rad}(\operatorname{Ann}_R N) \subseteq (\mathfrak{p}^l)_a^{(N)}.$$

Now, in view of Corollary 2.2, we can consider $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ a minimal primary decomposition of $(I^l)_a^{(N)}$ where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary and integrally closed with respect to N for every $i = 1, \ldots, n$. Then, there exists an integer l_i such that $\mathfrak{p}_i^{l_i} \subseteq \mathfrak{q}_i$ for $i = 1, \ldots, n$, and moreover, for some m_i, we have $S((I^{m_i})_a^{(N)}) \subseteq (\mathfrak{p}_i^{l_i})_a^{\langle N\rangle}$. Let $m = \max\{m_1, \ldots, m_n\}$. Then, we deduce that $S((I^m)_a^{(N)}) \subseteq (\mathfrak{p}_i^{l_i})_a^{\langle N\rangle}$ for each $1 \leq i \leq n$. On the other hand, we have

$$(\mathfrak{p}_i^{l_i})_a^{\langle N\rangle} \subseteq \bigcup_{s\in R\setminus \mathfrak{p}_i} ((\mathfrak{q}_i)_a^{(N)} :_R s) = \mathfrak{q}_i,$$

and therefore, $S((I^m)_a^{(N)}) \subseteq \cap_{i=1}^n \mathfrak{q}_i$. This completes the proof of (ii).

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow directly.

(iv) $\Rightarrow$ (v). It is sufficient to show that

$$S \subseteq R \setminus \bigcup\{\mathfrak{p} \in \overline{Q}^*(I, N)\}.$$

In order to do so, let $\mathfrak{p} \in \overline{Q}^*(I, N)$. Then, by Proposition 3.10, there exists an integer $k \geq 0$ such that $((I + \operatorname{Ann}_R N)^m :_R \langle \mathfrak{p}\rangle) \nsubseteq (\mathfrak{p}^k)_a^{\langle N\rangle}$ for all integers $m \geq 0$. On the other hand, by assumption, there is an integer $l \geq 0$ such that $S((I + \operatorname{Ann}_R N)^l) \subseteq (I^k)_a^{(N)}$. Therefore,

$$(I + \operatorname{Ann}_R N)^l :_R \langle \mathfrak{p}\rangle \nsubseteq S((I + \operatorname{Ann}_R N)^l).$$

Then, it is readily seen that $\mathfrak{p} \cap S = \emptyset$, as required.

Conclusions (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) follow directly. Finally, an argument similar to that used in the proof of the implication (iv) $\Rightarrow$ (v) shows that (vii) $\Rightarrow$ (ii) holds. $\square$

An immediate consequence of Theorem 3.11 is the following corollary.

Corollary 3.12. *Let R be a Noetherian ring, N a finitely generated R-module and I an ideal of R. Then the following conditions are equivalent*:

(i) $\overline{Q}^*(I,N) = \mathrm{mAss}_R\, N/IN$.

(ii) *The topologies defined by $\{(I^n)_a^{(N)}\}_{n\geq 0}$ and $\{(I^n)_a^{\langle N\rangle}\}_{n\geq 0}$ are equivalent.*

Proof. Let $S=R\backslash\cup\{\mathfrak{p}\in\mathrm{mAss}_R\, N/IN\}$. Then, $S((I^n)_a^{(N)})=(I^n)_a^{\langle N\rangle}$. Now, if

$$\overline{Q}^*(I,N) = \mathrm{mAss}_R\, N/IN,$$

then

$$S = R\setminus\cup\{\mathfrak{p}\in\overline{Q}^*(I,N)\}.$$

Hence, Theorem 3.11 implies that the topologies defined by $\{(I^n)_a^{(N)}\}_{n\geq 0}$ and $\{(I^n)_a^{\langle N\rangle}\}_{n\geq 0}$ are equivalent. Conversely, if these topologies are equivalent, then, it follows from Theorem 3.11 that $S\subseteq R\setminus\cup\{\mathfrak{p}\in\overline{Q}^*(I,N)\}$, and thus, $\overline{Q}^*(I,N)\subseteq\mathrm{mAss}_R\, N/IN$. On the other hand, using [**1,** Lemma 3.5], it is easily seen that $\mathrm{mAss}_R\, N/IN\subseteq\overline{Q}^*(I,N)$, and thus, $\overline{Q}^*(I,N)=\mathrm{mAss}_R\, N/IN$. This completes the proof. $\square$

4. Local cohomology and ideal topologies. The purpose of this section is to establish equivalence between the topologies defined by $\{(I^n)_a^{(N)}\}_{n\geq 1}$ and $\{S((I^n)_a^{(N)})\}_{n\geq 1}$ in terms of the vanishing of the top local cohomology module $H_I^{\dim N}(N)$. This will generalize the main result of Marti-Farre [**7**], as an extension of the main results of [**3,** Corollary 1.4], [**4**] and [**19,** Corollary 4.3].

Theorem 4.1. *Let $(R,\mathfrak{m})$ be a local (Noetherian) ring, N a finitely generated R-module of dimension d and I an ideal of R. Consider the following conditions*:

(i) *there exists a multiplicatively closed subset S of R such that $\mathfrak{m}\cap S\neq\emptyset$ and such that the topologies defined by $\{S((I^n)_a^{(N)})\}_{n\geq 0}$ and $\{(I^n)_a^{(N)}\}_{n\geq 0}$ are equivalent.*

(ii) $H_I^d(N)=0$.

