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# COMPACTNESS OF MULTIPLICATION, COMPOSITION AND WEIGHTED COMPOSITION OPERATORS BETWEEN SOME CLASSICAL SEQUENCE SPACES: A NEW APPROACH

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ABSTRACT. In [**11**], the author studied the compactness of multiplication, composition and weighted composition operators among some sequence spaces. We were motivated by these results and present two different approaches for obtaining some of the results in [**11**]. The first approach is to apply the theory of matrix transformations and the Hausdorff measure of noncompactness, and the second one is to use known results on multiplier spaces and the Hausdorff measure of noncompactness. We also use our techniques and methods from our proofs of the existing results to establish some new results related to the class of Fredholm operators and some classes of operators considered here and in [**11**].

**1. Introduction and notation.** As is standard, let $\omega$ be the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. By $\ell_\infty$, $c$, $c_0$ and $\phi$, we denote the sets of all bounded, convergent, null and finite sequences, respectively. We write

$$\ell_p = \left\{ x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty \right\} \quad \text{for } 1 \leq p < \infty,$$

and $e$ and $e^{(n)}$, $n = 0, 1, \ldots$, for the sequences with $e_k = 1$ for all $k$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$, respectively.

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers, $X$ and $Y$ subsets of $\omega$ and $x \in \omega$. We write $A_n = (a_{nk})_{k=0}^{\infty}$ for the sequence

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in the $n$th row of $A$,

$$A_n x = \sum_{k=0}^{\infty} a_{nk} x_k,$$

$Ax = (A_n x)_{n=0}^{\infty}$ (provided all of the series $A_n x$ converge), and $X_A = \{x \in \omega : Ax \in X\}$ for the *matrix domain* of $A$ in $X$. In addition, $(X, Y)$ is the class of all matrices $A$ such that $X \subset Y_A$; we write $(X) = (X, X)$, for short.

A Banach space $X \subset \omega$ is a $BK$ *space* if each projection $x \mapsto x_n$ on the $n$th coordinate is continuous. A $BK$ space $X \supset \phi$ is said to have $AK$ if

$$x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)} \longrightarrow x, \quad m \to \infty,$$

for every sequence $x = (x_k)_{k=0}^{\infty} \in X$.

If $X$ and $Y$ are Banach spaces, then we write, as usual, $\mathcal{B}(X, Y)$ for the set of all bounded linear operators $L : X \to Y$ with the operator norm $\| \cdot \|$ defined by $\|L\| = \sup_{\|x\|=1}\{\|L(x)\|\}$; we write $\mathcal{B}(X) = \mathcal{B}(X, X)$, for short.

The following result is very important for our research.

**Lemma 1.1.** *Let $X$ and $Y$ be $BK$ spaces.*

(a) *Then, we have $(X, Y) \subset \mathcal{B}(X, Y)$, that is, every $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$ where $L_A(x) = Ax$ for all $x \in X$* [**6,** Theorem 1.23], [**12,** Theorem 4.2.8].

(b) *If $X$ has $AK$, then we have $\mathcal{B}(X, Y) \subset (X, Y)$, that is, every $L \in \mathcal{B}(X, Y)$ is represented by a matrix $A \in (X, Y)$ such that $Ax = L(x)$ for all $x \in X$* [**4,** Theorem 1.9].

**Remark 1.2.** It is well known that the sets $\ell_p$, $1 \le p < \infty$, $c_0$, $c$ and $\ell_\infty$ are $BK$ spaces with their natural norms defined by

$$\|x\|_p = \left\{ \sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} \quad \text{for } x \in \ell_p$$

and

$$\|x\|_\infty = \sup_k |x_k| \quad \text{for } x \in c_0, c, \ell_\infty;$$

furthermore, $\ell_p$, $1 \leq p < \infty$, and $c_0$ have $AK$, every sequence $x = (x_k)_{k=0}^{\infty} \in c$ has a unique representation

$$x = \xi \cdot e + \sum_{k=0}^{\infty} (x_k - \xi) e^{(k)} \quad \text{where } \xi = \lim_{k \to \infty} x_k,$$

and $\ell_\infty$ has no Schauder basis. Therefore, from Lemma 1.1, we have

$$\mathcal{B}(X) = (X) \quad \text{for } X = c_0 \text{ or } X = \ell_p (1 \leq p < \infty),$$

and

$$(X) \subset \mathcal{B}(X) \quad \text{for } X = c \text{ or } X = \ell_\infty.$$

