

ON n -TRIVIAL EXTENSIONS OF RINGS

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ABSTRACT. The notion of trivial extension of a ring by a module has been extensively studied and used in ring theory as well as in various other areas of research such as cohomology theory, representation theory, category theory and homological algebra. In this paper, we extend this classical ring construction by associating a ring to a ring R and a family $M = (M_i)_{i=1}^n$ of n R -modules for a given integer $n \geq 1$. We call this new ring construction an n -trivial extension of R by M . In particular, the classical trivial extension will merely be the 1-trivial extension. Thus, we generalize several known results on the classical trivial extension to the setting of n -trivial extensions, and we give some new ones. Various ring-theoretic constructions and properties of n -trivial extensions are studied, and a detailed investigation of the graded aspect of n -trivial extensions is also given. We finish the paper with an investigation of various divisibility properties of n -trivial extensions. In this context, several open questions arise.

1. Introduction. Except for a brief excursion in Section 2, all rings considered in this paper are assumed to be commutative with an identity; in particular, R denotes such a ring, and all modules are assumed to be unitary left modules. Of course, left-modules over a commutative ring R are actually R -bimodules with $mr := rm$. Let \mathbb{Z} (respectively, \mathbb{N}) denotes the set of integers (respectively, natural numbers). The set $\mathbb{N} \cup \{0\}$ will be denoted by \mathbb{N}_0 . The ring $\mathbb{Z}/n\mathbb{Z}$ of the residues modulo an integer $n \in \mathbb{N}$ will be noted by \mathbb{Z}_n .

Recall that the trivial extension of R by an R -module M is the ring denoted by $R \ltimes M$ whose underlying additive group is $R \oplus M$ with multiplication given by $(r, m)(r', m') = (rr', rm' + mr')$. Since

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its introduction by Nagata in [39], the trivial extension of rings (also called idealization since it reduces questions about modules to ideals) has been used by many authors and in various contexts in order to produce examples of rings satisfying preassigned conditions (see, for instance, [9, 37]).

It is known that the trivial extension $R \ltimes M$ is related to the following two ring constructions (see, for instance, [9, Section 2]):

Generalized triangular matrix ring. Let $\mathcal{R} := (R_i)_{i=1}^n$ be a family of rings and $\mathcal{M} := (M_{i,j})_{1 \leq i < j \leq n}$ a family of modules such that, for each $1 \leq i < j \leq n$, $M_{i,j}$ is an (R_i, R_j) -bimodule. Assume that, for every $1 \leq i < j < k \leq n$, there exists an (R_i, R_k) -bimodule homomorphism

$$M_{i,j} \otimes_{R_j} M_{j,k} \longrightarrow M_{i,k},$$

denoted multiplicatively such that $(m_{i,j}m_{j,k})m_{k,l} = m_{i,j}(m_{j,k}m_{k,l})$ for every $(m_{i,j}, m_{j,k}, m_{k,l}) \in M_{i,j} \times M_{j,k} \times M_{k,l}$. Then, the set

$$\begin{pmatrix} R_1 & M_{1,2} & \cdots & \cdots & M_{1,n-1} & M_{1,n} \\ 0 & R_2 & \cdots & \cdots & M_{2,n-1} & M_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & R_{n-1} & M_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 & R_n \end{pmatrix}$$

consisting of matrices

$$\begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & \cdots & m_{1,n-1} & m_{1,n} \\ 0 & m_{2,2} & \cdots & \cdots & m_{2,n-1} & m_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & m_{n-1,n-1} & m_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 & m_{n,n} \end{pmatrix}$$

$$m_{i,i} \in R_i \quad \text{and} \quad m_{i,j} \in M_{i,j}, \quad 1 \leq i < j \leq n,$$

with the usual matrix addition and multiplication is a ring called a *generalized* (or *formal*) *triangular matrix ring* and denoted also by $T_n(\mathcal{R}, \mathcal{M})$ (see [16, 17]). Then the trivial extension $R \ltimes M$ is naturally

isomorphic to the subring of $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$ consisting of matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$ where $r \in R$ and $m \in M$ (note that, since R is commutative, $rm = mr$).

Symmetric algebra. Recall that the symmetric algebra associated to M is the graded ring quotient $S_R(M) := T_R(M)/H$, where $T_R(M)$ is the graded tensor R -algebra with $T_R^n(M) = M^{\otimes n}$ and H is the homogeneous ideal of $T_R(M)$ generated by $\{m \otimes n - n \otimes m \mid m, n \in M\}$. Note that

$$S_R(M) = \bigoplus_{n=0}^{\infty} S_R^n(M)$$

is a graded R -algebra with $S_R^0(M) = R$ and $S_R^1(M) = M$ and, in general, $S_R^i(M)$ is the image of $T_R^i(M)$ in $S_R(M)$. Then, $R \ltimes M$ and

$$S_R(M) / \bigoplus_{n \geq 2} S_R^n(M)$$

are naturally isomorphic as graded R -algebras.

It is also worth recalling that, when M is a free R -module with a basis B , the trivial extension $R \ltimes M$ is also naturally isomorphic to $R[\{X_b\}_{b \in B}] / (\{X_b\}_{b \in B})^2$, where $\{X_b\}_{b \in B}$ is a set of indeterminates over R . In particular, $R \ltimes R \cong R[X] / (X^2)$.

Inspired by the above facts, we introduce an extension of the classical trivial extension of rings to extensions associated to n modules for any integer $n \geq 1$.

In the literature, particular cases of such extensions have been used to solve some open questions. In [10], the authors introduced an extension for $n = 2$, and they used it to give a counterexample of the so-called Faith conjecture. Also, in the case $n = 2$, an extension is introduced in [31] to give an example of a ring which has a non-self-injective injective hull with compatible multiplication. This gave a negative answer to a question posed by Osofsky. In [42], the author introduced and studied a particular extension for the case $n = 3$ to obtain a Galois covering for the enveloping algebras of trivial extension algebras of triangular algebras. In addition, there is a master’s thesis [38] which introduced and studied factorization properties of an extension of the trivial extension of a ring by itself (i.e., self-idealization). In this paper, we introduce the following extension ring construction for an arbitrary integer $n \geq 1$.

Let $M = (M_i)_{i=1}^n$ be a family of R -modules and $\varphi = \{\varphi_{i,j}\}_{\substack{i+j \leq n \\ 1 \leq i,j \leq n-1}}$ a family of bilinear maps such that each $\varphi_{i,j}$ is written multiplicatively:

$$\begin{aligned} \varphi_{i,j} : M_i \times M_j &\longrightarrow M_{i+j} \\ (m_i, m_j) &\longmapsto \varphi_{i,j}(m_i, m_j) := m_i m_j. \end{aligned}$$

In particular, if all M_i are submodules of the same R -algebra L , then the bilinear maps, if they are unspecified, are merely the multiplication of L (see the examples in Section 2). The n - φ -trivial extension of R by M is the set denoted by $R \ltimes_{\varphi} M_1 \ltimes \cdots \ltimes M_n$, or simply $R \ltimes_{\varphi} M$, whose underlying additive group is $R \oplus M_1 \oplus \cdots \oplus M_n$ with multiplication given by

$$(m_0, \dots, m_n)(m'_0, \dots, m'_n) = \left(\sum_{j+k=i} m_j m'_k \right)$$

for all $(m_i), (m'_i) \in R \ltimes_{\varphi} M$. We could also define the product

$$\varphi_{i,j} : M_i \times M_j \longrightarrow M_{i+j}$$

as an R -bimodule homomorphism

$$\tilde{\varphi}_{i,j} : M_i \otimes M_j \longrightarrow M_{i+j},$$

see Section 2 for details. For the sake of simplicity, it is convenient to set $M_0 = R$. In what follows, if no ambiguity arises, the n - φ -trivial extension of R by M will simply be called an n -trivial extension of R by M and denoted by $R \ltimes_n M_1 \ltimes \cdots \ltimes M_n$ or $R \ltimes_n M$.

While, in general, $R \ltimes_n M$ need not be a commutative ring, in Section 2, we give conditions on the maps $\varphi_{i,j}$ that force $R \ltimes_n M$ to be a ring. Unless otherwise stated, we assume the maps $\varphi_{i,j}$ have been defined so that $R \ltimes_n M$ is a commutative associative ring with identity. Thus, $R \ltimes_n M$ is a commutative ring with the identity $(1, 0, \dots, 0)$. Moreover, $R \ltimes_n M$ is naturally isomorphic to the subring of the generalized triangular matrix ring

$$\begin{pmatrix} R & M_1 & M_2 & \cdots \cdots & M_n \\ 0 & R & M_1 & \cdots & M_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_1 \\ 0 & 0 & 0 & \cdots & R \end{pmatrix}$$

consisting of matrices¹

$$\begin{pmatrix} r & m_1 & m_2 & \cdots & m_n \\ 0 & r & m_1 & \cdots & m_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_1 \\ 0 & 0 & 0 & \cdots & r \end{pmatrix},$$

where $r \in R$ and $m_i \in M_i$ for every $i \in \{1, \dots, n\}$.

When, for every $k \in \{1, \dots, n\}$, $M_k = S_R^k(M_1)$, the ring $R \times_n M$ is naturally isomorphic to

$$S_R(M_1) / \bigoplus_{k \geq n+1} S_R^k(M_1).$$

In particular, if $M_1 = F$ is a free R -module with a basis B , then the n -trivial extension $R \times F \times S_R^2(F) \times \cdots \times S_R^n(F)$ is also naturally isomorphic to $R[\{X_b\}_{b \in B}] / (\{X_b\}_{b \in B})^{n+1}$, where $\{X_b\}_{b \in B}$ is a set of indeterminates over R , namely, when $F \cong R$,

$$R \times_n R \times \cdots \times R \cong R[X] / (X^{n+1}).$$

In addition, in [13], the trivial extension of a ring R by an ideal I is connected to the Rees algebra \mathcal{R}_+ associated to R and I , which is precisely the following graded subring of $R[t]$, where t is an indeterminate over R :

$$\mathcal{R}_+ := \bigoplus_{n \geq 0} I^n t^n.$$

Using [13, Lemma 1.2 and Proposition 1.3], we obtain, similar to [13, Proposition 1.4], the following diagram of extensions and isomorphisms of rings:

$$\begin{CD} R^C @>>> \mathcal{R}_+ / (I^{n+1}t^{n+1})^C @>>> R[t] / (t^{n+1}) \\ @| @VV \cong V @VV \cong V \\ R^C @>>> R \times_n I \times I^2 \times \cdots \times I^n^C @>>> R \times_n R \times \cdots \times R. \end{CD}$$

In this paper, we study some properties of the ring $R \times_n M$, extending well-known results on the classical trivial extension of rings. The paper is organized as follows.

In Section 2, we carefully define the n -trivial extension $R \times_n M$, giving conditions on the maps $\varphi_{i,j}$ so that $R \times_n M$ is actually a commutative ring with identity. In particular, we investigate the situation in greater generality, where R is not assumed to be commutative and M_i is an R -bimodule for $i = 1, \dots, n$. We conclude the section with a number of examples.

In Section 3, we investigate some ring-theoretic constructions of n -trivial extensions. We begin by showing that $R \times_n M$ may be considered as a graded ring for three different grading monoids, in particular, $R \times_n M$ may be considered as \mathbb{N}_0 -graded ring or \mathbb{Z}_{n+1} -graded ring. We then show how $R \times_n M$ behaves with respect to polynomials (Corollary 3.4) and power series (Theorem 3.5), extensions and localization (Theorem 3.7). In Theorem 3.9, we show that the n -trivial extension of a finite direct product of rings is a finite direct product of n -trivial extensions. We finish with two results on inverse limits and direct limits of n -trivial extensions (Theorems 3.10 and 3.11).

In Section 4, we present some natural ring homomorphisms related to n -trivial extensions (see Proposition 4.3). Also, we study some basic properties of $R \times_n M$, namely, we extend the characterization of prime and maximal ideals of the classical trivial extension to $R \times_n M$ (see Theorem 4.7). As a consequence, the nilradical and the Jacobson radical are determined (see Corollary 4.8). Finally, as an extension of [9, Theorems 3.5 and 3.7], the set of zero divisors, the set of units and the set of idempotents of $R \times_n M$ are also characterized (see Proposition 4.9).

In Section 5, we investigate the graded aspect of n -trivial extensions. The motivation behind this study is that, in the classical case (where $n = 1$), the study of trivial extensions as \mathbb{Z}_2 -graded rings has led to some interesting properties (see [9]) and has shed more light on the structure of ideals of the trivial extensions. In Section 5, we extend some of the results given in [9], and we provide some new ones, namely, among other results, we characterize the homogeneous ideals of $R \times_n M$ (Theorem 5.1) and we investigate some of their properties (Propositions 5.2 and 5.3). We devote the remainder of Section 5 to investigating the question “when is every ideal of a given class \mathcal{I} of ideals of $R \times_n M$ homogeneous?” (see the discussion after Proposition 5.3). In this context, various results and examples are established.

Section 6 is devoted to some classical ring-theoretic properties, namely, we characterize when $R \times_n M$ is, respectively, Noetherian, Artinian, (Manis) valuation, Prüfer, chained, arithmetical, a π -ring, a generalized ZPI-ring or a PIR. We conclude the section with a remark on a question posed in [2] concerning m -Boolean rings.

Finally, in Section 7, we study divisibility properties of n -trivial extensions. We are mainly interested in showing how the results could be extended on the classical trivial extension presented in [9, Section 5] to the context of n -trivial extensions.

2. The general n -trivial extension construction and some examples. The purpose of this section is to formally define the n -trivial extension ($n \geq 1$) $R \times_n M_1 \times \cdots \times M_n$, where R is a commutative ring with identity and each M_i is an R -module, and to give some interesting examples of n -trivial extensions. However, to better understand the construction and the underlying multiplication maps

$$\varphi_{i,j} : M_i \times M_j \longrightarrow M_{i+j},$$

we begin in the more general context of R being an associative ring (not necessarily commutative) with identity and the M_i 's being R -bimodules. In addition, since there is a significant difference in the cases $n = 1$, $n = 2$ and $n \geq 3$, we handle these three cases separately.

Let R be an associative ring with identity and M_1, \dots, M_n unitary R -bimodules (in the case where R is commutative, we will always assume that $rm = mr$ unless stated otherwise).

Case $n = 1$. $R \times_1 M_1 = R \times M_1 = R \oplus M_1$ is merely the trivial extension with multiplication $(r, m)(r', m') = (rr', rm' + mr')$. Here, $R \times_1 M_1$ is an associative ring with identity where the associative and distributive laws follow from the ring and R -bimodule axioms. For R commutative, we write $(r, m)(r', m') = (rr', rm' + r'm)$ as $r'm = mr'$. Now, $R \times_1 M_1$ is an \mathbb{N}_0 -graded or a \mathbb{Z}_2 -graded ring isomorphic to

$$T_R(M_1) / \bigoplus_{i \geq 2} T_R^i(M_1) \quad \text{or} \quad S_R(M_1) / \bigoplus_{i \geq 2} S_R^i(M_1)$$

and to the matrix ring representation mentioned in the introduction. Note that we could drop the assumption that R has an identity and M_1 is unitary. We then obtain that $R \times_1 M_1$ has an identity, namely, $(1, 0)$, if and only if R has an identity and M_1 is unitary.

Case $n = 2$. Here, $R \ltimes_2 M_1 \ltimes M_2 = R \oplus M_1 \oplus M_2$ with coordinate-wise addition and multiplication

$$(r, m_1, m_2)(r', m'_1, m'_2) = (rr', rm'_1 + m_1r', rm'_2 + m_1m'_1 + m_2r')$$

where $m_1m'_1 := \varphi_{1,1}(m_1, m'_1)$ with the map

$$\varphi_{1,1} : M_1 \times M_1 \longrightarrow M_2.$$

We readily see that $R \ltimes_2 M_1 \ltimes M_2$ satisfying the distributive laws is equivalent to $\varphi_{1,1}$ being additive in each coordinate. Since R is assumed to be associative and M_1 and M_2 to be R -bimodules, $R \ltimes_2 M_1 \ltimes M_2$ is associative precisely when

$$\begin{aligned} (rm_1)m'_1 &= r(m_1m'_1), \\ (m_1r)m'_1 &= m_1(rm'_1) \end{aligned}$$

and

$$(m_1m'_1)r = m_1(m'_1r)$$

for $r \in R$ and $m_1, m'_1 \in M_1$. This is equivalent to

$$\begin{aligned} \varphi_{1,1}(rm_1, m'_1) &= r\varphi_{1,1}(m_1, m'_1), \\ \varphi_{1,1}(m_1r, m'_1) &= \varphi_{1,1}(m_1, rm'_1) \end{aligned}$$

and

$$\varphi_{1,1}(m_1, m'_1)r = \varphi_{1,1}(m_1, m'_1r).$$

For R -bimodules M, N and L , we call a function

$$f : M \times N \longrightarrow L$$

a *pre-product map* if it is additive in each coordinate, is middle linear (i.e., $f(mr, m') = f(m, rm')$) and is left and right homogeneous (i.e., $f(rm, m') = rf(m, m')$ and $f(m, m'r) = f(m, m')r$). Note that a pre-product map $f : M \times N \rightarrow L$ uniquely corresponds to an R -bimodule homomorphism

$$\tilde{f} : M \otimes_R N \longrightarrow L$$

with $f(m, n) = \tilde{f}(m \otimes n)$. Thus, a pre-product map

$$\varphi_{1,1} : M_1 \times M_1 \longrightarrow M_2$$

corresponds to an R -bimodule homomorphism

$$\tilde{\varphi}_{1,1} : M_1 \otimes_R M_1 \longrightarrow M_2.$$

Hence, we may equivalently define $m_1 m'_1 := \tilde{\varphi}_{1,1}(m_1 \otimes m'_1)$.

Therefore, $R \ltimes_2 M_1 \ltimes M_2$ is an (associative) ring with identity precisely when $\varphi_{1,1}$ is a pre-product map or $\tilde{\varphi}_{1,1} : M_1 \otimes_R M_1 \rightarrow M_2$ is an R -bimodule homomorphism. We can identify $R \ltimes_2 M_1 \ltimes M_2$ with the matrix representation given in the introduction: (r, m_1, m_2) is identified with

$$\begin{pmatrix} r & m_1 & m_2 \\ 0 & r & m_1 \\ 0 & 0 & r \end{pmatrix}.$$

However, the relationship with a tensor or symmetric algebra is more difficult. When $R \ltimes_2 M_1 \ltimes M_2$ is an associative ring, we can define a ring epimorphism

$$T_R(M_1 \oplus M_2) / \bigoplus_{i \geq 3} T_R^i(M_1 \oplus M_2) \longrightarrow R \ltimes_2 M_1 \ltimes M_2$$

by

$$\begin{aligned} \left(r, (m_1, m_2), \sum_{i=1}^l (m_{1,i}, m_{2,i}) \otimes (m'_{1,i}, m'_{2,i}) \right) + \bigoplus_{i \geq 3} T_R^i(M_1 \oplus M_2) \\ \longmapsto \left(r, m_1, m_2 + \sum_{i=1}^l m_{1,i} m'_{1,i} \right). \end{aligned}$$

For the commutative case, we get a similar ring epimorphism

$$S_R(M_1 \oplus M_2) / \bigoplus_{i \geq 3} S_R^i(M_1 \oplus M_2) \longrightarrow R \ltimes_2 M_1 \ltimes M_2.$$

In order for $R \ltimes_2 M_1 \ltimes M_2$ to be a commutative ring with identity, R must be commutative with identity and $m_1 m'_1 = m'_1 m_1$ for $m_1, m'_1 \in M_1$, or $\varphi_{1,1}(m_1, m'_1) = \varphi_{1,1}(m'_1, m_1)$. Thus, for R commutative, $R \ltimes_2 M_1 \ltimes M_2$ is a commutative ring if and only if $\varphi_{1,1}$ is a symmetric R -bilinear map, or equivalently, $\tilde{\varphi}_{1,1}(m_1 \otimes m'_1) = \tilde{\varphi}_{1,1}(m'_1 \otimes m_1)$.

Case $n \geq 3$. Here again, R is an associative ring with identity and $M_1, \dots, M_n, n \geq 3$, are R -bimodules. Thus, $R \ltimes_n M_1 \ltimes \dots \ltimes M_n = R \oplus M_1 \oplus \dots \oplus M_n$ with coordinate-wise addition. Assume that we

have pre-product maps

$$\varphi_{i,j} : M_i \times M_j \longrightarrow M_{i+j},$$

or equivalently, the corresponding R -bimodule homomorphism

$$\tilde{\varphi}_{i,j} : M_i \otimes_R M_j \longrightarrow M_{i+j}$$

for $1 \leq i, j \leq n - 1$ with $i + j \leq n$. As usual, set

$$m_i m_j := \varphi_{i,j}(m_i, m_j) = \tilde{\varphi}_{i,j}(m_i \otimes m_j)$$

for $m_i \in M_i$ and $m_j \in M_j$. Setting $R = M_0$, we can write the multiplication in $R \times_n M_1 \times \cdots \times M_n$ as $(m_0, \dots, m_n)(m'_0, \dots, m'_n) = (m''_0, \dots, m''_n)$ where

$$m''_i = \sum_{j+k=i} m_j m'_k.$$

Then, $R \times_n M_1 \times \cdots \times M_n$ satisfies the distributive laws since the maps $\varphi_{i,j}$ are additive in each coordinate. Hence, $R \times_n M_1 \times \cdots \times M_n$ is a not necessarily associative ring with identity $(1, 0, \dots, 0)$ (see Example 2.2 for a case where $R \times_n M_1 \times \cdots \times M_n$ is not associative). Note that $R \times_n M_1 \times \cdots \times M_n$ is associative precisely when $(m_i m_j) m_k = m_i (m_j m_k)$ for $m_i \in M_i, m_j \in M_j$ and $m_k \in M_k$ with $1 \leq i, j, k \leq n - 2$ and $i + j + k \leq n$. In terms of the pre-product maps, this means that $\varphi_{i+j,k}(\varphi_{i,j}(m_i, m_j), m_k) = \varphi_{i,j+k}(m_i, \varphi_{j,k}(m_j, m_k))$, or equivalently,

$$\tilde{\varphi}_{i+j,k} \circ (\tilde{\varphi}_{i,j} \otimes \text{id}_{M_k}) = \tilde{\varphi}_{i,j+k} \circ (\text{id}_{M_i} \otimes \tilde{\varphi}_{j,k})$$

where id_{M_l} is the identity map on M_l for $l \in \{1, \dots, n\}$, in other words, the diagram below commutes:

$$\begin{CD} M_i \otimes M_j \otimes M_k @>{\text{id}_{M_i} \otimes \tilde{\varphi}_{j,k}}>> M_i \otimes M_{j+k} \\ @V{\tilde{\varphi}_{i,j} \otimes \text{id}_{M_k}}VV @VV{\tilde{\varphi}_{i,j+k}}V \\ M_{i+j} \otimes M_k @>{\tilde{\varphi}_{i+j,k}}>> M_{i+j+k} \end{CD}$$

We call a family $\{\varphi_{i,j}\}_{\substack{i+j \leq n \\ 1 \leq i,j \leq n-1}}$ (or $\{\tilde{\varphi}_{i,j}\}_{\substack{i+j \leq n \\ 1 \leq i,j \leq n-1}}$) of pre-product maps satisfying the previously stated associativity condition a *family of product maps*. Thus, when $\{\varphi_{i,j}\}_{\substack{i+j \leq n \\ 1 \leq i,j \leq n-1}}$ (or equivalently $\{\tilde{\varphi}_{i,j}\}_{\substack{i+j \leq n \\ 1 \leq i,j \leq n-1}}$) is a family of product maps, $R \times_n M_1 \times \cdots \times M_n$ is an associative

ring with identity. Further, for $R \times_n M_1 \times \cdots \times M_n$ to be a commutative ring with identity, R must be commutative with identity and $\varphi_{i,j}(m_i, m_j) = \varphi_{j,i}(m_j, m_i)$ for every $1 \leq i, j \leq n - 1$ with $i + j \leq n$, or equivalently, $\tilde{\varphi}_{i,j} = \tilde{\varphi}_{j,i} \circ \tau_{i,j}$ where

$$\tau_{i,j} : M_i \otimes M_j \longrightarrow M_j \otimes M_i$$

is the “flip” map defined by $\tau_{i,j}(m_i \otimes m_j) = m_j \otimes m_i$ for every $m_i \otimes m_j \in M_i \otimes M_j$, in other words, the diagram below commutes:

$$\begin{array}{ccc} M_i \otimes M_j & \xrightarrow{\tilde{\varphi}_{i,j}} & M_{i+j} \\ \tau_{i,j} \downarrow & \nearrow \tilde{\varphi}_{j,i} & \\ M_j \otimes M_i & & \end{array}$$

In this case, the family $\{\varphi_{i,j}\}_{\substack{i+j \leq n \\ 1 \leq i, j \leq n-1}}$ (or $\{\tilde{\varphi}_{i,j}\}_{\substack{i+j \leq n \\ 1 \leq i, j \leq n-1}}$) will be called a *family of commutative product maps*. Thus, when R is commutative and $\{\varphi_{i,j}\}_{\substack{i+j \leq n \\ 1 \leq i, j \leq n-1}}$ (or equivalently $\{\tilde{\varphi}_{i,j}\}_{\substack{i+j \leq n \\ 1 \leq i, j \leq n-1}}$) is a family of commutative product maps, $R \times_n M_1 \times \cdots \times M_n$ is a commutative ring with identity.

