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FINDING NEW SMALL DEGREE POLYNOMIALS WITH SMALL MAHLER MEASURE BY GENETIC ALGORITHMS

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ABSTRACT. In this paper, we propose a new application of genetic-type algorithms to find monic, irreducible, non-cyclotomic integer polynomials with *small degree* and Mahler measure less than 1.3, which do not appear in Mossinghoff's list of all known polynomials with degree at most 180 and Mahler measure less than 1.3 **[10]**. The primary focus lies in finding such polynomials of small degree. In particular, the list referred to above is known to be complete through degree 44, and we show that it is not complete from degree 46 on by supplying two new polynomials of small Mahler measure, of degrees 46 and 56. We also provide a large list of polynomials of small Mahler measure of degrees up to 180 which, although discovered by us through the use of a method described in Boyd and Mossinghoff **[3]** based on limit points of small Mahler measures, do not appear on Mossinghoff's list **[10]**. Finally, we verify that our new polynomials of degrees 46 and 56 cannot be produced from the known small limit points.

1. Introduction. The *Mahler measure* of a polynomial $P \in \mathbb{C}[x]$, where

$$P(x) = b_0x^d + b_1x^{d-1} + \cdots + b_d = b_0 \prod_{k=1}^{d} (x - \alpha_k), \quad b_0 \neq 0,$$

is defined by

$$M(P) := |b_0| \prod_{k=1}^{d} \max(1, |\alpha_k|).$$

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For an algebraic number α, we denote by $M(\alpha)$ the Mahler measure of its minimal polynomial in $\mathbb{Z}[x]$. For a general survey on Mahler measure, we refer to Smyth [**16**].

For $P \in \mathbb{Z}[x]$, we clearly have $M(P) \geq 1$. From Kronecker's first theorem [**7**], $M(P) = 1$ if and only if P is a product of cyclotomic polynomials and a power of x.

In 1933, Lehmer [**8**] asked whether there exists a positive number δ such that, if α is neither 0 nor a root of unity, then $M(\alpha) \geq 1+\delta$. This is an open problem known as *Lehmer's question.* He found the smallest known Mahler measure > 1, $M(P_0) = 1.176280\ldots$, where

$$P_0(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

A polynomial is said to be *reciprocal* if $P(x) = x^d P(1/x)$. An algebraic number is reciprocal if its minimal polynomial is reciprocal. In 1971, Smyth [**15**] proved that, if $\alpha \neq 0, 1$ is a nonreciprocal algebraic number, then $M(\alpha) \geq 1.324717\ldots$, which is the smallest Pisot number equal to $M(x^3 - x - 1) = \theta_0$.

Many computations have been done to obtain, for a fixed degree d, all the polynomials P with $M(P) < \theta_0$. Boyd [**1, 2**] computed all irreducible, noncyclotomic integer polynomials P with degree $d \leq 20$ having $M(P) < 1.3$, and Mossinghoff [**9**] used this same algorithm to extend the computation to $d \leq 24$. Flammang, Rhin and Sac-Épée [**5**] extended these computations for the polynomials P with $M(P) < \theta_0$ and $d \leq 36$, and polynomials P with $M(P) < 1.31$ and $d = 38$ or 40. These computations use a large family of explicit auxiliary functions to obtain better bounds on the coefficients of P. Using the same method, Mossinghoff, Rhin and Wu [**12**] computed all polynomials with measure less than 1.3 and degree at most 44. They also determined the minimal Mahler measure for each degree $\leq$ 54. In all of these computations, the polynomials P are primitive, that is to say, there is no polynomial Q such that $P(x) = Q(x^k)$ for some integer $k > 1$ (this implies $M(P) = M(Q)$).

Mossinghoff maintains a website [**10**] which contains all known irreducible polynomials $P \in \mathbb{Z}[x]$ with $\deg P \leq 180$ and $M(P) < 1.3$. All exhaustive searches are contained in this table. Several heuristic searches produced a large number of polynomials.

The *length* of P, denoted $L(P)$, is equal to

$$\sum_{0\le i\le d} |b_i|,$$

and the *height* of P, denoted $H(P)$, is

$$\max_{0\le i\le d} |b_i|.$$

One of the searches, due to Mossinghoff [**9**] and Lisonek, tested for a particular degree d and a fixed length l of all polynomials of height 1. This is very efficient since it finds all P with degree at most 40 and measure less than 1.3. Other techniques used include a numerical descent technique, which produced 19 polynomials with degree between 174 and 180 by Rhin and Sac-Épée [**14**], and a method slightly modifying cyclotomic polynomials by Mossinghoff, Pinner and Vaaler [**11**]. There were also polynomials associated with some small limit points of Mahler measure [**3**]. Surprisingly, the last to be added to Mossinghoff's list were three polynomials of degrees 46, 48 and 52 [**12**].

For the reader's convenience, we now provide some details regarding the genetic algorithms upon which our search method is based. We do not recall general theoretical results about genetic optimization strategies, but we focus on the practical process used to solve our particular problem.