Then, (i) $\Rightarrow$ (ii), *and these conditions are equivalent whenever N is quasi-unmixed.*

Proof. We begin with the proof of the implication (i) $\Rightarrow$ (ii). From Theorem 3.11, we have $S\subseteq R\backslash\cup\{\mathfrak{p}\in\overline{Q}^*(I,N)\}$. Then, $\mathfrak{m}\notin\overline{Q}^*(I,N)$. Therefore, for all $\mathfrak{q}\in\operatorname{mAss}_{\widehat{R}}\widehat{N}$, we have $\dim\widehat{R}/I\widehat{R}+\mathfrak{q}>0$. By the Lichtenbaum-Hartshorne theorem, see [**5,** Corollary 3.4], it follows that $H_I^d(N)=0$.

Now, assume that N is quasi-unmixed and that (ii) holds. We show that (i) is true. Toward this end, let $S=R\backslash\cup\{\mathfrak{p}\in\overline{Q}^*(I,N)\}$. Then, in view of Theorem 3.11, the topologies defined by $\{S((I^n)_a^{(N)})\}_{n\geq 0}$ and $\{(I^n)_a^{(N)}\}_{n\geq 0}$ are equivalent. Hence, it is sufficient to show that $\mathfrak{m}\cap S\neq\emptyset$. Suppose the contrary, namely, $\mathfrak{m}\cap S=\emptyset$. Then, $\mathfrak{m}\in\overline{Q}^*(I,N)$. Thus, there exists a $\mathfrak{q}\in\operatorname{mAss}_{\widehat{R}}\widehat{N}$ such that $\mathfrak{m}\widehat{R}=\operatorname{Rad}(I\widehat{R}+\mathfrak{q})$. As N is quasi-unmixed, it follows that $\dim\widehat{R}/I\widehat{R}+\mathfrak{q}=0$ for some $\mathfrak{q}\in\operatorname{mAss}_{\widehat{R}}\widehat{N}$ such that $\dim\widehat{R}/\mathfrak{q}=d$. Now, use [**5,** Corollary 3.4] to see that $H_I^d(N)\neq 0$, which is a contradiction. □

The final results will be a strengthened and generalized version of corresponding results by Marley [**6,** Corollaries 2.4 and 2.5] and Naghipour and Sedghi [**14,** Corollary 3.3].

Corollary 4.2. *Assume that R is a Noetherian ring. Let N be a finitely generated R-module of dimension d and I an ideal of R. Then,* $\operatorname{Supp}(H_I^d(N))\subseteq\overline{Q}^*(I,N)$. *Moreover, the equality holds whenever N is Cohen-Macaulay.*

Proof. Let $\mathfrak{p}\in\operatorname{Supp}(H_I^d(N))$. Then, $H_{IR_\mathfrak{p}}^d(N_\mathfrak{p})\neq 0$, and thus, $\dim N_\mathfrak{p}=d$. Hence, in view of the Lichtenbaum-Hartshorne theorem, [**5,** Corollary 3.4], there exists a $\mathfrak{q}\in\operatorname{mAss}_{\widehat{R}_\mathfrak{p}}\widehat{N}_\mathfrak{p}$ such that $\mathfrak{p}\widehat{R}_\mathfrak{p}=\operatorname{Rad}(I\widehat{R}_\mathfrak{p}+\mathfrak{q})$. Thus, $\mathfrak{p}\in\overline{Q}^*(I,N)$, and therefore, $\operatorname{Supp}(H_I^d(N))\subseteq\overline{Q}^*(I,N)$.

In order to prove the second assertion, let $\mathfrak{p} \in \overline{Q}^*(I, N)$. Then, there exists a $\mathfrak{q} \in \mathrm{mAss}_{\widehat{R}_\mathfrak{p}} \widehat{N}_\mathfrak{p}$ such that $\mathfrak{p}\widehat{R}_\mathfrak{p} = \mathrm{Rad}(I\widehat{R}_\mathfrak{p} + \mathfrak{q})$. Now, since N is Cohen-Macaulay, we deduce that $\dim \widehat{R}_\mathfrak{p}/\mathfrak{q} = \dim \widehat{N}_\mathfrak{p}$, whence, in view of the Lichtenbaum-Hartshorne theorem, $H^d_{IR_\mathfrak{p}}(N_\mathfrak{p}) \neq 0$, and thus, $\mathfrak{p} \in \mathrm{Supp}(H^d_I(N))$. □

Corollary 4.3. *Let* $(R, \mathfrak{m})$ *be a local* (*Noetherian*) *ring, N a finitely generated R-module of dimension d and I an ideal of R. Then*:

$$\mathrm{Supp}(H^{d-1}_I(N)) \subseteq \overline{Q}^*(I, N) \cup \{\mathfrak{m}\}.$$

Therefore, $\mathrm{Ass}_R H^{d-1}_I(N)$ *is a finite set.*

Proof. Let $\mathfrak{p} \in \mathrm{Supp}(H^{d-1}_I(N))$ be such that $\mathfrak{p} \neq \mathfrak{m}$. Then, $H^{d-1}_{IR_\mathfrak{p}}(N_\mathfrak{p}) \neq 0$, and thus, $\dim N_\mathfrak{p} = d - 1$. Now, from the proof of Corollary 4.2, we have $\mathfrak{p} \in \overline{Q}^*(I, N)$, as required. □

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