As in the case $n = 2$, when $R \times_n M_1 \times \cdots \times M_n$ is an (associative) ring with identity, we can identify $R \times_n M_1 \times \cdots \times M_n$ with the matrix representation given in the introduction: (r, m_1, \dots, m_n) is identified with

$$\begin{pmatrix} r & m_1 & m_2 & \cdots & m_n \\ 0 & r & m_1 & \cdots & m_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_1 \\ 0 & 0 & 0 & \cdots & r \end{pmatrix}.$$

Also, as in the case $n = 2$, when $R \times_n M_1 \times \cdots \times M_n$ is an associative ring, we can define a ring epimorphism

$$T_R(M_1 \oplus \cdots \oplus M_n) / \bigoplus_{i \geq n+1} T_R^i(M_1 \oplus \cdots \oplus M_n) \longrightarrow R \times_n M_1 \times \cdots \times M_n,$$

and we have a similar result concerning the symmetric algebra when $R \times_n M_1 \times \cdots \times M_n$ is commutative.

Remark 2.1.

(1) Let R_1 and R_2 be two rings and H an (R_1, R_2) -bimodule. It is well known that every generalized triangular matrix ring is naturally isomorphic to the trivial extension of $R_1 \times R_2$ by H where the actions of $R_1 \times R_2$ on H are defined as follows: $(r_1, r_2)h = r_1h$ and $h(r_1, r_2) = hr_2$ for every $(r_1, r_2) \in R_1 \times R_2$ and $h \in H$. Below, we see that an observation on the product of two matrices of the generalized triangular matrix ring shows that this fact can be extended to n -trivial extensions.

Consider the generalized triangular matrix ring

$$T_n(\mathcal{R}, \mathcal{M}) = \begin{pmatrix} R_1 & M_{1,2} & \cdots & \cdots & M_{1,n-1} & M_{1,n} \\ 0 & R_2 & \cdots & \cdots & M_{2,n-1} & M_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & R_{n-1} & M_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 & R_n \end{pmatrix},$$

where $\mathcal{R} = (R_i)_{i=1}^n$ is a family of rings and $\mathcal{M} = (M_{i,j})_{1 \leq i < j \leq n}$ is a family of modules such that, for each $1 \leq i < j \leq n$, $M_{i,j}$ is an (R_i, R_j) -bimodule. Assume, for every $1 \leq i < j < k \leq n$, that there exists an (R_i, R_k) -bimodule homomorphism

$$M_{i,j} \otimes_{R_j} M_{j,k} \longrightarrow M_{i,k}$$

denoted multiplicatively such that

$$(m_{i,j}m_{j,k})m_{k,l} = m_{i,j}(m_{j,k}m_{k,l})$$

for every $(m_{i,j}, m_{j,k}, m_{k,l}) \in M_{i,j} \times M_{j,k} \times M_{k,l}$.

Consider the finite direct product of rings $R = R_1 \times \cdots \times R_n$ and set, for $2 \leq i \leq n$, $M_i = M_{1,i} \times M_{2,i+1} \times \cdots \times M_{n-(i-1),n}$, for $i = n$, $M_n = M_{1,n}$. We must define an action of R on each M_i and a family of product maps so that $R \times_{n-1} M_2 \times \cdots \times M_n$ is an $n-1$ -trivial extension isomorphic to $T_n(\mathcal{R}, \mathcal{M})$.

First, note that, for every matrix $A = (a_{i,j})$ of $T_n(\mathcal{R}, \mathcal{M})$ and for every $2 \leq i \leq n$, the i th diagonal above the main diagonal of A naturally corresponds to the following $(n-i+1)$ -tuple $(a_{1,i}, a_{2,i+1}, \dots, a_{n-(i-1),n})$ of M_i . On the other hand, consider two matrices $A = (a_{i,j})$ and

$B = (b_{i,j})$ of $T_n(\mathcal{R}, \mathcal{M})$, and denote the product AB by $C = (c_{i,j})$. Then, using the above correspondence for $2 \leq i \leq n$, the i th diagonal above the main diagonal of C may be seen as the $(n - i + 1)$ -tuple $c_i = (c_{j,i+j-1})_j \in M_i$ such that, for every $1 \leq j \leq n - i + 1$,

$$c_{j,i+j-1} = \sum_{k=j}^{i+j-1} a_{j,k} b_{k,i+j-1} = \sum_{k=1}^i a_{j,k+j-1} b_{k+j-1,i+j-1}.$$

Then,

$$c_i = \left(\sum_{k=1}^i a_{j,k+j-1} b_{k+j-1,i+j-1} \right)_j = \sum_{k=1}^i (a_{j,k+j-1} b_{k+j-1,i+j-1})_j.$$

Thus, the cases $k = 1$ and $k = i$ allow us to define the left and right actions of R on M_i as follows: For every $(r_l)_l \in R$ and $(m_{j,i+j-1})_j \in M_i$,

$$(r_l)_l (m_{j,i+j-1})_j := (r_j m_{j,i+j-1})_j$$

and

$$(m_{j,i+j-1})_j (r_l)_l := (m_{j,i+j-1} r_{i+j-1})_j.$$

The other cases of k may be used to define the product maps $M_k \times M_{i-k} \rightarrow M_i$ as follows: fix k , $1 < k < i$, and consider $e_k = (e_{j,k+j-1})_{1 \leq j \leq n-k+1} \in M_k$ and $f_{i-k} = (f_{j,i-k+j-1})_{1 \leq j \leq n-i+k+1} \in M_{i-k}$. Then,

$$e_k f_{i-k} := (e_{j,k+j-1} f_{k+j-1,i+j-1})_{1 \leq j \leq n-i+1}.$$

Therefore, endowed with these products, $R \times_{n-1} M_2 \times \cdots \times M_n$ is an $n-1$ -trivial extension naturally isomorphic to the generalized triangular matrix ring $T_n(\mathcal{R}, \mathcal{M})$.

(2) It is known that the generalized triangular matrix ring $T_n(\mathcal{R}, \mathcal{M})$ can be seen as a generalized triangular 2×2 matrix ring,² namely, there is a natural ring isomorphism between $T_n(\mathcal{R}, \mathcal{M})$ and $T_2(S, N)$, where

$$S = \left(T_{n-1}((R_i)_{i=1}^{n-1}, (M_{i,j})_{1 \leq i < j \leq n-1}), R_n \right)$$

and

$$N = \begin{pmatrix} M_{1,n} \\ M_{2,n} \\ \vdots \\ M_{n-1,n} \end{pmatrix}.$$

However, an n -trivial extension is not necessarily a 1-trivial extension. For that, consider, for instance, the 2-trivial extension $S = \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. It may easily be verified that S cannot be isomorphic to any 1-trivial extension.

We conclude this section with a number of examples.

Example 2.2. Suppose that R is a commutative ring, and consider $R \rtimes_n R \rtimes \cdots \rtimes R$, $n \geq 1$, with a family of product maps

$$\varphi_{i,j} : Re_i \times Re_j \longrightarrow Re_{i+j},$$

where, for $k \in \{1, \dots, n\}$, $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the k +1th place.

For $n = 1$, $R \rtimes_1 R \cong R[X]/(X^2)$.

Suppose that $n = 2$ and $e_1^2 = r_{1,1}e_2$. Then,

$$R \rtimes_2 R \rtimes R \cong R[X, Y]/(X^2 - r_{1,1}Y, XY, Y^2),$$

where X and Y are commuting indeterminates. Thus, in the case where $r_{1,1} = 1$, we obtain

$$R \rtimes_2 R \rtimes R \cong R[X, Y]/(X^2 - Y, XY, Y^2) \cong R[X]/(X^3).$$

The case $n = 3$ is more interesting. Now, for $1 \leq i, j \leq 2$ with $i + j \leq 3$,

$$\varphi_{i,j} : R \times R \longrightarrow R$$

with $\varphi_{i,j}(r, s) = r\varphi_{i,j}(1, 1)s$. Set $\varphi_{i,j}(1, 1) = r_{i,j}$ such that $(re_i)(se_j) = rr_{i,j}se_{i+j}$. Now, $R \rtimes_3 R \rtimes R \rtimes R$ is commutative if and only if $e_1e_2 = e_2e_1$ or $r_{1,2} = r_{2,1}$; and, $R \rtimes_3 R \rtimes R \rtimes R$ is associative if and only if $(e_1e_1)e_1 = e_1(e_1e_1)$ or $r_{1,1}r_{2,1} = r_{1,2}r_{1,1}$. Thus, if $R \rtimes_3 R \rtimes R \rtimes R$ is commutative, it is also associative. However, if R is a commutative integral domain and $r_{1,1} \neq 0$, $R \rtimes_3 R \rtimes R \rtimes R$ is associative if and only if it is commutative. Thus, if we take $R = \mathbb{Z}$, $r_{1,1} = 1$, $r_{1,2} = 1$ and $r_{2,1} = 2$, $R \rtimes_3 R \rtimes R \rtimes R$ is a non-commutative, non-associative ring.

For $n = 4$, the reader can easily verify that $R \rtimes_4 R \rtimes R \rtimes R \rtimes R$ is commutative if and only if $r_{i,j} = r_{j,i}$ for $1 \leq i, j \leq 3$ with $i + j \leq 4$, and that $R \rtimes_4 R \rtimes R \rtimes R \rtimes R$ is associative if and only if $r_{1,1}r_{2,1} = r_{1,2}r_{1,1}$, $r_{2,1}r_{3,1} = r_{2,2}r_{1,1}$, $r_{1,2}r_{3,1} = r_{1,3}r_{2,1}$, and $r_{1,1}r_{2,2} = r_{1,3}r_{1,2}$. Thus, if R is a commutative integral domain with $r_{1,1} \neq 0$, then $r_{1,1}r_{2,1} = r_{1,2}r_{1,1}$ if and only if $r_{2,1} = r_{1,2}$. Hence, if $r_{1,1} \neq 0$ and $r_{1,2} \neq 0$, then $r_{1,2}r_{3,1} = r_{1,3}r_{2,1}$ if and only if $r_{3,1} = r_{1,3}$. Thus, if $r_{1,1} \neq 0$ and $r_{1,2} \neq 0$, then the fact that $R \rtimes_4 R \rtimes R \rtimes R \rtimes R$ is associative forces $R \rtimes_4 R \rtimes R \rtimes R \rtimes R$ to be commutative and, in this case, $R \rtimes_4 R \rtimes R \rtimes R \rtimes R$ is associative if and only if $r_{1,1}r_{2,2} = r_{1,3}r_{1,2}$. Therefore, if three of the numbers $r_{1,1}$, $r_{2,2}$, $r_{1,3}$ and $r_{1,2}$ are given and nonzero, then there is only one possible choice for the remaining $r_{i,j}$ for $R \rtimes_4 R \rtimes R \rtimes R \rtimes R$ to be associative. If we take $R = \mathbb{Z}$ and $r_{1,1} = 1$, $r_{2,1} = r_{1,2} = 2$, $r_{2,2} = 3$ and $r_{1,3} = r_{3,1} = 4$, then the resulting ring is commutative but not associative.

For $n \geq 5$, the reader can easily write conditions on the $r_{i,j} = \varphi_{i,j}(1, 1)$ for $R \rtimes_n R \rtimes \cdots \rtimes R$ to be commutative or associative.

Example 2.3. Let R be a commutative ring, and let N_1, \dots, N_n be R -submodules of an R -algebra T with $N_i N_j \subseteq N_{i+j}$ for $1 \leq i, j \leq n-1$ with $i+j \leq n$. Then, using the multiplication from T , $R \rtimes_n N_1 \rtimes \cdots \rtimes N_n$ is a ring which is commutative if T is commutative. Following are some interesting special cases:

(a) Let R be a commutative ring and I an ideal of R . Then, $R \rtimes_n I \rtimes I^2 \rtimes \cdots \rtimes I^n$ is the quotient of the Rees ring $R[It]/(I^{n+1}t^{n+1})$ mentioned in the introduction.

(b) Let R be a commutative ring, T an R -algebra and $J_1 \subseteq \cdots \subseteq J_n$ ideals of T . Then, $R \rtimes_n J_1 \rtimes \cdots \rtimes J_n$ is an example of an n -trivial extension since $J_i J_j \subseteq J_i \subseteq J_{i+j}$ for $i + j \leq n$. For example, we could take $R \rtimes_2 XR[X] \rtimes R[X]$.

(c) Suppose that $R_1 \subseteq \cdots \subseteq R_n$ are R -algebras, where R is a commutative ring. Let N be an R_{n-1} -submodule of R_n (in particular, we could take $N = R_n$). Then, $R \rtimes_n R_1 \rtimes \cdots \rtimes R_{n-1} \rtimes N$ with the multiplication induced by R_n is a ring. For example, we could take $\mathbb{Z} \rtimes_3 \mathbb{Q} \rtimes \mathbb{R} \rtimes N$ where N is the \mathbb{R} -submodule of $\mathbb{R}[X]$ of polynomials of degree ≤ 5 .

Example 2.4. Let R be a commutative ring and M an R -module. Let $S := R \times_n R \times \cdots \times R \times M$ with $\varphi_{i,j} : R \times R \rightarrow R$ be the usual ring product in R for $i + j \leq n - 1$, but, for $i + j = n$ and $i, j \geq 1$, $\varphi_{i,j}$ is the zero map. Thus,

$$\begin{aligned} &(r_0, \dots, r_{n-1}, m_n)(r'_0, \dots, r'_{n-1}, m'_n) \\ &= (r_0r'_0, r_0r'_1 + r_1r'_0, \dots, r_0r'_{n-1} + \cdots + r_{n-1}r'_0, r_0m'_n + r'_0m_n). \end{aligned}$$

Then, $S \cong \mathbb{R}[X]/(X^n) \times M$, where M is considered as an $\mathbb{R}[X]/(X^n)$ -module with $f(X)m = f(0)m$.

Example 2.5. Let R be a commutative ring and T an R -algebra. Let $J_1 \subseteq \cdots \subseteq J_n$ be ideals of T . Then, take $R \times_n T/J_1 \times \cdots \times T/J_n$, where the product

$$T/J_i \times T/J_j \longrightarrow T/J_{i+j}$$

is given by $(t_i + J_i)(t_j + J_j) = t_it_j + J_{i+j}$ for $i + j \leq n$.

Example 2.6. Let R be a commutative ring, N_1, \dots, N_{n-1} ideals of R and $N_n = Ra$ a cyclic R -module. Then, consider $R \times_n N_1 \times \cdots \times N_n$, where the products

$$N_i \times N_j \longrightarrow N_{i+j}$$

are the usual products for R when $i + j \leq n - 1$, and for $i + j = n$, define $n_in_j = n_in_ja$.

In what follows, we adopt the following notation.

Notation 2.7. Unless otherwise specified, R denotes a non-trivial ring and, for an integer $n \geq 1$, $M = (M_i)_{i=1}^n$ is a family of R -modules with bilinear maps as indicated in the definition of the n -trivial extension defined such that $R \times_n M$ is a commutative associative ring with identity. Thus, $R \times_n M$ is indeed a commutative ring with identity. Let S be a nonempty subset of R and $N = (N_i)_{i=1}^n$ a family of sets such that, for every i , $N_i \subseteq M_i$. Then, as a subset of $R \times_n M$, $S \times N_1 \times \cdots \times N_n$ will be denoted by $S \times_n N_1 \times \cdots \times N_n$ or simply $S \times_n N$.

3. Some ring-theoretic constructions of n -trivial extensions.

In this section, we investigate some ring-theoretic constructions of n -

trivial extensions. First, we investigate the graded aspect of n -trivial extensions.

For the reader's convenience, we recall the definition of graded rings. Let Γ be a commutative additive monoid. Recall that a ring S is said to be a Γ -graded ring if there is a family of subgroups of S , $(S_\alpha)_{\alpha \in \Gamma}$, such that $S = \bigoplus_{\alpha \in \Gamma} S_\alpha$ as an abelian group, with $S_\alpha S_\beta \subseteq S_{\alpha+\beta}$ for all $\alpha, \beta \in \Gamma$. In addition, an S -module N is said to be Γ -graded if $N = \bigoplus_{\alpha \in \Gamma} N_\alpha$ (as an abelian group) and $S_\alpha N_\beta \subseteq N_{\alpha+\beta}$ for all $\alpha, \beta \in \Gamma$. Note that S_0 is a subring of S and each N_α is an S_0 -module. When $\Gamma = \mathbb{N}_0$, a Γ -graded ring (respectively, a Γ -graded module) will simply be called a graded ring (respectively, a graded module). See, for instance, [40, 41] for more details regarding graded rings, although [40] deals with group graded rings.

Now,

$$R \times_n M_1 \times \cdots \times M_n = R \oplus M_1 \oplus \cdots \oplus M_n$$

may be considered as a graded ring for the following three different grading monoids:

As an \mathbb{N}_0 -graded ring. In this case, we set $M_k = 0$ for all $k \geq n+1$, and we extend the definition of $\varphi_{i,j}$ to all $i, j \geq 0$ as follows: for i or $j = 0$,

$$\begin{aligned} \varphi_{0,j} : R \times M_j &\longrightarrow M_j \\ (r, m_j) &\longmapsto \varphi_{0,j}(r, m_j) := rm_j \end{aligned}$$

and

$$\begin{aligned} \varphi_{i,0} : M_i \times R &\longrightarrow M_i \\ (m_i, r) &\longmapsto \varphi_{i,0}(m_i, r) := m_i r \end{aligned}$$

are just the multiplication of R when $i = j = 0$ or the R -actions on M_j and M_i , respectively, when $j > 0$ and $i > 0$, respectively. For $i, j \geq 0$ such that $i + j \geq n + 1$, we define

$$\varphi_{i,j} : M_i \times M_j \longrightarrow M_{i+j}$$

by $\varphi_{i,j}(m_i, m_j) = 0$ for all $(m_i, m_j) \in M_i \times M_j$. Thus, $R \times_n M_1 \times \cdots \times M_n$ is an \mathbb{N}_0 -graded ring $\bigoplus_{i=0}^\infty R_i$ where $R_0 = R$ and $R_i = M_i$ for $i \in \mathbb{N}$.

As a \mathbb{Z}_{n+1} -graded ring. In this case, we consider, for $a \in \mathbb{Z}$, the least nonnegative integer \widehat{a} with $\widehat{a} \equiv a \pmod{n+1}$, and we set $M_{\widehat{a}} := M_a$. Then, for $a, b \in \mathbb{Z}$, we define maps

$$\overline{\varphi}_{\widehat{a}, \widehat{b}} : M_{\widehat{a}} \times M_{\widehat{b}} \longrightarrow M_{\widehat{a+b}} \text{ by } \overline{\varphi}_{\widehat{a}, \widehat{b}} = \varphi_{\widehat{a}, \widehat{b}}$$

when $\widehat{a} + \widehat{b} \leq n$ and $\overline{\varphi}_{\widehat{a}, \widehat{b}}$ to be the zero map when $\widehat{a} + \widehat{b} > n$. Then, $R \times_n M_1 \times \cdots \times M_n$ is a \mathbb{Z}_{n+1} -graded ring $R_{\widehat{0}} \oplus R_{\widehat{1}} \oplus \cdots \oplus R_{\widehat{n}}$ where $R_{\widehat{0}} = R$ and $R_{\widehat{a}} = M_a$ for $a = 1, \dots, n$.

As a Γ_{n+1} -graded ring. Here, $\Gamma_{n+1} = \{0, 1, \dots, n\}$ is a commutative monoid with addition $i \widehat{+} j := i + j$ if $i + j \leq n$ and $i \widehat{+} j := 0$ if $i + j > n$ (thus, \mathbb{Z}_2 and Γ_2 are isomorphic). In this case, we define maps $\widehat{\varphi}_{i,j}$, for $i, j \in \Gamma_{n+1}$, by $\widehat{\varphi}_{i,j} = \varphi_{i,j}$ when $i = j = 0$ or $i \widehat{+} j \neq 0$ and

$$\widehat{\varphi}_{i,j} : M_i \times M_j \longrightarrow M_0 = R$$

to be the zero map when $i \widehat{+} j = 0$. Then, $R \times_n M_1 \times \cdots \times M_n$ is a Γ_{n+1} -graded ring $R_0 \oplus R_1 \oplus \cdots \oplus R_n$, where $R_0 = R$ and $R_i = M_i$ for $1 \leq i \leq n$.

Note that each of these gradings have the same set of homogeneous elements.

