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AN IDENTITY FOR COCYCLES ON COSET SPACES OF LOCALLY COMPACT GROUPS

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ABSTRACT. We prove here an identity for cocycles associated with homogeneous spaces in the context of locally compact groups. Mackey introduced cocycles (λ-functions) in his work on representation theory of such groups. For a given locally compact group G and a closed subgroup H of G, with right coset space G/H, a cocycle λ is a real-valued Borel function on $G/H \times G$ satisfying the cocycle identity

$$\lambda(x, st) = \lambda(x.s, t)\lambda(x, s),$$
$$\text{almost everywhere } x \in G/H,\ s, t \in G,$$

where the "almost everywhere" is with respect to a measure whose null sets pull back to Haar measure null sets on G. Let H and K be regularly related closed subgroups of G. Our identity describes a relationship among cocycles for G/H^x, G/K^y and $G/(H^x \cap K^y)$ for almost all $x, y \in G$. This also leads to an identity for modular functions of G and the corresponding subgroups.

1. Introduction and statement of results. The aim of this paper is to prove an identity for cocycles (Mackey's λ functions). The need for this identity arose in connection with problems on induced representations to be discussed in a later publication. Let G be a separable locally compact group and H a closed subgroup of G. In his treatment of induced representation on locally compact groups [**4**], Mackey introduced the concept of a cocycle λ as a real-valued positive function on $(G/H) \times G$ satisfying certain identities (see Section 2). Most importantly, such cocycles are associated with quasi-invariant measures; to each such cocycle there is a quasi-invariant measure μ on G/H so that the Radon-Nikodym derivative of the translated measure

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$\mu \cdot s$ with respect to μ is $\lambda(\cdot, s)$ for all $s \in G$. Once λ is specified, this measure is unique up to a positive scalar multiple. The basic properties of these functions are well established in the literature, cf., [**1, 4, 5**].

In order to state our results, we will need some notation and concepts. Let λ_H denote a cocycle corresponding to the homogeneous space G/H, and let $H^x = x^{-1}Hx$ for $x \in G$. Let Δ_H be the modular function of the group H. Closed subgroups H and K of G are said to be *regularly related* if the double coset space $H\backslash G/K$ is a standard Borel space, cf., [**4**]. We note that the double coset space formed by the right action of the diagonal subgroup $\Lambda = \{(x, x) : x \in G\}$ of $G \times G$ on the coset space $(H \times K)\backslash(G \times G)$ (that is, $H \times K\backslash G \times G/\Lambda$) is identified with $H\backslash G/K$ by the map

$$(x, y) \longmapsto xy^{-1}.$$

The regularly related property for H and K is equivalent to this action being smooth [**4**].

Our main result provides a link between cocycles for conjugates of regularly related subgroups. We will abuse notation and assume that the cocycle $\lambda_H(s, t)$ is actually defined on $G \times G$ and constant on the right cosets of H, rather than on $(G/H) \times G$.

Theorem 1.1. *Let G be a separable locally compact group and H and K closed subgroups of G. If H and K are regularly related, then, for each double coset $D(x, y) = (H \times K)(x, y)\Lambda$, there is a quasi-invariant measure $\mu_{x,y}$ on $G/(H^x \cap K^y)$, $x, y \in G$, and a corresponding cocycle $\lambda_{H^x \cap K^y}$ such that*

$$\lambda_{H^x}(ts^{-1}, s)\lambda_{K^y}(ts^{-1}, s)\lambda_{H^x\cap K^y}(t, s^{-1}) = 1, \tag{1.1}$$
$$s, t \in G, \ \textit{almost everywhere } (x, y) \in (G \times G)/(H \times K).$$

Moreover, $\lambda_{H^x \cap K^y}(t, s)$ is defined everywhere and continuous on $(G/(H^x \cap K^y)) \times G$.

