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THE REAL-ROOTEDNESS OF GENERALIZED NARAYANA POLYNOMIALS RELATED TO THE BOROS-MOLL POLYNOMIALS

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ABSTRACT. In this paper, we prove the real-rootedness of a family of generalized Narayana polynomials which arose in the study of the infinite log-concavity of the Boros-Moll polynomials. We establish certain recurrence relations for these Narayana polynomials, from which we derive the real-rootedness. In order to prove the real-rootedness, we use a sufficient condition due to Liu and Wang to determine whether two polynomials have interlaced zeros. The recurrence relations are verified with the help of the `Mathematica` package `HolonomicFunctions`.

1. Introduction. For any nonnegative integers n and m, let

$$N_n(x)=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\binom{n+1}{k+1}x^k, \tag{1.1}$$

$$N_{n,m}(x)=\sum_{k=0}^{n}\left(\binom{n}{k}\binom{m}{k}-\binom{n}{k+1}\binom{m}{k-1}\right)x^k. \tag{1.2}$$

The polynomial $N_n(x)$ is the classical Narayana polynomial. It is well known that $N_n(x)$ has only real zeros. Moreover, it is easy to verify that both $N_{n,n}(x)$ and $N_{n+1,n}(x)$ are simply the polynomial $N_n(x)$, while it seems that the polynomials $N_{n,m}(x)$ have not been studied well for general n and m. In this paper, we shall prove that the polynomials $N_{n,m}(x)$ have only real zeros for any n and m. We first review some of the background of the polynomials $N_{n,m}(x)$.

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The polynomials $N_{n,m}(x)$ arose in the study of the infinite log-concavity of the Boros-Moll polynomials. The Boros-Moll polynomials were introduced by Boros and Moll [**1**] in their study of a quartic integral, and they obtained the following expression for the Boros-Moll polynomials:

$$P_n(x) = 2^{-2n} \sum_j 2^j \binom{2n-2j}{n-j} \binom{n+j}{j} (x+1)^j.$$

Recall that a finite nonnegative sequence $\{a_k\}_{k=0}^n$ is said to be log-concave if

$$a_k^2 - a_{k+1}a_{k-1} \geq 0 \quad \text{for } 0 \leq k \leq n,$$

where, for convenience, we set $a_{-1} = 0$ and $a_{n+1} = 0$. We say that it is infinitely log-concave if, for any $i \geq 1$, the ith iterative sequence $\{\mathcal{L}^i(a_k)\}_{k=0}^n$ is nonnegative, where $\mathcal{L}$ is the operator acting on $\{a_k\}_{k=0}^n$ as

$$\mathcal{L}(a_k) = a_k^2 - a_{k+1}a_{k-1} \quad \text{for } 0 \leq k \leq n.$$

We say that a polynomial

$$f(x) = \sum_{k=0}^n a_k x^k$$

is infinitely log-concave if its coefficient sequence $\{a_k\}_{k=0}^n$ is infinitely log-concave. Boros and Moll proposed the following conjecture.

Conjecture 1.1 ([**1**]). *The polynomial $P_n(x)$ is infinitely log-concave.*

The log-concavity of $P_n(x)$ was first conjectured by Moll [**11**] and then proven by Kauers and Paule [**7**]. The 2-fold log-concavity of $P_n(x)$ was proven by Chen and Xia [**5**]. Brändén [**2**] proposed an innovative approach to the higher-fold log-concavity of $P_n(x)$. He conjectured the real-rootedness of some variations of $P_n(x)$, from which its 3-fold log-concavity can be deduced. Brändén's conjectures were later verified by Chen, Dou and Yang [**4**]. While Conjecture 1.1 is open, Brändén [**2**] has proved the infinite log-concavity of real-rooted polynomials, which was independently conjectured by Fisk [**6**] and Stanley, McNamara and Sagan [**10**].

Theorem 1.2 ([**2**]). *If*

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

is a real-rooted polynomial with nonnegative coefficients, then so is the polynomial

$$\sum_{k=0}^{n} (a_k^2 - a_{k-1}a_{k+1})x^k.$$

Newton's well-known inequality states that, if a polynomial $f(x)$ has only real zeros, then it must be log-concave. Therefore, Theorem 1.2 implies the infinite log-concavity of the real-rooted polynomials. Motivated by Brändén's theorem, we are led to study the real-rootedness of the following polynomial:

$$Q_n(x) = \sum_{k=0}^{n} (d_k(n)^2 - d_{k-1}(n)d_{k+1}(n))x^k,$$

where

$$d_k(n) = 2^{-2n} \sum_{j=k}^{n} 2^j \binom{2n-2j}{n-j}\binom{n+j}{j}\binom{j}{k}$$

is the coefficient of x^k in the Boros-Moll polynomial $P_n(x)$.