We have observed that $R \times_n M_1 \times \cdots \times M_n$ is an \mathbb{N}_0 -graded ring $\bigoplus_{i=0}^{\infty} R_i$, where $R_0 = R$, $R_i = M_i$ for $i = 1, \dots, n$ and $R_i = 0$ for $i > n$. Thus, $R \times_n M_1 \times \cdots \times M_n$ is a graded ring isomorphic to

$$\bigoplus_{i=0}^{\infty} R_i / \bigoplus_{i \geq n+1} R_i.$$

The next result presents the converse implication, namely, it shows that n -trivial extensions can be realized as quotients of graded rings.

Proposition 3.1. *Let $\bigoplus_{i=0}^{\infty} S_i$ be an \mathbb{N}_0 -graded ring and $m \in \mathbb{N}$. Then, $S_0 \times_m S_1 \times \cdots \times S_m$ with the product induced by $\bigoplus_{i=0}^{\infty} S_i$ is naturally an \mathbb{N}_0 -graded ring isomorphic to*

$$\bigoplus_{i=0}^{\infty} S_i / \bigoplus_{i \geq m+1} S_i.$$

Proof. Obvious. □

The next result presents a particular case of Proposition 3.1.

Proposition 3.2. *For an R -module N , we have the following two natural ring isomorphisms:*

$$T_R(N) / \bigoplus_{i \geq n+1} T_R^i(N) \cong R \times_n N \times T_R^2(N) \times \cdots \times T_R^n(N)$$

and

$$S_R(N) / \bigoplus_{i \geq n+1} S_R^i(N) \cong R \times_n N \times S_R^2(N) \times \cdots \times S_R^n(N).$$

Moreover, suppose that N is a free R -module with a basis B . Then, $R \times_n N \times S_R^2(N) \times \cdots \times S_R^n(N)$ is (graded) isomorphic to $R[\{X_b\}_{b \in B}] / (\{X_b\}_{b \in B})^{n+1}$. In particular, $R \times_n R \times \cdots \times R$ with the natural maps is isomorphic to $R[X] / (X^{n+1})$.

Proof. Obvious. □

Our next result shows that the n -trivial extension of a graded ring by graded modules has a natural grading. It is an extension of [9, Theorem 4.5].

Theorem 3.3. *Let Γ be a commutative additive monoid. Assume that*

$$R = \bigoplus_{\alpha \in \Gamma} R_\alpha$$

is Γ -graded and

$$M_i = \bigoplus_{\alpha \in \Gamma} M_\alpha^i$$

is Γ -graded as an R -module for every $i \in \{1, \dots, n\}$, such that

$$\varphi_{i,j}(M_\alpha^i, M_\beta^j) \subseteq M_{\alpha+\beta}^{i+j}.$$

Then, $R \times_n M_1 \times \cdots \times M_n$ is a Γ -graded ring with

$$(R \times_n M_1 \times \cdots \times M_n)_\alpha = R_\alpha \oplus M_\alpha^1 \oplus \cdots \oplus M_\alpha^n.$$

Proof. Similar to the proof of [9, Theorem 4.5]. □

In the case where R is either a polynomial ring or a Laurent polynomial ring, we get the following result in which the first assertion is an extension of [9, Corollary 4.6 (1)].

Corollary 3.4. *The following statements are true.*

1. $(R \rtimes_n M_1 \times \cdots \times M_n)[\{X_\alpha\}] \cong R[\{X_\alpha\}] \rtimes_n M_1[\{X_\alpha\}] \times \cdots \times M_n[\{X_\alpha\}]$ for any set of indeterminates $\{X_\alpha\}$ over R .
2. $(R \rtimes_n M_1 \times \cdots \times M_n)[\{X_\alpha^{\pm 1}\}] \cong R[\{X_\alpha^{\pm 1}\}] \rtimes_n M_1[\{X_\alpha^{\pm 1}\}] \times \cdots \times M_n[\{X_\alpha^{\pm 1}\}]$ for any set of indeterminates $\{X_\alpha\}$ over R .

In addition, as in the classical case, we get the related (but not graded) power series case. It is a generalization of [9, Corollary 4.6 (2)]. First recall that, for a given set of analytic indeterminates $\{X_\alpha\}_{\alpha \in \Lambda}$ over R , we can consider three types of power series rings (see [43] for further details about generalized power series rings):

$$R[[\{X_\alpha\}_{\alpha \in \Lambda}]]_1 \subseteq R[[\{X_\alpha\}_{\alpha \in \Lambda}]]_2 \subseteq R[[\{X_\alpha\}_{\alpha \in \Lambda}]]_3.$$

Here,

$$R[[\{X_\alpha\}_{\alpha \in \Lambda}]]_1 = \cup \{R[[\{X_{\alpha_1}, \dots, X_{\alpha_n}\}]] \mid \{\alpha_1, \dots, \alpha_n\} \subseteq \Lambda\},$$

$$R[[\{X_\alpha\}_{\alpha \in \Lambda}]]_2 = \left\{ \sum_{i=0}^{\infty} f_i \mid f_i \in R[\{X_\alpha\}_{\alpha \in \Lambda}] \text{ is homogeneous of degree } i \right\}$$

and

$$\begin{aligned} &R[[\{X_\alpha\}_{\alpha \in \Lambda}]]_3 \\ &= \left\{ \sum_{i=0}^{\infty} f_i \mid f_i \text{ is a possibly infinite sum of monomials of degree } i \right. \\ &\quad \left. \text{with at most one monomial of the form } r_{\alpha_1, \dots, \alpha_n} X_{\alpha_1}^{i_1} \cdots X_{\alpha_n}^{i_n} \right. \\ &\quad \left. \text{for each set } \{\alpha_1, \dots, \alpha_n\} \text{ with } i_1 + \cdots + i_n = i \right\}. \end{aligned}$$

More generally, given a partially ordered additive monoid $(S, +, \leq)$, the *generalized power series ring* $R[[X, S^{\leq}]]$ consists of all formal sums

$$f = \sum_{s \in S} a_s X^s,$$

where $\text{supp}(f) = \{s \in S \mid a_s \neq 0\}$ is Artinian and narrow (i.e., has no infinite family of incomparable elements) where addition and multiplication are carried out in the standard manner. If Λ is a well-ordered set,

$$S = \bigoplus_{\lambda \in \Lambda} \mathbb{N}_0,$$

and \leq is the reverse lexicographic order on S , then $R[[X, S^{\leq}]] \cong R[[\{X_\alpha\}]]_3$.

Note that, in a similar manner, we can define three types of power series over a module. The routine proof of the next theorem is left to the reader.

Theorem 3.5.

(1) *Let $\{X_\alpha\}_{\alpha \in \Lambda}$ be a set of analytic indeterminates over R . Then, for $i = 1, 2, 3$,*

$$(R \times_n M)[[\{X_\alpha\}_{\alpha \in \Lambda}]]_i \cong R[[\{X_\alpha\}_{\alpha \in \Lambda}]]_i \times_n M_1[[\{X_\alpha\}_{\alpha \in \Lambda}]]_i \\ \times \cdots \times M_n[[\{X_\alpha\}_{\alpha \in \Lambda}]]_i.$$

(2) *Let $(S, +, \leq)$ be a partially ordered additive monoid. Then*

$$(R \times_n M)[[X, S^{\leq}]] \cong R[[X, S^{\leq}]] \times_n M_1[[X, S^{\leq}]] \\ \times \cdots \times M_n[[X, S^{\leq}]].$$

Now, we give, as an extension of [9, Theorem 4.1], the following result which investigates the localization of an n -trivial extension. For this, we need the next technical lemma.

Lemma 3.6. *For every $(m_i) \in R \times_n M$ and $k \in \{1, \dots, n\}$,*

$$(m_0, 0, \dots, 0, m_k, m_{k+1}, \dots, m_n)(m_0, 0, \dots, 0, -m_k, 0, \dots, 0) \\ = (m_0^2, 0, \dots, 0, e_{k+1}, \dots, e_n)$$

where $e_l = m_0 m_l - m_k m_{l-k}$ for every $l \in \{k+1, \dots, n\}$. Consequently, there is an element (f_i) of $R \times_n M$ such that

$$(m_i)(f_i) = (m_0^{2^n}, 0, \dots, 0).$$

We will denote the element (f_i) in Lemma 3.6 by (\widetilde{m}_i) ; thus, $(m_i)(\widetilde{m}_i) = (m_0^{2^n}, 0, \dots, 0)$.

Theorem 3.7. *Let S be a multiplicatively closed subset of R and $N = (N_i)$ a family of R -modules where N_i is a submodule of M_i for each $i \in \{1, \dots, n\}$ and $N_i N_j \subseteq N_{i+j}$ for every $1 \leq i, j \leq n - 1$ and $i + j \leq n$. Then, the set $S \times_n N$ is a multiplicatively closed subset of $R \times_n M$, and we have a ring isomorphism*

$$(R \times_n M)_{S \times_n N} \cong R_S \times_n M_S,$$

where $M_S = (M_{iS})$.

Proof. It is trivial to show that $S \times_n N$ is a multiplicatively closed subset of $R \times_n M$. Now, in order to show the desired isomorphism, we need to make, as done in the proof of [9, Theorem 4.1 (1)], the following observation: let $(m_i) \in R \times_n M$ and $(s_i) \in S \times_n N$. Then, using the notation of Lemma 3.6,

$$\frac{(m_i)}{(s_i)} = \frac{(m_i)(\widetilde{s}_i)}{(S_0, 0, \dots, 0)} = \frac{(m'_i)}{(S_0, 0, \dots, 0)}$$

where $(m'_i) = (m_i)(\widetilde{s}_i)$ and $S_0 = s_0^{2^n}$. Then, the map

$$f : (R \times_n M)_{S \times_n N} \longrightarrow R_S \times_n M_S$$

$$\frac{(m_i)}{(s_i)} \longmapsto \left(\frac{m'_0}{S_0}, \frac{m'_1}{S_0}, \dots, \frac{m'_n}{S_0} \right)$$

is the desired isomorphism. □

As a simple, but important, specific case of Theorem 3.7, we get the following result which extends [9, Theorem 4.1, Corollary 4.7]. In Theorem 4.7, we will show that, if P is a prime ideal of R , then $P \times_n M$ is a prime ideal of $R \times_n M$. This fact is used in the next result to show that the localization of an n -trivial extension at a prime ideal is isomorphic to an n -trivial extension. In what follows, we use $T(A)$ to denote the total quotient ring of a ring A . In Proposition 4.9, we will prove that $S \times_n M$, where

$$S = R - (Z(R) \cup Z(M_1) \cup \dots \cup Z(M_n))$$

is the set of all regular elements of $R \times_n M$. Thus,

$$T(R \times_n M) = (R \times_n M)_{S \times_n M}.$$

Corollary 3.8. *The following assertions are true.*

(1) *Let P be a prime ideal of R . Then, we have a ring isomorphism*

$$(R \times_n M)_{P \times_n M} \cong R_P \times_n M_P,$$

where $M_P = (M_i)_P$.

(2) *We have a ring isomorphism*

$$T(R \times_n M) \cong R_S \times_n M_S,$$

where $S = R - (Z(R) \cup Z(M_1) \cup \dots \cup Z(M_n))$.

(3) *For an indeterminate X over R , we have a ring isomorphism*

$$(R \times_n M_1 \times \dots \times M_n)(X) \cong R(X) \times_n M_1(X) \times \dots \times M_n(X).$$

Proof. All of the proofs are similar to those corresponding to the classical case. □

Our next result generalizes [9, Theorem 4.4]. It shows that the n -trivial extension of a finite direct product of rings is a finite direct product of n -trivial extensions. For the reader's convenience, we recall some known facts on the structure of modules over a finite direct product of rings. Let

$$R = \prod_{i=1}^s R_i$$

be a finite direct product of rings where $s \in \mathbb{N}$. For $j \in \{1, \dots, s\}$, we set

$$\overline{R}_j := 0 \times \dots \times 0 \times R_j \times 0 \times \dots \times 0$$

and, for an R -module N , $N_j := \overline{R}_j N$. Then, N_j is a submodule of N , and we have $N = N_1 \oplus \dots \oplus N_s$, namely, every element x in N can be written in the form $x = x_1 + \dots + x_s$ where $x_j = e_j x \in N_j$ for every $j \in \{1, \dots, s\}$ (here, $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the j th place). Note that each N_j is also an R_j -module, and $N_1 \times \dots \times N_s$ is

an R -module isomorphic to N via the following R -isomorphisms:

$$N \longrightarrow N_1 \times \cdots \times N_s$$

$$\sum e_j x = x \longmapsto (e_1 x_1, \dots, e_s x_s)$$

and

$$N_1 \times \cdots \times N_s \longrightarrow N$$

$$(y_1, \dots, y_s) \longmapsto \sum y_j.$$

Now, consider the family of commutative product maps $\varphi = \{\varphi_{i,j} \}_{\substack{i+j \leq n \\ 1 \leq i, j \leq n-1}}$, and define the following maps:

$$\varphi_{j,i,k} : M_{j,i} \times M_{j,k} \longrightarrow M_{j,i+k}$$

$$(m_{j,i}, m_{j,k}) \longmapsto \varphi_{j,i,k}(m_{j,i}, m_{j,k}) = e_j \varphi_{i,k}(m_{j,i}, m_{j,k}),$$

where $M_{j,i} := \overline{R}_j M_i$ for $j \in \{1, \dots, s\}$ and $i \in \{1, \dots, n\}$. It is easily checked that, for every $j \in \{1, \dots, s\}$, $\varphi_j = \{\varphi_{j,i,k} \}_{\substack{i+k \leq n \\ 1 \leq i, k \leq n}}$ is a family of commutative product maps and

$$R_j \rtimes_{\varphi_j} M_{j,1} \rtimes \cdots \rtimes M_{j,n}$$

is an n - φ_j -trivial extension. Furthermore,

$$\varphi_{i,k} : M_i \times M_k \longrightarrow M_{i+k}$$

$$(m_i, m_k) \longmapsto \varphi_{i,k}(m_i, m_k) = \sum_{j=1}^s \varphi_{j,i,k}(m_{j,i}, m_{j,k}).$$

With this notation in mind, we are ready to give the desired result.

Theorem 3.9. *Let*

$$R = \prod_{i=1}^s R_i$$

be a finite direct product of rings where $s \in \mathbb{N}$. Then,

$$R \rtimes_{\varphi} M_1 \rtimes \cdots \rtimes M_n \cong \prod_{i=1}^s (R_i \rtimes_{\varphi_i} M_{i,1} \rtimes \cdots \rtimes M_{i,n}).$$

Proof. It is easily verified that the map

$$(r, m_1, \dots, m_n) \mapsto ((r_j, m_{j,1}, \dots, m_{j,n}))_{1 \leq j \leq s}$$

is an isomorphism. □

We conclude this section with two results which investigate the inverse and direct limits of a system of n -trivial extensions, namely, we show that, under some conditions, the inverse limit or direct limit of a system of n -trivial extensions is isomorphic to an n -trivial extension. The inverse limit case is a generalization of [9, Theorem 4.11].

Let Γ be a directed set, and let $\{M_\alpha; f_{\alpha\beta}\}$ be an inverse system of abelian groups over Γ (thus, for $\alpha \leq \beta$, $f_{\alpha\beta} : M_\beta \rightarrow M_\alpha$). We know that the inverse limit $\varprojlim M_\alpha$ is isomorphic to the following subset of the direct product $\prod_\alpha M_\alpha$:

$$M_\infty := \{(x_\alpha)_{\alpha \in \Gamma} \mid \lambda \leq \mu \implies x_\lambda = f_{\lambda\mu}(x_\mu)\}.$$

In the next result, by $\varprojlim M_\alpha$, we mean exactly the set M_∞ .

Theorem 3.10. *Let Γ be a directed set and $n \geq 1$ an integer. Consider a family of inverse systems $\{M_{i,\alpha}; f_{i,\alpha,\beta}\}$ over Γ (for $i \in \{0, \dots, n\}$) which satisfy the following conditions:*

- (1) *for every $\alpha \in \Gamma$, $M_{0,\alpha} = R_\alpha$ is a ring;*
- (2) *for every $\alpha \in \Gamma$ and $i \in \{1, \dots, n\}$, $M_{i,\alpha}$ is an R_α -module; and*
- (3) *for every $\alpha \in \Gamma$, $R_\alpha \times_n M_{1,\alpha} \times \dots \times M_{n,\alpha}$ is an n -trivial extension with a family of commutative product maps:*

$$\varphi_{i,j,\alpha} : M_{i,\alpha} \times M_{j,\alpha} \longrightarrow M_{i+j,\alpha}$$

which satisfy, for every $\alpha \leq \beta$,

$$\varphi_{i,j,\alpha}(f_{i,\alpha,\beta}(m_{i,\beta}), f_{j,\alpha,\beta}(m_{j,\beta})) = f_{i+j,\alpha,\beta}(\varphi_{i,j,\beta}(m_{i,\beta}, m_{j,\beta})).$$

Then $\varprojlim R_\alpha \times_n \varprojlim M_{1,\alpha} \times \dots \times \varprojlim M_{n,\alpha}$ is an n -trivial extension with the following family of well-defined commutative product maps:

$$\begin{aligned} \varphi_{i,j,\alpha} : \varprojlim M_{i,\alpha} \times \varprojlim M_{j,\alpha} &\longrightarrow \varprojlim M_{i+j,\alpha} \\ ((m_{i,\alpha})_\alpha, (m_{j,\alpha})_\alpha) &\longmapsto (\varphi_{i,j,\alpha}(m_{i,\alpha}, m_{j,\alpha}))_\alpha. \end{aligned}$$

Moreover, there is a natural ring isomorphism:

$$\varprojlim (R_\alpha \times_n M_{1,\alpha} \times \cdots \times M_{n,\alpha}) \cong \varprojlim R_\alpha \times_n \varprojlim M_{1,\alpha} \times \cdots \times \varprojlim M_{n,\alpha}.$$

Proof. The result follows using a standard argument. □

Let Γ be a directed set and $\{M_\gamma; f_{\gamma\lambda}\}$ a direct system of abelian groups over Γ (thus, for $\gamma \leq \lambda$, $f_{\gamma\lambda} : M_\gamma \rightarrow M_\lambda$). We know that the direct limit $\varinjlim M_\gamma$ is isomorphic to $\bigoplus_\gamma M_\gamma / S$, where S is generated by all elements $\lambda_\beta(f_{\alpha\beta}(a_\alpha)) - \lambda_\alpha(a_\alpha)$ where $\alpha \leq \beta$, and

$$\lambda_\lambda : M_\lambda \longrightarrow \bigoplus_\gamma M_\gamma$$

is the natural inclusion map for $\lambda \in \Gamma$. Since Γ is directed, every element of $\bigoplus_\gamma M_\gamma / S$ has the form $\lambda_\alpha(a_\alpha) + S$ for some $\alpha \in \Gamma$ and $a_\alpha \in M_\alpha$.

Theorem 3.11. *Let Γ be a directed set and $n \geq 1$ an integer. Consider a family of direct systems $\{M_{i,\alpha}; f_{i,\alpha,\beta}\}$ over Γ (for $i \in \{0, \dots, n\}$) which satisfy the following conditions:*

- (1) *for every $\alpha \in \Gamma$, $M_{0,\alpha} = R_\alpha$ is a ring;*
- (2) *For every $\alpha \in \Gamma$ and $i \in \{1, \dots, n\}$, $M_{i,\alpha}$ is an R_α -module; and*
- (3) *For every $\alpha \in \Gamma$, $R_\alpha \times_n M_{1,\alpha} \times \cdots \times M_{n,\alpha}$ is an n -trivial extension with a family of commutative product maps:*

$$\varphi_{i,j,\alpha} : M_{i,\alpha} \times M_{j,\alpha} \longrightarrow M_{i+j,\alpha},$$

which satisfy, for every $\beta \leq \alpha$,

$$\varphi_{i,j,\alpha}(f_{i,\beta,\alpha}(m_{i,\beta}), f_{j,\beta,\alpha}(m_{j,\beta})) = f_{i+j,\beta,\alpha}(\varphi_{i,j,\beta}(m_{i,\beta}, m_{j,\beta})).$$

Then,

$$\varinjlim R_\alpha \times_n \varinjlim M_{1,\alpha} \times \cdots \times \varinjlim M_{n,\alpha}$$

is an n -trivial extension with the following family of well-defined commutative product maps:

$$\begin{aligned} \varphi_{i,j,\alpha} : \varinjlim M_{i,\alpha} \times \varinjlim M_{j,\alpha} &\longrightarrow \varinjlim M_{i+j,\alpha} \\ ((m_{i,\alpha})_\alpha, (m_{j,\alpha})_\alpha) &\longmapsto (\varphi_{i,j,\alpha}(m_{i,\alpha}, m_{j,\alpha}))_\alpha. \end{aligned}$$

Moreover, there is a natural ring isomorphism:

$$\varinjlim (R_\alpha \times_n M_{1,\alpha} \times \cdots \times M_{n,\alpha}) \cong \varinjlim R_\alpha \times_n \varinjlim M_{1,\alpha} \times \cdots \times \varinjlim M_{n,\alpha}.$$

Proof. The result follows using a standard argument. □

4. Some basic algebraic properties of $R \times_n M$. In this section, we give some basic properties of n -trivial extensions. Before giving the first result, we make the following observations regarding situations where a subfamily of M is trivial.

Observation 4.1.

(1) If there is an integer $i \in \{1, \dots, n - 1\}$ such that $M_j = 0$ for every $j \in \{i + 1, \dots, n\}$, then there is a natural ring isomorphism

$$R \times_n M_1 \times \cdots \times M_i \times 0 \times \cdots \times 0 \cong R \times_i M_1 \times \cdots \times M_i.$$

If $M_1 = \cdots = M_{n-1} = 0$, then $R \times_n M$ can be represented as $R \times_1 M_n$. However, if $n \geq 3$ and there is an integer $i \in \{1, \dots, n - 2\}$ such that, for $j \in \{1, \dots, n\}$, $M_j = 0$ if and only if $j \in \{1, \dots, i\}$, then, in general, $R \times_n 0 \times \cdots \times 0 \times M_{i+1} \times \cdots \times M_n$ cannot be represented as an $n - i$ -trivial extension as above. Indeed, if, for example, i satisfies $2i + 2 \leq n$, then $R \times_{n-i} M_{i+1} \times \cdots \times M_n$ is nonsensical (since $\varphi_{i+1, i+1}(M_{i+1}, M_{i+1})$ is a subset of M_{2i+2} , not of M_{i+2}).

