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PARTIAL REPRESENTATIONS AND THEIR DOMAINS

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In memory of Boris Novikov

ABSTRACT. We study the structure of the partially ordered set of the elementary domains of partial (linear or projective) representations of groups. This provides an important information on the lattice of all domains. Some of these results are obtained through structural facts on the ideals of the semigroup $\mathcal{S}_3(G)$, a quotient of Exel's semigroup $\mathcal{S}(G)$, which plays a crucial role in the theory of partial projective representations. We also fill a gap in the proof of an earlier result on the structure of partial group representations.

1. Introduction. Partial group representations were introduced in the theory of operator algebras by Exel [**13**], see also [**12**], and independently by Quigg and Raeburn [**20**], as an important ingredient of a new approach to C^*-algebras generated by partial isometries (see [**2**]). Similarly as in the case of usual representations, there is an algebra, called the partial group algebra of a group G, which governs the partial representations of G. A decomposition result for partial group algebras of finite groups was obtained in [**4**], see also [**8**], whereas the structure of partial representations of an arbitrary G were studied in [**11**], where it was shown that the so-called elementary partial representations, together with the irreducible (indecomposable) representations, of subgroups of G are building pieces from which the irreducible (indecomposable) partial representations of G can be constructed. In [**5, 6, 7**], the theory of partial projective representations was developed, in particular, the structure of the partial Schur multiplier was investigated.

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Other results on partial group algebras, partial representations and the partial Schur multiplier were obtained in [**3, 9, 10, 15, 17, 18**], (also see the survey [**19**]).

According to [**6**], the partial Schur multiplier $pM(G)$ is a semilattice of abelian groups $pM_X(G)$, called components, where X runs over the domains of the partial projective representations of G. It was shown that the domains X are exactly the $\mathcal{T}$-invariant subsets of $G \times G$, where $\mathcal{T}$ is a semigroup of order 25 acting on $G \times G$. The structure of $\mathcal{T}$ does not depend upon G, and it is a disjoint union of the symmetric group S_3 and an ideal which is a completely 0-simple semigroup. A rather good understanding of the components $pM_X(G)$ was achieved in [**6, 7**], in particular, any $pM_X(G)$ is an epimorphic image of the total component $pM_{G\times G}(G)$. The latter contains the classical Schur multiplier $M(G)$ of G; however, $pM_{G\times G}(G)$ is essentially larger than $M(G)$. In regards to the domains, a lattice is formed with respect to the set-theoretic inclusion, intersection and union. We know from [**6**] that the domains X of partial projective representations of a finite group G are exactly the finite unions of elementary domains. An elementary domain is understood to be the domain of an elementary partial representation (which, obviously can be seen as a projective partial representation with idempotent factor set).

Our main purpose is to heighten our understanding of the lattice $C(G)$ of domains of partial (projective or linear) representations by concentrating on their structural pieces, the elementary domains. The article is organized as follows. In the first part of Section 2, see subsection 2.1, we present a correction to the proof of a structural result [**11,** Theorem 2.2] on partial representations. In the second part, subsection 2.2, we give some preliminary facts and clarify a remark already mentioned by experts in semigroups, according to which the structural result on the partial group algebra of a finite group given in [**4**] can be obtained by using the theory of semigroup algebras. This is seen in detail in Remark 2.5, and we do not claim novelty, neither for the idea nor for the proof. The latter is based on the fact that the partial group algebra is the semigroup algebra of Exel's semigroup $\mathcal{S}(G)$, introduced by Exel in [**13**] to deal with partial actions and partial representations. The semigroup $\mathcal{S}(G)$ is also known to be isomorphic to the Birget-Rhodes expansion of G, see subsection 2.2.

The theory of partial projective representations depends heavily upon the semigroup $\mathcal{S}_3(G)$, which is a quotient of $\mathcal{S}(G)$. There is an anti-isomorphism between $C(G)$ and the lattice of the non-trivial ideals of $\mathcal{S}_3(G)$, which we recall at the beginning of Section 3, where we also give some facts regarding the ideals of $\mathcal{S}_3(G)$ for further use. This anti-isomorphism is applied in the following sections to translate results on ideals in $\mathcal{S}_3(G)$ into facts about $C(G)$, and vice versa.

The main part of the paper begins in subsection 3.1, in which we characterize the minimal and irreducible elementary domains in terms of $\mathcal{T}$-orbits. An irreducible domain is called a *block* if it is not minimal, i.e., it is not an atom in the lattice of the domains. We prove the uniqueness of the decomposition of a domain into blocks and atoms in Corollary 3.19, which is a consequence of the more general Proposition 3.18. Corollary 3.21 translates the latter result into the language of ideals in $\mathcal{S}_3(G)$, implying, in particular, the uniqueness of a decomposition of a non-zero ideal in $\mathcal{S}_3(G)$ into an intersection of meet-prime ideals. Section 4 is dedicated to the decomposition of elementary domains in the case of a finite G. In Theorem 4.3, we give the decomposition of a non-minimal elementary domain into blocks and prove its uniqueness under the assumption of minimality. Corollary 4.8 states the corresponding fact for the ideals in $\mathcal{S}_3(G)$. Proposition 4.14 characterizes the finite groups G with irreducible elementary domains. In Section 5, Theorem 5.1 determines some invariants of $C(G)$, giving the formulas for the number of minimal elementary domains as well as for the numbers of blocks according to their type. The results are used in Proposition 5.4 to correct [**7,** Proposition 6.1]. Finally, Proposition 5.11 gives invariants for the partially ordered set of the $\mathcal{T}$-orbits, and we list some properties of the lattice $C(G)$ in Proposition 6.2 of Section 6.

2. Partial representations and the partial group algebra $K_{\text{par}}G$. Let G be a group, K a field and V a vector space over K. We recall from [**4**] that a partial representation of G on V is a map

$$\pi : G \longrightarrow \operatorname{End}_K(V),$$

which sends 1_G to the identity operator and such that

$$\pi(g^{-1})\pi(gh) = \pi(g^{-1})\pi(g)\pi(h)$$

and

$$\pi(gh)\pi(h^{-1}) = \pi(g)\pi(h)\pi(h^{-1}),$$

for all $g, h \in G$.

It is well known that the group algebra KG controls the theory of K-representations of G; similarly, the *partial group algebra* $K_{\mathrm{par}}G$, which is the semigroup algebra $K\mathcal{S}(G)$, governs the partial representations of G. For the reader's convenience, we recall that $\mathcal{S}(G)$ is the monoid generated by the symbols $\{[g] \mid g \in G\}$ with defining relations:

$$[g^{-1}][g][h] = [g^{-1}][gh],$$
$$[g][h][h^{-1}] = [gh][h^{-1}]$$

and

$$[g][1] = [g],$$

(it follows that $[1][h] = [h]$). This monoid was introduced by Exel in **[13]**.

2.1. On the structure of partial representations. Let G be an arbitrary group. In order to study the structure of partial representations of G, it is useful to consider the groupoid $\Gamma(G)$ which was introduced in **[4]** for the case of a finite G and with non-necessarily finite G as used in **[11]** to investigate the partial representations of G of finite degree. The groupoid $\Gamma(G)$ is the small category whose objects are the subsets $A \subseteq G$ containing 1 and whose morphisms are (A, g), where $g \in G$ and A is a subset of G containing 1 and g^{-1}. The composition rule in $\Gamma(G)$ is defined for the pairs (A, g) and (B, h), such that $A = hB$, in which we define $(hB, g) \cdot (B, h) = (B, gh)$.

It follows that the identity morphisms in $\Gamma(G)$ are of the form $(A, 1)$, and the inverse of (A, g) is (gA, g^{-1}). The groupoid $\Gamma(G)$ can be considered as an oriented graph whose vertices are labeled by the objects of $\Gamma(G)$, and each morphism (A, g) in $\Gamma(G)$ gives an arrow

$$A \longrightarrow gA.$$

Given an object (vertex) A of $\Gamma(G)$, we take the connected component of $\Gamma(G)$ containing A and denote its set of vertices by $\mathcal{V}_A$. Evidently, the stabilizer

$$H = \mathrm{St}\, A = \{g \in G \mid gA = A\}$$

is a subset of A, and A is a union of right cosets of H, i.e.,

$$A = \bigcup_{i \in \mathcal{I}} Hg_i$$

for some index set $\mathcal{I}$ and some transversal

$$\{g_i \mid i \in \mathcal{I}\} \subseteq A$$

containing 1. Clearly, the cardinality $|\mathcal{V}_A|$ of $\mathcal{V}_A$ equals $|\mathcal{I}|$, and $\mathcal{V}_A$ consists of the vertices of the form $g_i^{-1}A$, $i \in \mathcal{I}$.

For a small category Γ, the category algebra $K\Gamma$ is defined by taking the free K-module whose basis is formed by the morphisms of Γ and defining the multiplication in the following way:

$$\gamma_1 \cdot \gamma_2 = \begin{cases} \gamma_1\gamma_2 & \text{if the composition } \gamma_1\gamma_2 \in \Gamma \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Let Δ be a connected component of $\Gamma = \Gamma(G)$ with a finite number of vertices. Define the map

$$\lambda_\Delta : G \longrightarrow K\Delta$$

by:

$$\lambda_\Delta(g) = \begin{cases} \sum_{A \in \mathcal{V}_\Delta,\, A \ni g^{-1}} (A, g) & \text{if there is an } A \in \mathcal{V}_\Delta \text{ with } g^{-1} \in A, \\ 0 & \text{otherwise.} \end{cases}$$

The next fact from **[11]** describes the structure of the partial representations of G of finite degree.

Theorem 2.1. **[11,** Theorem 2.2]. *Let G be an arbitrary group and V a vector space over a field K. For any connected component Δ of $\Gamma(G)$ with a finite number of vertices, the map*

$$\lambda_\Delta : G \longrightarrow K\Delta$$

is a partial representation of G in $K\Delta$. Moreover:

(i) *For every irreducible (respectively, indecomposable) K-representation*

$$\varphi : K\Delta \longrightarrow \mathrm{End}(V)$$

of finite degree, $\varphi \circ \lambda_\Delta$ is an irreducible (respectively, indecomposable) partial representation of G.

(ii) *Conversely, for every irreducible (respectively, indecomposable) partial K-representation*

$$\pi : G \longrightarrow \mathrm{End}(V)$$

of finite degree, there is a unique connected component Δ of $\Gamma(G)$ with a finite number of vertices and a unique representation

$$\widetilde{\pi} : K\Delta \longrightarrow \mathrm{End}(V)$$

such that $\widetilde{\pi} \circ \lambda_\Delta = \pi$.

Next, we fill a gap in the proof of [**11,** Theorem 2.2]. In order to do this, we state a fact which was used in the proof of the above result but was not properly justified, see [**11,** pages 314, 315].

Proposition 2.2. *Using the notation of Theorem* 2.1*, let Δ be a connected component of $\Gamma(G)$, the number of vertices of which $|\mathcal{V}_\Delta|$ is finite. Then, the K-algebra $\mathcal{A}$ generated by the elements*

$$\{\lambda_\Delta(g) \mid g \in G\}$$

coincides with $K\Delta$.