We have the following conjecture.

Conjecture 1.3. *For any $n \geq 1$, the polynomial $Q_n(x)$ has only real zeros.*

Since the log-concavity of $P_n(x)$ is known, from Theorem 1.2, Conjecture 1.3 would imply Conjecture 1.1. Note that the polynomial $Q_n(x)$ may be rewritten as

$$Q_n(x) = \sum_{i=0}^{n}\sum_{j=0}^{n} 2^{i+j}\binom{2n-2i}{n-i}\binom{2n-2j}{n-j}\binom{n+i}{i}\binom{n+j}{j} N_{i,j}(x),$$

where $N_{i,j}(x)$ is the Narayana polynomial defined by (1.2). The numerical evidence suggests that the polynomial $N_{n,m}(x)$ has only real zeros for any n and m.

Our main result is as follows.

Theorem 1.4. *For any m, $n \geq 0$, the polynomial $N_{m,n}(x)$ has only real zeros.*

The remainder of this paper is organized as follows. In the next section, we shall give an overview of some tools which will be used to prove Theorem 1.4. In Section 3, we shall establish some interlacing properties concerning the polynomials $N_{n,m}(x)$, from which we derive Theorem 1.4.

2. Preliminaries. The results contained in this section serve as a reference point used in Section 3.

First, we introduce the definition of interlacing. Given two real-rooted polynomials $f(x)$ and $g(x)$ with positive leading coefficients, we say that $g(x)$ *interlaces* $f(x)$, denoted $g(x) \preceq f(x)$, if

$$\cdots \leq s_2 \leq r_2 \leq s_1 \leq r_1,$$

where $\{r_i\}$ and $\{s_j\}$ are the sets of zeros of $f(x)$ and $g(x)$, respectively.

Liu and Wang [**9**] obtained the following sufficient condition to determine whether two polynomials have interlaced zeros.

Theorem 2.1 ([**9,** Theorem 2.3])**.** *Let $F(x), f(x), g_1(x), \ldots, g_k(x)$ be polynomials with real coefficients satisfying the following conditions.*

(a) *There exist some polynomials $\phi(x), \varphi_1(x), \ldots, \varphi_k(x)$ with real coefficients such that*

$$F(x) = \phi(x)f(x) + \varphi_1(x)g_1(x) + \cdots + \varphi_k(x)g_k(x) \tag{2.1}$$

and $\deg F(x) = \deg f(x)$ *or* $\deg F(x) = \deg f(x) + 1$;

(b) *the polynomials $f(x), g_1(x), \ldots, g_k(x)$ are real-rooted polynomials, and moreover, $g_j(x) \preceq f(x)$ for each $1 \leq j \leq k$;*

(c) *the leading coefficients of $F(x)$ and $g_j(x)$ have the same sign for each $1 \leq j \leq k$.*

Suppose that $\varphi_j(r) \leq 0$ for each j and each zero r of $f(x)$. Then, $F(x)$ has only real zeros and $f(x) \preceq F(x)$.

We shall use the above result to prove the real-rootedness of $N_{n,m}(x)$. The key point is to prove certain recurrence relations related to these polynomials. As will be shown later, the coefficients of these recurrence relations look complicated. Due to Zeilberger's holonomic systems approach to special function identities, Koutschan (private communication) pointed out that these recurrence relations can be easily verified by using the `Mathematica` package `HolonomicFunctions`, see [**8, 13**]. The Ore algebras introduced in [**12**] serve as a unifying framework to represent such recurrence relations. These algebras were obtained by applying Ore extensions to some base rings, also called Ore polynomial rings. Let S_n denote the shift operator with respect to n. Let $\mathbb{R}(n,x)$ denote the field of rational functions in n and x over the field $\mathbb{R}$ of real numbers. The Ore algebra used throughout this paper could be considered as $\mathbb{R}(n,x)\langle S_n\rangle$, which consists of all linear operators of the form