As mentioned in the abstract the compactness of operators will be treated by applying the theory of matrix transformations and the Hausdorff measure of noncompactness. Hence, we recall the definition of the Hausdorff measure of noncompactness of operators. Let $X$ be a complete metric space, and let $\mathcal{M}_X$ denote the class of bounded subsets of $X$. Then, the function

$$\chi : \mathcal{M}_X \longrightarrow [0, \infty),$$

defined by

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ can be covered by finitely many open balls of radii} < \varepsilon\}$$

is called the Hausdorff measure of noncompactness; $\chi(Q)$ is called the Hausdorff measure of noncompactness of the set $Q \in \mathcal{M}_X$.

Let $\chi_1$ and $\chi_2$ be Hausdorff measures of noncompactness on the Banach spaces $X$ and $Y$. An operator $L : X \to Y$ is said to be $(\chi_1, \chi_2)$-bounded if $L(Q) \in \mathcal{M}_Y$ for all $Q \in \mathcal{M}_X$, and there exists a nonnegative real number $c$ such that

$$\chi_2(L(Q)) \leq c \cdot \chi_1(Q) \quad \text{for all } Q \in \mathcal{M}_X. \tag{1.1}$$

If an operator $L$ is $(\chi_1, \chi_2)$-bounded, then the number

$$\|L\|_\chi = \inf\{c \geq 0 : (1.1) \text{ holds}\}$$

is called the Hausdorff measure of noncompactness of $L$.

For more properties and results of the Hausdorff measure of noncompactness, the reader is referred to **[3, 6, 8]**. Here, we give just a few essential results.

**Theorem 1.3** ([**6,** Theorem 2.25]). *Let $X$ and $Y$ be Banach spaces, $A \in (X, Y)$, and $S_X = \{x \in X : \|x\| = 1\}$ and $\overline{B}_X = \{x \in X : \|x\| \leq 1\}$ denote the unit sphere and closed unit ball in $X$. Then, the Hausdorff measure of noncompactness of the operator $L_A$, denoted by $\|L_A\|_\chi$, is given by*

$$\|L_A\|_\chi = \chi(L_A(\overline{B}_X)) = \chi(L_A(S_X)).$$

It is known that, if $X$ and $Y$ are infinite-dimensional Banach spaces and $L \in \mathcal{B}(X, Y)$, then

$$L \text{ is compact if and only if } \|L\|_\chi = 0, \tag{1.2}$$

[**6,** Corollary 2.26 (2.58)].

**Theorem 1.4** ([**6,** Theorem 2.23]). *Let $X$ be a Banach space with a Schauder basis $(b_k)_{k=0}^\infty$, $Q \in \mathcal{M}_X$, $\mathcal{P}_n : X \to X$ the projector onto the linear span of $\{b_0, b_1, \ldots, b_n\}$ and $\mathcal{R}_n = I - \mathcal{P}_n$, where $I$ is the identity on $X$. Then, we have*

$$\frac{1}{a} \limsup_{n\to\infty} \left( \sup_{x\in Q} \|\mathcal{R}_n(x)\| \right) \leq \chi(Q) \leq \limsup_{n\to\infty} \left( \sup_{x\in Q} \|\mathcal{R}_n(x)\| \right),$$

*where $a = \limsup_{n\to\infty} \|\mathcal{R}_n\|$.*

**Theorem 1.5** ([**6,** Theorem 2.15]). *Let $Q \in \mathcal{M}_X$, where $X = \ell_p$ $(1 \leq p < \infty)$ or $X = c_0$. If $\mathcal{R}_n : X \to X$ is the operator defined by $\mathcal{R}_n(x) = (0, 0, \ldots, x_{n+1}, x_{n+2}, \ldots)$ for $x = (x_k)_{k=0}^\infty \in X$, then we have*

$$\chi(Q) = \lim_{n\to\infty} \left( \sup_{x\in Q} \|\mathcal{R}_n(x)\| \right).$$

In **[11]**, the authors considered special kinds of operators: multiplication, composition and weighted composition operators. Here, we note their definitions.