Proof. In what follows, if $(A, g) \in \Gamma(G)$, then A will be called the *support set* of (A, g). Denote by $B_1, \ldots, B_m$ the vertices of the component Δ, and let (B, h) be an arbitrary element of this component. Let $t_2, \ldots, t_k$ be the only indices t_i for which $B \setminus B_{t_i} \neq \emptyset$, and choose an element $x_i \in B \setminus B_{t_i}$ for each $2 \leq i \leq k$. For convenience, also define $x_0 = x_1 = 1$ and $x_{k+1} = h^{-1}$. Then,

$$A_0 = \{x_1, x_2, \ldots, x_{k+1}\} \subseteq B.$$

Define $g_1, g_2, \ldots, g_{k+1} \in G$ by means of the equality $g_i = x_i^{-1} x_{i-1}$ for $i \in \mathcal{I} = \{1, \ldots, k+1\}$. In particular, we have $g_1 = x_1 = 1$, as well as

$g_2 = x_2^{-1}$ and $g_{k+1} = hx_k$. Note that, if $i \in \mathcal{I}$, then g_i^{-1} belongs to $x_{i-1}^{-1}B \in \mathcal{V}_\Delta$. Consequently,

$$\begin{aligned}\lambda_\Delta(g_{k+1})\cdots\lambda_\Delta(g_1) &= \sum_{\substack{g_i^{-1}\in A_i\in\mathcal{V}_\Delta\\ i\in\mathcal{I}}} (A_{k+1}, g_{k+1})\cdot\cdots\cdot(A_1, g_1)\\ &= \sum_{\substack{g_1^{-1}\in A_1\in\mathcal{V}_\Delta\\ g_2^{-1}\in g_1A_1\\ \vdots\\ g_{k+1}^{-1}\in g_k\cdots g_1A_1}} (A_1, g_{k+1}\cdot\cdots\cdot g_1)\\ &= \sum_{\substack{g_1^{-1}\in A_1\in\mathcal{V}_\Delta\\ (g_2g_1)^{-1}\in A_1\\ \vdots\\ (g_{k+1}\cdots g_1)^{-1}\in A_1.}} (A_1, h)\end{aligned}$$

Since $g_1 = 1 = x_1^{-1}$ and $(g_2g_1) = x_2^{-1}, \ldots, (g_{k+1}\cdots g_1) = h = x_{k+1}^{-1}$, we have

$$\lambda_\Delta(g_{k+1})\cdots\lambda_\Delta(g_1) = \sum_{A_0\subseteq A\in\mathcal{V}_\Delta}(A,h) = \sum_{B\subseteq A\in\mathcal{V}_\Delta}(A,h). \tag{2.1}$$

The last equality in (2.1) is a consequence of the fact that $A_0 \not\subseteq B_{t_i}$ for every $i \in \{2, \ldots, k\}$, by definition of A_0, and thus, in the above sum every $A \supseteq B$.

We will complete the proof using induction on $d(B)$, where the function $d : \mathcal{V}_\Delta \to \mathbb{N}$ is defined below. If A is maximal, that is, if

$$A \subseteq A' \text{ for some } A' \in \mathcal{V}_\Delta \text{ implies } A = A',$$

we set $d(A) = 0$. Otherwise, we define

$$\begin{aligned}d(A) = \max\{n \in \mathbb{N} \mid \text{there exists an } A_1, \ldots, A_n \in \mathcal{V}_\Delta,\\ \text{with } A_n \supsetneq \cdots \supsetneq A_1 \supsetneq A\}.\end{aligned}$$

If the support of (B, h) matches the first case, then B is not contained in the support of any other element of Δ, and therefore, (2.1) is reduced to

$$\lambda_\Delta(g_{k+1})\cdots\lambda_\Delta(g_1) = (B,h).$$

Thus, the algebra $\mathcal{A}$ contains every (B,h) whose support is maximal.

Let $d(B) = k > 0$, and assume by induction that every $(A,g) \in \Delta$ satisfying

$$0 \le d(A) < k$$

is in $\mathcal{A}$. Then, writing (2.1) in the form:

$$\lambda_\Delta(g_{k+1})\cdots\lambda_\Delta(g_1) = (B,h) + \sum_{B\subsetneq A\in\mathcal{V}_\Delta}(A,h),$$

it is easily seen that, if $(A,h) \in \Delta$ and $B \subsetneq A$, then $d(A) < k = d(B)$. By induction, it follows that $(A,h) \in \mathcal{A}$ and

$$(B,h) = \lambda_\Delta(g_{k+1})\cdots\lambda_\Delta(g_1) - \sum_{B\subsetneq A\in\mathcal{V}_\Delta}(A,h) \in \mathcal{A}. \quad \square$$

2.2. On the structure of $K_{\mathrm{par}}(G)$. Here, we show that the structural result from [**4,** Section 3] on partial group algebras can be obtained using the theory of semigroup algebras. This was previously mentioned in private communications by experts in semigroups, so this idea is not new; nevertheless, we prefer to give some details here in order to make this clear. First, we record some basic facts and notation from semigroup theory. Let S be a semigroup, and $a \in S$. Then:

- $\mathcal{I}_a = S^1 a S^1$, is the *principal ideal* of S *generated by* a, and $\mathcal{J}_a$ is the $\mathcal{J}$*-class* containing a, i.e., the set of elements of $\mathcal{I}_a$ which generate $\mathcal{I}_a$. Let $J(a) = \mathcal{I}_a \setminus \mathcal{J}_a$. Then, $J(a)$ is an ideal of S [**1,** page 72], and each factor semigroup $\mathcal{J}_a^0 = \mathcal{I}_a/J(a)$ is called a *principal factor* of S. In particular, each principal factor of an inverse semigroup is inverse. The $\mathcal{J}$-classes will be ordered by $\mathcal{J}_a \le \mathcal{J}_b$ if, and only if, $\mathcal{I}_a \subseteq \mathcal{I}_b$.
- $E(S) = \{e \in S \mid e^2 = e\}$, and there is a partial order defined in $E(S)$ by

$$e \le f \Longleftrightarrow e = ef = fe.$$

- If S is a semigroup with 0, then the quotient $K_0S = KS/K\theta$ is called the *contracted semigroup algebra*, where

$$K\theta = \{\lambda 0 \mid \lambda \in K\}.$$

If S has no 0, we shall write $K_0 S = KS$.

- By the Rees-Sushkevich theorem, see [**1**], all *completely 0-simple semigroups* are described by the following construction.

  Let H be a group, $\mathcal{I}$ and Δ arbitrary sets, 0 a symbol and $P = (p_{\lambda,i})_{(\lambda,i)\in\Delta\times\mathcal{I}}$ a matrix with entries in $H \cup 0$ such that each row and each column contains an element from H. Denote by $S^0(H;\mathcal{I},\Delta;P)$ the set

  $$\{\langle i,h,\lambda\rangle \mid i\in\mathcal{I}, \lambda\in\Delta, h\in H\} \cup 0,$$

  with multiplication

  $$\langle i,g,\lambda\rangle \cdot \langle j,h,\mu\rangle = \begin{cases} \langle i, gp_{\lambda,j}h, \mu\rangle & \text{if } p_{\lambda,j} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

  $$0 \cdot \langle i,g,\lambda\rangle = \langle i,g,\lambda\rangle \cdot 0 = 0 \cdot 0 = 0.$$

  In particular, completely 0-simple inverse semigroups are those where $\mathcal{I}$ and Δ can be chosen to be equal, and P to be Id, the identity matrix, where the entries in the main diagonal of P are equal to the identity element of H, and all other entries are 0. Note that a completely simple inverse semigroup is necessarily a group.
- Similarly, one defines a Munn algebra $\mathfrak{M}(R;\mathcal{I},\Delta;P)$ whose elements are matrices $\mathcal{I}\times\Delta$ over a K-algebra R with finitely many non-zero entries. The product of A and B is defined as APB, for all $A,B\in\mathfrak{M}(R;\mathcal{I},\Delta;P)$, whereas the addition and multiplication by scalars are defined as usual.
- If $\mathcal{I}$ and Δ are finite of orders m and n, respectively, one writes $S^0(H;\mathcal{I},\Delta;P) = S^0(H;m,n;P)$ and $\mathfrak{M}(R;\mathcal{I},\Delta;P) = \mathfrak{M}(R;m,n;P)$.

Let G be a finite group. It follows from [**4,** Corollary 2.7 and Theorem 3.2] that there is a K-algebra isomorphism

$$\psi : K_{\mathrm{par}}G \longrightarrow \bigoplus M_m(KH), \tag{2.2}$$

where the H's are subgroups of G and $m\in\mathbb{N}$. Every subgroup H of G appears in the above decomposition, and summands with the same m

and H may repeat several times; the interested reader should consult [**8,** Theorem 2.1] for the details on multiplicities.

As suggested in [**6,** Remark 1], it is possible to establish (2.2) by means of the theory of semigroup algebras. We shall note this in Remark 2.5, after stating some facts.

Lemma 2.3. *Let S be a semigroup. Then, each ideal I of S is a join of the principal ideals of S contained in I, and, for every $a, b \in S$, we have*:

- $\mathcal{I}_a = \bigcup\{\mathcal{J}_x \mid x \in S \text{ such that } \mathcal{J}_x \leq \mathcal{J}_a\}$.
- $\mathcal{J}_{ab} \leq \mathcal{J}_a$ *and* $\mathcal{J}_{ab} \leq \mathcal{J}_b$.

If S is an inverse semigroup, then:

- $\mathcal{J}_a = \mathcal{J}_{aa^{-1}} = \mathcal{J}_{a^{-1}a} = \mathcal{J}_{a^{-1}}$, *for every $a \in S$, and thus, each principal ideal is generated by an idempotent.*
- $\mathcal{J}_e \leq \mathcal{J}_f$ *for every $e, f \in E(S)$ with $e \leq f$, and consequently,*

$$\mathcal{I}_e = \bigcup\{\mathcal{J}_{e'} \mid e' \in E(S) \text{ such that } e' \leq e\}.$$

It is shown [**14**] that there is a semigroup isomorphism

$$\mathcal{S}(G) \simeq \widetilde{G}^{\mathcal{R}} = \{(A, g) \mid \{1, g\} \subseteq A \subseteq G, |A| < \infty\},$$

where the multiplication in $\widetilde{G}^{\mathcal{R}}$ is given by the rule $(A, g)(B, h) = (A \cup gB, gh)$.

The isomorphism is induced by the map

$$[g] \longmapsto (\{1, g\}, g).$$

The latter implies that $\mathcal{S}(G)$ is isomorphic to the Birget-Rhodes expansion of G, see [**14, 21**] for details. In particular, $\mathcal{S}(G)$ is an inverse semigroup. From this point on, $\mathcal{S}(G)$ stands for $\widetilde{G}^{\mathcal{R}}$.

Lemma 2.4. *For any group G, the following statements hold*:

- $E(\mathcal{S}(G)) = \{(A, 1) \mid \{1\} \subseteq A \subseteq G, |A| < \infty\}$, *and* $(A, 1) \leq (B, 1)$ *if, and only if,* $A \supseteq B$.