(2) If $M_{2k} = 0$ for every $k \in \mathbb{N}$ with $1 \leq 2k \leq n$, then $R \times_n M$ can be represented as the trivial extension of R by the R -module $M_1 \times M_3 \times \cdots \times M_{2n'+1}$, where $2n' + 1$ is the largest odd integer in $\{1, \dots, n\}$, namely, there is a natural ring isomorphism

$$R \times_n M \cong R \times_1 (M_1 \times M_3 \times \cdots \times M_{2n'+1}).$$

(3) If $M_{2k+1} = 0$ for every $k \in \mathbb{N}$ with $1 \leq 2k + 1 \leq n$, then there is a natural ring isomorphism

$$R \times_n M \cong R \times_{n''} M_2 \times M_4 \times \cdots \times M_{2n''},$$

where $2n''$ is the biggest even integer in $\{1, \dots, n\}$. In general, for every cyclic submonoid G of Γ_{n+1} generated by an element $g \in \{1, \dots, n\}$, if $M_i = 0$ if and only if $i \notin G$, then there is a natural ring isomorphism

$$R \times_n M \cong R \times_s M_g \times M_{2g} \times \cdots \times M_{sg},$$

where sg is the biggest integer in $G \cap \{1, \dots, n\}$.

As observed above, whether a subfamily of M is or is not trivial leads to various situations. Thus, for the sake of simplicity, we state the following convention.

Convention 4.2. *Unless explicitly stated otherwise, when we consider an n -trivial extension for a given n , we then implicitly suppose that $M_i \neq 0$ for every $i \in \{1, \dots, n\}$. This will be used in the sequel without explicit mention.*

Note also that the nature of the maps $\varphi_{i,j}$ can affect the structure of the n -trivial extension. For example, in case where $n = 2$, if $\varphi_{1,1} = 0$, then

$$R \rtimes_2 M_1 \times M_2 \cong R \times (M_1 \times M_2).$$

Additionally, if $I \subseteq J$ is an extension of ideals of R , then

$$R \rtimes_2 I \times R/J \cong R \times (I \times R/J).$$

We begin with the following result which presents some (easily established) relations among n -trivial extensions.

Proposition 4.3. *The following assertions are true.*

(1) *Let G be a submonoid of Γ_{n+1} , and consider the family of R -modules $M' = (M'_i)_{i=1}^n$ such that $M'_i = M_i$ if $i \in G$ and $M'_i = 0$ if $i \notin G$. Then, we have the following (natural) ring extensions:*

$$R \hookrightarrow R \rtimes_n M' \hookrightarrow R \rtimes_n M.$$

In particular, for every $m \in \{1, \dots, n\}$, we have the following (natural) ring extensions:

$$R \hookrightarrow R \rtimes_n 0 \times \dots \times 0 \times M_m \times \dots \times M_n \hookrightarrow R \rtimes_n M_1 \times \dots \times M_n.$$

The extension $R \hookrightarrow R \rtimes_n M_1 \times \dots \times M_n$ will be denoted by i_n .

(2) *For every $m \in \{1, \dots, n\}$,*

$$0 \rtimes_n 0 \times \dots \times 0 \times M_m \times \dots \times M_n$$

is an ideal of $R \rtimes_n M$ and an $R \rtimes_j M_1 \times \cdots \times M_j$ -module for every $j \in \{n - m, \dots, n\}$ via the action

$$\begin{aligned} &(x_0, x_1, \dots, x_j)(0, \dots, 0, y_m, \dots, y_n) \\ &:= (x_0, x_1, \dots, x_j, 0, \dots, 0)(0, \dots, 0, y_m, \dots, y_n) \\ &= (x_0, x_1, \dots, x_{n-m}, 0, \dots, 0)(0, \dots, 0, y_m, \dots, y_n). \end{aligned}$$

Moreover, the structure of $0 \rtimes_n 0 \times \cdots \times 0 \rtimes M_m \times \cdots \times M_n$ as an ideal of $R \rtimes_n M$ is the same as the $R \rtimes_j M_1 \times \cdots \times M_j$ -module structure for every $j \in \{n - m, \dots, n\}$. In particular, the structure of the ideal $0 \rtimes_n 0 \times \cdots \times 0 \rtimes M_n$ is the same as that of the R -module M_n .

(3) For every $m \in \{1, \dots, n\}$, we have the following natural ring isomorphism:

$$R \rtimes_n M / 0 \rtimes_n 0 \times \cdots \times 0 \rtimes M_m \times \cdots \times M_n \cong R \rtimes_{m-1} M_1 \times \cdots \times M_{m-1}$$

obtained from the natural ring homomorphism:

$$\begin{aligned} \pi_{m-1} : R \rtimes_n M &\longrightarrow R \rtimes_{m-1} M_1 \times \cdots \times M_{m-1} \\ (r, x_1, \dots, x_n) &\longmapsto (r, x_1, \dots, x_{m-1}) \end{aligned}$$

where, for $m = 1$, $R \rtimes_{m-1} M_1 \times \cdots \times M_{m-1} = R$.

In order to give another example for assertion (1), it can be shown that, for $n = 3$, $\{0, 2\}$ is a submonoid of Γ_4 . Then, we have the following (natural) ring extensions:

$$R \hookrightarrow R \rtimes M_2 \hookrightarrow R \rtimes_3 M_1 \times M_2 \times M_3.$$

Remark 4.4. We have seen that, in the case of $n = 1$, the ideal structure of $0 \rtimes_1 M_1$ is the same as the R -module structure of $0 \rtimes_1 M_1$. Specifically, Nagata [39] used this to reduce the proofs of module-theoretic results to the ideal case. However, for $n \geq 2$, the R -module structure of $0 \rtimes_n M_1 \times \cdots \times M_n$ need not be the same as the ideal structure. For instance, consider the 2-trivial extension $\mathbb{Z} \rtimes_2 \mathbb{Z} \times \mathbb{Z}$ (with maps induced by the multiplication in \mathbb{Z}). Then,

$$\mathbb{Z}(0, 1, 1) = \{(0, m, m) \mid m \in \mathbb{Z}\},$$

while the ideal of $\mathbb{Z} \times_2 \mathbb{Z} \times \mathbb{Z}$ generated by $(0, 1, 1)$ is $0 \times_2 \mathbb{Z} \times \mathbb{Z}$. However, according to Proposition 4.3 (2),

$$(\mathbb{Z} \times_1 \mathbb{Z})(0, 1, 1) = (\mathbb{Z} \times_2 \mathbb{Z} \times \mathbb{Z})(0, 1, 1).$$

The notion of extensions of ideals under ring homomorphisms is a natural way to construct examples of ideals. In this context, we use the ring homomorphism i_m (indicated in Proposition 4.3) to give such examples.

Proposition 4.5. *For an ideal I of R , we have the following assertions:*

(1) *The ideal $I \times_n IM_1 \times \cdots \times IM_n$ of $R \times_n M$ is the extension of I under the ring homomorphism i_n , and we have the following natural ring isomorphism:*

$$(R \times_n M)/(I \times_n IM_1 \times \cdots \times IM_n) \cong (R/I) \times_n (M_1/IM_1) \times \cdots \times (M_n/IM_n)$$

where the multiplication is well defined as follows:

$$\begin{aligned} \bar{\varphi}_{i,j} : M_i/IM_i \times M_j/IM_j &\longrightarrow M_{i+j}/IM_{i+j} \\ (\overline{m_i}, \overline{m_j}) &\longmapsto \overline{m_i m_j} := \overline{\varphi_{i,j}(\overline{m_i}, \overline{m_j})} \\ &:= \overline{\varphi_{i,j}(m_i, m_j)} = \overline{m_i m_j}. \end{aligned}$$

(2) *The ideal $I \times_n IM_1 \times \cdots \times IM_n$ is finitely generated if and only if I is finitely generated.*

Proof.

(1) The proof is straightforward.

(2) Using π_0 , it is clear that, if $I \times_n IM_1 \times \cdots \times IM_n$ is generated by the elements $(r_j, m_{j,1}, \dots, m_{j,n})$ with $j \in E$ for some set E , then I is generated by the r_j s. Conversely, if I is generated by elements r_j with $j \in E$ for some set E , then $I \times_n IM_1 \times \cdots \times IM_n$ is generated by the $(r_j, 0, \dots, 0)$ s. □

Now, we determine the radical, prime and maximal ideals of $R \times_n M$. As in the classical case, we show that these ideals are specific cases of those which are homogenous, characterized in the next section.

However, we provide these particular cases here due to their simplicity, which is reflected, using the following lemma, in the fact that they contain the nilpotent ideal $0 \times_n M$ (of index $n + 1$).

Lemma 4.6. *Every ideal of $R \times_n M$ which contains $0 \times_n M$ has the form $I \times_n M$ for some ideal I of R . In this case, we have the following natural ring isomorphism:*

$$(R \times_n M)/(I \times_n M) \cong R/I.$$

Proof. Let J be an ideal of $R \times_n M$ which contains $0 \times_n M$ and consider the ideal $I = \pi_0(J)$ of R where π_0 is the surjective ring homomorphism used in Proposition 4.3. Then $J \subseteq I \times_n M$ and by the fact that $0 \times_n M \subseteq J$, we deduce that $J = I \times_n M$. Finally, using π_0 and the fact that $\pi_0^{-1}(I) = J$, we get the desired isomorphism. \square

The following result is an extension of [9, Theorem 3.2].

Theorem 4.7. *Radical ideals of $R \times_n M$ have the form $I \times_n M$ where I is a radical ideal of R . In particular, the maximal (respectively, prime) ideals of $R \times_n M$ have the form $\mathcal{M} \times_n M$ (respectively, $P \times_n M$) where \mathcal{M} (respectively, P) is a maximal (respectively, prime) ideal of R .*

Proof. Using Lemma 4.6, it is sufficient to note that every radical ideal contains $0 \times_n M$ since $(0 \times_n M)^{n+1} = 0$. \square

Theorem 4.7 allows us easily to determine both the Jacobson radical and the nilradical of $R \times_n M$.

Corollary 4.8. *The Jacobson radical $J(R \times_n M)$ (respectively, the nilradical $\text{Nil}(R \times_n M)$) of $R \times_n M$ is $J(R) \times_n M$ (respectively, $\text{Nil}(R) \times_n M$), and the Krull dimension of $R \times_n M$ is equal to that of R .*

We end this section with an extension of [9, Theorems 3.5, 3.7] which determines, respectively, the set of zero divisors $Z(R \times_n M)$, the set of units $U(R \times_n M)$ and the set of idempotents $\text{Id}(R \times_n M)$ of $R \times_n M$. It is worth noting that trivial extensions have been used to construct examples of rings with zero divisors which satisfy certain properties. As mentioned in the introduction, particular 2-trivial extensions are used

to settle some questions. Recently, in [14], a 2-trivial extension is used in the context of zero-divisor graphs to give an appropriate example.

Proposition 4.9. *The following assertions are true.*

(1) *The set of zero divisors of $R \times_n M$ is*

$$Z(R \times_n M) = \{(r, m_1, \dots, m_n) \mid r \in Z(R) \cup Z(M_1) \cup \dots \cup Z(M_n), \\ m_i \in M_i \text{ for } 1 \leq i \leq n\}.$$

Hence, $S \times_n M$ where $S = R - (Z(R) \cup Z(M_1) \cup \dots \cup Z(M_n))$ is the set of regular elements of $R \times_n M$.

(2) *The set of units of $R \times_n M$ is $U(R \times_n M) = U(R) \times_n M$.*

(3) *The set of idempotents of $R \times_n M$ is $\text{Id}(R \times_n M) = \text{Id}(R) \times_n 0$.*

Proof. All of the proofs are similar to the corresponding ones for the classical case. For completeness, we give a proof of the first assertion.

Let $(r, m_1, \dots, m_n) \in R \times_n M$ be such that $r \in Z(R) \cup Z(M_1) \cup \dots \cup Z(M_n)$. If $r = 0$, then $(0, m_1, \dots, m_n)(0, \dots, m'_n) = (0, \dots, 0)$ for every $m'_n \in M_n$. Hence, $(r, m_1, \dots, m_n) \in Z(R \times_n M)$. Suppose that $r \neq 0$. If $r \in Z(R)$, then there exists a nonzero element $s \in R$ such that $rs = 0$; thus, $(r, 0, \dots, 0)(s, 0, \dots, 0) = (0, \dots, 0)$, and hence, $(r, 0, \dots, 0) \in Z(R \times_n M)$. If $r \in Z(M_i)$, then, for some $i \in \{1, \dots, n\}$, there exists a nonzero element m''_i of M_i such that $rm''_i = 0$. Therefore,

$$(r, 0, \dots, 0)(0, \dots, 0, m''_i, 0, \dots, 0) = (0, \dots, 0).$$

Hence, $(r, 0, \dots, 0) \in Z(R \times_n M)$. Now, since $Z(R \times_n M)$ is a union of prime ideals, $\text{Nil}(R \times_n M)$ is contained in each prime ideal and using the fact that $(0, m_1, \dots, m_n) \in \text{Nil}(R \times_n M)$, we conclude that

$$(r, m_1, \dots, m_n) = (r, 0, \dots, 0) + (0, m_1, \dots, m_n) \in Z(R \times_n M).$$

This shows the first inclusion.

Conversely, let $(r, m_1, \dots, m_n) \in Z(R \rtimes_n M)$. Then, there is an $(s, m'_1, \dots, m'_n) \in R \rtimes_n M - \{(0, \dots, 0)\}$ such that

$$\begin{aligned} (0, \dots, 0) &= (r, m_1, \dots, m_n)(s, m'_1, \dots, m'_n) \\ &= \left(rs, rm'_1 + sm_1, rm'_2 + m_1m'_1 + sm_2, \dots, \right. \\ &\qquad \left. rm'_n + \sum_{i+j=n} m_i m'_j + sm_n \right). \end{aligned}$$

If $s \neq 0$, then $r \in Z(R)$, and if $s = 0$, we obtain $r \in Z(M_1)$ if $m'_1 \neq 0$. Otherwise, we pass to m'_2 , and so on, until we arrive at $s = 0$ and $m'_i = 0$ for all $i \in \{1, \dots, n - 1\}$. Then, $rm'_n = 0$ and $m'_n \neq 0$; thus, $r \in Z(M_n)$. This gives the desired inclusion. \square

5. Homogeneous ideals of n -trivial extensions. The study of the classical trivial extension as a graded ring established some interesting properties (see, for instance, [9, Section 3]), namely, in [9], studying homogeneous ideals of the trivial extension shed more light on the structure of their ideals. Naturally, it would be ideal to extrapolate this study to the context of n -trivial extensions. In this section, we extend this study to the context of n -trivial extensions, where here $R \rtimes_n M$ is a (\mathbb{N}_0) -graded ring with, as indicated in Section 3, $(R \rtimes_n M)_0 = R$, $(R \rtimes_n M)_i = M_i$, for every $i \in \{1, \dots, n\}$, and $(R \rtimes_n M)_i = 0$ for every $i \geq n + 1$. Note that we could also consider $R \rtimes_n M$ as a \mathbb{Z}_{n+1} -graded ring or Γ_{n+1} -graded ring as mentioned in Section 3.

For that, it is convenient to recall the following definitions. Let Γ be a commutative additive monoid and

$$S = \bigoplus_{\alpha \in \Gamma} S_\alpha$$

a Γ -graded ring. Let

$$N = \bigoplus_{\alpha \in \Gamma} N_\alpha$$

be a Γ -graded S -module. For every $\alpha \in \Gamma$, the elements of N_α are said to be *homogeneous of degree α* . A submodule N' of N is said to be *homogeneous* if one of the following equivalent assertions is true.

- (1) N' is generated by homogeneous elements;
- (2) if

$$\sum_{\alpha \in G'} n_\alpha \in N',$$

where G' is a finite subset of Γ and each n_α is homogeneous of degree α , then $n_\alpha \in N'$ for every $\alpha \in G'$; or

- (3)

$$N' = \bigoplus_{\alpha \in \Gamma} (N' \cap N_\alpha).$$

In particular, an ideal J of $R \rtimes_n M$ is homogeneous if and only if

$$J = (J \cap R) \oplus (J \cap M_1) \oplus \cdots \oplus (J \cap M_n).$$

Note that $I := J \cap R$ is an ideal of R and, for $i \in \{1, \dots, n\}$, $N_i := J \cap M_i$ is an R -submodule of M_i which satisfies $IM_i \subseteq N_i$ and $N_i M_j \subseteq N_{i+j}$ for every $i, j \in \{1, \dots, n\}$.

The next result extends [9, Theorem 3.3 (1)], namely, it determines the structure of the homogeneous ideals of n -trivial extensions.

In what follows, we use the ring homomorphism $\Pi_0 := \pi_0$ (used in Proposition 4.3) and, for $i \in \{1, \dots, n\}$, the following homomorphism of R -modules:

$$\begin{aligned} \Pi_i : R \rtimes_n M_1 \rtimes \cdots \rtimes M_n &\longrightarrow M_i \\ (r, m_1, \dots, m_n) &\longmapsto m_i. \end{aligned}$$

Theorem 5.1. *The following assertions are true.*

(1) *Let I be an ideal of R , and let $C = (C_i)_{i \in \{1, \dots, n\}}$ be a family of R -modules such that $C_i \subseteq M_i$ for every $i \in \{1, \dots, n\}$. Then, $I \rtimes_n C$ is a (homogeneous) ideal of $R \rtimes_n M$ if and only if $IM_i \subseteq C_i$ and $C_i M_j \subseteq C_{i+j}$ for all $i, j \in \{1, \dots, n\}$ with $i + j \leq n$. Thus, if $I \rtimes_n C$ is an ideal of $R \rtimes_n M$, then M_i/C_i is an R/I -module for every $i \in \{1, \dots, n\}$, and we have a natural ring isomorphism*

$$(R \rtimes_n M)/(I \rtimes_n C_1 \rtimes \cdots \rtimes C_n) \cong (R/I) \rtimes_n (M_1/C_1) \rtimes \cdots \rtimes (M_n/C_n)$$

where the multiplication is well defined as:

$$\begin{aligned} \overline{\varphi}_{i,j} : \frac{M_i}{C_i} \times \frac{M_j}{C_j} &\longrightarrow \frac{M_{i+j}}{C_{i+j}} \\ (\overline{m_i}, \overline{m_j}) &\longmapsto \overline{m_i m_j}. \end{aligned}$$

In particular,

$$(R \rtimes_n M)/(0 \rtimes_n C_1 \times \cdots \times C_n) \cong R \rtimes_n (M_1/C_1) \times \cdots \times (M_n/C_n).$$

(2) Let J be an ideal of $R \rtimes_n M$, and consider $K := \Pi_0(J)$ and $N_i := \Pi_i(J)$ for every $i \in \{1, \dots, n\}$. Then:

(a) K is an ideal of R , and N_i is a submodule of M_i for every $i \in \{1, \dots, n\}$ such that $KM_i \subseteq N_i$ and $N_i M_j \subseteq N_{i+j}$ for every $j \in \{1, \dots, n\}$ with $i+j \leq n$. Thus, $K \rtimes_n N_1 \times \cdots \times N_n$ is a homogeneous ideal of $R \rtimes_n M_1 \times \cdots \times M_n$.

(b) $J \subseteq K \rtimes_n N_1 \times \cdots \times N_n$.

(c) The ideal J is homogeneous if and only if $J = K \rtimes_n N_1 \times \cdots \times N_n$.

Proof.

(1) If $I \rtimes_n C_1 \times \cdots \times C_n$ is an ideal of $R \rtimes_n M$, then

$$\begin{aligned} (R \rtimes_n M_1 \times \cdots \times M_n)(I \rtimes_n C_1 \times \cdots \times C_n) \\ = I \rtimes_n (IM_1 + C_1) \times (IM_2 + C_2 + C_1 M_1) \\ \times \cdots \times \left(IM_n + C_n + \sum_{i+j=n} C_i M_j \right). \end{aligned}$$

Thus, $IM_i \subseteq C_i$ and $C_i M_j \subseteq C_{i+j}$ for every $i, j \in \{1, \dots, n\}$.

Conversely, suppose that we have $IM_i \subseteq C_i$ and $C_i M_j \subseteq C_{i+j}$ for all $i, j \in \{1, \dots, n\}$ with $i+j \leq n$. Then, M_i/C_i is an R/I -module for every $i \in \{1, \dots, n\}$, and the map

$$\begin{aligned} f : R \rtimes_n M &\longrightarrow (R/I) \rtimes_n (M_1/C_1) \times \cdots \times (M_n/C_n) \\ (r, m_1, \dots, m_n) &\longmapsto (r + I, m_1 + C_1, \dots, m_n + C_n) \end{aligned}$$

is a well-defined surjective homomorphism with $\text{Ker } f = I \rtimes_n C_1 \times \cdots \times C_n$; thus, $I \rtimes_n C_1 \times \cdots \times C_n$ is an ideal of $R \rtimes_n M_1 \times \cdots \times M_n$, and

$$(R \rtimes_n M)/(I \rtimes_n C_1 \times \cdots \times C_n) \cong (R/I) \rtimes_n (M_1/C_1) \times \cdots \times (M_n/C_n).$$

In particular,

$$(R \rtimes_n M_1 \rtimes \cdots \rtimes M_n)/(0 \rtimes_n C_1 \rtimes \cdots \rtimes C_n) \cong R \rtimes_n (M_1/C_1) \rtimes \cdots \rtimes (M_n/C_n).$$

(2) All three statements are easily verified. □

The next result presents some properties of homogeneous ideals of $R \rtimes_n M$. It is an extension of both [9, Theorem 3.2 (3) and Theorem 3.3 (2), (3)]. In particular, we determine, as an extension of [9, Theorem 3.3 (3)], the form of homogeneous principal ideals. In fact, the characterization of homogeneous principal ideals plays a key role in studying homogeneous ideals. This is due to the (easily verified) fact that an ideal I of a graded ring is homogeneous if every principal ideal generated by an element of I is homogeneous.

Proposition 5.2. *The following assertions are true.*

1. Let $I \rtimes_n N_1 \rtimes \cdots \rtimes N_n$ and $I' \rtimes_n N'_1 \rtimes \cdots \rtimes N'_n$ be two homogeneous ideals of $R \rtimes_n M$. Then, we have the following homogeneous ideals of $R \rtimes_n M$:

(a) $(I \rtimes_n N_1 \rtimes \cdots \rtimes N_n) + (I' \rtimes_n N'_1 \rtimes \cdots \rtimes N'_n) = (I + I') \rtimes_n (N_1 + N'_1) \rtimes \cdots \rtimes (N_n + N'_n),$

(b) $(I \rtimes_n N_1 \rtimes \cdots \rtimes N_n) \cap (I' \rtimes_n N'_1 \rtimes \cdots \rtimes N'_n) = (I \cap I') \rtimes_n (N_1 \cap N'_1) \rtimes \cdots \rtimes (N_n \cap N'_n),$

(c) $(I \rtimes_n N_1 \rtimes \cdots \rtimes N_n)(I' \rtimes_n N'_1 \rtimes \cdots \rtimes N'_n) = II' \rtimes_n (IN'_1 + I'N_1) \rtimes (IN'_2 + I'N_2 + N_1N'_1) \rtimes \cdots \rtimes (IN'_n + I'N_n + \sum_{i+j=n} N_iN'_j),$ and

(d) $(I \rtimes_n N_1 \rtimes \cdots \rtimes N_n) : (I' \rtimes_n N'_1 \rtimes \cdots \rtimes N'_n) = ((I :_R I') \cap (N_1 :_R N'_1) \cap \cdots \cap (N_n :_R N'_n)) \rtimes_n ((N_1 :_{M_1} I') \cap (N_2 :_{M_1} N'_1) \cap \cdots \cap (N_n :_{M_1} N'_{n-1})) \rtimes \cdots \rtimes (N_n :_{M_n} I')$ where $(N_{i+j} :_{M_i} N'_j) := \{m_i \in M_i \mid m_i N'_j \subseteq N_{i+j}\}$ for every $i, j \in \{0, \dots, n\}$ with $i + j \leq n$ (here $M_0 = R, N_0 = I$ and $N'_0 = I'$).