$$\sum_{i=0}^{r} p_i S_n^i,$$

where $r \geq 0$ and $p_i \in \mathbb{R}(n,x)$. Suppose that the polynomial sequence $\{f_n(x)\}_{n\geq 0}$ satisfies a certain recurrence relation

$$\sum_{i=0}^{k} a_i f_{i+n}(x) = 0,$$

where $a_i \in \mathbb{R}(n,x)$, and then the Ore polynomial of such a recurrence relation is given by

$$\sum_{i=0}^{k} a_i S_n^i.$$

For each $f \in \mathbb{R}(n,x)$, the annihilator of f with respect to $\mathbb{R}(n,x)\langle S_n\rangle$ is defined by

$$\mathrm{Ann}_{\mathbb{R}(n,x)\langle S_n\rangle}(f) = \{P \in \mathbb{R}(n,x)\langle S_n\rangle \mid P(f) = 0\},$$

which is a left ideal in $\mathbb{R}(n,x)\langle S_n\rangle$. For more information on the Ore algebras and the Ore polynomials, see Koutschan [**8**] and Ore [**12**].

In order to be self-contained, we give an example illustrating the use of this package. It is well known that the classical Narayana polynomials $N_n(x)$ given in (1.1) satisfy the following recurrence relation:

$$(n+3)N_{n+1}(x) = (2n+3)(x+1)N_n(x) - n(x-1)^2 N_{n-1}(x).$$

This may be proven in the following way by using the package:

1. Convert the above recurrence to an Ore polynomial in the Ore algebra:

$$\begin{aligned}\text{In}[1] := \mathbf{rec} = \mathbf{ToOrePolynomial}[&\boldsymbol{(2*n+3)*(x+1)*\mathrm{f}[n]-n}\\ &\boldsymbol{*(x-1)^2*\mathrm{f}[n-1]-(n+3)*\mathrm{f}[n+1],\mathrm{f}[n]}]\end{aligned}$$

$$\begin{aligned}\text{Out}[1] = \{(4+n)S_n^2 + (-5-2n-5x-2nx)S_n&\\ + (1+n-2x-2nx+x^2+nx^2)\}.&\end{aligned}$$

2. Generate a (Gröbner) basis of the annihilator A of the input, i.e., the set of all recurrence/differential relations that the input satisfies, using the command Annihilator:

$$\begin{aligned}\text{In}[2] := \mathbf{ann} = \mathbf{Annihilator}[&\mathbf{Sum}[\mathbf{Binomial}[\boldsymbol{n+1,k}]\\ &\boldsymbol{*}\,\mathbf{Binomial}[\boldsymbol{n+1,k+1}]\boldsymbol{*x^k/(n+1),\{k,0,n\}}],\mathbf{S}[\boldsymbol{n}]]\end{aligned}$$

$$\begin{aligned}\text{Out}[2] = \{(4+n)S_n^2 + (-5-2n-5x-2nx)S_n&\\ + (1+n-2x-2nx+x^2+nx^2)\}.&\end{aligned}$$

3. Reduce the Ore polynomial *rec* modulo A using the command OreReduce. If it returns 0, then *rec* is an element of the set A, and hence, the recurrence relation is valid.

$$\begin{aligned}\text{In}[3] &:= \mathbf{OreReduce}[\mathbf{rec},\mathbf{ann}]\\ \text{Out}[3] &= 0.\end{aligned}$$

3. Proof. The objective of this section is to give a proof of Theorem 1.4, namely, the real-rootedness of the polynomial $N_{n,m}(x)$. We first derive certain recurrence relations for these polynomials. For nonnegative integers t and n, let

$$\underline{N}_n^{(t)}(x) = N_{n,n+t}(x), \qquad \overline{N}_n^{(t)}(x) = N_{n+t,n}(x). \tag{3.1}$$

The polynomials $\underline{N}_n^{(t)}(x)$ satisfy the following recurrence relation.