- **[6,** Corollary 2]. *The $\mathcal{J}$-class of an element $(A, g) \in \mathcal{S}(G)$ is the set*

$$\mathcal{J}_A = \{(a^{-1}A, a^{-1}b) \in \mathcal{S}(G) \mid a, b \in A\},$$

and thus, $\mathcal{J}_A = \mathcal{J}_B$ for some B with $\{1\} \subseteq B \subseteq G$ and $|B| < \infty$ if, and only if, $B = a^{-1}A$ for some $a \in A$.
- **[6,** Lemma 3]. *The principal factor $\mathcal{J}_A^0$ of $\mathcal{S}(G)$ corresponding to the $\mathcal{J}$-class $\mathcal{J}_A$ is isomorphic to $S^0(H; m, m; \mathrm{Id})$, where*

$$H = \mathrm{St}\, A = \{g \in G \mid gA = A\} \subseteq A$$

and $m = |A|/|\mathrm{St}\, A|$.
- *With the notation of the above item, we have*

$$K_0\mathcal{J}_A^0 \simeq K_0S^0(H; m, m; \mathrm{Id}) \simeq M_m(KH).$$

Proof. We only prove the last item. From **[1,** Lemma 5.17], we obtain $K_0S^0(G; m, n; P) \simeq \mathfrak{M}(KG; m, n; P)$ for any group G. In particular, taking $G = H$, $m = n = |A|/|H|$ and $P = \mathrm{Id}$, we see that $\mathfrak{M}(KH; m, m, \mathrm{Id}) = M_m(KH)$, and

$$\phi : K_0S^0(H; m, m, \mathrm{Id}) \longrightarrow M_m(KH),$$

given by

$$\phi\bigg(\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{g\in H} c_{i,j,g}\langle i, g, j\rangle\bigg) = \sum_{i=1}^{m}\sum_{j=1}^{m}\bigg(\sum_{g\in H} c_{i,j,g}\, g\bigg)e_{i,j},$$

is an isomorphism, where $e_{i,j}$ stands for the matrix unit. □

Remark 2.5. It is known **[16,** Chapter 5, Corollary 27] that, if S is a finite inverse semigroup, then

$$K_0S \simeq \bigoplus_{i=1}^{r} K_0T_i,$$

where $T_1, \dots, T_r$ are the non-zero principal factors of S. In particular **[4]**, for a finite group G,

$$K_{\mathrm{par}}(G) \simeq \bigoplus_{i\in\mathcal{I}} M_{m_i}(KH_i),$$

and, in this case, H_i is a subgroup of G for every $i \in \mathcal{I}$. Indeed, for $A \subseteq G$ containing 1, it follows from the third and fourth items of Lemma 2.4 that, for the principal factor $\mathcal{J}_A^0$, we have that $K_0\mathcal{J}_A^0$ is isomorphic to

$$K_0 S^0(H; m, m, \mathrm{Id}) \simeq M_m(KH)$$

for some $m \in \mathbb{N}$, where H is a subgroup of G.

3. Domains of partial factor sets and some remarks on the ideal structure of the semigroup $\mathcal{S}_3(G)$. In [**5,** page 261], the authors introduced the monoid $\mathcal{T}$ generated by symbols u, v and t with relations $u^2 = v^2 = (uv)^3 = 1$, $t^2 = t$, $ut = t$, $tuvt = tvuv$, $tvt = 0$. Then, there is a disjoint union

$$\mathcal{T} = \mathcal{S} \cup t\mathcal{S} \cup vt\mathcal{S} \cup uvt\mathcal{S} \cup 0,$$

where

$$\mathcal{S} = \langle u, v \mid u^2 = v^2 = (uv)^3 = 1 \rangle$$

is a group isomorphic to the symmetric group S_3.

Given an arbitrary group G, there is a left action of $\mathcal{T}$ on $G \times G$ defined by means of the following transformations:

$$u(x,y) = (xy, y^{-1}), \quad v(x,y) = (y^{-1}, x^{-1}), \quad t(x,y) = (x, 1). \tag{3.1}$$

Then, $0(x,y) = (1,1)$, for all $x, y \in G$, and there is an action of S_3 on $G \times G$ induced by $\mathcal{T}$. The S_3-orbit of a pair $(x,y) \in G \times G$ is of the form

$$\begin{aligned} S_3(x,y) = \{&(x,y), (xy, y^{-1}), (y, y^{-1}x^{-1}), \\ &(y^{-1}, x^{-1}), (y^{-1}x^{-1}, x), (x^{-1}, xy)\}, \end{aligned} \tag{3.2}$$

for all $x, y \in G$.

Remark 3.1. The importance of the monoid $\mathcal{T}$ resides in the fact that the $\mathcal{T}$-subsets D of $G \times G$, that is, the elements of

$$C(G) = \{D \subseteq G \times G \mid \mathcal{T}D \subseteq D\} = \{D \subseteq G \times G \mid \mathcal{T}D = D\},$$

are precisely the domains of the partial factor sets of G, see [**5,** Corollary 7 and Theorem 5].

Note that $(C(G), \cap, \cup)$ is a complete lattice, and it follows from (3.1) that $\{(1,1)\}$ and $G \times G$ are the zero and the unity elements of $C(G)$, respectively.

Our goal is to describe the lattice $(C(G), \cap, \cup)$. For this, we consider the quotient semigroup

$$\mathcal{S}_3(G) = \{(A, a) \in \mathcal{S}(G) \mid |A| \leq 3\} \cup \{0\}$$

of $\mathcal{S}(G)$, see [**6,** page 447]. Denote by $Y^*(\mathcal{S}_3(G))$ the set of ideals of $\mathcal{S}_3(G)$ different from $\mathcal{S}_3(G)$ and the empty set. Then, from [**7,** Proposition 5.3], there is a lattice isomorphism

$$\iota : (C(G), \cap, \cup) \ni X \longrightarrow \iota(X) \in (Y^*(\mathcal{S}_3(G)), \cup, \cap), \tag{3.3}$$

where X and $\iota(X)$ are related by

$$(\{1, a, ab\}, ab) \in \iota(X) \Longleftrightarrow (a, b) \notin X. \tag{3.4}$$

Observe that

$$(\{1, a\}, a) \in \iota(X) \Longleftrightarrow (a, 1) \notin X$$

for any $a \in G$.

From the second item of Lemma 2.4, the $\mathcal{J}$-classes of $\mathcal{S}(G)$ can be indexed by subsets of G containing the identity. Furthermore, if $A = \{1, a\}$, we denote $\mathcal{J}_A$ by $\mathcal{J}_a$, and, if $A = \{1, a, b\}$ has three elements, $\mathcal{J}_A$ is denoted by $\mathcal{J}_{a,b}$. Then, by Lemma 2.3, we have the following.

Proposition 3.2. *Let $a, b \in G$. Then,*

- *$\mathcal{J}_a = \mathcal{J}_b$ if and only if $a = b$ or $ab = 1$.*
- *If $a^2 \neq 1$, then $\mathcal{J}_{a,a^2} = \mathcal{J}_{a^{-1},a} = \mathcal{J}_{a^2,a^{-1}} < \mathcal{J}_a$.*
- *If $b \notin \{a, a^{-1}, a^2\}$, then*

$$\mathcal{J}_{a,\ b} = \mathcal{J}_{a^{-1},a^{-1}b} = \mathcal{J}_{b^{-1},b^{-1}a} < \mathcal{J}_a, \mathcal{J}_b, \mathcal{J}_{a^{-1}b}.$$

- *All other non-zero $\mathcal{J}$-classes in $\mathcal{S}_3(G)$ are incomparable.*

Proposition 3.2 yields

$$I_{a,b} = \mathcal{J}_{a,b} \cup \{0\} \quad \text{and} \quad I_a = \bigcup_{x \in G \setminus \{1,a\}} \mathcal{J}_{a,x} \cup \mathcal{J}_a \cup \{0\}, \tag{3.5}$$

for any $a, b \in G \setminus \{1\}$ with $a \neq b$, where I_a and $I_{a,b}$ are the ideals of $\mathcal{S}_3(G)$ related to the $\mathcal{J}$-classes $\mathcal{J}_a$ and $\mathcal{J}_{a,b}$, respectively. It follows from (3.5) that the zero ideal

$$\mathbf{z} = \bigcup_{a \in G \setminus \{1\}} I_a = \mathcal{S}_3(G) \setminus (\{1\}, 1)$$

is the maximum of the poset $(Y^*(\mathcal{S}_3(G)), \subseteq)$. Moreover, in the poset of principal ideals of $\mathcal{S}_3(G)$, the ideals $I_{a,b}$, where $a, b \in G \setminus \{1\}$ with $a \neq b$ are minimal, and those of the form I_a are maximal. Furthermore, I_a is also minimal if and only if $G = C_2$.

Definition 3.3. An element I of $Y^*(\mathcal{S}_3(G))$ will be called *maximal* if it is maximal in the poset $(Y^*(\mathcal{S}_3(G)) \setminus \{\mathbf{z}\}, \subseteq)$.

Let $X \subseteq G$ and $I \in Y^*(\mathcal{S}_3(G))$. To simplify notation, we write

$$I_X = \bigcup_{x \in X} I_x,$$

in particular, $\mathbf{z} = I_{G \setminus \{1\}}$.

Proposition 3.4. *If I is a maximal element of $Y^*(\mathcal{S}_3(G))$, then there is an $a \in G$ such that*

$$I = I_{G \setminus \{1,a\}} \cup I_{a,a^{-1}},$$

where $I_{a,a^{-1}} = \emptyset$ if $a = a^{-1}$.

Proof. Let I be a maximal ideal from $Y^*(\mathcal{S}_3(G))$, and take $(A, 1) \in \mathbf{z} \setminus I$, where $A = \{1, a, b\}$. Then, one of the elements $(\{1, a\}, 1)$ or $(\{1, b\}, 1)$ is not in I. Without loss of generality, suppose that $(\{1, a\}, 1) \notin I$. Since I is maximal, and for any $y \in G \setminus \{1, a\}$, we have

$$I \subseteq I \cup I_y \subsetneq \mathbf{z}.$$

We conclude that

$$I \supseteq (\mathbf{z} \setminus I_a) = I_{G \setminus \{1,a\}}.$$

If $I \supsetneq I_{G \setminus \{1,a\}}$, there exists a $(B, b) \in I \setminus I_c$ for all $c \neq a$, which implies $\{1, c\} \nsubseteq B$ and $\{1, c^{-1}\} \nsubseteq B$. Thus, c, $c^{-1} \notin B$ for all $c \neq a$, and, since $|B| \geq 2$, we have $B = \{1, x, y\}$, where $x, y \in \{a, a^{-1}\}$. If

$x = y$, then $B = \{1, x\}$ and $I_a = I_x \subseteq I$, a contradiction. Hence, $x \neq y$, $B = \{1, a, a^{-1}\}$, $o\,(a) > 2$ and $I = (\mathbf{z} \setminus I_a) \cup I_{a,a^{-1}}$, which completes the proof. $\square$

3.1. Minimal and irreducible domains. Let G be a finite group, Pr the projection of $\bigoplus M_m(KH)$ onto the matrix algebra $M_m(KH)$, and consider the map

$$[\,]\colon G \ni g \longmapsto [g] \in K_{\mathrm{par}}G.$$

A function of the form

$$\varphi = \mathrm{Pr} \circ \psi \circ [\,] : G \longrightarrow M_l(KH),$$

where ψ is given by (2.2), is called an *elementary partial representation* of G, and the set

$$D = \{(x, y) \in G \times G \mid \varphi(x)\, \varphi(y) \neq 0\}$$

is called an *elementary domain.*[1]

Here, we recall an algorithm that determines the elementary partial representations of G (for more details, see [**4,** Section 3] or [**17,** Section 2]).

Let H be a group. We denote by $e_{i,j}(h)$ the matrix in $M_m(KH)$ whose (i, j)-entry is h and all other entries are zero. Below, we shall write $e_{i,j} = e_{i,j}(1_H)$.