2. A principal ideal $\langle (a, m_1, \dots, m_n) \rangle$ of $R \rtimes_n M$ is homogeneous if and only if

$$\langle (a, m_1, \dots, m_n) \rangle = aR \rtimes_n (Rm_1 + aM_1) \rtimes (Rm_2 + aM_2 + m_1M_1) \rtimes \cdots \rtimes \left(Rm_n + aM_n + \sum_{i+j=n} m_iM_j \right).$$

3. For an ideal J of $R \times_n M$, $\sqrt{J} = \sqrt{\Pi_0(J)} \times_n M$. In particular, if $I \times_n C_1 \times \cdots \times C_n$ is a homogeneous ideal of $R \times_n M$, then

$$\sqrt{I \times_n C_1 \times \cdots \times C_n} = \sqrt{I} \times_n M.$$

Proof.

1. The proof for each of the first three statements is similar to that corresponding to [37, Theorem 25.1 (2)]. The last statement easily follows from the fact that the residual of two homogeneous ideals is again homogeneous.

2. Apply assertion (1) and Theorem 5.1 (1).

3. The proof is similar to that of [9, Theorem 3.2 (3)]. □

It is a well-known fact that, in the case where $n = 1$, even if a homogeneous ideal $I \times C$ is finitely generated, the R -module C is not necessarily finitely generated (consider $\mathbb{Z} \times \mathbb{Q}$ and the principal ideal $\langle (2, 0) \rangle = 2\mathbb{Z} \times \mathbb{Q}$ as an example). The next result presents, in this context, some specific cases obtained using standard arguments.

Proposition 5.3. *The following assertions are true.*

(1) *The ideal $0 \times_n M$ of $R \times_n M$ is finitely generated if and only if each R -module M_i is finitely generated.*

(2) *If a homogeneous ideal $I \times_n C_1 \times \cdots \times C_n$ of $R \times_n M$ is finitely generated, then I is a finitely generated ideal of R . The converse implication is true when C_i is a finitely generated R -module for every $i \in \{1, \dots, n\}$.*

From the previous section, it may be seen that every radical (hence, prime) ideal of $R \times_n M$ is homogeneous. However, it is well known that the ideals of classical trivial extensions are not, in general, homogeneous (see [9]). The following natural questions arise.

Question 5.4. *When is every ideal of a given class \mathcal{I} of ideals of $R \times_n M$ homogeneous?*

Question 5.5. *For a given ring R and a family of R -modules $M = (M_i)_{i=1}^n$, what is the class of all homogeneous ideals of $R \times_n M$?*

Clearly, these questions depend upon the structure of both R and each M_i . For instance, for $n = 1$, if R is a quasi-local ring with maximal m , then a proper homogeneous ideal of $R \times R/m$ either has the form $I \times R/m$ or $I \times 0$ where I is a proper ideal of R . In addition, a proper homogeneous principal ideal of $R \times R/m$ either has the form $0 \times R/m$ or $I \times 0$ where I is a principal ideal of R . Then, for instance, a principal ideal of $R \times R/m$ generated by an element (a, e) , where a and e are both nonzero with $a \in m$, is not homogeneous.

Question 5.4 was investigated in [9] for the case where \mathcal{S} is the class of regular ideals of $R \times_1 M$ [9, Theorem 3.9]. Also, under the condition that R is an integral domain, a characterization of trivial extension rings over which every ideal is homogeneous is given (see [9, Theorem 3.3 and Corollary 3.4]). Our aim in the remainder of this section is to extend this study to n -trivial extensions. It is worth noting that, in the classical case (where $n = 1$), ideals J with $\Pi_0(J) = 0$ are homogeneous. This shows that the condition that all ideals J with $\Pi_0(J) \neq 0$ are homogeneous implies that all ideals of $R \times_1 M$ are homogeneous. In the context of $R \times_n M$ for $n \geq 2$, we show that more situations can occur.

We begin with the class of ideals J of $R \times_n M$ with $\Pi_0(J) \cap S \neq \emptyset$ for a given subset S of regular elements of R .

Recall that a ring S is said to be *présimplifiable* if, for every a and b in S : $ab = a$ implies $a = 0$ or $b \in U(S)$. Présimplifiable rings were introduced and studied by Bouvier in a series of papers (see references), and they have also been investigated in [7, 8]. In [9], the notion of a présimplifiable ring is used when homogeneous ideals of the classical trivial extensions were studied. For example, we have that, if R is présimplifiable but not an integral domain, then every ideal of $R \times_1 M$ is homogeneous if and only if $M_1 = 0$ (see [9, Theorem 3.3 (4)]). This is the reason we first consider only subsets of regular elements.

Theorem 5.6. *Let S be a nonempty subset of $R - Z(R)$, and let \mathcal{S} be the class of ideals J of $R \times_n M$ with $\Pi_0(J) \cap S \neq \emptyset$. Then, the following assertions are equivalent.*

- (1) *Every ideal in \mathcal{S} is homogeneous.*
- (2) *Every principal ideal in \mathcal{S} is homogeneous.*
- (3) *For every $s \in S$ and $i \in \{1, \dots, n\}$, $sM_i = M_i$.*

(4) Every principal ideal $\langle (s, m_1, \dots, m_n) \rangle$ with $s \in S$ has the form $I \times_n M$ where I is a principal ideal of R with $I \cap S \neq \emptyset$.

(5) Every ideal in \mathcal{I} has the form $I \times_n M$ where I is an ideal of R with $I \cap S \neq \emptyset$.

Proof.

(1) \Rightarrow (2). Obvious.

(2) \Rightarrow (3). Let $s \in S$ and $i \in \{1, \dots, n\}$. We only need prove that $M_i \subseteq sM_i$. Consider an element m_i of M_i . Since $s \in S$, $\langle (s, 0, \dots, 0, m_i, 0, \dots, 0) \rangle$ is homogeneous. Then, $(s, 0, \dots, 0) \in \langle (s, 0, \dots, 0, m_i, 0, \dots, 0) \rangle$; thus, there is an $(x, e_1, \dots, e_n) \in R \times_n M_1 \times \dots \times M_n$ such that

$$(s, 0, \dots, 0, m_i, 0, \dots, 0)(x, e_1, \dots, e_n) = (s, 0, \dots, 0).$$

Since s is regular, $x = 1$. Then, $m_i = (-s)e_i$, as desired.

(3) \Rightarrow (4). Let $\langle (s, m_1, \dots, m_n) \rangle$ be a principal ideal of $R \times_n M$ with $s \in S$. From (3),

$$(s, m_1, \dots, m_n)(0 \times_n 0 \times \dots \times 0 \times M_n) = 0 \times_n 0 \times \dots \times 0 \times M_n.$$

This implies that $0 \times_n 0 \times \dots \times 0 \times M_n \subset \langle (s, m_1, \dots, m_n) \rangle$. Using this inclusion and (3), we obtain

$$0 \times_n 0 \times \dots \times 0 \times M_{n-1} \times 0 \subset \langle (s, m_1, \dots, m_n) \rangle.$$

Then, inductively, we have

$$0 \times_n 0 \times \dots \times 0 \times M_i \times 0 \times \dots \times 0 \subset \langle (s, m_1, \dots, m_n) \rangle$$

for every $i \in \{1, \dots, n\}$. Thus, $0 \times_n M_1 \times \dots \times M_n \subset \langle (s, m_1, \dots, m_n) \rangle$. Therefore, by Lemma 4.6 and Proposition 5.2 2, $\langle (s, m_1, \dots, m_n) \rangle$ has the form $I \times_n M$ where $I = sR$.

(4) \Rightarrow (5). Consider an ideal J in \mathcal{I} . Then, there is an element $(s, m_1, \dots, m_n) \in J$ such that $s \in \Pi_0(J) \cap S$. Therefore, using (4) and Lemma 4.6, we obtain the desired result.

(5) \Rightarrow (1). Obvious. □

As an example, we may consider the trivial extension $S := \mathbb{Z} \times_2 \mathbb{Z}_W \times \mathbb{Q}$ where \mathbb{Z}_W is the ring of fractions of \mathbb{Z} with respect to the multiplicatively closed subset $W = \{2^k \mid k \in \mathbb{N}\}$ of \mathbb{Z} . Then, the principal ideal $\langle (3, 1, 0) \rangle$ of S is inhomogeneous. Suppose this is not true.

We must have $(3, 0, 0) \in \langle (3, 1, 0) \rangle$. Thus, there is an $(a, e, f) \in S$ such that $(3, 0, 0) = (3, 1, 0)(a, e, f)$. However, this implies that $a = 1$, and then, $e = -1/3$, which is absurd.

The following result is an extension of [9, Theorem 3.9]. Recall that an ideal is said to be *regular* if it contains a regular element. Here, from Proposition 4.9, an ideal of $R \rtimes_n M$ is regular if and only if it contains an element (s, m_1, \dots, m_n) with $s \in R - (Z(R) \cup Z(M_1) \cup \dots \cup Z(M_n))$.

Corollary 5.7. *Let $S = R - (Z(R) \cup Z(M_1) \cup \dots \cup Z(M_n))$. Then, the following assertions are equivalent.*

- (1) *Every regular ideal of $R \rtimes_n M$ is homogeneous.*
- (2) *Every principal regular ideal of $R \rtimes_n M$ is homogeneous.*
- (3) *For every $s \in S$ and $i \in \{1, \dots, n\}$, $sM_i = M_i$ (or equivalently, $M_{iS} = M_i$).*
- (4) *Every principal ideal $\langle (s, m_1, \dots, m_n) \rangle$ with $s \in S$ has the form $I \rtimes_n M$, where I is a principal ideal of R with $I \cap S \neq \emptyset$.*
- (5) *Every regular ideal of $R \rtimes_n M$ has the form $I \rtimes_n M$, where I is an ideal of R with $I \cap S \neq \emptyset$.*

Consequently, if $R \rtimes_n M$ is root closed (in particular, integrally closed), then every regular ideal of $R \rtimes_n M$ has the form given in (5).

Proof. The proof is similar to that of [9, Theorem 3.9]. □

Compare the following result with [9, Corollary 3.4].

Corollary 5.8. *Assume that R is an integral domain. Then, the following assertions are equivalent.*

- (1) *Every ideal J of $R \rtimes_n M$ with $\Pi_0(J) \neq 0$ is homogeneous.*
- (2) *Every principal ideal J of $R \rtimes_n M$ with $\Pi_0(J) \neq 0$ is homogeneous.*
- (3) *For every $i \in \{1, \dots, n\}$, M_i is divisible.*
- (4) *Every principal ideal $\langle (s, m_1, \dots, m_n) \rangle$ of $R \rtimes_n M$ with $s \neq 0$ has the form $I \rtimes_n M$, where I is a nonzero principal ideal of R .*
- (5) *Every ideal J of $R \rtimes_n M$ with $\Pi_0(J) \neq 0$ has the form $I \rtimes_n M$, where I is a nonzero ideal of R .*

(6) Every ideal of $R \rtimes_n M$ is comparable to $0 \rtimes_n M$.

Proof. Equivalence (5) \Leftrightarrow (6) is a simple consequence of Lemma 4.6. □

The proof of Theorem 5.6 shows that another situation may be considered. This is given in the following result. We use $\text{Ann}_R(H)$ to denote the annihilator of an R -module H .

Theorem 5.9. *Let \mathcal{I} be the class of ideals J of $R \rtimes_n M$ with $\Pi_0(J) \cap S \neq \emptyset$, where S is a nonempty subset of $R - \{0\}$ such that, for every $s \in S$, $\text{Ann}_R(s) \subseteq \text{Ann}_R(M_i)$. Then, the following assertions are equivalent.*

- (1) Every ideal in \mathcal{I} is homogeneous.
- (2) Every principal ideal in \mathcal{I} is homogeneous.
- (3) For every $s \in S$ and $i \in \{1, \dots, n\}$, $sM_i = M_i$.
- (4) Every principal ideal $\langle (s, m_1, \dots, m_n) \rangle$ with $s \in S$ has the form $I \rtimes_n M$, where I is a principal ideal of R with $I \cap S \neq \emptyset$.
- (5) Every ideal in \mathcal{I} has the form $I \rtimes_n M$, where I is an ideal of R with $I \cap S \neq \emptyset$.

Proof. We only need prove the implication (2) \Rightarrow (3). Let $s \in S$, $i \in \{1, \dots, n\}$, and consider an element m_i of $M_i - \{0\}$. Since $s \in S$, $\langle (s, 0, \dots, 0, m_i, 0, \dots, 0) \rangle$ is homogeneous. Then, $(s, 0, \dots, 0) \in \langle (s, 0, \dots, 0, m_i, 0, \dots, 0) \rangle$; thus, there is an $(x, e_1, \dots, e_n) \in R \rtimes_n M_1 \times \dots \times M_n$ such that $(s, 0, \dots, 0, m_i, 0, \dots, 0)(x, e_1, \dots, e_n) = (s, 0, \dots, 0)$. Next, $sx = s$ and, by the hypothesis on S , $(x - 1)m_i = 0$. Therefore, $m_i = xm_i = (-s)e_i$, as desired. □

For an example of a ring which satisfies the condition of the previous result, consider a ring R with an idempotent $e \in R - \{1, 0\}$, and set $S = \{e\}$ and $M_i = Re$ for every $i \in \{1, \dots, n\}$. Thus, since $eM_i = M_i$ for every $i \in \{1, \dots, n\}$, every ideal J of $R \rtimes_n M$ with $e \in \Pi_0(J)$ is homogeneous.

Unlike the classical case (where $n = 1$), the fact that, for every $i \in \{1, \dots, n\}$, M_i is divisible does not necessarily imply that every ideal is homogeneous. For that, we consider the 2-trivial extension

$S := k \rtimes_2 (k \times k) \rtimes (k \times k)$, where k is a field. Then, the principal ideal $\langle (0, (1, 0), (0, 1)) \rangle$ of S is inhomogeneous. Indeed, if it were homogeneous, we would have $(0, (1, 0), (0, 0)) \in \langle (0, (1, 0), (0, 1)) \rangle$. Thus, there is an $(a, (e, f), (e', f')) \in S$ such that

$$(0, (1, 0), (0, 0)) = (0, (1, 0), (0, 1))(a, (e, f), (e', f')).$$

However, this implies that $(a, 0) = (1, 0)$ and $(0, 0) = (e, a)$, which is absurd.

This example naturally leads us to investigate when every ideal J of $R \rtimes_n M$ with $\Pi_0(J) = 0$ is homogeneous. In this context, the notion of a présimplifiable module is used. For that, recall that an R -module H is called R -présimplifiable if, for every $r \in R$ and $h \in H$, $rh = h$ implies $h = 0$ or $r \in U(R)$. For example, over an integral domain, every torsion-free module is présimplifiable (see [3, 7]).

In studying the question when every ideal J of $R \rtimes_n M$ with $\Pi_0(J) = 0$ is homogeneous, several different cases occur. We use the following lemma for these.

Lemma 5.10. *Let J be an ideal of $R \rtimes_n M$ such that, for $i \in \{1, \dots, n\}$, $\Pi_0(J) = 0, \dots, \Pi_{i-1}(J) = 0$ and $\Pi_i(J) \neq 0$. Then, the following assertions are true.*

(1) *For $i = n$, the ideal J is homogeneous, and it has the form $0 \rtimes_n 0 \rtimes \dots \rtimes 0 \rtimes \Pi_n(J)$.*

(2) *For $i \neq n$, if $0 \rtimes_n 0 \rtimes \dots \rtimes 0 \rtimes M_{i+1} \rtimes \dots \rtimes M_n \subset J$, then J is homogeneous, and it has the form $0 \rtimes_n 0 \rtimes \dots \rtimes 0 \rtimes \Pi_i(J) \rtimes M_{i+1} \rtimes \dots \rtimes M_n$.*

Proof. Straightforward. □

Theorem 5.11. *Assume that $n \geq 2$, and M_j is présimplifiable for a given $j \in \{1, \dots, n - 1\}$. Let \mathcal{S} be the class of ideals J of $R \rtimes_n M$ with $\Pi_i(J) = 0$ for every $i \in \{0, \dots, j - 1\}$ and $\Pi_j(J) \neq 0$. Then, the following assertions are equivalent.*

- (1) *Every ideal in \mathcal{S} is homogeneous.*
- (2) *Every principal ideal in \mathcal{S} is homogeneous.*

(3) For every $k \in \{j + 1, \dots, n\}$ and every $m_j \in M_j - \{0\}$, $M_k = m_j M_{k-j}$.

(4) Every principal ideal $\langle (0, 0, \dots, 0, m_j, \dots, m_n) \rangle$ with $m_j \neq 0$ has the form $0 \times_n 0 \times \dots \times 0 \times N \times M_{j+1} \times \dots \times M_n$, where N is a nonzero cyclic submodule of M_j .

(5) Every ideal in \mathcal{I} has the form $0 \times_n 0 \times \dots \times 0 \times N \times M_{j+1} \times \dots \times M_n$, where N is a nonzero submodule of M_j .

(6) Every ideal in \mathcal{I} contains $0 \times_n 0 \times \dots \times 0 \times M_{j+1} \times \dots \times M_n$.

Proof. The implication (3) \Rightarrow (4) is proved similarly to the implication (3) \Rightarrow (4) of Theorem 5.6. The implication (6) \Rightarrow (1) is a simple consequence of Lemma 5.10. Then, only the implication (2) \Rightarrow (3) needs a proof. Let $k \in \{j+1, \dots, n\}$, $m_j \in M_j - \{0\}$ and $m_k \in M_k - \{0\}$. Further, the principal ideal $p = \langle (0, \dots, 0, m_j, 0, \dots, 0, m_k, 0, \dots, 0) \rangle$ is homogeneous. This implies that $(0, \dots, 0, m_j, 0, \dots, 0) \in p$, and thus, there exists an $(r, e_1, \dots, e_n) \in R \times_n M$ such that

$$(0, \dots, 0, m_j, 0, \dots, 0) = (r, e_1, \dots, e_n)(0, \dots, 0, m_j, 0, \dots, 0, m_k, 0, \dots, 0).$$

Then, $rm_j = m_j$ and $rm_k + e_{k-j}m_j = 0$. Since M_j is présimplifiable, r is invertible, and then $m_k = -r^{-1}e_{k-j}m_j$, as desired. \square

For examples of rings which satisfy the conditions of the previous result, we can consider the following two 2-trivial extensions: $\mathbb{Z} \times_2 \mathbb{Z}_W \times \mathbb{Q}$ and $\mathbb{Z} \times_2 \mathbb{Z}_W \times \mathbb{Z}_W$, where \mathbb{Z}_W is the ring of fractions of \mathbb{Z} with respect to the multiplicatively closed subset $W = \{2^k \mid k \in \mathbb{N}\}$ of \mathbb{Z} .

The following specific cases are of interest.

Corollary 5.12. *Assume that $n \geq 2$ and M_{n-1} is présimplifiable. Let \mathcal{I} be the class of ideals J of $R \times_n M$ with $\Pi_i(J) = 0$ for every $i \in \{0, \dots, n - 2\}$. Then, the following assertions are equivalent.*

- (1) Every ideal in \mathcal{I} is homogeneous.
- (2) For every $m_{n-1} \in M_{n-1} - \{0\}$, $M_n = m_{n-1}M_1$.
- (3) Every ideal in \mathcal{I} is comparable to $0 \times_n 0 \times \dots \times 0 \times M_n$.

Proof.

(1) \Rightarrow (2). This is a specific case of the corresponding one in Theorem 5.11.

(2) \Rightarrow (3). Let I be an ideal of $R \rtimes_n M$ in \mathcal{S} . If $\Pi_{n-1}(I) \neq 0$, then Theorem 5.11 shows that I contains $0 \rtimes_n 0 \rtimes \cdots \rtimes 0 \rtimes M_n$. Otherwise, $\Pi_{n-1}(I) = 0$, which means that $0 \rtimes_n 0 \rtimes \cdots \rtimes 0 \rtimes M_n$ contains I .

(3) \Rightarrow (1). Let I be a nonzero ideal of $R \rtimes_n M$ in \mathcal{S} . If $\Pi_{n-1}(I) \neq 0$, then Theorem 5.11 shows that I is homogeneous. The other case is a consequence of assertion (1) of Lemma 5.10. \square

When $n = 2$, we obtain the following specific case of Corollary 5.12.

Corollary 5.13. *Assume that M_1 is présimplifiable, $n = 2$. Let \mathcal{S} be the class of ideals J of $R \rtimes_2 M$ with $\Pi_0(J) = 0$. Then, the following assertions are equivalent.*

- (1) *Every ideal in \mathcal{S} is homogeneous.*
- (2) *For every $m_1 \in M_1 - \{0\}$, $M_2 = m_1 M_1$.*
- (3) *Every ideal in \mathcal{S} is comparable to $0 \rtimes_2 0 \rtimes M_2$.*

When $j = 1$ in Theorem 5.11, there are additional conditions equivalent to (1)–(6). The study of this case leads us to introduce the following notion in order to avoid trivial situations.

Definition 5.14. Assume that $n \geq 2$. For $i \in \{1, \dots, n - 1\}$ and $j \in \{2, \dots, n\}$ with product $ij \leq n$, M_i is said to be φ - j -integral (where $\varphi = \{\varphi_{i,j}\}_{\substack{i+j \leq n \\ 1 \leq i, j \leq n-1}}$ is the family of multiplication) if, for any j elements m_{i_1}, \dots, m_{i_j} of M_i , if the product $m_{i_1} \cdots m_{i_j} = 0$. Then, at least one of the m_{i_k} s is zero. If no ambiguity arises, M_i is simply called the j -integral.

Corollary 5.15. *Assume that $n \geq 2$, M_1 is présimplifiable and k -integral for every $k \in \{2, \dots, n - 1\}$. Let \mathcal{S} be the class of ideals J of $R \rtimes_n M$ with $\Pi_0(J) = 0$ and $\Pi_1(J) \neq 0$. Then, the following assertions are equivalent.*

- (1) *Every ideal in \mathcal{S} is homogeneous.*
- (2) *Every principal ideal in \mathcal{S} is homogeneous.*
- (3) *For every $k \in \{2, \dots, n\}$ and every $m_1 \in M_1 - \{0\}$, $M_k = m_1 M_{k-1}$.*

- (4) For every $k \in \{2, \dots, n\}$ and every nonzero element $m_1, \dots, m_{1_{k-1}} \in M_1 - \{0\}$, $M_k = m_{1_1} \cdots m_{1_{k-1}} M_1$.
- (5) For every $k \in \{2, \dots, n\}$ and every nonzero element $m \in M_1 - \{0\}$, $M_k = m^{k-1} M_1$.
- (6) Every principal ideal $\langle (0, m_1, \dots, m_n) \rangle$ with $m_1 \neq 0$ has the form $0 \times_n N \times M_2 \cdots \times M_n$, where N is a nonzero cyclic submodule of M_1 .
- (7) Every ideal in \mathcal{I} has the form $0 \times_n N \times M_2 \cdots \times M_n$, where N is a nonzero submodule of M_1 .
- (8) Every ideal in \mathcal{I} contains $0 \times_n 0 \times M_2 \times \cdots \times M_n$.