Theorem 3.1. *For nonnegative integers t and $n \geq 1$, we have*

$$\underline{N}_{n+1}^{(t)}(x) = \frac{a_0 + a_1 x + a_2 x^2}{(n+t+3)(n+1)(c_0 + c_1 x)} \underline{N}_n^{(t)}(x) - \frac{n(n+t)(x-1)^2(b_0 + b_1 x)}{(n+t+3)(n+1)(c_0 + c_1 x)} \underline{N}_{n-1}^{(t)}(x), \tag{3.2}$$

where

$$\begin{aligned}
a_0 &= -(2n^3 + (2t+5)n^2 + (2t+3)n),\\
a_1 &= (2t(t+2)n^3 + 3t(t+2)^2 n^2 + (t(t+2)(t^2+5t+5))n\\
&\qquad + (t(t+1)(t+2)(t+3)/2)),\\
a_2 &= (t+1)((2t+2)n^3 + (3t^2+9t+5)n^2 + (2t+3)(t^2+3t+1)n\\
&\qquad + t(t+1)(t+2)(t+3)/2),\\
b_0 &= -(n+1),\\
b_1 &= (t+1)^2 n + (t+1)(t^2+4t+2)/2,\\
c_0 &= -n,\\
c_1 &= (t+1)^2 n + t(t+1)(t+2)/2.
\end{aligned}$$

Proof. We shall prove an equivalent form of this recurrence relation, obtained by multiplying $(n+t+3)(n+1)(c_0+c_1 x)$ on both sides of (3.2). This could be converted into an Ore polynomial as follows:

```
In[4] := rec = ToOrePolynomial[(a0 + a1 * x + a2 * x^2)
      * f[n] − (n * (n + t) * (x − 1)^2 * (b0 + b1 * x))
      * f[n − 1] − (n + t + 3) * (n+1) * (c0 + c1 * x) * f[n+1]/.
        MapThread[Rule, {{a0, a1, a2, b0, b1, c0, c1}, {−(2 * n^3,
      + (2 * t + 5) * n^2 + (2 * t + 3) * n), 2 * t * (t + 2) * n^3 + 3
      * t * (t + 2)^2 * n^2 + (t * (t + 2) * (t^2 + 5 * t + 5)) * n + t
      * (t + 1) * (t + 2) * ((t + 3)/2), (t + 1) * ((2 * t + 2) * n^3
      + (3 * t^2 + 9 * t + 5) * n^2 + (2 * t + 3) * (t^2 + 3 * t + 1)
      * n + t * (t + 1) * (t + 2) * ((t + 3)/2)), −(n + 1), (t + 1)^2
      * n + (t + 1) * ((t^2 + 4 * t + 2)/2), −n, (t + 1)^2
      * n + t * (t + 1) * ((t + 2)/2)}}], f[n]];
```

Then, compute a (Gröbner) basis *ann* of the set of all recurrence/differential relations that $\underline{N}_n^{(t)}(x)$ satisfies, and reduce the Ore polynomial *rec* modulo *ann*:

In[5] := **ann = Annihilator[Sum[(Binomial[$\boldsymbol{n, k}$]**
$*$ Binomial[$\boldsymbol{n+t, k}$] − Binomial[$\boldsymbol{n, k+1}$]
$*$ Binomial[$\boldsymbol{n+t, k-1}$]) $*$ $\boldsymbol{x^k}$, {$\boldsymbol{k}$, 0, $\boldsymbol{n}$}], S[$\boldsymbol{n}$]];

In[6] := **OreReduce[rec,ann]**

Out[6] = 0

We have obtained the desired output. This completes the proof. □

Next we turn to proving the real-rootedness of $\underline{N}_n^{(t)}(x)$.

Theorem 3.2. *For any $t \geq 0$ and $n \geq 0$, the polynomial $\underline{N}_n^{(t)}(x)$ has only real zeros, and moreover, we have $\underline{N}_n^{(t)}(x) \preceq \underline{N}_{n+1}^{(t)}(x)$.*

Proof. We use induction on n. It is straightforward to verify that

$$\underline{N}_0^{(t)}(x) = 1, \quad \underline{N}_1^{(t)}(x) = 1 + (t+1)x, \quad \underline{N}_0^{(t)}(x) \preceq \underline{N}_1^{(t)}(x).$$

Assume that $\underline{N}_{n-1}^{(t)}(x) \preceq \underline{N}_n^{(t)}(x)$. We see that the recurrence relation (3.2) is of the form (2.1) in Theorem 2.1 with $k = 1$, where

$$F(x) = \underline{N}_{n+1}^{(t)}(x),$$
$$f(x) = \underline{N}_n^{(t)}(x),$$
$$g_1(x) = \underline{N}_{n-1}^{(t)}(x),$$

$$\phi(x) = \frac{a_0 + a_1x + a_2x^2}{(n+t+3)(n+1)(c_0 + c_1x)},$$
$$\varphi_1(x) = -\frac{n(n+t)(x-1)^2(b_0 + b_1x)}{(n+t+3)(n+1)(c_0 + c_1x)}.$$