Let A be a vertex in the groupoid $\Gamma(G)$, $H = \mathrm{St}\, A$, and let

$$A = \bigcup_{i=1}^{m} H g_i$$

be a disjoint union, where $g_1 = 1$. Denote $\tau = \{g_1, \ldots, g_m\}$. We say that τ is a right transversal of H in A. The elementary partial representation

$$\varphi_{A,\tau}\colon G \longrightarrow M_m(KH)$$

is obtained as follows. For any $x \in G$, set

$$\mathcal{I}_x^{A,\tau} = \{i \in \{1, \ldots, m\} \mid g_i x \in A\}.$$

If $\mathcal{I}_x^{A,\tau} = \emptyset$, define $\varphi_{A,\tau}(x) = 0$. If $\mathcal{I}_x^{A,\tau} \neq \emptyset$, then, for each $i \in \mathcal{I}_x^{A,\tau}$, there is a unique element $j = j_{i,x}$ in $\{1, \ldots, m\}$ and $h = h_{i,x} \in H$ such that $g_i x = hg_j$. In this case, we have

$$\varphi_{A,\tau}(x) = \sum_{i \in \mathcal{I}_x^{A,\tau}} e_{i,j}(h). \tag{3.6}$$

Let $D_{A,\tau}$ be the elementary domain corresponding to $\varphi_{A,\tau}$.

Lemma 3.5. *Let G be a finite group and A and A' two vertices lying in the same connected component of $\Gamma(G)$. Furthermore, let τ be a right transversal of $\operatorname{St} A$ in A, and let τ' be a right transversal of $\operatorname{St} A'$ in A'. Then, $D_{A,\tau} = D_{A',\tau'}$.*

Proof. We have that $A' \in \mathcal{V}_A$ and $A' = g_k^{-1} A$ for some $g_k \in \tau = \{g_1 = 1, g_2, \dots, g_m\}$. Then, $H' = \operatorname{St} A' = g_k^{-1}(\operatorname{St} A) g_k$, and $\tau'' = g_k^{-1}\tau$ is a right transversal of $\operatorname{St} A'$ in A'. It is readily seen that

$$\varphi_{A',\tau''} \colon G \longrightarrow M_m(KH')$$

is related to

$$\varphi_{A,\tau} \colon G \longrightarrow M_m(KH)$$

by the formula

$$\varphi_{A',\tau''}(x) = \operatorname{diag}(g_k^{-1}, g_k^{-1}, \dots g_k^{-1})\, \varphi_{A,\tau}(x)\, \operatorname{diag}(g_k, g_k, \dots g_k),$$

for all $x \in G$, and it follows that $D_{A,\tau} = D_{A',\tau''}$. From **[11,** Lemma 3.1], the representations

$$\varphi_{A',\tau''}, \varphi_{A',\tau'} \colon G \longrightarrow M_m(KH')$$

are equivalent, which implies that $D_{A',\tau''} = D_{A',\tau'}$ and from which we conclude that $D_{A,\tau} = D_{A',\tau'}$. □

In view of Lemma 3.5, $D_{A,\tau}$ and $\mathcal{I}^{A,\tau}$ will be denoted by D_A and $\mathcal{I}^A$, respectively.

When G is finite, Lemma 3.5 implies that every connected component of $\Gamma(G)$ gives an elementary domain, which can be obtained from an arbitrary vertex of the component. We shall prove that, for the vertex $A = \{1, a, b\}$, the domain D_A is $\mathcal{T}(a^{-1}, b)$. First, we need some preliminary results.

Lemma 3.6. *Let G be a finite group and A a vertex in $\Gamma(G)$. Then, $(x,y) \in D_A$ if, and only if, there is an $A' \in \mathcal{V}_A$ such that $\{x^{-1}, y\} \subseteq A'$.*

Proof. Let $x, y \in G$. Then, by definition, $(x,y) \in D_A$ is equivalent to $\varphi_{A,\tau}(x)\varphi_{A,\tau}(y) \neq 0$, where τ is a right transversal of $\operatorname{St} A$ in A. Furthermore, the equality $g_i x = h g_j$ holds if and only if $g_j x^{-1} = h^{-1} g_i$, and using (3.6), we obtain

$$(x,y) \in D_A \iff \text{there exists } j \in \mathcal{I}^A_{x^{-1}} \cap \mathcal{I}^A_y \iff g_j x^{-1} \in A$$

and

$$g_j y \in A \iff \{x^{-1}, y\} \subseteq g_j^{-1} A. \qquad \square$$

The next result easily follows from the definition of $\mathcal{T}$.

Lemma 3.7. *Let $x \in G$. Then,*

$$\mathcal{T}(x,1) = \{(1,1), (x,1), (1,x^{-1}), (x^{-1},x), (x^{-1},1), (1,x), (x,x^{-1})\}$$

and

$$\mathcal{T}(x,1) = \mathcal{T}(x^{-1},1) = \mathcal{T}(1,x^{-1}) = \mathcal{T}(1,x) = \mathcal{T}(x^{-1},x) = \mathcal{T}(x,x^{-1}).$$

Now, we calculate the elementary domain associated to the vertex $\{1,a,b\}$.

Proposition 3.8. *Let G be a finite group and $a, b \in G$. Then,*

$$D_{\{1,a,b\}} = \mathcal{T}(1,a) \cup \mathcal{T}(1,b) \cup \mathcal{T}(1,b^{-1}a) \cup S_3(b^{-1},a) = \mathcal{T}(b^{-1},a).$$

In particular, any $\mathcal{T}$-orbit is an elementary domain.

Proof. For the first equality, if $A = \{1,a,b\}$, then

$$\mathcal{V}_A = \{A, \{1, a^{-1}, a^{-1}b\}, \{1, b^{-1}, b^{-1}a\}\},$$

(with repetitions if and only if $A \cong C_3$ or $|A| \leq 2$). From Lemma 3.6, a pair (u,v) is in $D_{\{1,a,b\}}$ if, and only if, $\{u^{-1}, v\} \subseteq \{1,a,b\}$, $\{u^{-1}, v\} \subseteq \{1, a^{-1}, a^{-1}b\}$ or $\{u^{-1}, v\} \subseteq \{1, b^{-1}, b^{-1}a\}$. Checking case by case, we get

$$\begin{aligned} D_{\{1,a,b\}} = \{&(1,1),(1,a),(1,b),(a^{-1},1),(a^{-1},a),(a^{-1},b),(b^{-1},1),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (b^{-1},a),(b^{-1},b)\}\\ \cup\,\{&(1,a^{-1}),(1,a^{-1}b),(a,1),(a,a^{-1}),(a,a^{-1}b),(b^{-1}a,1),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (b^{-1}a,a^{-1}),(b^{-1}a,a^{-1}b)\}\\ \cup\,\{&(1,b^{-1}),(1,b^{-1}a),(b,1),(b,b^{-1}),(b,b^{-1}a),(a^{-1}b,1),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (a^{-1}b,b^{-1}),(a^{-1}b,b^{-1}a)\}. \end{aligned}$$

From Lemma 3.7 and (3.2) we conclude that

$$D_{\{1,a,b\}} = \mathcal{T}(1,a) \cup \mathcal{T}(1,b) \cup \mathcal{T}(1,b^{-1}a) \cup S_3(b^{-1},a).$$

Now, we verify that $D_{\{1,a,b\}} = \mathcal{T}(b^{-1},a)$. From the above, we have $(b^{-1},a) \in D_{\{1,a,b\}}$, and from Remark 3.1, we obtain $\mathcal{T}(b^{-1},a) \subseteq D_{\{1,a,b\}}$. For the opposite inclusion, keeping in mind that $S_3 \simeq \mathcal{S} \subseteq \mathcal{T}$, and the fact that $\mathcal{T}(b^{-1},a)$ is $\mathcal{T}$-invariant, it is sufficient to note that $(1,a) = vtv(b^{-1},a), (1,b) = vt(b^{-1},a)$ and $(1,b^{-1}a) = vtuv(b^{-1},a)$. $\square$

Definition 3.9. Let G be an arbitrary group and $D \in C(G)$.

- D will be called *irreducible* if $D = D_1 \cup D_2$, where $D_1, D_2 \in C(G)$, implies $D_1 = D$ or $D = D_2$, otherwise, D will be called *reducible.*
- $D \neq \{(1,1)\}$ is called *minimal* or an *atom* if $D \supseteq D' \supsetneq \{(1,1)\}$ implies $D = D'$ for any domain D'.

Observe that, in the above definition, if G is finite, it may be assumed that any minimal domain is elementary since any domain is a union of elementary domains, see [**6,** Theorem 4].

Remark 3.10. In a decomposition

$$D = \bigcup_{i\in I} D_i,$$

there could be indices $j, k \in I$ such that $D_j \subseteq D_k$. In such a case, the domain D_j would be superfluous. Then, we consider only minimal decompositions, i.e., decompositions in which $D_j \nsubseteq D_k$ for $j \neq k$.

Example 3.11. There is a decomposable elementary domain for C_5. Indeed, it is computed in [**17,** Section 3] that

$$C_5 \times C_5 = \mathcal{T}(a, a^3) \cup \mathcal{T}(a^2, a^2),$$

and the domain $C_5 \times C_5$ is decomposable. Obviously, any minimal domain is irreducible.

Proposition 3.12. *A domain D is minimal if, and only if, $D = \mathcal{T}(a, 1)$, for some $a \in G \setminus \{1\}$.*

Proof. A domain D is minimal if, and only if, $\iota(D)$ is maximal in the poset $Y^*(\mathcal{S}_3(G))$, where ι is the isomorphism given in (3.3). Then, by Proposition 3.4, D is minimal if, and only if, there exists an $a \in G$ such that

$$\iota(D) = I_{G\setminus\{1,a\}} \cup I_{a,a^{-1}},$$

where $I_{a,a^{-1}} = \emptyset$ if $a = a^{-1}$. The proof will be complete if we show that

$$\iota(\mathcal{T}(a, 1)) = I_{G\setminus\{1,a\}} \cup I_{a,a^{-1}},$$

and this follows from (3.4) and Lemma 3.7. □

Combining Propositions 3.8 and 3.12 we obtain the following.

Corollary 3.13. *Let D be a domain for a finite group G. Then D is minimal if and only if there exists $1 \neq x \in G$ such that $D = D_{\{1,x\}}$.*

Proposition 3.14. *A domain is irreducible if, and only if, it is a $\mathcal{T}$-orbit.*

Proof. Suppose that there are (elementary if G is finite) domains D_1 and D_2 such that

$$\mathcal{T}(a, b) = D_1 \cup D_2.$$

Since $(a, b) \in \mathcal{T}(a, b)$, without loss of generality, we suppose that $(a, b) \in D_1$. Thus, $\mathcal{T}(a, b) \subseteq D_1 \subseteq \mathcal{T}(a, b)$, that is, $\mathcal{T}(a, b) = D_1$. Therefore, $\mathcal{T}(a, b)$ is irreducible. □

Definition 3.15. An ideal $I \in Y^*(\mathcal{S}_3(G))$ will be called *meet-prime* (or *meet-irreducible*) if $I = I_1 \cap I_2$, where $I_1, I_2 \in Y^*(\mathcal{S}_3(G))$ implies $I = I_1$ or $I = I_2$.

Remark 3.16. It is easy to verify that the zero ideal is a meet-prime element of $Y^*(\mathcal{S}_3(G))$ if, and only if, $|G| \leq 4$.