Proof. The equivalences (3) \Leftrightarrow (4) \Leftrightarrow (5) are easily proved. □

The following result shows that, in fact, the conditions of Corollary 5.15 above are necessary and sufficient to show that every ideal J of $R \times_n M$ with $\Pi_0(J) = 0$ is homogeneous. Note that Corollary 5.13 presents the case $n = 2$. Thus, in the following result, we may assume that $n \geq 3$.

Corollary 5.16. *Assume that $n \geq 3$ and M_1 is présimplifiable and k -integral for every $k \in \{2, \dots, n - 1\}$. Then, the following assertions are equivalent.*

- (1) Every ideal J of $R \times_n M$ with $\Pi_0(J) = 0$ and $\Pi_1(J) \neq 0$ is homogeneous.
- (2) For every $j \in \{1, \dots, n - 1\}$, M_j is présimplifiable, and every ideal J of $R \times_n M$ with $\Pi_0(J) = 0$ is homogeneous.

Proof. We only need prove that (1) \Rightarrow (2). Let $j \in \{1, \dots, n - 1\}$, and consider $m_j \in M_j - \{0\}$. Let $r \in R$ be such that $rm_j = m_j$. From Corollary 5.15 (4), there are $m_1, \dots, m_{1_j} \in M_1 - \{0\}$ such that $m_j = m_{1_1} \cdots m_{1_j}$. Then, $rm_{1_1} \cdots m_{1_j} = m_{1_1} \cdots m_{1_j}$, which implies that $(rm_{1_1} - m_{1_1})m_{1_2} \cdots m_{1_j} = 0$. Now, since M_1 is k -integral for every $k \in \{2, \dots, n - 1\}$, $rm_{1_1} - m_{1_1} = 0$. Therefore, r is invertible since M_1 is présimplifiable. Thus, M_j is présimplifiable.

Now, to prove that every ideal J of $R \times_n M$ with $\Pi_0(J) = 0$ is homogeneous, it suffices to prove that $M_k = m_j M_{k-j}$ for every $k \in \{2, \dots, n\}$, every $j \in \{1, \dots, k - 1\}$ and every $m_j \in M_j - \{0\}$ (by

Theorem 5.11). The case where $k = 2$ is trivial. Thus, fix $k \in \{3, \dots, n\}$ and $j \in \{1, \dots, k - 1\}$. Consider $m_j \in M_j - \{0\}$ and $m_k \in M_k - \{0\}$. We prove that $m_k = m_j m_{k-j}$ for some $m_{k-j} \in M_{k-j} - \{0\}$. From Corollary 5.15 (5), $m_j = m^{j-1} m_1$ for some $m, m_1 \in M_1 - \{0\}$. And, from Corollary 5.15 (3), $m_k = m_1 m_{k-1}$ for some $m_{k-1} \in M_{k-1} - \{0\}$. In addition, from Corollary 5.15 (5), $m_{k-1} = m^{k-2} m'_1$ for some $m'_1 \in M_1 - \{0\}$. Then, $m_k = m^{k-2} m_1 m'_1 = (m^{j-1} m_1)(m^{k-j-1} m'_1) = m_j m_{k-j}$, where $m_{k-j} = m^{k-j-1} m'_1 \in M_{k-j} - \{0\}$, as desired. \square

Finally, we give a case when we can characterize rings in which every ideal is homogeneous. Note that, when R is a ring with $aM_i = M_i$ for every $i \in \{1, \dots, n - 1\}$, $a \in R - \{0\}$, and $M_i = m^{i-1} M_1$ for every $i \in \{2, \dots, n\}$ and nonzero element $m \in M_1 - \{0\}$, then R is an integral domain and M_i must be torsion-free for every $i \in \{1, \dots, n - 1\}$.

Corollary 5.17. *Suppose that $n \geq 2$ and R is an integral domain. Assume that M_i is torsion-free, for every $i \in \{1, \dots, n - 1\}$, and that M_1 is k -integral for every $k \in \{2, \dots, n - 1\}$. Then, the following assertions are equivalent.*

- (1) *Every ideal of $R \rtimes_n M$ is homogeneous.*
- (2) *The following two conditions are satisfied:*
 - (i) *for every $i \in \{1, \dots, n\}$, M_i is divisible; and*
 - (ii) *for every $i \in \{2, \dots, n\}$ and $m_1 \in M_1 - \{0\}$, $M_i = m_1 M_{i-1}$.*

Proof. Simply use Corollaries 5.8 and 5.15 and Theorem 5.11. \square

It is simple to show that the two n -trivial extensions $\mathbb{Z} \rtimes_n \mathbb{Q} \times \dots \times \mathbb{Q}$ and $\mathbb{Z} \rtimes_n \mathbb{Q} \times \dots \times \mathbb{Q} \times \mathbb{Q}/\mathbb{Z}$ satisfy the conditions of the above result, and thus, every ideal of these rings is homogeneous.

We end this section with the following specific case.

Corollary 5.18. *Suppose that $n \geq 2$. Consider the n -trivial extension $S := k \rtimes_n E_1 \times \dots \times E_n$ where k is a field and, for $i \in \{1, \dots, n\}$, E_i is a k -vector space. Suppose that E_1 is k -integral for every $k \in \{2, \dots, n - 1\}$. Then, the following assertions are equivalent.*

- (1) *Every ideal of S is homogeneous.*

(2) For every $k \in \{2, \dots, n\}$, $j \in \{1, \dots, k - 1\}$ and $e_j \in E_j - \{0\}$, $E_k = e_j E_{k-j}$.

As a specific case, we may consider a field extension $K \subseteq F$. Then, every ideal of $S := K \rtimes_n F \rtimes \dots \rtimes F$ is homogeneous, namely, every proper ideal of S has the form $0 \rtimes_n 0 \rtimes \dots \rtimes 0 \rtimes N \rtimes F \rtimes \dots \rtimes F$, where N is a K -subspace of F .

6. Some ring-theoretic properties of $R \rtimes_n M$. In this section, we determine when $R \rtimes_n M$ has certain ring properties such as being Noetherian, Artinian, Manis valuation, Prüfer, chained, arithmetical, a π -ring, a generalized ZPI-ring or a PIR. We conclude the section with a remark on a question posed in [2] concerning m -Boolean rings.

We begin by characterizing when the n -trivial extensions are Noetherian (respectively, Artinian). The next result extends [9, Theorem 4.8].

Theorem 6.1. *The ring $R \rtimes_n M$ is Noetherian (respectively, Artinian) if and only if R is Noetherian (respectively, Artinian) and, for every $i \in \{1, \dots, n\}$, M_i is finitely generated.*

Proof. Similar to the proof of [9, Theorem 4.8]. □

The following result is an extension of [9, Theorem 4.2, Corollary 4.3]. It investigates the integral closure of $R \rtimes_n M$ in the total quotient ring $T(R \rtimes_n M)$ of $R \rtimes_n M$.

Theorem 6.2. *Let*

$$S = R - (Z(R) \cup Z(M_1) \cup \dots \cup Z(M_n)).$$

If R' is the integral closure of R in $T(R)$, then

$$(R' \cap R_S) \rtimes_n M_{1S} \rtimes \dots \rtimes M_{nS}$$

is the integral closure of $R \rtimes_n M$ in $T(R \rtimes_n M)$. In particular,

(1) *If R is an integrally closed ring, then $R \rtimes_n M_{1S} \rtimes \dots \rtimes M_{nS}$ is the integral closure of $R \rtimes_n M_1 \rtimes \dots \rtimes M_n$ in $T(R \rtimes_n M_1 \rtimes \dots \rtimes M_n)$, and*

(2) if $Z(M_i) \subseteq Z(R)$ for all $i \in \{1, \dots, n\}$, then $R \times_n M_{1S} \times \cdots \times M_{nS}$ is integrally closed if and only if R is integrally closed.

Proof. All statements may be proved similarly to those of [9, Theorem 4.2, Corollary 4.3]. □

It is worth noting, as in the classical case, that $R \times_n M$ can be integrally closed without R being integrally closed (see the example given after [9, Corollary 4.3]).

Similar to the classical case [9, Theorem 4.16 (1), (2)], as a consequence of Corollary 5.7 and Theorem 6.2, we give the following result which characterizes when $R \times_n M$ is a (Manis) valuation and when it is Prüfer. First, recall these two notions.

Let S be a subring of a ring T , and let P be a prime ideal of S . Then, (S, P) is called a *valuation pair* on T (or S is merely a *valuation ring* on T) if there is a surjective valuation

$$v : T \longrightarrow G \cup \{\infty\} \{v(xy) = v(x) + v(y), v(x + y) \geq \min\{v(x), v(y)\}, v(1) = 0 \text{ and } v(0) = \infty\},$$

where G is a totally ordered Abelian group, with

$$S = \{x \in T \mid v(x) \geq 0\} \quad \text{and} \quad P = \{x \in T \mid v(x) > 0\}.$$

This is equivalent to the following: if $x \in T - S$, then there exists an $x' \in P$ with $xx' \in S - P$. A valuation ring S is called a (*Manis*) *valuation ring* if $T = T(S)$. Also, S is called a *Prüfer ring* if every finitely generated regular ideal of S is invertible. This is equivalent to every overring of S being integrally closed (see [37] for more details).

Corollary 6.3. *Let $S = R - (Z(R) \cup Z(M_1) \cup \cdots \cup Z(M_n))$.*

(1) $R \times_n M$ is a Manis valuation ring if and only if R is a valuation ring on R_S and $M_i = M_{iS}$ for every $i \in \{1, \dots, n\}$.

(2) $R \times_n M$ is a Prüfer ring if and only if, for every finitely generated ideal I of R with $I \cap S \neq \emptyset$, I is invertible and $M_i = M_{iS}$ for every $i \in \{1, \dots, n\}$.

Now, as an extension of [9, Theorem 4.16 (3)], we characterize when $R \times_n M$ is a chained ring. Recall that a ring S is said to be *chained* if the set of ideals of S is totally ordered by inclusion.

As an exception to Convention 4.2, in the following results (Lemma 6.4, Theorem 6.5, Corollary 6.6, Lemma 6.7 and Theorem 6.8), a module in the family associated to an n -trivial extension can be zero.

The proof of the desired result uses the next lemma, which gives another characterization of a particular n -trivial extension with the property that every ideal is homogeneous.

Lemma 6.4. *Assume that R is quasi-local with maximal ideal m . Suppose also that at least one of the modules of the family M is nonzero. Then, every ideal of $R \times_n M$ is homogeneous if and only if the following three conditions are satisfied:*

- (1) R is an integral domain.
- (2) For every $i \in \{1, \dots, n\}$, M_i is divisible.
- (3) For every $1 \leq i \leq j \leq n$ (when $n \geq 2$), if $M_i \neq 0$ and $M_j \neq 0$, then $M_{j-i} \neq 0$ and $eM_i = M_j$ for every $e \in M_{j-i}$.

In this case, each ideal has one of the forms $I \times_n M$, for some ideal I of R , or

$$0 \times_n 0 \times \dots \times 0 \times N \times M_{j+1} \times \dots \times M_n,$$

where N is a nonzero submodule of M_j for some $j \in \{1, \dots, n\}$.

Proof.

\Rightarrow . Clearly, the first assertion is a simple consequence of the second one. Then, we only need prove the second and third assertions.

(2) Let $r \in R - \{0\}$ and $i \in \{1, \dots, n\}$. Consider an element $m_i \in M_i$. If $r \notin m$, the maximal ideal of R , then r is invertible, and trivially, we get the result. Next, assume that $r \in m$. By hypothesis, the ideal $\langle (r, 0, \dots, 0, m_i, 0, \dots, 0) \rangle$ is homogeneous, so there is an (r', m'_1, \dots, m'_n) such that

$$(r, 0, \dots, 0) = (r, 0, \dots, 0, m_i, 0, \dots, 0)(r', m'_1, \dots, m'_n).$$

Then, $rr' = r$ and $0 = rm'_i + r'm_i$. Thus, r' cannot be in m , so r' is invertible, and thus, $m_i = -(r')^{-1}rm'_i$, as desired.

(3) Let $1 \leq i \leq j \leq n$ be such that $M_i \neq 0$ and $M_j \neq 0$. Consider $m_i \in M_i - \{0\}$ and $m_j \in M_j - \{0\}$. By hypothesis, $\langle (0, \dots, 0, m_i, 0, \dots, 0, m_j, 0, \dots, 0) \rangle$ is homogeneous. Then,

$$(0, \dots, 0, m_j, 0, \dots, 0) = (0, \dots, 0, m_i, 0, \dots, 0, m_j, 0, \dots, 0)(r', m'_1, \dots, m'_n)$$

for some $(r', m'_1, \dots, m'_n) \in R \times_n M$. This implies that

$$r'm_i = 0 \quad \text{and} \quad r'm_j + m_im'_{j-i} = m_j.$$

If $M_{j-i} = 0$, we obtain $r'm_i = 0$ and $(r' - 1)m_j = 0$. This is impossible since either r' or $r' - 1$ is invertible. Then, $M_{j-i} \neq 0$. Now, suppose that $r' \neq 0$. From (2), there exists an $m''_{j-i} \in M_{j-i}$ such that $m'_{j-i} = r'm''_{j-i}$. Hence, using the fact that $r'm_i = 0$, the equality $r'm_j + m_im'_{j-i} = m_j$ becomes $r'm_j = m_j$. As in the previous case, this is impossible. Therefore, $r' = 0$, and this gives the desired result.

\Leftarrow . We only need to prove that every principal ideal $\langle (s, m_1, \dots, m_n) \rangle$ of $R \times_n M$ is homogeneous. For this, distinguish two cases $s \neq 0$, and $s = 0$, and follow an argument similar to that of Theorem 5.6 (3) \Rightarrow (4). □

Theorem 6.5. *Assume that $n \geq 2$ and that at least one of the modules of the family M is nonzero. Then, the ring $R \times_n M$ is chained if and only if the following conditions are satisfied:*

- (1) R is a valuation domain;
- (2) for every $i \in \{1, \dots, n\}$, M_i is divisible;
- (3) for every $1 \leq i \leq j \leq n$, if $M_i \neq 0$ and $M_j \neq 0$, then $M_{j-i} \neq 0$ and $eM_i = M_j$ for every $e \in M_{j-i}$; and
- (4) for every $i \in \{1, \dots, n\}$, the set of all (cyclic) submodules of M_i is totally ordered by inclusion.

Proof.

\Rightarrow . First, we prove that R is a chained ring. Consider two ideals I and J of R . Then, $I \times_n M$ and $J \times_n M$ are two ideals of $R \times_n M$. In addition, they are comparable and so are I and J , as desired. A similar argument may be used to prove the last assertion.

Now, we prove that every ideal of $R \times_n M$ is homogeneous. Then, by Lemma 6.4, we get the other assertions. Consider a nonzero ideal K of $R \times_n M$. If $\Pi_0(K) \neq 0$, then necessarily, $0 \times_n M \subset K$. Then,

from Lemma 4.6, K is homogeneous. Now, let $i \geq 1$ be the smallest integer such that $\Pi_i(K) \neq 0$. If $i = n$, then, by the first assertion of Lemma 5.10, K is homogeneous. If $i \neq n$, then necessarily,

$$0 \times_n \cdots \times 0 \times M_{i+1} \times \cdots \times M_n \subset K.$$

Thus, by the second assertion of Lemma 5.10, K is homogeneous, as desired.

\Leftarrow . Using Lemma 5.10, we deduce that any two ideals I and J of $R \times_n M$ have the forms

$$I = 0 \times_n \cdots \times 0 \times I_i \times M_{i+1} \times \cdots \times M_n$$

and

$$J = 0 \times_n \cdots \times 0 \times J_j \times M_{j+1} \times \cdots \times M_n$$

for some $i, j \in \{0, \dots, n\}$, where I_i and J_j are submodules of M_i and M_j , respectively (here, $M_0 = R$). If $i \neq j$, then obviously, I and J are comparable. If $i = j$, then, using the first and the last assertion, we can show that I_i and J_j are comparable and so are I and J , as desired. \square

Using Corollary 3.8 and Theorem 6.5, we obtain an extension of [9, Theorem 4.16 (4)], which characterizes when $R \times_n M$ is arithmetical. Recall that a ring S is *arithmetical* if and only if S_P is chained for each prime (maximal) ideal P of S . Also, recall that, for a ring S , an S -module H is called *arithmetical* if, for each prime (maximal) ideal P of S , the set of submodules of H_P is totally ordered by inclusion. Finally, recall that the support of an S -module H , $\text{supp}(H)$, over a ring S is the set of all prime ideals P of S such that $H_P \neq 0$.

Corollary 6.6. *The ring $R \times_n M$ is arithmetical if and only if the following conditions are satisfied:*

- (1) R is arithmetical;
- (2) for every $i \in \{1, \dots, n\}$, M_i is an arithmetical R -module;
- (3) for every $P \in \cup_i \text{supp}(M_i)$, R_P is a valuation domain;
- (4) for every $i \in \{1, \dots, n\}$ and $P \in \text{supp}(M_i)$, M_{iP} is a divisible R_P -module; and
- (5) for every $1 \leq i \leq j \leq n$, if $P \in \text{supp}(M_i) \cap \text{supp}(M_j)$, then $P \in \text{supp}(M_{j-i})$ and $eM_{iP} = M_{jP}$ for every $e \in M_{(j-i)P}$.

Recall that a ring S is called a *generalized ZPI-ring* (respectively, a π -ring) if every proper ideal (respectively, proper principal ideal) of S is a product of prime ideals. An integral domain which is a π -ring is called a π -domain. Clearly, a generalized ZPI-domain is nothing but a Dedekind domain. It is well known (for example, see [28, Sections 39, 46]) that S is a π -ring (respectively, a generalized ZPI-ring, a principal ideal ring (PIR)) if and only if S is a finite direct product of the following types of rings: (1) π -domains (respectively, Dedekind domains, PIDs) which are not fields, (2) special principal ideal rings (SPIRs) and (3) fields.

Our next results extend [9, Lemma 4.9, Theorem 4.10]. They characterize when $R \times_n M$ is a π -ring, a generalized ZPI-ring or a PIR.

Lemma 6.7. *If $R \times_n M$ is a π -ring (respectively, a generalized ZPI-ring, a PIR), then R is a π -ring (respectively, a generalized ZPI-ring, a PIR). Hence, $R = R_1 \times \cdots \times R_s$, where R_i is either (1) a π -domain (respectively, a Dedekind domain, a PID) but not a field, (2) an SPIR, or (3) a field. Let*

$$M_{j,i} = (0 \times \cdots \times 0 \times R_j \times 0 \times \cdots \times 0)M_i,$$

where $1 \leq i \leq n$ and $1 \leq j \leq s$. If R_i is a domain or SPIR, but not a field, then $M_{j,i} = 0$, while, if R_i is a field, $M_{j,i} = 0$ or $M_{j,i} \cong R_i$.

Conversely, if $R = R_1 \times \cdots \times R_s$ and $M_i = M_{1,i} \times \cdots \times M_{s,i}$ are as above and R is a π -ring (respectively, a generalized ZPI-ring, a PIR), then $R \times_n M$ is a π -ring (respectively, a generalized ZPI-ring, a PIR).

Proof. Using Theorem 3.9, the proof is similar to that of [9, Lemma 4.9]. □

Theorem 6.8. *$R \times_n M$ is a π -ring (respectively, a generalized ZPI-ring, a PIR) if and only if R is a π -ring (respectively, a generalized ZPI-ring, a PIR) and M_i is cyclic with annihilator $P_{i_1} \cdots P_{i_s}$, where P_{i_1}, \dots, P_{i_s} are some idempotent maximal ideals of R (if $i_s = 0$, $\text{Ann}(M_i) = R$, that is, $M_i = 0$).*

Proof. Similar to the proof of [9, Theorem 4.10]. □

We end the section with a remark on a question posed in [2]. Recall that a ring R is called m -Boolean for some $m \in \mathbb{N}$, if $\text{char } R = 2$ and

$$x_1 x_2 \cdots x_m (1 + x_1) \cdots (1 + x_m) = 0$$

for all $x_1, \dots, x_m \in R$. Thus, Boolean rings are merely 1-Boolean rings. It is shown in [2, Theorem 10] that 2-Boolean rings can be represented as trivial extensions, namely, it is proved that, if R is 2-Boolean, then $R \cong B \rtimes \text{Nil}(R)$, where $B = \{b \in R \mid b^2 = b\}$ [2, Theorem 10]. Based on this result, the following natural question is posed (see [2, page 74]): can [2, Theorem 10] be extended to m -Boolean rings for $m \geq 2$?

It may be asked whether the n -trivial extension is the suitable construction for solving this question. Using [2, Theorem 6], it may be shown that the amalgamated algebras along an ideal (introduced in [32]) partially resolve this question. Recall that, given a ring homomorphism $f : A \rightarrow B$ and an ideal J of B , the *amalgamation* of A with B along J with respect to f is the following subring of $A \times B$:

$$A \bowtie^f B = \{(a, f(a) + j) \mid a \in A, j \in J\}.$$

Note that $A \bowtie^f B \cong A \dot{\oplus} J$, where $A \dot{\oplus} J \subseteq A \times B$ is the ring whose underlying group is $A \oplus J$ with multiplication given by $(a, x)(a', x') = (aa', ax' + a'x + xx')$ for all $a, a' \in A$ and $x, x' \in J$. Here, J is an A -module via f , and then, $ax' := f(a)x'$ and $a'x := f(a')x$ (see [32] for more details). Now, if R is m -Boolean for $m \geq 2$, then, from [2, Theorems 6, 7], $R = B \dot{\oplus} \text{Nil}(R)$, where $B = \{b \in R \mid b^2 = b\}$. Then,

$$R \cong B \dot{\oplus} \text{Nil}(R) \cong B \bowtie^i \text{Nil}(R),$$

where $i : B \hookrightarrow R$ is the canonical injection.

From a practical standpoint, any n -trivial extension $R \rtimes_n M$ may be seen as the amalgamation of R with $R \rtimes_n M$ along $0 \rtimes_n M$ with respect to the canonical injection. This leads to the following question for every $m \geq 2$: is any m -Boolean ring an m -trivial extension?