Theorem 1.1 leads to an identity relating the modular functions corresponding to the subgroups involved:

Corollary 1.2. *For $(x, y) \in G \times G$ such that (1.1) holds, and for $s \in H^x \cap K^y$,*

$$\frac{\Delta_{H^x}(s)\Delta_{K^y}(s)}{\Delta_G(s)\Delta_{H^x\cap K^y}(s)} = 1.$$

A necessary and sufficient condition for the existence of an invariant measure on a quotient group G/H of G is that $\Delta_H(x) = \Delta_G(x)$ for all $x \in H$. We say that $G \supset H$ is *comodular* if this occurs. Now, we have the following straightforward consequence of Corollary 1.2.

Corollary 1.3. *Let H and K be regularly related closed subgroups of a separable locally compact group G and $(x, y) \in G \times G$ such that identity* (1.1) *holds. If $G \supset H$ is comodular, then $K \supset (H^z \cap K)$ is comodular for almost all $z \in G$.*

2. Preliminaries on cocycles. In order to avoid measure theoretic complications, we assume throughout that G is a locally compact separable group and H a closed subgroup. We denote the right-invariant Haar measure on G by ν_G, with e denoting the identity of the group. The canonical mapping from G to the set of right-cosets G/H is denoted by p_H. Throughout this section, X denotes the set of right cosets G/H of H with the standard *right* action of G. A *left* action by any other (closed) subgroup K of G gives rise to orbits in one-to-one correspondence with the double cosets $H\backslash G/K$, and the stabilizer of $Hx \in X$ under the action of K is $H^x \cap K$.

The next brief list provides the key results on cocycles, quasi-invariant measures and related concepts from our perspective. The interested reader may refer to [**1, 4, 5**].

• There is a *regular Borel section* $B \subset G$, that is, a Borel set B that intersects each right G coset in exactly one point such that $(p_H^{-1}(p_H(K))) \cap B$ has a compact closure for each compact subset K of G.

• A strictly positive, real-valued continuous function ρ_H exists on G satisfying

$$\rho_H(hx) = (\Delta_H(h)/\Delta_G(h))\rho_H(x), \quad x \in G,\ h \in H.$$

• Such a *ρ-function* gives rise to a unique Borel cocycle λ_ρ on $X \times G$ such that

$$\lambda_\rho(p_H(s), y) = \frac{\rho(sy)}{\rho(s)}, \quad s, y \in G,$$

with the properties:

(a) for all $x \in X$ and $s, t \in G$, $\lambda_\rho(x, st) = \lambda_\rho(x.s, t)\lambda_\rho(x, s)$;
(b) for all $h \in H, \lambda_\rho(p_H(e), h) = \Delta_H(h)/\Delta_G(h)$;
(c) for $t \in G$, $\lambda_\rho(p_H(e), t)$ is bounded on compact sets as a function of t. For $x, t \in G$ and for almost all $v \in G/H$, $\lambda_{H^x}(x^{-1}v, t) = \lambda_H(v, t)$.

• For each ρ-function on G there is a quasi-invariant measure μ on X such that, for all $y \in G$, the corresponding cocycle λ_ρ has the property that $\lambda_\rho(\cdot, y)$ is a Radon-Nikodym derivative of the translation measure $\mu \cdot y$ with respect to the measure μ.

• For $x \in G$, let $\dot{x} = p_H(x)$. If μ denotes the quasi-invariant measure corresponding to the function ρ, then

$$\int_G f(x)\rho(x)\, d\nu_G(x) = \int_X \int_H f(hx)\, d\nu_H(h)\, d\mu(\dot{x}), \quad f \in C_{00}(G),$$

where $C_{00}(G)$ denotes the continuous functions on G with compact support. We write $\mu \succ \lambda$ to denote that, for all $y \in G$, $\lambda(\cdot, y)$ is the cocycle which is the Radon-Nikodym derivative of the translate

$$E \longmapsto \mu([E]y)$$

with respect to μ. The following facts on λ and corresponding ρ-functions may be found in many references in the literature (see, for example, [**1, 4**]).