Here, $a_0, a_1, a_2, b_0, b_1, c_0, c_1$ are given by (3.2). Note that, for any n, $t \geq 0$, the coefficients of $\underline{N}_{n+1}^{(t)}(x)$ are nonnegative, since, for any

$0 \le k \le n$, the coefficient of x^k in $\underline{N}_{n+1}^{(t)}(x)$ is

$$\begin{aligned}[x^k]\underline{N}_{n+1}^{(t)}(x) &= \binom{n}{k}\binom{n+t}{k} - \binom{n}{k+1}\binom{n+t}{k-1} \\ &= \left(1 - \frac{k}{k+1} \cdot \frac{n-k}{n-k+t+1}\right)\binom{n}{k}\binom{n+t}{k} > 0.\end{aligned}$$

It is clear that, for any $x \le 0$, we have $\varphi_1(x) \le 0$. From Theorem 2.1, the polynomial $\underline{N}_{n+1}^{(t)}(x)$ is real-rooted, and moreover, $\underline{N}_{n}^{(t)}(x) \preceq \underline{N}_{n+1}^{(t)}(x)$. $\square$

We have the following recurrence relation for $\overline{N}_{n}^{(t)}(x)$.

Theorem 3.3. *For nonnegative integers t and $n \ge 1$, we have*

$$\begin{aligned}\overline{N}_{n+1}^{(t)}(x) &= \frac{a_0 + a_1 x + a_2 x^2}{(n+t+1)(n+3)(c_0 + c_1 x)}\overline{N}_{n}^{(t)}(x) \\ &\quad - \frac{n(n+t)(x-1)^2(b_0 + b_1 x)}{(n+t+1)(n+3)(c_0 + c_1 x)}\overline{N}_{n-1}^{(t)}(x),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}a_0 &= -(2n+3)(n+t)(n+t+1), \\ a_1 &= 3t(t-2)(t+1)^2/2 + t(t-2)(t^2+7t+5)n \\ &\quad + 3t(t-2)(t+2)n^2 + 2t(t-2)n^3, \\ a_2 &= t^2(t-1)(t+1)^2/2 + (t-1)(2t^3+3t^2+t-3)n \\ &\quad + (t-1)(3t^2+3t-5)n^2 + 2(t-1)^2 n^3, \\ b_0 &= -n-1-t, \\ b_1 &= (t-1)^2 n + (t-1)t^2/2 + (t-1)^2, \\ c_0 &= -n-t, \\ c_1 &= (t-1)^2 n + (t-1)t^2/2.\end{aligned}$$

Proof. The proof is similar to that of Lemma 3.1. We need to prove an equivalent form of (3.3) obtained by multiplying $(n+t+1)(n+3)(c_0+c_1 x)$ on both sides. This may be converted into an Ore polynomial as follows:

$$\begin{aligned}\mathrm{In}[7] := {} & \mathbf{rec} = \mathbf{ToOrePolynomial}[(\boldsymbol{a0} + \boldsymbol{a1} * \boldsymbol{x} + \boldsymbol{a2} * \boldsymbol{x}^{\mathbf{2}}) \\ & * \mathrm{f}[\boldsymbol{n}] - (\boldsymbol{n} * (\boldsymbol{n} + \boldsymbol{t}) * (\boldsymbol{x} - \mathbf{1})^{\mathbf{2}} * (\boldsymbol{b0} + \boldsymbol{b1} * \boldsymbol{x})) \\ & * \mathrm{f}[\boldsymbol{n} - \mathbf{1}] - (\boldsymbol{n}{+}\mathbf{3}) * (\boldsymbol{n} + \boldsymbol{t} + \mathbf{1}) * (\boldsymbol{c0} + \boldsymbol{c1} * \boldsymbol{x}) * \mathrm{f}[\boldsymbol{n}{+}\mathbf{1}]/. \\ & \quad \mathbf{MapThread}[\mathbf{Rule}, \{\{\boldsymbol{a0}, \boldsymbol{a1}, \boldsymbol{a2}, \boldsymbol{b0}, \boldsymbol{b1}, \boldsymbol{c0}, \boldsymbol{c1}\}, \\ & \quad \{-(\mathbf{2} * \boldsymbol{n}{+}\mathbf{3}) * (\boldsymbol{n} + \boldsymbol{t}) * (\boldsymbol{n} + \boldsymbol{t} + \mathbf{1}), \mathbf{3} * \boldsymbol{t} * (\boldsymbol{t} - \mathbf{2}) \\ & * (\boldsymbol{t} + \mathbf{1})^{\mathbf{2}}/\mathbf{2} + \boldsymbol{t} * (\boldsymbol{t} - \mathbf{2}) * (\boldsymbol{t}^{\mathbf{2}} + \mathbf{7}\boldsymbol{t} + \mathbf{5}) * \boldsymbol{n} + \mathbf{3} * \boldsymbol{t} \\ & * (\boldsymbol{t} - \mathbf{2}) * (\boldsymbol{t} + \mathbf{2}) * \boldsymbol{n}^{\mathbf{2}} + \mathbf{2} * \boldsymbol{t} * (\boldsymbol{t} - \mathbf{2}) * \boldsymbol{n}^{\mathbf{3}}, \boldsymbol{t}^{\mathbf{2}} * (\boldsymbol{t} - \mathbf{1}) \\ & * (\boldsymbol{t} + \mathbf{1})^{\mathbf{2}}/\mathbf{2} + (\boldsymbol{t} - \mathbf{1}) * (\mathbf{2} * \boldsymbol{t}^{\mathbf{3}} + \mathbf{3} * \boldsymbol{t}^{\mathbf{2}} + \boldsymbol{t} - \mathbf{3}) \\ & * \boldsymbol{n} + (\boldsymbol{t} - \mathbf{1}) * (\mathbf{3} * \boldsymbol{t}^{\mathbf{2}} + \mathbf{3} * \boldsymbol{t} - \mathbf{5}) * \boldsymbol{n}^{\mathbf{2}} + \mathbf{2} * (\boldsymbol{t} - \mathbf{1})^{\mathbf{2}} * \boldsymbol{n}^{\mathbf{3}}, \\ & - \boldsymbol{n} - \mathbf{1} - \boldsymbol{t}, (\boldsymbol{t} - \mathbf{1})^{\mathbf{2}} * \boldsymbol{n} + (\boldsymbol{t} - \mathbf{1}) * \boldsymbol{t}^{\mathbf{2}}/\mathbf{2} + (\boldsymbol{t} - \mathbf{1})^{\mathbf{2}}, \\ & - \boldsymbol{n} - \boldsymbol{t}, (\boldsymbol{t} - \mathbf{1})^{\mathbf{2}} * \boldsymbol{n} + (\boldsymbol{t} - \mathbf{1}) * \boldsymbol{t}^{\mathbf{2}}/\mathbf{2}\}\}], \mathrm{f}[\boldsymbol{n}]];\end{aligned}$$

Then, compute a (Gröbner) basis *ann* of the set of all recurrence/differential relations that $\overline{N}_n^{(t)}(x)$ satisfies, and reduce the Ore polynomial *rec* modulo *ann*:

$$\begin{aligned}\mathrm{In}[8] := {} & \mathbf{ann} = \mathbf{Annihilator}\ [\mathbf{Sum}[(\mathbf{Binomial}[\boldsymbol{n} + \boldsymbol{t}, \boldsymbol{k}] \\ & * \mathbf{Binomial}[\boldsymbol{n}, \boldsymbol{k}] - \mathbf{Binomial}[\boldsymbol{n} + \boldsymbol{t}, \boldsymbol{k} + \mathbf{1}] \\ & * \mathbf{Binomial}[\boldsymbol{n}, \boldsymbol{k} - \mathbf{1}]) * \boldsymbol{x}^{\boldsymbol{k}}, \{\boldsymbol{k}, \mathbf{0}, \boldsymbol{n} + \boldsymbol{t}\}], \mathrm{S}[\boldsymbol{n}]]; \\ \mathrm{In}[9] := {} & \mathbf{OreReduce[rec{,}ann]} \\ \mathrm{Out}[9] = {} & 0\end{aligned}$$

The output is zero, as desired. This completes the proof. □

We now prove the real-rootedness of $\overline{N}_n^{(t)}(x)$.