7. Divisibility properties of $R \rtimes_n M$. Factorization in commutative rings with zero divisors was first investigated in a series of papers by Bouvier, Fletcher and Billis (see References), where the focus had been on the unicity property. Papers [1, 7, 8] marked the start of a systematic study of factorization in commutative rings with zero divisors. Since then, this theory has attracted the interest of a number of

authors. The study of divisibility properties of the classical trivial extension has led to some interesting examples and then to solutions for several questions (see [9, Section 5]). In this section, we are interested in extending a part of this study to the context of n -trivial extensions.

First, we recall the following definitions. Let S be a commutative ring and H an S -module. Two elements $e, f \in H$ are said to be *associates* (written $e \sim f$) (respectively, *strong associates* (written $e \approx f$), *very strong associates* (written $e \cong f$)) if $Se = Sf$ (respectively, $e = uf$ for some $u \in U(S)$, $e \sim f$ and either $e = f = 0$ or $e = rf$ implies $r \in U(S)$). Taking $H = S$ gives the notions of “associates” in S . We say that H is *strongly associate* if, for every $e, f \in H$, $e \sim f \Rightarrow e \approx f$. When S is strongly associate as an S -module, we also say that S is *strongly associate*. Finally, recall that H is said to be *S -présimplifiable* if, for $r \in S$ and $e \in H$,

$$re = e \implies r \in U(S)$$

or $e = 0$. If S is S -présimplifiable, we simply say that S is *présimplifiable*.

We begin with an extension of [9, Theorem 5.1].

Proposition 7.1. *Let $R \subseteq S$ be a ring extension such that $U(S) \cap R = U(R)$.*

(1) *If S is présimplifiable, then every R -submodule of S is présimplifiable. In particular, R is présimplifiable.*

(2) *Suppose that $S = R \oplus N$ as an R -module, where N is a nilpotent ideal of S which satisfies either $N^2 = 0$ or*

$$N = \bigoplus_{i \in \mathbb{N}} N_i$$

as an R -module, where

$$S = R \oplus N_1 \oplus N_2 \oplus \dots$$

is a graded ring. Then, S is présimplifiable if R is présimplifiable and N is R -présimplifiable.

Proof.

(1) Let H be an R -submodule of S . Consider $e = xe$ with $e \in H - \{0\}$ and $x \in R - \{0\}$. Since S is présimplifiable, $x \in U(S)$, and thus, $x \in U(S) \cap R = U(R)$.

(2) Let $x = r_x + n_x \neq 0$ and $y = r_y + n_y$ be two elements of

$$R \oplus N = S,$$

where $r_x, r_y \in R$ and $n_x, n_y \in N$, such that $x = yx$. Assume that $r_x \neq 0$. Then, $r_x = r_y r_x$ implies that $r_y \in U(R) \subseteq U(S)$, and, since N is nilpotent, $y = r_y + n_y$ is invertible in S , as desired. Next, assume that $r_x = 0$. Then, $n_x \neq 0$, and thus, $n_x = r_y n_x + n_y n_x$. In the case $N^2 = 0$, we have $n_x = r_y n_x$. Hence, $r_y \in U(R)$ since N is présimplifiable, and, as above, $y \in U(S)$. Finally, in the case where

$$S = R \oplus N_1 \oplus N_2 \oplus \dots$$

is a graded ring, we may set

$$n_x = n_{i_1} + \dots + n_{i_m}$$

with $\{i_1, \dots, i_m\} \subset \mathbb{N}$ and $m \in \mathbb{N}$ such that $i_1 \leq \dots \leq i_m$ and $n_{i_1} \neq 0$. Then, $n_{i_1} = r_y n_{i_1}$, which implies that $r_y \in U(R)$ and similarly, as above, $y \in U(S)$. □

Proposition 7.2. *Let $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ be a graded ring.*

(1) *If R is strongly associate, then R_0 is a strongly associate ring and R_i is a strongly associate R_0 -module for every $i \in \mathbb{N}$.*

(2) *Suppose that there exists an $n \in \mathbb{N}$ such that $R_i = 0$ for every $i \geq n + 1$, that is,*

$$R = R_0 \times_n R_1 \times \dots \times R_n,$$

and assume that R_0 is a présimplifiable ring and R_1, \dots, R_{n-1} are présimplifiable R_0 -modules. Then, R is strongly associate if and only if R_n is strongly associate.

Proof.

(1) Let $x_i, y_i \in R_i - \{0\}$ for $i \in \mathbb{N}_0$ be such that $R_0 x_i = R_0 y_i$. Then, $R x_i = R y_i$. Hence, there is a

$$u = u_0 + u_1 + \dots \in U(R)$$

such that $x_i = uy_i$. Then, $u_0 \in U(R_0)$ and $x_i = u_0y_i$, as desired.

(2) Let

$$x = x_m + \cdots + x_n \quad \text{and} \quad y = y_m + \cdots + y_n$$

be two associate elements of R where $m \in \{0, \dots, n\}$ and $x_i, y_i \in R_i$ for $i \in \{m, \dots, n\}$ such that x_m and y_m are nonzero. It follows that $x_m \sim y_m$. In particular, there is an

$$\alpha = \alpha_0 + \cdots + \alpha_n$$

such that $x = \alpha y$. Then, $x_m = \alpha_0 y_m$. Hence, two cases occur.

Case $m \neq n$. Since R_m is pré-simplifiable, $\alpha_0 \in U(R_0)$. Then, $\alpha \in U(R)$, as desired.

Case $m = n$ (i.e., $x = x_m$ and $y = y_m$). Here, the result follows since R_n is strongly associate. □

Now, we can give the extension of [9, Theorem 5.1] to the context of n -trivial extensions.

Corollary 7.3. *The following assertions are true.*

(1) $R \times_n M_1 \times \cdots \times M_n$ is pré-simplifiable if and only if R, M_1, \dots, M_n are pré-simplifiable.

(2) If $R \times_n M_1 \times \cdots \times M_n$ is strongly associate, then R, M_1, \dots, M_n are strongly associate.

(3) Suppose that R, M_1, \dots, M_n are pré-simplifiable. Then,

$$R \times_n M_1 \times \cdots \times M_n$$

is strongly associate if and only if M_n is strongly associate.

Now, we investigate the extension of [9, Theorem 5.4]. It is convenient to recall the following definitions. Let S be a commutative ring. A nonunit $a \in S$ is said to be *irreducible* or an *atom* (respectively, *strongly irreducible*, *very strongly irreducible*) if $a = bc$ implies $a \sim b$ or $a \sim c$ (respectively, $a \approx b$ or $a \approx c$, $a \cong b$ or $a \cong c$), and a is said to be *m-irreducible* if Sa is a maximal element of the set of proper principal

ideals of S . Note that, for a nonzero nonunit $a \in S$,

$$\begin{aligned} \text{a very strongly irreducible} &\implies a \text{ is } m\text{-irreducible} \\ &\implies a \text{ is strongly irreducible} \implies a \text{ is irreducible;} \end{aligned}$$

however, none of these implications can be reversed. In the case of an S -module H , we say that $e \in H$ is S -primitive (respectively, strongly S -primitive, very strongly S -primitive) if, for $a \in S$ and $f \in H$,

$$e = af \implies e \sim f$$

(respectively, $e \approx f$, $e \cong f$). In addition, e is S -superprimitive if $be = af$ for $a, b \in S$ and $f \in H$ implies $a \mid b$. Note that:

- (1) e is S -primitive $\Leftrightarrow Se$ is a maximal cyclic S -submodule of H ,
- (2) e is S -superprimitive $\Rightarrow e$ is very strongly S -primitive $\Rightarrow e$ is strongly S -primitive $\Rightarrow e$ is S -primitive,
- (3) if $\text{Ann}(e) = 0$, e is S -primitive $\Rightarrow e$ is very strongly S -primitive, and
- (4) e is S -superprimitive $\Rightarrow \text{Ann}(e) = 0$.

In the following results, the homogeneous element $(0, \dots, 0, m_i, 0, \dots, 0)$ of $R \times_n M$, where $i \in \{1, \dots, n\}$ and $m_i \in M_i - \{0\}$, is denoted by \underline{m}_i . The next result extends [9, Theorem 5.4 (1)].

Proposition 7.4. *Let $i \in \{1, \dots, n\}$ and $m_i, n_i \in M_i - \{0\}$. Then, $m_i \sim n_i$ (respectively, $m_i \approx n_i$, $m_i \cong n_i$) in M_i if and only if $\underline{m}_i \sim \underline{n}_i$ (respectively, $\underline{m}_i \approx \underline{n}_i$, $\underline{m}_i \cong \underline{n}_i$) in $R \times_n M$.*

Proof. The assertion is proved similarly to the corresponding classical one. □

It is worth noting that the analog of assertion (4) of [9, Theorem 5.4] does not hold in the context of n -trivial extensions with $n \geq 2$. Indeed, consider the 2-trivial extension $S = \mathbb{Z}_4 \times_2 \mathbb{Z}_4 \times \mathbb{Z}_4$. It is simple to show that $\bar{1}$ is superprimitive in the \mathbb{Z}_4 -module \mathbb{Z}_4 . However, $(\bar{0}, \bar{0}, \bar{1})$ is not very strongly irreducible in S (since $(\bar{2}, \bar{1}, \bar{2})^2 = (\bar{0}, \bar{0}, \bar{1})$). Moreover, even if we assume that R is an integral domain, we still do not have the desired analog. For this, take $S = \mathbb{Z} \times_2 \mathbb{Z} \times \mathbb{Z}$. We have that 1 is superprimitive in the \mathbb{Z} -module \mathbb{Z} . However, $(0, 0, 1)$ is not very strongly irreducible in S (since $(0, 1, 0)^2 = (0, 0, 1)$). The last example

also shows that assertion (2) of [9, Theorem 5.4] does not hold in the context of n -trivial extensions with $n \geq 2$, namely, if $0 \neq m_i = m_j m_k$ where $(m_i, m_j, m_k) \in M_i \times M_j \times M_k$, $i \geq 2$, and $j, k \in \{1, \dots, i - 1\}$ with $j + k = i$, then $\underline{m_i}$ cannot be irreducible. Indeed, $\underline{m_i} = \underline{m_j m_k}$, but neither $\underline{m_j}$ nor $\underline{m_k}$ are in

$$\langle \underline{m_i} \rangle \subseteq 0 \times_n \cdots \times 0 \times M_i \times \cdots \times M_n.$$

In order to extend [9, Theorem 5.1 (2)], we need to introduce the following definitions.

Definition 7.5. Assume that $n \geq 2$ and each multiplication in the family $\varphi = \{\varphi_{i,j} \mid \substack{i+j \leq n \\ 1 \leq i,j \leq n-1}$ is not trivial. Let $i \in \{2, \dots, n\}$. An element $m_i \in M_i - \{0\}$ is said to be φ -*indecomposable* (or indecomposable relative to the family of multiplications φ) if, for every $(m_j, m_k) \in M_j \times M_k$ (where $j, k \in \{1, \dots, i - 1\}$ with $j + k = i$), $m_i \neq m_j m_k$. If no ambiguity can arise, φ -indecomposable elements are simply called *indecomposables*.

For example, in $\mathbb{Z} \times_2 \mathbb{Z} \times \mathbb{Q}$, every element in $\mathbb{Q} - \mathbb{Z}$ is indecomposable. However, every element $x \in \mathbb{Z}$ (\mathbb{Z} as a submodule of \mathbb{Q}) is decomposable (since $(0, 1, 0)(0, x, 0) = (0, 0, x)$).

Definition 7.6. Let $i \in \{2, \dots, n\}$. The R -module M_i is said to be φ -*integral* (or integral relative to the family of multiplications φ) if, for every $(m_j, m_k) \in M_j \times M_k$ (where $j, k \in \{1, \dots, i - 1\}$ with $j + k = i$), $m_j m_k = 0$ implies that $m_j = 0$ or $m_k = 0$. If no ambiguity can arise, a φ -integral R -module is simply called *integral*.

For example, for $\mathbb{Z} \times_2 \mathbb{Z} \times \mathbb{Z}$, $M_2 = \mathbb{Z}$ is integral. And, for $\mathbb{Z} \times_2 \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ is not integral since, for instance, $\varphi_{1,1}(1, 2) = \bar{1} \bar{2} = \bar{0}$.

Proposition 7.7. Assume that $n \geq 2$. Let $i \in \{1, \dots, n\}$ and $m_i \in M_i - \{0\}$. If $\underline{m_i}$ is irreducible (respectively, strongly irreducible, very strongly irreducible) in $R \times_n M$, then m_i is primitive (respectively, strongly primitive, very strongly primitive) in M_i .

Conversely, three cases occur:

Case $i = 1$. *The reverse implication holds if R is an integral domain and M_j is torsion-free for every $j \in \{2, \dots, n\}$.*

Case $i = 2$ (here, $n \geq 2$). *The reverse implication holds if R is an integral domain, M_j is torsion-free for every $j \in \{1, \dots, n\} - \{2\}$ and m_2 is indecomposable.*

Case $i \geq 3$ (here, $n \geq 3$). *The reverse implication holds if R is an integral domain, M_j is torsion-free for every $j \in \{1, \dots, n\} - \{i\}$, M_j is integral for every $j \in \{2, \dots, i - 1\}$, and m_i is indecomposable.*

Proof. We prove only the primitive (irreducible) case. The other two cases are proved similarly.

\Rightarrow . Suppose that m_i is irreducible, and let $m_i = an_i$ for some $a \in R$ and $n_i \in M_i$. Then, $\underline{m_i} = (a, 0, \dots, 0)\underline{n_i}$ and thus, $(R \times_n M)\underline{m_i} = (R \times_n M)\underline{n_i}$. This implies that $Rm_i = Rn_i$, as desired.

\Leftarrow . Let $\underline{m_i} = (a_j)(n_j)$ for some $(a_j), (n_j) \in R \times_n M$. Then, $a_0n_0 = 0$. First, we show that the case $a_0 = 0$ and $n_0 = 0$ is impossible. Cases $i = 1, 2$ are easy and are left to the reader. Thus, assume $i \geq 3$. Suppose that $a_0 = 0$ and $n_0 = 0$. Then, we have the following equalities: for $j \in \{2, \dots, i - 1\}$,

$$a_1n_{j-1} + a_2n_{j-2} + \dots + a_{j-1}n_1 = 0$$

and

$$a_1n_{i-1} + a_2n_{i-2} + \dots + a_{i-1}n_1 = m_i.$$

A recursive argument on these equalities shows that, for $l \in \{2, \dots, i\}$, there is a $k \in \{0, \dots, l - 1\}$ such that $(a_0, \dots, a_k) = (0, \dots, 0)$ and $(n_0, \dots, n_{l-(k+1)}) = (0, \dots, 0)$. Indeed, it is clear this is true for $l = 2$. Then, suppose this is true for a given $l \in \{2, \dots, i - 1\}$. Thus, the equality

$$a_1n_{l-1} + a_2n_{l-2} + \dots + a_{l-1}n_1 = 0$$

becomes $a_{k+1}n_{l-(k+1)} = 0$. Then, since M_l is integral, we obtain the desired result for $l + 1$. Thus, for $l = i$, we get $a_{k+1}n_{i-(k+1)} = m_i$, which is absurd since m_i is indecomposable.

Now, we may assume that $a_0 \neq 0$ and $n_0 = 0$; thus, $a_0n_1 = 0$. Since M_1 is torsion-free, $n_1 = 0$. Recursively, we obtain $n_j = 0$ for $j \in \{1, \dots, i - 1\}$. Then, $a_0n_i = m_i$, and, since m_i is primitive, there

exists a $b_0 \in R$ such that $n_i = b_0 m_i$. It remains to show that there is a $b_j \in M_j$ for $j \in \{1, \dots, n - i\}$ such that $n_{i+j} = b_j m_i$, and this implies that

$$(n_i) = (0, \dots, 0, n_i, n_{i+1}, \dots) = (b_0, \dots, b_{n-i}, 0, \dots, 0) \underline{m}_i,$$

as desired. We have $a_0 n_{i+1} + a_1 n_i = 0$. Then, using both $a_0 n_i = m_i$ and $n_i = b_0 m_i$, we get $a_0 n_{i+1} + a_1 b_0 a_0 n_i = 0$. Next, $a_0(n_{i+1} + a_1 b_0 n_i) = 0$, so $n_{i+1} + a_1 b_0 n_i = 0$ (since M_{i+1} is torsion-free). Then,

$$n_{i+1} = -a_1 b_0^2 m_i,$$

and we set $b_1 = -a_1 b_0^2$. This leads to $n_{i+1} = b_1 m_i$. Similarly, using the equality

$$a_0 n_{i+2} + a_1 n_{i+1} + a_2 n_i = 0$$

with the equalities $a_0 n_i = m_i$ and $n_i = b_0 m_i$, we obtain

$$a_0 n_{i+2} + a_1 b_1 a_0 n_i + a_2 b_0 a_0 n_i = 0.$$

Therefore, $n_{i+2} + a_1 b_1 n_i + a_2 b_0 n_i = 0$; then, $n_{i+2} = b_2 m_i$, where $b_2 = -a_1 b_1 b_0 - a_2 b_0^2$. Finally, a recursive argument gives the desired result. □

The following result extends [9, Theorem 5.4 (3)].

Proposition 7.8. *Suppose that R has a nontrivial idempotent. Then, for every $i \in \{1, \dots, n\}$ and $m_i \in M_i - \{0\}$, \underline{m}_i is not irreducible in $R \times_n M$.*

Proof. The assertion is proved similarly to the corresponding classical one. □

Now, we are interested in finding some factorization properties. Recall that a ring S is called *atomic* if every (nonzero) nonunit of S is a product of irreducible elements (atoms) of S . Note that, as in the domain case, the ascending chain condition on principal ideals (ACCP) implies atomic.

We begin with an extension of [9, Theorem 5.5 (2)] which characterizes when a trivial extension of a ring satisfies ACCP. For this, we need the next lemma.

Lemma 7.9. *Let $i \in \{0, \dots, n\}$, and consider two elements $a = (0, \dots, 0, a_i, a_{i+1}, \dots, a_n)$ and $b = (0, \dots, 0, b_i, b_{i+1}, \dots, b_n)$ of $R \rtimes_n M$ with $a_i \neq 0$. Then, the implication*

$$\langle a \rangle \subsetneq \langle b \rangle \implies b_i \neq 0 \quad \text{and} \quad \langle a_i \rangle \subsetneq \langle b_i \rangle$$

is true if either (1) $0 \leq i \leq n - 1$ and M_i is présimplifiable (here $M_0 = R$) or (2) $i = n$.

Proof. Since $\langle a \rangle \subsetneq \langle b \rangle$, there is a $c = (c_0, \dots, c_n) \in R \rtimes_n M - U(R \rtimes_n M)$ such that $a = cb$. Then, $a_i = c_0 b_i$ and $c_0 \notin U(R)$. This shows that $\langle a_i \rangle \subsetneq \langle b_i \rangle$ in both cases. □

Theorem 7.10. *Assume that $n \geq 2$. Suppose that M_i is présimplifiable for every $i \in \{0, \dots, n - 1\}$ (here $M_0 = R$). Then, $R \rtimes_n M$ satisfies ACCP if and only if R satisfies ACCP and, for every $i \in \{1, \dots, n\}$, M_i satisfies ACC on cyclic submodules.*

Proof. The proof of the direct implication is easy. We prove the converse. Suppose that $R \rtimes_n M$ admits a strictly ascending chain of principal ideals

$$\langle \langle a_{1,i} \rangle \rangle \subsetneq \langle \langle a_{2,i} \rangle \rangle \subsetneq \dots$$

If there exists a $j_0 \in \mathbb{N}$ such that $a_{j_0,0} \neq 0$, then, for every $k \geq j_0$, $a_{k,0} \neq 0$. In addition, from Lemma 7.9, we obtain the following strictly ascending chain of principal ideals of R :

$$\langle a_{j_0,0} \rangle \subsetneq \langle a_{j_0+1,0} \rangle \subsetneq \dots$$

This is absurd since R satisfies ACCP. Now, suppose that $a_{j,0} = 0$ for every $j \in \mathbb{N}$ and that there exists a $j_1 \in \mathbb{N}$ such that $a_{j_1,1} \neq 0$. From Lemma 7.9, we obtain the following strictly ascending chain of cyclic submodules of M_1 :

$$\langle a_{j_1,1} \rangle \subsetneq \langle a_{j_1+1,1} \rangle \subsetneq \dots$$

This is absurd since M_1 satisfies ACC on cyclic submodules. We continue in this manner until the case is reached where we may suppose that $a_{j,i} = 0$ for every $i \in \{1, \dots, n - 1\}$ and $j \in \mathbb{N}$. Therefore, from Lemma 7.9, we obtain the desired result. □

Now, we investigate when $R \times_n M$ is atomic, namely, we give an extension of [9, Theorem 5.5 (4)]. Recall that an R -module N is said to satisfy MCC if every cyclic submodule of N is contained in a maximal (not necessarily proper) cyclic submodule of N .

Theorem 7.11. *Assume that $n \geq 2$. Suppose that M_i is présimplifiable for every $i \in \{0, \dots, n - 1\}$ (here, $M_0 = R$). Then, $R \times_n M$ is atomic if R satisfies ACCP, M_i satisfies ACC on cyclic submodules, for every $i \in \{1, \dots, n - 1\}$, and M_n satisfies MCC.*

Proof. The proof is slightly more technical than that of [9, Theorem 5.5 (4)]. Here, we need to break the proof into the following $n + 1$ steps such that, in the step number $k \in \{1, \dots, n + 1\}$, we prove that every nonunit element $(m_i) \in R \times_n M$ with $m_0 = 0, \dots, m_{k-2} = 0$ and $m_{k-1} \neq 0$ is a product of irreducibles.

We use an inductive argument for the first n steps.

Step 1. Suppose that there is a nonunit element (m_i) of $R \times_n M$ with $m_0 \neq 0$ and such that (m_i) cannot be factored into irreducibles. Then, there exist $(a_{1,i}), (b_{1,i}) \in R \times_n M - U(R \times_n M)$ such that $(m_i) = (a_{1,i})(b_{1,i})$, and neither (m_i) and $(a_{1,i})$ nor (m_i) and $(b_{1,i})$ are associate. Since

$$0 \neq m_0 = a_{1,0}b_{1,0},$$

$a_{1,0} \neq 0$ and $b_{1,0} \neq 0$. Clearly, $(a_{1,i})$ or $(b_{1,i})$ must be reducible, say $(a_{1,i})$. Also, for $(a_{1,i})$, there are $(a_{2,i}), (b_{2,i}) \in R \times_n M - U(R \times_n M)$ such that $(a_{1,i}) = (a_{2,i})(b_{2,i})$, and neither $(a_{1,i})$ and $(a_{2,i})$ nor $(a_{1,i})$ and $(b_{2,i})$ are associate. As above, $a_{2,0} \neq 0$ and $b_{2,0} \neq 0$, and say $(a_{2,i})$, are reducible. Thus we continue until we obtain a strictly ascending chain

$$\langle (m_i) \rangle \subsetneq \langle (a_{1,i}) \rangle \subsetneq \langle (a_{2,i}) \rangle \subsetneq \dots$$

Using Lemma 7.9, we get a strictly ascending chain of principal ideals of R

$$\langle m_0 \rangle \subsetneq \langle a_{1,0} \rangle \subsetneq \langle a_{2,0} \rangle \subsetneq \dots$$

This is absurd since R satisfies ACCP.