There are quasi-invariant measures on X, any two of which are absolutely continuous. Null sets for such measures are exactly those sets E for which $p_H^{-1}(E)$ has Haar measure zero. The relations $\mu \succ \lambda$ and $\lambda = \lambda_\rho$ among quasi-invariant measures, cocycles, and ρ-functions have the following properties.

(i) Every cocycle is of the form λ_ρ; $\lambda_{\rho_1} = \lambda_{\rho_2}$ if and only if ρ_1/ρ_2 is a constant.

(ii) For every cocycle there is a quasi-invariant measure μ such that $\mu \succ \lambda$; if $\mu_1 \succ \lambda$ and $\mu_2 \succ \lambda$, then μ_1 is a constant multiple of μ_2.

(iii) For every quasi-invariant measure μ there is a cocycle λ such that $\mu \succ \lambda$. If $\mu \succ \lambda_1$ and $\mu \succ \lambda_2$, then, for all t, $\lambda_1(\cdot, t) = \lambda_2(\cdot, t)$ almost everywhere.

(iv) If $\mu \succ \lambda_{\rho_1}$ and $\mu \succ \lambda_{\rho_2}$, then ρ_1/ρ_2 is constant almost everywhere.

3. Proofs of the results. First, we recall some standard results on the disintegration of measures. Let X be a separable, locally compact space supporting a finite measure μ, and let R be an equivalence relation on X where $r(x)$ is the equivalence class containing x. The relation R is *measurable* if there exists a countable family $E_1, E_2, \ldots$ of subsets of X/R such that $r^{-1}(E_i)$ is measurable for each i and such that each point in X/R is the intersection of the E_i containing it, cf., [**2, 4**].

It is well known (see, for example, [**4,** page 124, Lemma 11.1]) that the measure μ is decomposable as an integral over X/R of measures μ_y concentrated on the equivalence classes.

If $\widetilde{\mu}$ is the "push-forward" measure on X/R from the measure μ on X, i.e., $\widetilde{\mu}(E) = \mu(r^{-1}(E))$, then, for each y in X/R, there exists a finite Borel measure μ_y on X such that $\mu_y(X - r^{-1}(\{y\})) = 0$ and

$$\int f(y) \int g(x)\, d\mu_y(x)\, d\widetilde{\mu}(y) = \int f(r(x))g(x)\, d\mu(x), \tag{3.1}$$

whenever $f \in L_1(X/R, \widetilde{\mu})$ and g is bounded and measurable on X. If μ is quasi-invariant, then, in the disintegration of μ in (3.1) above, μ_y is also quasi-invariant under the action of G almost everywhere y [**4**].

Proof of Theorem 1.1. It is clear that, if H and K are regularly related, then the orbits of G/H under the action of K outside of a set of measure zero form the equivalence classes of a measurable equivalence relation. The right action of the diagonal subgroup $\Lambda = \{(x, x) : x \in G\}$ of $G \times G$ on the coset space $(G \times G)/(H \times K)$ has stabilizer

$$H^x \times K^y \cap \Lambda = (H \times K)^{(x,y)} \cap \Lambda$$

at (Hx, Ky), and the orbit of this point is the double coset $(H \times K)(x, y)\Lambda$. We write Υ for the set of all double cosets

$$(H \times K)\backslash G \times G/\Lambda \simeq H\backslash G/K.$$

As noted earlier, the regularly related property for H and K is equivalent to this orbit space being smooth [**4**]. Writing $D(x,y)$ for the double coset to which (x,y) belongs, for a fixed finite measure ν_0 on $G\times G$ equivalent to the Haar measure, we define a measure $\mu_{(H,K)}$ on Υ by

$$\mu_{(H,K)}(F)=\nu_0(D^{-1}(F)).$$

Such a measure is called an *admissible measure* by Mackey.

Fix a finite product measure $\nu_0=\nu_1\times\nu_2$ on $(G\times G)$ equivalent to the Haar measure. Let $\mu_{H\times K}$ be the image of ν_0 under $p_{H\times K}$ and μ_H,μ_K the images of ν_1,ν_2 under p_H and p_K, respectively. Let $\mu_{H,K}$ be an admissible measure in Υ corresponding to ν_0.