Theorem 3.4. *For any $n,\ t \geq 0$, the polynomial $\overline{N}_n^{(t)}(x)$ has only real zeros. If $t \geq 2$, then $\overline{N}_n^{(t)}(x)$ has one and only one positive zero.*

Proof. Note that both the polynomials $\overline{N}_n^{(0)}(x)$ and $\overline{N}_n^{(1)}(x)$ are the classical Narayana polynomial, which is known to be real-rooted.

We proceed to consider the case of $t \geq 2$. We first prove that $\overline{N}_n^{(t)}(x)$ has one and only one positive zero. Note that, for any $n \geq 0$ and $t \geq 2$,

$\overline{N}_n^{(t)}(x)$ is polynomial in x of degree $n+1$, and, for any $0 \le k \le n+1$, the coefficient of x^k in $\overline{N}_n^{(t)}(x)$ is

$$\binom{n+t}{k}\binom{n}{k} - \binom{n+t}{k+1}\binom{n}{k-1} = \frac{n+1-kt}{(n+1)(k+1)}\binom{n+t}{k}\binom{n+1}{k}.$$

Therefore, the number of changes in sign of the coefficients is 1. By Descartes' rule, the polynomial $\overline{N}_n^{(t)}(x)$ has at most one positive zero. Moreover, we see that

$$[x^0]\overline{N}_n^{(t)}(x) = 1 > 0, \qquad [x^{n+1}]\overline{N}_n^{(t)}(x) = -\binom{n+t}{n+2} < 0.$$

Thus, the polynomial $\overline{N}_n^{(t)}(x)$ has one and only one positive zero.

Next, we claim that $\overline{N}_n^{(t)}(x)$ has n negative zeros, and moreover, for any $n \ge 1$,

$$r_{n+1}^{(n+1)} < r_n^{(n)} < r_n^{(n+1)} < r_{n-1}^{(n)} < \cdots < r_2^{(n)} < r_2^{(n+1)} < r_1^{(n)} < r_1^{(n+1)} < 0,$$

where $\{r_k^{(n)}\}_{k=0}^n$ and $\{r_k^{(n+1)}\}_{k=0}^{n+1}$ are the negative zeros of $\overline{N}_n^{(t)}(x)$ and $\overline{N}_{n+1}^{(t)}(x)$, respectively.

Before proving the above claim, we note the following property: for any $x < 0$, $n \ge 1$ and $t \ge 2$, clearly we have

$$-\frac{n(n+t)(x-1)^2(b_0+b_1x)}{(n+t+1)(n+3)(c_0+c_1x)} < 0.$$

In order to prove the claim, we use induction on n. We first prove the base case of $n=1$. We have already shown that $\overline{N}_1^{(t)}(x)$ has one and only one positive zero. Since $\overline{N}_1^{(t)}(x)$ is of degree 2 and $[x^0]\overline{N}_1^{(t)}(x) = 1$, it also has one negative zero $r_1^{(1)}$. By the recurrence (3.3), we see that $\overline{N}_2^{(t)}(r_1^{(1)}) < 0$ since $\overline{N}_0^{(t)}(r_1^{(1)}) > 0$. Moreover, we have $\overline{N}_2^{(t)}(0) = 1 > 0$ and $\overline{N}_2^{(t)}(-\infty) > 0$. Thus, $\overline{N}_2^{(t)}(x)$ has two negative zeros $r_1^{(2)}$, $r_2^{(2)}$, and moreover, $r_2^{(2)} < r_1^{(1)} < r_1^{(2)} < 0$, as claimed.

Assume that the claim is true for n. We proceed to show that it is also true for $n+1$. From (3.3), we deduce that

$$(-1)^k\overline{N}_{n+1}^{(t)}(r_k^{(n)}) > 0, \quad \text{for any } 1 \le k \le n.$$

Moreover, we have $\overline{N}_{n+1}^{(t)}(0) = 1 > 0$ and $(-1)^{n+1}\overline{N}_{n+1}^{(t)}(-\infty) > 0$. Thus, the polynomial $\overline{N}_{n+1}^{(t)}(x)$ has $n+1$ negative zeros $\{r_k^{(n+1)}\}_{k=0}^{n+1}$, and moreover, for each $1 \le k \le n$, we have $r_{k+1}^{(n+1)} < r_k^{(n)} < r_k^{(n+1)}$, as claimed. This completes the proof. □

Combining Theorems 3.2 and 3.4, the proof of Theorem 1.4 is complete.

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