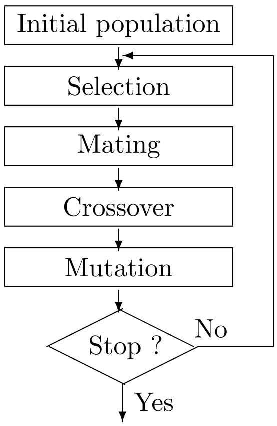
2. A brief review of genetic algorithms. In the framework of the genetic optimization strategies, a polynomial is a point in the optimization *phase-space*, which is defined by its *genes*. For our purposes, the genes of a polynomial are merely its integer coefficients. The goal of the optimization process is to improve the *quality*, i.e., the position, of the points in the phase-space. To give an example, suppose that we wish to find polynomials

$$x^{16} + a_{15}x^{15} + a_{14}x^{14} + \cdots + a_{14}x^2 + a_{15}x + 1$$

of degree 16 with Mahler measure less than 1.3. These polynomials are reciprocal [**15**] due to their small Mahler measure. The principle of our genetic algorithm is to consider the Mahler measure as a function

$$\mathfrak{M}(a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15})$$

that we wish to minimize over a given domain $\mathfrak{D}$, in other words, the goal is to find values of the integer variables $a_8, \ldots, a_{15}$ which minimize the function $\mathfrak{M}$. Below, we summarize the outline of the method. See [**6**] for a detailed presentation of the principle.



- **Initial population:** An initial population of polynomials is created randomly.
- **Selection:** Each polynomial of the current generation (the initial population in the first step) will be copied as often to the new generation, the lower the Mahler measure, compared to the average Mahler measure of the entire current sample.
- **Mating and crossover:** In addition to *Elite children*, i.e., polynomials with a very low Mahler measure, which are automatically still present in the following generation, *Crossover children* are generated by combining the genes, i.e., the coefficients, of pairs of *parents* (polynomials of the previous generation).
- **Mutation:** *Mutation children* are generated via random changes of a small number of their genes.

The mating/crossover process on one hand and the mutation process on the other hand are both essential to the proper functioning of the algorithm. The mating/crossover allows for selection of the best genes stemming from the previous generation, whereas the mutation introduces an element of chance for improving the diversity of the current generation.

3. Implementation and results. In order to implement the genetic optimization strategies in our context, we make use of the optimization toolbox available in the `MATLAB` software environment [**4**], which allows taking into account some particular constraints since our variables only have integer values.

Due to the fact that the genetic algorithms are known to be rather slow, we choose to utilize only the values -1, 0 and 1 for our variables. When the algorithm supplies a polynomial, the Mahler measure of which is lower than 1.3, it is necessary to check that this polynomial is irreducible. This test is made via the `GP-PARI` number theory library [**13**]. When the polynomial is irreducible, we check whether or not it is in Mossinghoff's list [**10**]. If it is not irreducible, we examine its irreducible factors, and we also compare them with Mossinghoff's list. Such factors supply the polynomials with high coefficients appearing in our list.

Our two new polynomials of small Mahler measure of degrees 46 and 56 (with their Mahler measures and their coefficients) are given below. Since the polynomials which follow are necessarily reciprocal, we only provide half of the coefficients of each polynomial. Overall, our list is comprised of 51 new polynomials, 49 of which have a degree greater than 174.

- **Degree 46:**
  $\hookrightarrow$**1.286061466752** 1 -2 3 -3 3 -3 3 -2 0 1 -2 3 -5 6 -6 5 -4 3 -1 -1 3 -4 5 -5
- **Degree 56:**
  $\hookrightarrow$**1.2839804468828** 1 -2 2 -2 1 1 -2 3 -3 2 -1 0 1 -1 1 -1 1 -1 1 0 -1 2 -3 3 -2 0 2 -4 5

4. Comments on the results. Another effective method for finding monic, irreducible, non-cyclotomic integer polynomials with degree at most 180 and Mahler measure less than 1.3 is to use [**3**], which gives a list of 48 limit points of small Mahler measures. For example, the smallest known limit point is

$$M(1+x+y(x^2+x+1)+y^2(x^2+x)) = 1.25543\ldots.$$

The one-variable Mahler measures close to these limit points may be obtained by substituting $\pm x^i$ for x and $\pm x^j$ for y (with i and j coprime integers) and by taking their noncyclotomic factors. It therefore needed to be determined whether our polynomials could or could not be constructed by this method. Using the 48 two-variable polynomials corresponding to the 48 limit points, we then tested our 51 polynomials, and it appeared that 49 of them were of this type. In fact, only polynomials

$$p_1 = 1 + x + y(x^2 + x + 1) + (x^2 + x)y^2$$

and

$$p_2 = 1 + y(x^2 + x + 1) + y^2x^2$$

were sufficient for constructing these 49 polynomials. This was no surprise since the two corresponding limit points are the smallest and are less than 1.3. It was even made much clearer for us after additional intensive computing: denoting

$$P_{a,b}(x, y) = x^{\max(a-b,0)}(\varphi_a(x) + \varphi_b(x)y + x^{b-a}\varphi_a(x)y^2) \quad \textbf{[3]}$$

($\varphi_a(x)$ is the polynomial $(x^a - 1)/(x - 1)$), we sought to find missing polynomials in Mossinghoff's list by using each set of coefficients given in **[3]**

$$(P_{2,3}, P_{7,12}, P_{8,15} \ldots),$$

and this method provided a large number of polynomials, as may be seen at `http://iecl.univ-lorraine.fr/~Jean-Marc.Sac-Epee/SMM3.txt`. As the results in this file indicate, only $P_{2,3}$ and $P_{1,3}$ (or $P_{2,1}$) provided valuable polynomials, contrary to the other sets of coefficients $(P_{3,5}, P_{7,12}, P_{8,15} \ldots)$.

It is also interesting to note that constructing polynomials from limit points seems to supply only those of high degree (at least 174). Therefore, our genetic method has the benefit of providing small degree polynomials (degrees 46 and 56) which cannot be constructed from the known limit points.

5. Final remarks. In order to use genetic algorithms to handle our problem of minimization was effective since this method works even if the functions to be minimized are very complex with, furthermore, in our case, the constraint linked to integer variables. However, the genetic algorithms have the disadvantage of being rather slow, and it would be interesting to explore the possibilities of developing hybrid algorithms to accelerate this minimization method.

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