For a function f on $(G/H)\times(G/K)$ for which

$$\int_{G/H}\int_{G/K} f(x,y)\,d\mu_H(x)\,d\mu_K(y)$$

is integrable, using the change of variables

$$x\longmapsto xs \quad\text{and}\quad y\longmapsto ys,$$

we obtain

$$\begin{aligned}
&\int_{G/H}\int_{G/K} f(x,y)\,d\mu_H(x)\,d\mu_K(y)\\
&\quad=\int_{G/H}\int_{G/K}\lambda_H(x,s)\lambda_K(y,s)f(xs,ys)\,d\mu_H(x)\,d\mu_K(y)\\
&\quad=\int_{(G\times G)/(H\times K)}\lambda_H(x,s)\lambda_K(y,s)f(xs,ys)\,d\mu_{H\times K}(x,y).
\end{aligned}$$

For $(x,y)\in(G\times G)/(H\times K)$, write $r(x,y)=D(p_{H\times K}^{-1}(x,y))$; this defines a measurable equivalence relation since H and K are regularly related. The measure $\mu_{H\times K}$ is disintegrated into an integral of measures $\mu_{x,y}$, where $D(x,y)\in\Upsilon$, with respect to the measure $\mu_{H,K}$ on Υ. Also, each $\mu_{x,y}$ is a quasi-invariant measure on the orbit $r^{-1}(D(x,y))$, cf., (3.1). Using this disintegration, we have

$$\begin{aligned}
&\int_{(G\times G)/(H\times K)}\lambda_H(x,s)\lambda_K(y,s)f(xs,ys)\,d\mu_{H\times K}(x,y)\\
&\quad=\int_{D\in\Upsilon}\int_{\underline{t}\in\Lambda/(H\times K)^{(x,y)}\cap\Lambda}\lambda_H(xt,s)\lambda_K(yt,s)f(xts,yts)\,d\mu_{x,y}(\underline{t})\,d\mu_{H,K}(D),
\end{aligned}$$

where (x, y) is a coset representative of the coset $D(x, y)$. Identifying the space $\Lambda/((H \times K)^{(x,y)} \cap \Lambda)$ with $G/(H^x \cap K^y)$, we can regard $\mu_{x,y}$ as a measure on $G/(H^x \cap K^y)$. Then, we have

$$\begin{aligned} &\int_{(G\times G)/(H\times K)} \lambda_H(x,s)\lambda_K(y,s)f(xs,ys)\,d\mu_{H\times K}(x,y) \\ &\quad = \int_{D\in\Upsilon}\int_{t\in G/(H^x\cap K^y)} \lambda_H(xt,s)\lambda_K(yt,s)f(xts,yts)\,d\mu_{x,y}(t)\,d\mu_{H,K}(D). \end{aligned} \tag{3.2}$$

Changing variables $t \mapsto ts^{-1}$ in the integral on the right-hand side, we find that

$$\begin{aligned} &\int_{(G\times G)/(H\times K)} \lambda_H(x,s)\lambda_K(y,s)f(xs,ys)\,d\mu_{H\times K}(x,y) \\ &\quad = \int_{D\in\Upsilon}\int_{t\in G/(H^x\cap K^y)} \lambda_H(xts^{-1},s)\lambda_K(yts^{-1},s)f(xt,yt) \\ &\qquad\qquad \lambda_{H^x\cap K^y}(t,s^{-1})\,d\mu_{x,y}(t)\,d\mu_{H,K}(D). \end{aligned} \tag{3.3}$$

On the other hand, if we begin with

$$\int_{(G\times G)/(H\times K)} f(x,y)\,d\mu_{H\times K}(x,y),$$

and use disintegration, we have

$$\begin{aligned} &\int\int_{(G\times G)/(H\times K)} f(x,y)\,d\mu_{H\times K}(x,y) \\ &\quad = \int_{D\in\Upsilon}\int_{\underline{t}\in\Lambda/(H\times K)^{(x,y)}\cap\Lambda} f(xt,yt)\,d\mu_{x,y}(t,t)\,d\mu_{(H,K)}(D) \\ &\quad = \int_{D\in\Upsilon}\int_{t\in G/(H^x\cap K^y)} f(xt,yt)\,d\mu_{x,y}(t)\,d\mu_{(H,K)}(D). \end{aligned} \tag{3.4}$$