For $x \in G$, we denote

$$\mathfrak{i}(x) = (\{1, x\}, x) \cup (\{1, x^{-1}\}, x^{-1}) \cup (\{1, x\}, 1) \cup (\{1, x^{-1}\}, 1).$$

Corollary 3.17. *An element I in $Y^*(\mathcal{S}_3(G))$ is meet-prime if, and only if, it is maximal or has the form*

$$I = \mathcal{S}_3(G) \setminus \Bigg(\bigcup_{x \in \{1, a, b, a^{-1}b\}} \mathfrak{i}(x) \cup I_{a,b} \Bigg),$$

where $a, b \in G$ and $1 \notin \{a, b, a^{-1}b\}$.

Proof. Take I in $Y^*(\mathcal{S}_3(G))$. Then, by Proposition 3.14, I is meet-prime if, and only if, there are $a, b \in G$ such that $I = \iota(\mathcal{T}(b^{-1}, a))$. Now, if $1 \in \{a, b, a^{-1}b\}$, Proposition 3.12 implies that I is maximal. Suppose that $1 \notin \{a, b, a^{-1}b\}$. From (3.4), we have that, for $x, y \in G$, the element $(\{1, x, xy\}, xy)$ is not in I if, and only if, $(x, y) \in \mathcal{T}(b^{-1}, a)$. Now, using the second equality of Proposition 3.8, (3.5) and the third item of Proposition 3.2, we obtain the desired result. □

A non-minimal $\mathcal{T}$-orbit is called a *block*. Hence, for any domain D, there are blocks $\{B_i \mid i \in \mathcal{I}\}$ and minimal domains $\{M_j \mid j \in \mathcal{J}\}$ in $C(G)$ such that

$$D = \bigcup_{i \in \mathcal{I}} B_i \cup \bigcup_{j \in \mathcal{J}} M_j.$$

We may also remove from such a decomposition any minimal domain M_j which is a subset of some B_i, in other words, we always assume that, in the decomposition of a domain D, there are no i, j such that $B_i \supseteq M_j$.

Proposition 3.18. *Let G be an arbitrary group. Suppose that the domains $D_1, D_2 \in C(G)$ are decomposed as*

$$D_1 = \bigcup_{i\in\mathcal{I}} B_i \cup \bigcup_{j\in\mathcal{J}} M_j \quad \text{and} \quad D_2 = \bigcup_{k\in\mathcal{K}} C_k \cup \bigcup_{l\in\mathcal{L}} N_l,$$

with $M_j \nsubseteq B_i$ and $N_l \nsubseteq C_k$, where the B_i's and C_k's are blocks and the M_j's and N_k's are minimal. If $D_1 \subseteq D_2$, then

- $\{B_i \mid i \in \mathcal{I}\} \subseteq \{C_k \mid k \in \mathcal{K}\}$.
- *For any $j \in \mathcal{J}$, we have that $M_j \subseteq C_k$ for some $k \in \mathcal{K}$ or $M_j = N_l$ for some $l \in \mathcal{L}$.*

Proof. From Proposition 3.12, the condition $M_j \nsubseteq B_i$ for each $i \in \mathcal{I}$ implies

$$M_j \nsubseteq \bigcup_{i\in\mathcal{I}} B_i,$$

analogously,

$$N_l \nsubseteq \bigcup_{k\in\mathcal{K}} C_k.$$

Let $i \in \mathcal{I}$. From Lemma 3.7 and Proposition 3.8, we have that $B_i = \mathcal{T}(x_i, y_i)$, where $1 \notin \{x_i, y_i, x_i y_i\}$, and $(x_i, y_i) \in D_2$. Since, for any $j \in \mathcal{J}$, $N_j = \mathcal{T}(1, a_j)$ for some $a_j \neq 1$, each element $(u, v) \in N_j$ satisfies $1 \in \{u, v, uv\}$. Consequently, there exists a $k \in \mathcal{K}$ such that $(x_i, y_i) \in C_k$, and this implies $C_k = \mathcal{T}(x_i, y_i) = B_i$ and

$$\{B_i \mid i \in \mathcal{I}\} \subseteq \{C_k \mid k \in \mathcal{K}\}.$$

For the second item, we have that $M_j = \mathcal{T}(1, z_j)$, for some $z_j \neq 1$, and $j \in \mathcal{J}$. Then, if M_j is not contained in any block appearing in the decomposition of D_2, we obtain

$$M_j \subseteq \bigcup_{l\in\mathcal{L}} N_l.$$

Since, for each l, we have $N_l = \mathcal{T}(1, z_l)$, with $z_l \neq 1$, we get $\mathcal{T}(1, z_j) \subseteq \mathcal{T}(1, z_{l_0})$ for some l_0. Finally, by Proposition 3.12, $M_j = N_{l_0}$. □

Corollary 3.19. *Let G be an arbitrary group. If a domain $D \in C(G)$ is decomposed as*

$$\bigcup_{i\in\mathcal{I}} B_i \cup \bigcup_{j\in\mathcal{J}} M_j \quad \textit{and} \quad \bigcup_{k\in\mathcal{K}} C_k \cup \bigcup_{l\in\mathcal{L}} N_l,$$

with $M_j \nsubseteq B_i$ and $N_l \nsubseteq C_k$, where the B_is and C_ks are blocks and the M_js and N_ks are atoms, then

$$\{B_i \mid i \in \mathcal{I}\} = \{C_k \mid k \in \mathcal{K}\}$$

and

$$\{M_j \mid j \in \mathcal{J}\} = \{N_l \mid l \in \mathcal{L}\}.$$

Since every element in $C(G)$ can be written as a union of elements of $\mathcal{T}$-orbits, the isomorphism (3.3) implies that each element in $Y^*(\mathcal{S}_3(G))$ is an intersection of maximal ideals and ideals which correspond to blocks.

Definition 3.20. An element $I \in Y^*(\mathcal{S}_3(G))$ will be called a *block ideal* if $I = \iota(D)$ for some block $D \in \mathcal{T}(G)$.

By Proposition 3.14, the block ideals are meet-prime, and their description is given in Corollary 3.17. Next, we write Proposition 3.18 in terms of maximal and block ideals.

Corollary 3.21. *Let G be an arbitrary group and $I_1, I_2 \in Y^*(\mathcal{S}_3(G))$. Let*

$$I_1 = \bigcap_{i\in\mathcal{I}} \mathfrak{m}_i \cap \bigcap_{j\in\mathcal{J}} \mathfrak{n}_j$$

and

$$I_2 = \bigcap_{k\in\mathcal{K}} \mathfrak{a}_k \cap \bigcap_{l\in\mathcal{L}} \mathfrak{b}_l,$$

with $\mathfrak{n}_j \nsubseteq \mathfrak{m}_i$ and $\mathfrak{b}_l \nsubseteq \mathfrak{a}_k$, where the $\mathfrak{m}_i$s and $\mathfrak{a}_k$s are maximal ideals and the $\mathfrak{n}_j$s and $\mathfrak{b}_k$s are block ideals. If $I_1 \supseteq I_2$, then

- $\{\mathfrak{b}_l \mid l \in \mathcal{L}\} \supseteq \{\mathfrak{n}_j \mid j \in \mathcal{J}\}$.
- *For any $i \in \mathcal{I}$, we have that $\mathfrak{b}_l \subseteq \mathfrak{m}_i$ for some $l \in \mathcal{L}$ or $\mathfrak{m}_i = \mathfrak{a}_k$ for some $k \in \mathcal{K}$.*

Clearly, Corollary 3.21 implies the uniqueness of a decomposition of an ideal into an intersection of maximal and block ideals in a sense similar to Corollary 3.19.

4. Decomposition of elementary domains. In this section, G will denote a finite group. From Propositions 3.8 and 3.14 every $\mathcal{T}$-orbit is an irreducible elementary domain; moreover, the union of $\mathcal{T}$-orbits is an element of $C(G)$ but not necessarily an elementary domain. For instance, in $C_4 = \{1, a, a^2, a^3\}$, the domain $X = \mathcal{T}(1, a) \cup \mathcal{T}(1, a^2)$ is not elementary. Since each elementary domain D is a union of $\mathcal{T}$-orbits

$$D = \bigcup_{(x,y)\in D} \mathcal{T}(x, y),$$

it is natural to consider the following problem.

Given an elementary domain D, find a set $\mathcal{I}$ with minimal cardinality such that

$$D = \bigcup_{i\in\mathcal{I}} \mathcal{T}(x_i, y_i).$$

In order to solve our problem, we use the following.

Lemma 4.1. *Let A and B be vertices in $\Gamma(G)$ such that $A \subseteq B$. Then, $D_A \subseteq D_B$.*

Proof. Let $a, b \in G$. From Lemma 3.6, we know that $(a, b) \in D_A$ if, and only if, there is a $g_j \in A$ with $\{a^{-1}, b\} \subseteq g_j^{-1}A$. Then, $g_j \in B$ and

$$\{a^{-1}, b\} \subseteq g_j^{-1}B \in \mathcal{V}_B.$$

Again, from Lemma 3.6, we obtain $(a, b) \in D_B$. $\square$

Remark 4.2. The converse of Lemma 4.1 is not true. For example, by [**17,** Corollary 6.2], if $|G| > 3$ and A is a vertex in $\Gamma(G)$, such that $|A| + 1 = |G|$, then $D_G = D_A$.

Setting $G_n = \{A \subseteq G \mid 1 \in A, |A| = n\}$, we define an equivalence relation on G_n by

$$A \sim B \Longleftrightarrow \mathcal{V}_A = \mathcal{V}_B.$$

Then, the equivalence class of a vertex A is exactly $\mathcal{V}_A$.

Let $\mathcal{C}_n$ be a full set of representatives of the equivalence classes of $\sim$. The next result gives a decomposition of the elementary domain induced by a vertex $B \in \Gamma(G)$, with $|B| \geq 3$.[2]

Theorem 4.3. *Let B be a vertex in $\Gamma(G)$ with $|B| \geq 3$, and consider a transversal $\tau = \{b_1 = 1, b_2, \ldots, b_m\} \subseteq B$ such that*

$$B = \bigcup_{i=1}^{m} \operatorname{St} Bb_i.$$

Then,

$$D_B = \bigcup_{i=1}^{m} \bigcup_{\substack{A \in \mathcal{C}_3 \\ A \subseteq b_i^{-1}B}} D_A. \tag{4.1}$$

Moreover, the above decomposition is minimal in the sense that no block D_A appearing in (4.1) *can be omitted, and the decomposition is unique up to permutations of blocks.*

Proof. It follows from Corollary 4.1 and Lemma 3.5 that, for any $b \in B$ and $A \subseteq b^{-1}B$, $D_A \subseteq D_{b^{-1}B} = D_B$. Consequently,

$$D_B \supseteq \bigcup_{i=1}^{m} \bigcup_{\substack{A \in \mathcal{C}_3 \\ A \subseteq b_i^{-1}B}} D_A.$$

In order to prove the other inclusion, suppose that $(u, v) \in D_B$. From Lemma 3.6, there is a vertex $B' \in \mathcal{V}_B$ containing the set $\{u^{-1}, v\}$. Since $|B'| = |B| \geq 3$, there exists a subset E of B' such that $\{1, u^{-1}, v\} \subseteq E$ and $|E| = 3$. It again follows from Lemma 3.6 that $(u, v) \in D_E$, and we obtain (4.1). The minimality of (4.1) is a consequence of the next two claims.