Step j. $1 \leq j \leq n$. Suppose that there is a nonunit element $(m_i) \in R \times_n M$ with $m_0 = 0, \dots, m_{j-2} = 0$ and $m_{j-1} \neq 0$ which is not a product of irreducibles. Then, there are $(a_{1,i}), (b_{1,i}) \in$

$R \rtimes_n M - U(R \rtimes_n M)$ such that $(m_i) = (a_{1,i})(b_{1,i})$, and neither (m_i) and $(a_{1,i})$ nor (m_i) and $(b_{1,i})$ are associate. Then,

$$a_{1,0}b_{1,j-1} + a_{1,1}b_{1,j-2} + \cdots + a_{1,j-2}b_{1,1} + a_{1,j-1}b_{1,0} = m_{j-1} \neq 0.$$

If $a_{1,k} = 0$ for every $k \in \{0, \dots, j-2\}$, then necessarily, $b_{1,0} \neq 0$. Hence, by the preceding steps, $(b_{1,i})$ is a product of irreducibles, and then, by the hypothesis on (m_i) , $(a_{1,i})$ is reducible. If $a_{1,k} \neq 0$ for some $k \in \{0, \dots, j-2\}$, then $(a_{1,i})$ is a product of irreducibles and $(b_{1,i})$ is reducible. Thus, by symmetry, we may assume that $(a_{1,i})$ is reducible, and it is not a product of irreducibles. Thus, necessarily, $a_{1,0} = 0, \dots, a_{1,j-2} = 0$ and $a_{1,j-1} \neq 0$. The last argument is repeated so that we obtain a strictly ascending chain of principal ideals of $R \rtimes_n M$

$$\langle (m_i) \rangle \subsetneq \langle (a_{1,i}) \rangle \subsetneq \langle (a_{2,i}) \rangle \subsetneq \cdots$$

such that, for every $k \in \mathbb{N}$, $a_{k,0} = 0, \dots, a_{k,j-2} = 0$ and $a_{k,j-1} \neq 0$. Then, from Lemma 7.9, we obtain a strictly ascending chain of cyclic submodules of M_{j-1}

$$\langle m_{j-1} \rangle \subsetneq \langle a_{1,j-1} \rangle \subsetneq \langle a_{2,j-1} \rangle \subsetneq \cdots,$$

which is absurd by hypothesis on M_{j-1} , as desired.

Step $n+1$. It remains to prove that every element of $R \rtimes_n M$ of the form $(0, \dots, 0, m_n)$ with $m_n \neq 0$ is a product of irreducibles. Since M_n satisfies MCC, $Rm_n \subseteq Rm$, where Rm is a maximal cyclic submodule of M_n . Then, $m_n = am$ for some $a \in R - \{0\}$, and thus,

$$(0, \dots, 0, m_n) = (a, 0, \dots, 0)(0, \dots, 0, m).$$

Now, $a \neq 0$ shows that $(a, 0, \dots, 0)$ is a product of irreducibles (by Step 1) and Rm is maximal shows that either $(0, \dots, 0, m)$ is irreducible or $(0, \dots, 0, m) = (a_i)(b_i)$, where $a_k \neq 0$ and $b_l \neq 0$ for some $k, l \in \{0, \dots, n-1\}$. Then, by the preceding steps, (a_i) and (b_i) are products of irreducibles, and hence, so is $(0, \dots, 0, m)$. This concludes the proof. \square

A ring S is said to be a *bounded factorial ring* (BFR) if, for each nonzero nonunit $x \in S$, there is a natural number $N(x)$ such that, for any factorization $x = x_1 \cdots x_s$ where each x_i is a nonunit, we have $s \leq N(x)$. For domains, we say BFD instead of BFR. Recall that an S -module H is said to be a *BF-module* if, for each nonzero $h \in H$,

there exists a natural number $N(h)$ so that

$$h = a_1 \cdots a_{s-1} h_s \text{ (each } a_i \text{ a nonunit)} \implies s \leq N(h).$$

Our next theorem, which is a generalization of [9, Theorem 5.5 (4)], investigates when $R \rtimes_n M$ is BFR. It is based on the next lemma.

Lemma 7.12. *For $j \in \mathbb{N} - \{1\}$, a product of j elements of $R \rtimes_n M$ of the form $(0, x_1, \dots, x_n)$ is of the form $(0, \dots, 0, y_j, \dots, y_n)$ (where, if $j \geq n + 1$ the product is zero).*

Theorem 7.13. *Assume that $n \geq 2$, R is an integral domain and M_i is torsion-free for every $i \in \{1, \dots, n - 1\}$. Then, $R \rtimes_n M$ is a BFR if and only if R is a BFD, and M_i is a BF-module for every $i \in \{1, \dots, n\}$.*

Proof.

\implies . Clear.

\Leftarrow . Let (m_i) be a nonzero nonunit element of $R \rtimes_n M$, and suppose that we have a factorization into nonunits $(m_i) = (a_{1,i}) \cdots (a_{s,i})$ for some $s \in \mathbb{N}$. If $m_0 \neq 0$, $m_0 = a_{1,0} \cdots a_{s,0}$ implies that $s \leq N(m_0)$. Otherwise, there is a $j \in \{1, \dots, n\}$ such that $m_0 = 0, \dots, m_{j-1} = 0$ and $m_j \neq 0$. We may assume that $s \geq j + 1$. Since R is an integral domain, by Lemma 7.12, we may assume that there is a $k \in \{1, \dots, j\}$ such that $a_{l,0} = 0$ for every $l \in \{1, \dots, k\}$ and $a_{l,0} \neq 0$ for every $l \in \{k + 1, \dots, s\}$. Let

$$(0, \dots, 0, b_k, \dots, b_n) = \prod_{l=1}^k (a_{l,i}) \quad \text{and} \quad (c_0, \dots, c_n) = \prod_{l=k+1}^s (a_{l,i}).$$

Since M_i is torsion-free for every $i \in \{1, \dots, j - 1\}$ and

$$c_0 = \prod_{l=k+1}^s a_{l,0} \neq 0, b_k = 0, \dots, b_{j-1} = 0,$$

$b_j \neq 0$. Then,

$$m_j = c_0 b_j = \prod_{l=k+1}^s a_{l,0} b_j.$$

Therefore, $s \leq N(m_j) + k - 1$ (since M_j is a BF-module). □

Now, we investigate the notion of a U -factorization. It was introduced by Fletcher [33, 34] and developed by Axtell, et al., in [11, 12]. Let S be a ring, and consider a nonunit $a \in S$. By a *factorization* of a , we mean $a = a_1 \cdots a_s$, where each a_i is a nonunit. Recall from [33] that, for $a \in S$,

$$\begin{aligned} U(a) &= \{r \in S \mid \text{there exists an } s \in S \text{ with } rsa = a\} \\ &= \{r \in S \mid r(a) = (a)\}. \end{aligned}$$

A U -factorization of a is a factorization $a = a_1 \cdots a_s b_1 \cdots b_t$, where, for every $1 \leq i \leq s$, $a_i \in U(b_1 \cdots b_t)$ and, for every $1 \leq i \leq t$, $b_i \notin U(b_1 \cdots \widehat{b_i} \cdots b_t)$. We denote this U -factorization by $a = a_1 \cdots a_s [b_1 \cdots b_t]$ and call a_1, \dots, a_s (respectively, b_1, \dots, b_t) the *irrelevant* (respectively, the *relevant*) factors.

Our next result investigates when an n -trivial extension is a U -FFR. First, recall the following definitions.

A ring S is called a *finite factorization ring* (FFR) (respectively, a *U -finite factorization ring* (U -FFR)) if every nonzero nonunit of S has only a finite number of factorizations (respectively, U -factorizations) up to order and associates (respectively, associates on the relevant factors). A ring S is called a *weak finite factorization ring* (WFFR) (respectively, a *U -weak finite factorization ring* (U -WFFR)) if every nonzero nonunit of R has only a finite number of nonassociate divisors (respectively, nonassociate relevant factors). We have $\text{FFR} \Rightarrow \text{WFFR}$, and the converse holds in the domain case. But, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a WFFR that is not an FFR. However, from [11, Theorem 2.9],

$$U\text{-FFR} \iff U\text{-WFFR}.$$

The study of the above notions on the classical trivial extensions leads to consideration of the following notion, see [11]. Let N be an S -module. For a nonzero element $x \in N$, we say that $Sd_1 d_2 \cdots d_s x$ is a *reduced submodule factorization* if, for every $j \in \{1, \dots, s\}$, $d_j \notin U(S)$ and, with no canceling and reordering of the d_j s, it is the case that

$$Sd_1 d_2 \cdots d_s x = Sd_1 d_2 \cdots d_t x,$$

where $t < s$. The module N is said to be a U -FF module if, for every nonzero element $x \in N$, there exist only finitely many reduced submodule factorizations $Sx = Sd_1 d_2 \cdots d_t x_k$, up to order

and associates on the d_i , as well as up to associates on the x_k . In this context, we introduce the following definition.

Definition 7.14. Assume $n \geq 2$, and consider $i \in \{1, \dots, n\}$.

(1) Let $m_i \in M_i - \{0\}$, $s \in \mathbb{N}$, and $(d_{i_1}, \dots, d_{i_s}) \in M_{i_1} \times \dots \times M_{i_s}$, where $\{i_1, i_2, \dots, i_s\} \subseteq \{0, \dots, n\}$ with $i_1 + \dots + i_s = i$. We say that

$$Rd_{i_1}d_{i_2} \cdots d_{i_s}m_i \subseteq M_i$$

is a φ -reduced submodule factorization if, for every $j \in \{1, \dots, s\}$ such that $i_j = 0$, $d_{i_j} \notin U(R)$ and, with no canceling and reordering of the d_j 's, it is the case that

$$Rd_{i_1}d_{i_2} \cdots d_{i_s} = Rd_{i_1}d_{i_2} \cdots d_{i_t},$$

where $t < s$. If no ambiguity can arise, a φ -reduced submodule factorization is simply called a *reduced submodule factorization*.

(2) The R -module M_i is said to be a φ - U -FF module (or simply U -FF module) if, for every nonzero element $x \in M_i$, there exist only finitely many reduced submodule factorizations $Rx = Rd_{i_1}d_{i_2} \cdots d_{i_s}$, up to order and associates on the d_{i_j} .

It is clear that, for $i = 1$, the notion of the U -FF module defined here is the same as that of Axtell.

Based on the proof of [11, Theorem 4.2], it is asserted in [12, Theorem 3.6] that, if $R \rtimes_1 M_1$ is a U -FFR, then, for every nonzero nonunit $d \in R$, there are only finitely many distinct principal ideals $\langle\langle d, m \rangle\rangle$ in $R \rtimes_1 M_1$. However, a careful reading of this proof shows that the case of ideals $\langle\langle d, m \rangle\rangle$ with $dM_1 = 0$ should also be treated. The validity of this assertion may be confirmed for reduced rings. However, the context of n -trivial extensions seems to be more complicated. Nevertheless, under some certain conditions, we next investigate when $R \rtimes_n M$ is a U -FFR.

Lemma 7.15. Assume that $n \geq 2$ and M_n is integral. Then, for every nonzero nonunit $d \in R$, the following assertions are true.

(1) For every $i \in \{1, \dots, n - 1\}$, the following assertions are equivalent:

- (a) $dM_i = 0$.
- (b) $dm_i = 0$ for some $m_i \in M_i - \{0\}$.
- (c) $dM_{n-i} = 0$.
- (d) $dm_{n-i} = 0$ for some $m_{n-i} \in M_{n-i} - \{0\}$.

(2) *The following assertions are equivalent:*

- (a) $dM_i = 0$ for some $i \in \{1, \dots, n - 1\}$.
- (b) $dM_i = 0$ for every $i \in \{1, \dots, n - 1\}$.

(3) *If $dM_n = 0$, then $dM_i = 0$ for every $i \in \{1, \dots, n - 1\}$.*

(4) *If M_n is torsion-free, then M_i is torsion-free for every $i \in \{1, \dots, n - 1\}$.*

Proof.

(1) For the implications (1)(a) \Rightarrow (1)(b) and (1)(c) \Rightarrow (1)(d), there is nothing to prove.

(1)(b) \Rightarrow (1)(c). Let $m \in M_{n-i}$. Then, $dm_i m = 0 \in M_n$. Therefore, $dm = 0$ (since M_n is integral and $m_i \neq 0$).

(1)(d) \Rightarrow (1)(a). Similar to the previous proof.

(2) For the implication (2)(b) \Rightarrow (2)(a), there is nothing to prove.

(2)(a) \Rightarrow (2)(b). First, we prove that $dM_1 = 0$. For every $m_1 \in M_1 - \{0\}$, $dm_1^i = 0 \in M_i$, and thus, $dm_1^n = 0 \in M_n$. Therefore, $dm_1 = 0$ (since M_n is integral and $m_1 \neq 0$). Now, consider any $j \in \{1, \dots, n - 1\}$ and any $m_j \in M_j - \{0\}$. Then, for every $m_1 \in M_1 - \{0\}$, $dm_j m_1^{n-j} = 0 \in M_n$ which shows that $dm_j = 0$.

(3) This is proved as above.

(4) If there are $m_1 \in M_1 - \{0\}$ and $r \in R - \{0\}$ such that $rm_1 = 0$, then $rm_1^n = 0 \in M_n$. Since M_n is torsion-free and $r \neq 0$, $m_1^n = 0 \in M_n$ so $m_1 = 0$ (since M_n is integral). This is absurd since $m_1 \neq 0$. Finally, by assertions (1) and (2), we conclude that M_i is torsion-free for every $i \in \{1, \dots, n - 1\}$. □

Theorem 7.16. *Assume that $n \geq 2$. If $R \rtimes_n M$ is a U -FFR (equivalently, a U -WFFR), then the following conditions are satisfied:*

- (1) R is an FFR.
- (2) M_i is a U -FF module for every $i \in \{1, \dots, n\}$.

Moreover, if R is an integral domain and M_n is integral and torsion-free, then:

(3) for every nonzero nonunit $d \in R$, there are only finitely many distinct principal ideals $\langle (d, m_1, \dots, m_n) \rangle$ in $R \times_n M$.

(4) For every $i \in \{1, \dots, n - 1\}$ and $m \in M_i - \{0\}$, there are only finitely many distinct principal ideals of the form $\langle (0, \dots, 0, m, m_{i+1}, \dots, m_n) \rangle$ in $R \times_n M$.

Conversely, if R is an integral domain and M_n is integral and torsion-free, then assertions (1)–(4) imply that $R \times_n M$ is a U -FFR.

Proof. The proof of the “converse” part is similar to that of [11, Theorem 4.2].

\Rightarrow . The proof of each (1) and (2) is similar to that given in [11, Theorem 4.2].

(3) Suppose, by contradiction, there exists a nonzero nonunit $d \in R$ for which there is a family of distinct principal ideals of the form $\langle (d, m_{j,1}, \dots, m_{j,n}) \rangle$, where j is in an infinite indexing set Γ . We prove this is impossible by showing that, for every $j \neq k$ in Γ , there exists $(1, x_1, \dots, x_n) \in R \times_n M$ such that

$$(d, m_{j,1}, \dots, m_{j,n}) = (1, x_1, \dots, x_n)(d, m_{k,1}, \dots, m_{k,n}).$$

A recursive argument shows that the fact that every equation $dX = b_i$, with $b_i \in M_i$, admits a solution $X \in M_i$, implying the existence of the desired $(1, x_1, \dots, x_n)$. Note that, from Lemma 7.15, M_i is torsion-free for every $i \in \{1, \dots, n\}$. First, consider an element $b_n \in M_n - \{0\}$. For every $j \in \Gamma$, $(d, m_{j,1}, \dots, m_{j,n})(0, \dots, 0, b_n) = (0, \dots, 0, db_n)$. Then,

$$(0, \dots, 0, db_n) = (d, m_{j,1}, \dots, m_{j,n})[(0, \dots, 0, b_n)]$$

is the only possible corresponding U -factorization of $(0, \dots, 0, db_n)$ (since $R \times_n M$ is a U -FFR), so there exists an $r \in R$ such that $b_n = drb_n$. This shows that the above equation admits a solution for $i = n$. Now, consider $k \in \{1, \dots, n - 1\}$ and any $b_k \in M_k$. For every $b_{n-k} \in M_{n-k} - \{0\}$, $b_k b_{n-k} \in M_n - \{0\}$, and thus, there is an $r \in R$ such that $b_k b_{n-k} = drb_k b_{n-k}$. Then, $(b_k - drb_k)b_{n-k} = 0$. Therefore, $b_k = drb_k$ (since M_n is integral).

(4) Let $i \in \{1, \dots, n - 1\}$. Suppose, by contradiction, that there exists an $m \in M_i - \{0\}$, for which there is a family of distinct

principal ideals of the form $\langle(0, \dots, 0, m, m_{j,i+1}, \dots, m_{j,n})\rangle$, where j is in an infinite indexing set Γ . Let $m_{n-i} \in M_{n-i} - \{0\}$. Necessarily, $mm_{n-i} \neq 0$. Then,

$$(0, \dots, 0, m, m_{j,i+1}, \dots, m_{j,n})(0, \dots, 0, m_{n-i}, 0, \dots, 0) = (0, \dots, 0, mm_{n-i}),$$

and

$$(0, \dots, 0, mm_{n-i}) = (0, \dots, 0, m, m_{j,i+1}, \dots, m_{j,n})[(0, \dots, 0, m_{n-i}, 0, \dots, 0)]$$

is the only possible corresponding U -factorization of $(0, \dots, 0, mm_{n-i})$ (since $R \times_n M$ is a U -FFR); thus, there exists an $(r_i) \in R \times_n M$ such that

$$\begin{aligned} (0, \dots, 0, m, m_{j,i+1}, \dots, m_{j,n})(r_i)(0, \dots, 0, m_{n-i}, 0, \dots, 0) \\ = (0, \dots, 0, m_{n-i}, 0, \dots, 0), \end{aligned}$$

equivalently $(0, \dots, 0, r_0 mm_{n-i}) = (0, \dots, 0, m_{n-i}, 0, \dots, 0)$, which is absurd. \square

A ring S is called a U -bounded factorization ring (U -BFR) if, for each nonzero nonunit $x \in S$, there is a natural number $N(x)$ such that, for any factorization $x = a[b_1 \cdots b_t]$, we have $t \leq N(x)$. An S -module H is said to be a U -BF module if, for every $h \in H - \{0\}$, there exists a natural number $N(h)$ such that, if $Sh = Sd_1 \cdots d_t h'$, where $d_j \notin U(S)$, $t > N(h)$ and $h' \in H$, then, after cancelation and reordering of some of the d_j 's, we have $Sh = Sd_1 \cdots d_s h'$ for some $s \leq N(h)$.

The question of when the classical trivial extension is a U -BFR is still open. However, there is an answer to this question for an integral domain D [11, Theorem 4.4]: for a D -module N , $D \times N$ is a U -BFR if and only if D is a BFD and N is a U -BF R -module. Two more general results for the direct implication were established in [12, Theorem 3.7, Lemma 3.8]. Here, we extend these results to our context. For this, we need to introduce the following definition.

Definition 7.17. Assume that $n \geq 2$, and consider $i \in \{1, \dots, n\}$. The R -module M_i is said to be a φ - U -BF module (or simply a U -BF module) if, for every nonzero element $x \in M_i$, there exists a natural

number $N(x)$ so that, if

$$Rx = Rd_{i_1}d_{i_2}\cdots d_{i_t} \quad \text{where } t \in \mathbb{N},$$

$$(d_{i_1}, \dots, d_{i_t}) \in M_{i_1} \times \cdots \times M_{i_t}$$

for some $\{i_1, i_2, \dots, i_t\} \subseteq \{0, \dots, n\}$, with $i_1 + \cdots + i_t = i$, $d_{i_j} \notin U(R)$ when $i_j = 0$, and $t > N(x)$, then, after cancelation and reordering of some of the d_{i_j} 's in R , we have $Rx = Rd_{i_1}d_{i_2}\cdots d_{i_s}$ for some $s \leq N(h)$.

Theorem 7.18. *If $R \rtimes_n M$ is a U -BFR, then R is a U -BFR and M_i is a U -BF module for every $i \in \{1, \dots, n\}$. Moreover, if R is *présimplifiable*, then R is a BFR.*

Conversely, assume R to be an integral domain. If R is a BFD and, for every $i \in \{1, \dots, n\}$, M_i is a U -BF module, then $R \rtimes_n M$ is a U -BFR.

Proof. Similar to the classical case. □

A ring S is called *U -atomic* if every nonzero nonunit element of S has a U -factorization in which all the relevant factors are irreducibles. The question of when the classical trivial extension is U -atomic is still unsolved. In [11, Theorem 4.6], Axtell gave an answer to this question for an integral domain D with ACCP: for a D -module N , $D \times N$ is atomic if and only if $D \times N$ is U -atomic. In [12, Theorem 3.15], it is shown that the condition that the ring is an integral domain could be replaced by the ring is *présimplifiable*. The following result extends [12, Theorem 3.15] to the context of n -trivial extensions.

Theorem 7.19. *Assume that $n \geq 2$. Suppose that M_i is *présimplifiable* for every $i \in \{0, \dots, n - 1\}$ (here, $M_0 = R$), R satisfies ACCP and M_i satisfies ACC on cyclic submodules for every $i \in \{1, \dots, n - 1\}$. Then, $R \rtimes_n M$ is atomic if and only if $R \rtimes_n M$ is U -atomic.*

Proof.

\Rightarrow . Clear.

\Leftarrow . Suppose that $R \rtimes_n M$ is not atomic. Then, by the proof of Theorem 7.11, there exists an

$$\underline{m}_n := (0, \dots, 0, m_n) \in R \rtimes_n M$$

with $m_n \neq 0$ which is not a product of irreducibles. Since $R \times_n M$ is U -atomic, \underline{m}_n admits a U -factorization of the form

$$\underline{m}_n = a_1 \cdots a_s [b_1 \cdots b_t]$$

such that the b_i s are irreducibles. Since \underline{m}_n cannot be a product of irreducibles and, by the proof of Theorem 7.11, necessarily $s = 1$ and a_1 has the form $\underline{x}_n := (0, \dots, 0, x_n)$. However, $\underline{x}_n \langle b_1 \cdots b_t \rangle = \langle b_1 \cdots b_t \rangle$, and thus, $b_1 \cdots b_t$ has the form $\underline{y}_n := (0, \dots, 0, y_n)$. This is impossible since $\underline{m}_n = \underline{x}_n \underline{y}_n = 0$. \square

ENDNOTES

1. When R is a field and $M_i = R$ for every $i \in \{1, \dots, n\}$, these matrices are well known as upper triangular Toeplitz matrices. In [38], the author used the same terminology for such matrices with entries in a commutative ring.

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