Now, (3.3) and (3.4) yield

$$\begin{aligned} &\lambda_H(xts^{-1},s)\lambda_K(yts^{-1},s)\lambda_{H^x\cap K^y}(t,s^{-1}) = 1, \\ &s\in G,\ \text{almost everywhere } t\in G/(H^x\cap K^y); \end{aligned} \tag{3.5}$$

or, using cocycle property (c) in Section 2,

$$\lambda_{H^x}(ts^{-1},s)\lambda_{K^y}(ts^{-1},s)\lambda_{H^x\cap K^y}(t,s^{-1}) = 1, \tag{3.6}$$

for $s \in G$, almost everywhere $t \in G/(H^x \cap K^y)$ and for almost all $(x,y) \in (G \times G)/(H \times K)$.

Fixing such an (x,y) in $(G \times G)/(H \times K)$, and invoking continuity of λ_H and λ_K, we see that (3.6) is true for all $t \in G/(H^x \cap K^y)$. Furthermore, (3.6) implies that $\lambda_{H^x \cap K^y}(t,s)$ is defined everywhere and continuous on $(G/(H^x \cap K^y)) \times G$. □

Proof of Corollary 1.2. Setting $t = s$ in (1.1) and using property (a) of the cocycles in Section 2, we obtain

$$\lambda_{H^x}(e,s)\lambda_{K^y}(e,s) = \lambda_{H^x \cap K^y}(e,s). \tag{3.7}$$

Now, we use property (b) of the cocycles in Section 2 to obtain

$$\frac{\Delta_{H^x}(s)}{\Delta_G(s)} \frac{\Delta_{K^y}(s)}{\Delta_G(s)} = \frac{\Delta_{H^x \cap K^y}(s)}{\Delta_G(s)} \tag{3.8}$$

This leads to the required equality

$$\frac{\Delta_{H^x}(s)}{\Delta_G(s)} \frac{\Delta_{K^y}(s)}{\Delta_{H^x \cap K^y}(s)} = 1. \tag{3.9}$$ □

Remarks 3.1.

• We emphasize that the result is an almost everywhere statement on the product space $G/H \times G/K$. If $H = K$, the diagonal $\{(x,x) : x \in G/H\}$ will normally have zero measure. Indeed, if it has non-zero measure, so that our results allow us to make statements regarding the comodularity of

$$G \supset H^x = H^x \cap H^x,$$

the quotient space G/H is discrete, and thus, H is an open subgroup. In that case, it is already well known (and trivial) that $\Delta_G(h) = \Delta_{H^x}(h)$ for all $h \in H^x$ and x.

• If we consider the special case where $K = e$, we have $K^y = e$ for all $y \in G$, giving $s = e$. The conclusion from Corollary 1.2 in this case is trivial.

• If H is a normal subgroup of G, then $H^x = H$ for all $x \in G$, and we have $\Delta_H(s) = \Delta_G(s)$ in consequence of the normality. Here, with an application of Fubini's theorem, cf., [**3,** page 153], Corollary 1.2 becomes

$$\Delta_{H^x \cap K^y}(s) = \Delta_{K^y}(s) \quad \text{for } s \in H \cap K^y.$$

However, this remains a fact, since $H \cap K^y$ is normal in K^y.

Proof of Corollary 1.3. If $G \supset H$ is comodular, then so is $G \supset H^x$ for all $x \in G$. An application of Corollary 1.2 implies that $H^x \supset H^x \cap K^y$ is comodular for almost all x and y. By conjugating with y^{-1} and using Fubini's theorem, it then follows by conjugation that

$$K \supset H^z \cap K$$

is comodular for almost all $z \in G$. □

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