Claim 4.4. *Let A and B_1 be vertices in $\Gamma(G)$ with $A, B_1 \in G_3$. Then, $\mathcal{V}_A = \mathcal{V}_{B_1}$ if, and only if, $D_A = D_{B_1}$.*

From Lemma 3.5, all vertices from a given component of $\Gamma(G)$ induce the same elementary domain. For the converse, let $A = \{1, a_1, a_2\}$ and $B_1 = \{1, b_1, b_2\}$, with $|A| = |B_1| = 3$. We have

(4.2)

$$\begin{aligned}\mathcal{V}_A = \mathcal{V}_{B_1} &\iff B_1 = A \quad \text{or} \quad B_1 = a_1^{-1}A \quad \text{or} \quad B_1 = a_2^{-1}A \\ &\iff \{b_1, b_2\} \in \{\{a_1, a_2\}, \{a_1^{-1}, a_1^{-1}a_2\}, \{a_2^{-1}, a_2^{-1}a_1\}\} \\ &\overset{(3.2)}{\iff} (b_1^{-1}, b_2) \in S_3(a_1^{-1}, a_2).\end{aligned}$$

We use Proposition 3.8 to prove that $\mathcal{T}(a_1^{-1}, a_2) = \mathcal{T}(b_1^{-1}, b_2)$ implies (4.2). If $\mathcal{T}(a_1^{-1}, a_2) = \mathcal{T}(b_1^{-1}, b_2)$, then $(b_1^{-1}, b_2) \in \mathcal{T}(a_1^{-1}, a_2)$, and there is

$$\gamma \in \mathcal{T} = \mathcal{S} \cup t\mathcal{S} \cup vt\mathcal{S} \cup uvt\mathcal{S} \cup 0$$

such that $(b_1^{-1}, b_2) = \gamma(a_1^{-1}, a_2)$. Moreover, by (3.1) and (3.2), we see that

$$tS_3(x, y) = \{(x, 1), (xy, 1), (y^{-1}, 1), (y^{-1}x^{-1}, 1), (y, 1), (x^{-1}, 1)\},$$

for every $(x, y) \in G \times G$. Hence, the pairs (a, b) in $(t\mathcal{S} \cup vt\mathcal{S} \cup 0)(x, y)$ have a 1 in its second coordinate. Since $1 \notin \{b_1, b_2\}$, it follows that

$$\gamma \notin tS_3 \cup vtS_3 \cup 0.$$

Similarly, $\gamma \notin uvtS_3$ since the pairs in $uvtS_3(a_1^{-1}, a_2)$ are of the form (x^{-1}, x); however, by hypothesis, $b_1 \neq b_2$. Therefore, $\gamma \in \mathcal{S}$, and $(b_1^{-1}, b_2) \in S_3(a_1^{-1}, a_2)$, which is (4.2). This completes the proof.

Claim 4.5. *If $A, B_1, \ldots, B_m \in G_3$ are vertices in different connected components of $\Gamma(G)$, then*

$$D_A \not\subseteq \bigcup_{j=1}^{m} D_{B_j}.$$

Fix an arbitrary $j \in \{1, \ldots, m\}$, and write $A = \{1, a_1, a_2\}$ and $B_j = \{1, b_1, b_2\}$. Then, from Claim 4.4 and Proposition 3.8, we obtain

$$D_A = \mathcal{T}(a_1^{-1}, a_2) \neq \mathcal{T}(b_1^{-1}, b_2) = D_{B_j}.$$

For

$$(x, y) \in \mathcal{T}(a_1^{-1}, a_2) \cap \mathcal{T}(b_1^{-1}, b_2),$$

we have

$$\mathcal{T}(x,y) \subseteq \mathcal{T}(a_1^{-1},a_2) \cap \mathcal{T}(b_1^{-1},b_2).$$

On the other hand, by Proposition 3.8, the non-trivial domains properly contained in a $\mathcal{T}$-orbit are minimal. Consequently, $1 \in \{x,y,xy\}$, and therefore, for any $(u,v) \in D_A$ such that $1 \notin \{u,v,uv\}$, we have $(u,v) \notin D_{B_j}$. In particular, $(a_1^{-1},a_2) \notin D_{B_j}$. Since j is arbitrary, it follows that

$$D_A = \mathcal{T}(a_1^{-1},a_2) \nsubseteq \bigcup_{j=1}^{m} D_{B_j}.$$

Finally, Corollary 3.19 implies that decomposition (4.1) is unique. $\square$

Remark 4.6. Our decomposition theorem gives a criterion to decide whether an element $D \in C(G)$ is an elementary domain. Indeed, let

$$D = \bigcup_{i\in\mathcal{I}} \mathcal{T}(x_i,y_i)$$

with $|\mathcal{I}| \geq 2$ and B as in Theorem 4.3. If there exists an $i \in \mathcal{I}$ such that every vertex in $\mathcal{V}_{\{1,y_i,x_i^{-1}\}}$ is not contained in $b_j^{-1}B$ for all $1 \leq j \leq m$, we have $D \neq D_B$.

Definition 4.7. We say that an ideal I of $Y^*(\mathcal{S}_3(G))$ is elementary if $I = \iota(D)$ for some elementary domain D of G.

For elementary ideals, we have the following version of Theorem 4.3.

Corollary 4.8. *Let G be a finite group and I_B the ideal of $\mathcal{S}_3(G)$ corresponding to an elementary domain D_B, with $|B| > 3$, and consider a transversal $\tau = \{b_1 = 1, b_2, \ldots, b_m\} \subseteq B$ such that*

$$B = \bigcup_{i=1}^{m} \operatorname{St} Bb_i.$$

Then,

$$I_B = \bigcap_{i=1}^{m} \bigcap_{\substack{A\in\mathcal{C}_3 \\ A\subseteq b_i^{-1}B}} I_A.$$

Moreover, this decomposition is minimal, i.e., no ideal I_A can be omitted, and it is unique up to permutations of ideals.

Definition 4.9. We say that a vertex A is *total*[3] if $D_A = G \times G$.

Another consequence of Theorem 4.3 is given in the following.

Corollary 4.10. *Let G be a group such that $|G| \geq 3$. Then,*

- $D_G = G \times G = \bigcup_{A \in \mathcal{C}_3} D_A$. *Furthermore, for any vertex B of $\Gamma(G)$, D_B is not total if, and only if, there exists an $A \in \mathcal{C}_3$ such that D_A is not contained in D_B.*
- $0 = \bigcap_{A \in \mathcal{C}_3} I_A$, *where $I_A = \iota(D_A)$ and ι is the isomorphism defined in* (3.3).

Proposition 4.11. *If A is a vertex of $\Gamma(G)$ such that $|\mathrm{St}\, A|(|G|^2-1) > |A|(|A|^2-1)$, then D_A is not total.*

Proof. From Lemma 3.6, $(x, y) \in D_A$ if and only if there exists a vertex $A' \in \mathcal{V}_A$ which contains x^{-1} and y. This corresponds to the following pair of arrows in the groupoid $\Gamma(G)$:

$$y^{-1}A' \xrightarrow{y} A' \xrightarrow{x} xA'.$$

For each vertex A', there are exactly $|A|^2$ different pairs of such arrows. Since one of these pairs is always $(1, 1)$, and $|\mathcal{V}_A| = |A|/|\mathrm{St}\, A|$, we conclude that D_A has at most

$$\frac{|A|}{|\mathrm{St}\, A|}(|A|^2 - 1) + 1$$

elements. In particular, for any A which is total, we have

$$|G|^2 = |D_A| \leq \frac{|A|}{|\mathrm{St}\, A|}(|A|^2 - 1) + 1. \qquad \square$$

Remark 4.12. The following assertions hold for a finite group G:

- A vertex is total if and only if $I_A = \iota(D_A) = 0$ and, by [**17,** Theorem 6.1], this occurs when $|A| = |G| - k$, for some $0 < k < |G|$ and $|G| > k(2|\mathrm{St} A| + 1)$.

- From Proposition 4.11, $I_A = 0$ also when $|\mathrm{St}\, A|(|G|^2 - 1) \leq |A|(|A|^2 - 1)$. Note that, in the above cases, A has a "small" stabilizer. Moreover,
- for any subgroup $H \subsetneq G$, the domain D_H is not total.
- If A is a vertex of $\Gamma(G)$, where $|G| \geq 6$ and $|A| \leq 3$, then D_A is not total due to Proposition 4.11.

Definition 4.13. Let G be a group.

- We say that an ideal I of $Y^*(\mathcal{S}_3(G))$ is elementary if $I = \iota(D)$, for some elementary domain D for G.
- The domains $\{(1,1)\}$ and $G \times G$ will be called the *trivial domains* of G.

Proposition 4.14. *Let G be a finite group. Then, the following are equivalent*:

(i) $1 \leq |G| \leq 4$;

(ii) *the zero ideal is a meet-prime element of* $Y^*(\mathcal{S}_3(G))$;

(iii) *every elementary domain for G is irreducible*;

(iv) *all non-trivial elementary domains for G are minimal*;

(v) *every elementary ideal from* $Y^*(\mathcal{S}_3(G))$ *is meet-prime*;

(vi) *all non-trivial elementary ideals from* $Y^*(\mathcal{S}_3(G))$ *are maximal.*

Proof.

(i) $\Leftrightarrow$ (ii). Observed in Remark 3.16.

(i) $\Rightarrow$ (iii). If $1 \leq |G| \leq 4$, it is easily seen that every non trivial elementary domain for G induced by $A \in \mathcal{V}_A$ is minimal, and D_G is irreducible.

(iii) $\Rightarrow$ (i). Suppose that $|G| > 4$. If $|G| \geq 6$, it follows from the last item of Remark 4.12 and Proposition 3.8 that $D_G = G \times G$ is not a $\mathcal{T}$-orbit, and consequently, is reducible. Finally, if $|G| = 5$, Example 3.11 shows that there are reducible elementary domains for G.

(iv) $\Rightarrow$ (iii). This is evident if D_G is irreducible. On the other hand, if D_G is reducible, then it can easily be verified that $|G| \geq 5$ and G contains elements $a \neq 1$ and $b \neq 1$ with $ab \neq 1$. Then, the domain

$\mathcal{T}(a,b) \supsetneq \mathcal{T}(a,1)$ is not minimal. Finally, (v) and (vi) are equivalent to (iii) and (iv), respectively, via ι. $\square$

5. Some invariants of $C(G)$. We compute some numbers with respect to the minimal elementary domains and blocks. For $n \in \mathbb{N}$, the subset of elements of order n in G will be denoted by $o_n(G)$. Note that Proposition 3.8 implies that any block contains one, two or three minimal elementary domains. We shall calculate the number of such blocks in each of these cases, as well as the number of minimal domains.

Theorem 5.1. *Let G be a finite group.*

(i) *The number* $\min(G)$ *of minimal elementary domains for G is given by*:

$$\min(G) = \frac{|G| + |o_2(G)| - 1}{2}.$$

(ii) *Denote by* $\mathrm{Bl}_1(G)$ *the set of blocks which contain exactly one minimal elementary domain. Then,*

$$|\mathrm{Bl}_1(G)| = |\{H \textit{ subgroup of } G \mid H \simeq C_3\}| = \frac{|o_3(G)|}{2}.$$

(iii) *Let* $\mathrm{Bl}_2(G)$ *be the set of blocks which contains exactly two minimal elementary domains. Then,*

$$|\mathrm{Bl}_2(G)| = \frac{|\{a \in G \mid o(a) > 3\}|}{2}.$$

(iv) *Write* $\mathrm{Bl}(G)$ *for the set of all blocks. Then,*

$$|\mathrm{Bl}(G)| = \frac{\binom{|G|-1}{2} + |o_3(G)|}{3}.$$

(v) *Let* $\mathrm{Bl}_3(G)$ *be the set of blocks which contains exactly three minimal elementary domains. Then,*

$$|\mathrm{Bl}_3(G)| = \frac{2\binom{|G|-1}{2} - |o_3(G)| - 3|\{a \in G | o(a) > 3\}|}{6}.$$

Proof.

(i) Any $x \in G \setminus \{1\}$ induces an elementary domain $\mathcal{T}(x,1) = \mathcal{T}(x^{-1},1)$, and Proposition 3.12 guarantees that these are the only domains which are minimal. On the other hand, from Lemma 3.7, we have $\mathcal{T}(y,1) = \mathcal{T}(x,1)$ if, and only if, $x = y$ or $x = y^{-1}$. Consequently, the number of minimal elementary domains $\mathcal{T}(x,1)$ for which $x \in o_2(G)$ is exactly $|o_2(G)|$. On the other hand, each subset $\{y, y^{-1}\}$ of $G \setminus (o_2(G) \cup \{1\})$ induces the minimal elementary domain $\mathcal{T}(y,1)$, and the number of such subsets is

$$\frac{|G| - (|o_2(G)| + 1)}{2}.$$

Therefore,

$$\min(G) = |o_2(G)| + \frac{|G| - (|o_2(G)| + 1)}{2} = \frac{|G| + |o_2(G)| - 1}{2}.$$

(ii) For any $a \in G$ such that $o\,(a) = 3$, we have from Lemma 3.7 and Proposition 3.8 that

$$\begin{aligned}\mathcal{T}(a,a) &= D_{\{1,a,a^2\}}\\ &= \mathcal{T}(1,a) \cup \mathcal{T}(1,a^2) \cup \mathcal{T}(1,a^{-2}a) \cup S_3(a^{-2},a)\\ &= \mathcal{T}(1,a) \cup S_3(a,a).\end{aligned}$$

Thus, any

$$H = \langle a \mid a^3 = 1\rangle \subseteq G$$

determines a block $\mathcal{T}(a,a)$ containing exactly one minimal elementary domain. Moreover, the fact that for $a, b \in G$ with $a \neq b \neq a^{-1}$ implies $\mathcal{T}(1,a) \neq \mathcal{T}(1,b)$, then $\mathcal{T}(a,a) \neq \mathcal{T}(b,b)$ and this correspondence is injective. We conclude that

$$|\mathrm{Bl}_1(G)| \geq |\{H < G \mid H \simeq C_3\}|.$$

On the other hand, if

$$D_{\{1,a,b\}} = \mathcal{T}(1,a) \cup \mathcal{T}(1,b) \cup \mathcal{T}(1,b^{-1}a) \cup S_3(b^{-1},a)$$

contains only one minimal elementary domain, we have $\mathcal{T}(1,a) = \mathcal{T}(1,b) = \mathcal{T}(1,b^{-1}a)$. This implies $b = a^{-1} = b^{-1}a$; hence, $b = a^2$ and $a^3 = 1$. Therefore, $\{1,a,b\} = \{1,a,a^2\} \simeq C_3$, and (ii) follows.

(iii) Let $\{1,a,b\}$ be a set of three elements such that $D_{\{1,a,b\}}$ is a block containing exactly two minimal elementary domains. The component $\mathcal{V}_{\{1,a,b\}}$ of $\Gamma(G)$ has three vertices or one vertex. In the latter case, by the proof of item (ii), $\{1,a,b\}$ is a group of order 3, the associated block of which lies in $\mathrm{Bl}_1(G)$. Consequently, $\mathcal{V}_{\{1,a,b\}}$ has three vertices and

$$D_{\{1,a,b\}} = D_{\{1,a^{-1},a^{-1}b\}} = D_{\{1,b^{-1},b^{-1}a\}}.$$

Without loss of generality, we may assume that

$$\mathcal{T}(1,a) = \mathcal{T}(1,b) \neq \mathcal{T}(1,b^{-1}a).$$

This implies $b = a^{-1}$ and $b \neq b^{-1}a$. Then, $a^{-1} = b \neq a^2$, and we conclude that $o(a) \notin \{1,2,3\}$. Conversely, it is clear that any block of the form $D_{\{1,a,a^{-1}\}}$, where $a \in G$ is such that $o(a) > 3$ contains exactly two minimal domains. Then, $|\mathrm{Bl}_2(G)|$ equals the number of subsets $\{a, a^{-1}\}$ of G for which $o(a) > 3$. The latter is

$$|\{a \in G \mid o(a) > 3\}|/2,$$

proving (iii).

(iv) Since, from Lemma 3.5, vertices lying in the same component of the groupoid $\Gamma(G)$ induce the same elementary domain, we merely need to count the number of components defined by the vertices A of three elements. If such an A is not a subgroup, then $\mathrm{St}\,A = \{1\}$, and its component in $\Gamma(G)$ has exactly three vertices. Otherwise, $A = \mathrm{St}\,A$ is one of the $|o_3(G)|/2$ subgroups of G isomorphic to C_3. Thus, the number of blocks is

$$\frac{|o_3(G)|}{2} + \frac{\binom{|G|-1}{2} - |o_3(G)|/2}{3},$$

and (iv) follows.

(v) From Proposition 3.8,

$$|\mathrm{Bl}(G)| = |\mathrm{Bl}_1(G)| + |\mathrm{Bl}_2(G)| + |\mathrm{Bl}_3(G)|,$$

and (v) follows from (ii), (iii) and (iv). $\square$

Remark 5.2. Let G be a finite group. From (3.2), each S_3-orbit contains 1, 2, 3 or 6 elements, see [**7,** page 216], where the notation $A_{(a,b)}$ is used for S_3-orbits. The orbits with 2 or 6 elements are the so-called effective orbits, see [**15**]. Then:

- from the proof of item (ii) of Theorem 5.1, the blocks in $\mathrm{Bl}_1(G)$ correspond to connected components of $\Gamma(G)$ which have exactly one vertex A and such that $|\mathrm{St}\,A| = 3$, and also are in one-to-one correspondence with the S_3-orbits containing exactly 2-elements.

- From Proposition 3.12, the S_3-orbits containing exactly three elements correspond to the minimal domains of G, and the blocks correspond to the effective orbits.

Example 5.3. Let $n \in \mathbb{N}$ and $C_n = \langle a \mid a^n = 1\rangle$. If $3 \mid n$, then $a^{n/3}$ and $a^{2n/3}$ are the only elements of C_n of order 3; otherwise, C_n has no element of order 3. Therefore, from Theorem 5.1 (iv),

$$|\mathrm{Bl}(C_n)| = \frac{\binom{n-1}{2} + |o_3(G)|}{3} = \begin{cases} \frac{(n-1)(n-2)}{6} & \text{if } 3 \nmid n, \\ \frac{(n-1)(n-2)+4}{6} & \text{if } 3 \mid n. \end{cases}$$

If $n \geq 4$ is not a multiple of 3, then $|\mathrm{Bl}(C_n)|$ is equal to the number

$$P_n = \left\lfloor \frac{n-1}{3} \right\rfloor \left(n - \frac{3}{2} \left\lfloor \frac{n-1}{3} \right\rfloor - \frac{3}{2} \right)$$

obtained after the proof of [**7,** Proposition 6.1]. Indeed, it is readily verified that $|\mathrm{Bl}(C_n)| = P_n$ if n is of the form $3k+1$ or $3k+2$, with $k \in \mathbb{N}$. However, for $n = 3k$, the number of effective S_3-orbits, which is the same as the number of blocks, is actually $|\mathrm{Bl}(C_n)| = P_n + 1$. Thus, [**7,** Proposition 6.1] must be restated as follows.

Proposition 5.4. *Let $n \geq 3$ and $\sigma \in pm'_{C_n \times C_n}(C_n)$. Then, σ is uniquely determined by its values $\sigma(i,j)$, where $1 \leq i \leq \lfloor n - 1/3 \rfloor$, $i \leq j \leq n - 2i - 1$ and $\sigma(n/3, n/3)$ if $3 \mid n$.*

In view of Example 2 in [**7,** Corollaries 6.2, 6.4] P_n should be replaced by $|\mathrm{Bl}(C_n)|$.

Now, we turn our attention to the question of determining the number of different blocks $D_{\{1,x,y\}}$ which contain a given minimal elementary domain $\mathcal{T}(1,a)$. As a first step in this direction, we have the following.

Lemma 5.5. *Let G be an arbitrary group and $a \in G \setminus \{1\}$. Then, $\mathcal{T}(1,a) \subseteq \mathcal{T}(x,y)$, where $1 \notin \{x,y,xy\}$ if, and only if, there is an element $z \in G$ such that $\mathcal{T}(x,y) = \mathcal{T}(z,a)$.*

Proof. From (3.1), it is clear that $(1, a) \in \mathcal{T}(z, a)$ for any $z \in G$. On the other hand, from Proposition 3.8, we have

$$\mathcal{T}(x, y) = \mathcal{T}(1, y) \cup \mathcal{T}(1, x) \cup \mathcal{T}(1, xy) \cup S_3(x, y),$$

and, from (3.2), any element (a, b) in $S_3(x, y)$ is such that $1 \notin \{a, b, ab\}$. Therefore, if $\mathcal{T}(1, a) \subseteq \mathcal{T}(x, y)$, then

$$(1, a) \in \mathcal{T}(1, x) \cup \mathcal{T}(1, y) \cup \mathcal{T}(1, xy),$$

and it follows from Lemma 3.7 that

$$a \in \{x, x^{-1}, y, y^{-1}, xy, y^{-1}x^{-1}\},$$

in other words,

$$\mathcal{T}(x, y) = \begin{cases} \mathcal{T}(a, y) = \mathcal{T}(y^{-1}a^{-1}, a), \text{ or} \\ \mathcal{T}(a^{-1}, y) = \mathcal{T}(y^{-1}, a), \text{ or} \\ \mathcal{T}(x, a), \text{ or} \\ \mathcal{T}(x, a^{-1}) = \mathcal{T}(xa^{-1}, a), \text{ or} \\ \mathcal{T}(x, x^{-1}a) = \mathcal{T}(x^{-1}, a), \text{ or} \\ \mathcal{T}(x, x^{-1}a^{-1}) = \mathcal{T}(x^{-1}a^{-1}, a). \end{cases}$$

In any case, we have $\mathcal{T}(x, y) = \mathcal{T}(z, a)$, for some $z \in G$. $\square$

Remark 5.6. Let $x, y \in G$. It follows from the proof of Lemma 5.5 that, for any $u, v \in G$, the following assertions are equivalent:

- $\mathcal{T}(x, y) \cap \mathcal{T}(u, v) = \{(1, 1)\}$;
- $\{u, v, uv\} \cap \{x, x^{-1}, y, y^{-1}, xy, y^{-1}x^{-1}\} = \emptyset$.

Next, we characterize different sets, say $\{1, a, y\}$ and $\{1, a, z\}$, which determine the same block.

Lemma 5.7. *If* $|\{1, a, y, z\}| = 4$, *then* $D_{\{1,a,y\}} = D_{\{1,a,z\}}$ *if, and only if, one of the following conditions holds*:

- $a^2 = 1$ *and* $z = ay$; *or*
- $\{y, z\} = \{a^{-1}, a^2\}$.

Proof. From Claim 4.4, we have $D_{\{1,a,y\}} = D_{\{1,a,z\}}$ if, and only if,

$$\{1, a, z\} = \{1, a^{-1}, a^{-1}y\}$$

or

$$\{1, a, z\} = \{1, y^{-1}, y^{-1}a\}.$$

This is equivalent to

$$\{a, z\} = \{a^{-1}, a^{-1}y\}$$

or

$$\{a, z\} = \{y^{-1}, y^{-1}a\},$$

which implies that $a^2 = 1$ and $z = ay$ or $\{y, z\} = \{a^{-1}, a^2\}$. $\square$

Now, we are ready to give the number of blocks which contain a given minimal elementary domain.

Proposition 5.8. *Let $a \in G \setminus \{1\}$, and denote by* $\mathrm{Bl}(1, a)$ *the set of blocks containing the minimal elementary domain $\mathcal{T}(1, a)$. Then,*

$$|\mathrm{Bl}(1, a)| = \begin{cases} (|G| - 2)/2 & \text{if } a^2 = 1, \\ |G| - 2 & \text{if } a^3 = 1, \\ |G| - 3 & \text{if } a^2 \neq 1 \neq a^3. \end{cases}$$

Proof. We use Lemma 5.7 and consider three cases. If $a^2 = 1$, then, for each $y \in G \setminus \{1, a\}$, we have two equal blocks, $D_{\{1,a,y\}} = D_{\{1,a,ay\}}$. Consequently,

$$|\mathrm{Bl}(1, a)| = (|G| - 2)/2.$$

If $a^3 = 1$, the condition $\{y, z\} = \{a^2, a^{-1}\}$ only occurs for $y = z$. Therefore, $|\mathrm{Bl}(1, a)| = |G| - 2$, in this case. Finally, if $a^2 \neq 1 \neq a^3$, then $D_{\{1,a,y\}} = D_{\{1,a,z\}}$ for $y, z \in G \setminus \{1, a\}$ only if $\{y, z\} = \{a^{-1}, a^2\}$. Thus, $|\mathrm{Bl}(1, a)| = |G| - 3$. $\square$

Write

$$\mathcal{T}(G) = \{\mathcal{T}(x, y) \mid (x, y) \in G \times G\}.$$

Then, $(\mathcal{T}(G), \subseteq)$ is a partially ordered set, and, by Corollary 3.19, any element of $C(G)$ may be uniquely written by taking unions of elements of $\mathcal{T}(G)$.

Proposition 5.9. *If G and H are groups, then $\mathcal{T}(G) \simeq \mathcal{T}(H)$ as partially ordered sets, if and only if $C(G) \simeq C(H)$ as lattices.*

Proof. Let

$$\phi : \mathcal{T}(G) \longrightarrow \mathcal{T}(H)$$

be an isomorphism of partially ordered sets. Since any $D \in C(G)$ has a unique decomposition

$$D = \bigcup_{i \in I} \mathcal{T}(a_i, b_i),$$

we may extend ϕ to $C(G)$ by setting

$$\varphi(D) = \bigcup_{i \in I} \phi(\mathcal{T}(a_i, b_i)) \in C(H).$$

Clearly, the above decomposition for $\varphi(D)$ is unique. Thus, φ is a one-to-one correspondence between $C(G)$ and $C(H)$ which preserves inclusions, and this implies that φ is a lattice isomorphism.

Conversely, a lattice isomorphism

$$\varphi : C(G) \longrightarrow C(H)$$

takes irreducible domains for G into irreducible domains for H, and, since the irreducible domains are exactly the $\mathcal{T}$-orbits, restricting φ to $\mathcal{T}(G)$, we obtain an isomorphism

$$\mathcal{T}(G) \longrightarrow \mathcal{T}(H). \qquad \square$$

Denote by $Y_{\mathcal{T}}(G)$ the subset of $Y^*(\mathcal{S}_3(G))$ consisting of maximal and block ideals. This leads to:

Proposition 5.10. *Let G and H be groups such that*

$$(Y_{\mathcal{T}}(G), \subseteq) \cong (Y_{\mathcal{T}}(H), \subseteq)$$

as partially ordered sets. Then, the lattices $Y^*(\mathcal{S}_3(G))$ and $Y^*(\mathcal{S}_3(H))$ are isomorphic.

We have the following.

Proposition 5.11. *If $\mathcal{T}(G) \simeq \mathcal{T}(G')$ as partially ordered sets, then $|G| = |G'|$, $|o_2(G)| = |o_2(G')|$ and $|o_3(G)| = |o_3(G')|$.*

Proof. For any group G, we have

$$G = \{1\} \cup o_2(G) \cup o_3(G) \cup \{a \in G \mid o\,(a) > 3\},$$

and, from Theorem 5.1, we obtain

$$|G| = 1 + |o_2(G)| + 2|\mathrm{Bl}_1(G)| + 2|\mathrm{Bl}_2(G)|.$$

The equalities $|\mathrm{Bl}_1(G)| = |\mathrm{Bl}_1(G')|$ and $|\mathrm{Bl}_2(G)| = |\mathrm{Bl}_2(G')|$ imply

$$|G| - |o_2(G)| = |G'| - |o_2(G')|. \tag{5.1}$$

Moreover, the isomorphism $\mathcal{T}(G) \simeq \mathcal{T}(G')$ also implies $\min(G) = \min(G')$, and Proposition 5.11 follows from Theorem 5.1. $\square$

It is natural to ask the following:

Question 5.12. *Are there non-isomorphic groups G and H such that $C(G) \simeq C(H)$?*

We checked some groups of small order and did not find non-isomorphic G and H with $C(G) \simeq C(H)$. It is clear from Proposition 5.11 that $C(C_n) \simeq C(C_m)$ implies $n = m$.

6. Some final remarks on the lattice $(C(G), \cap, \cup)$. Finally, we shall point out some properties of the lattice $(C(G), \cap, \cup)$ for any group G. For the reader's convenience, we recall the following notions.

Definition 6.1. Let $\mathcal{L} = (L, \wedge, \vee)$ be a lattice. Then:

- if $\mathcal{L}$ is complete, an element $x \in L$ is called *compact* if, whenever $x \leq \bigvee A$, $A \subseteq L$, then there exists a finite subset F of A such that $x \leq \bigvee F$. The set of all compact elements of $\mathcal{L}$ is denoted by $\mathcal{L}^c$.
- We say that $\mathcal{L}$ is *algebraic* if $\mathcal{L}$ is complete, and $x = \bigvee(\downarrow x \cap \mathcal{L}^c)$, for any $x \in L$, where $\downarrow x = \{a \in L \mid a \leq x\}$.
- Suppose that $\mathcal{L}$ has 0 and 1. For $x \in L$, we say that $y \in L$ is a *complement* if $x \wedge y = 0$ and $x \vee y = 1$. (Note that, in a distributive lattice, an element has at most one complement.)

- $\mathcal{L}$ is called *Boolean* if $\mathcal{L}$ is distributive, there exist 0 and 1, and any $x \in L$ has a complement.

Proposition 6.2. *Let G be a group. Then,*

- *the set of compact elements of $C(G)$ is*

$$C(G)^c = \{D \in C(G) \mid D \text{ is a finite union of } \mathcal{T}\text{-orbits}\}. \tag{6.1}$$

- $(C(G), \cap, \cup)$ *is algebraic.*
- $(C(G), \cap, \cup)$ *is Boolean if, and only if,* $|G| \leq 2$.

Proof. Let

$$D = \bigcup_{i=1}^{m} \mathcal{T}(x_i, y_i)$$

be an element of the right-hand side of (6.1), and let A be a subset of $C(G)$, if $D \leq \bigcup A$. Then, for any i, there exists a $D_i \in A$ such that $\mathcal{T}(x_i, y_i) \in D_i$. Thus,

$$D \subseteq \bigcup_{i=1}^{m} D_i,$$

and $D \in C(G)^c$. Conversely, if

$$D = \bigcup_{i \in \mathcal{I}} \mathcal{T}(x_i, y_i) \in C(G)^c,$$

then there exists a finite subset F of $\mathcal{I}$ such that

$$D \leq \bigcup_{i \in F} \mathcal{T}(x_i, y_i).$$

Since, from Proposition 3.8, every $\mathcal{T}$-orbit contains at most three minimal domains, Proposition 3.18 implies that $\mathcal{I}$ is finite.

Next, take $X \in C(G)$. Then, we have

$$\downarrow X \cap C(G)^c = \{D \in C(G) \mid D \subseteq X \text{ is a finite union of } \mathcal{T}\text{-orbits}\}.$$

Clearly, $X \supseteq \bigcup(\downarrow X \cap C(G)^c)$ and, given $(x_i, y_i) \in X$, the $\mathcal{T}$-orbit $\mathcal{T}(x_i, y_i)$ is in $\downarrow X \cap C(G)^c$. We obtain

$$X = \bigcup(\downarrow X \cap C(G)^c),$$

which shows that $C(G)$ is algebraic.

It is easy to verify that, if $|G| \leq 4$, then $(C(G), \cap, \cup)$ is Boolean if and only if $|G| \leq 2$. Thus, we need to prove that $(C(G), \cap, \cup)$ is not Boolean when $|G| \geq 5$. The list of elementary domains for $G = C_5$ is given in [**17,** page 74], from which we readily see that $(C(C_5), \cap, \cup)$ is not Boolean (the domain D_4 has no complement). Assume that $|G| \geq 8$. We will check that, in this case $\mathcal{T}(x, y)$, where $1 \notin \{x, y, xy\}$ has no complement. Let τ be a full set of representatives of $\mathcal{T}$-orbits of $G \times G$, and write

$$\Lambda = \{x, x^{-1}, y, y^{-1}, xy, y^{-1}x^{-1}\}.$$

Take $X \in C(G)$, given by

$$X = \bigcup_{\substack{(u,v)\in\tau \\ \{u,v,uv\}\cap\Lambda=\emptyset}} \mathcal{T}(u, v).$$

From Remark 5.6, X is the greatest element in $C(G)$ such that

$$X \cap \mathcal{T}(u, v) = \{(1, 1)\}.$$

Since $|G| \geq 8$, there exist $z \in G \setminus \Lambda$, $z \neq 1$, and the pair $(x, z) \notin \mathcal{T}(x, y) \cup X$ such that $C(G)$ is not Boolean in this case. Finally, the latter argument also works for G with $|G| \in \{6, 7\}$ by taking $(x, y) = (a, a)$ if $G = \langle a \mid a^n = 1\rangle$, where $n \in \{6, 7\}$ or

$$G = S_3 = \langle a, b \mid a^3 = b^2 = 1, b^{-1}ab = a^{-1}\rangle. \qquad \square$$

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ENDNOTES

1. In [**11**], elementary partial representations were defined for arbitrary groups.

2. By our results, for every vertex B such that $|B| < 3$, we have that D_B is a $\mathcal{T}$-orbit.

3. A partial factor set of an arbitrary group G is called *total* if its domain is $G \times G$. When working over algebraically closed fields, it is shown in [**7,** Corollary 5.8 iv)] that there is a group epimorphism from $pM_{G\times G}(G)$ to any other component of the partial Schur multiplier $pM(G)$, given by restriction.

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