Let $\lambda = (\lambda_n)_{n=0}^\infty \in \omega$ be given.

The operator

$$T : \omega \longrightarrow \omega,$$

defined by

$$T(x) = \lambda \cdot x = (\lambda_n x_n)_{n=0}^{\infty},$$

is called a *multiplication operator*. If

$$\sigma : \mathbf{N}_0 \longrightarrow \mathbf{N}_0$$

is a permutation map, then the operator $T : \omega \to \omega$, defined by

$$T(x) = (x_{\sigma(n)})_{n=0}^{\infty},$$

is called a *composition operator*.

A *weighted composition operator* is the composition of a multiplication and a composition operator, that is, the operator $T : \omega \to \omega$, defined by

$$T(x) = (\lambda_n x_{\sigma(n)})_{n=0}^{\infty}.$$

## 2. Main results: Compactness of some classes of operators I.

In this section, we consider the above-mentioned classes of operators on the classical sequence spaces and establish necessary and sufficient conditions for the operators to be compact. For that purpose, we will use characterizations of the corresponding classes of matrix transformations and the Hausdorff measure of noncompactness. Although these results already exist **[11]**, we present totally different methods of proofs. We also demonstrate the applicability of our techniques for obtaining new research results concerning Fredholm operators and some of the classes considered here and in **[11]**.

### 2.1. Some remarks on multiplication, composition and weighted composition operators.

Let $\lambda = (\lambda_n)_{n=0}^{\infty} \in \omega$ be given, and let $D(\lambda) = d_{nk}(\lambda)_{n,k=0}^{\infty}$ be the diagonal matrix with the sequence $\lambda$ on its diagonal. It is clear that the matrix $D(\lambda)$ defines the multiplication operator $T$ with $T(x) = D(\lambda)x$ for all $x \in \omega$.

Similarly, if the entries of the infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$ are defined by

$$a_{nk} = \begin{cases} 1 & \text{if } k = \sigma(n) \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

$n = 0, 1, \ldots$, then the associated operator $T$ with $T(x) = Ax$ for all $x \in \omega$ is a composition operator.

Finally, the composition of a multiplication and a composition operator gives a weighted composition operator. Thus, if the entries of the infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$ are defined by

$$a_{nk} = \begin{cases} \lambda_n & \text{if } k = \sigma(n) \\ 0 & \text{otherwise}, \end{cases} \tag{2.2}$$

$n = 0, 1, \ldots$, then the associated weighted composition operator $T$ is defined by $T(x) = Ax$ for all $x \in \omega$.

Now, we treat our problems by applying the theory of matrix transformations and sequence spaces.

**2.2. The space $c_0$.** From Remark 1.2, the characterization of the class $\mathcal{B}(c_0)$ is equivalent to the characterization of the class $(c_0)$, and it is known [**12,** 8.4.5A] that $A \in (c_0)$ if and only if

$$\|A\|_{(c_0)} = \sup_n \|A_n\|_1 = \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty \tag{2.3}$$

and

$$Ae^{(k)} \in c_0 \quad \text{for each } k. \tag{2.4}$$

It is also known [**6**] that

$$\|L\| = \|A\|_{(c_0)} \quad \text{for all } L \in \mathcal{B}(c_0), \tag{2.5}$$

where $A$ is the matrix that represents $L$.

In addition, we note that, from the famous Silverman-Toeplitz theorem [**12,** Theorem 1.3.8], $A \in (c)$ if and only if (2.3) holds and the conditions

$$Ae^{(k)} \in c \quad \text{for each } k \tag{2.6}$$

and

$$Ae \in c, \tag{2.7}$$

are satisfied; in addition, $\|L_A\| = \|A\|_{(c_0)}$ for all $A \in (c)$.

Finally, from [**1,** Corollary 5], if $L \in \mathcal{B}(c_0)$, then

$$\|L\|_\chi = \limsup_{n\to\infty} \|A_n\|_1 = \limsup_{n\to\infty} \sum_{k=0}^{\infty} |a_{nk}|, \tag{2.8}$$

and $L$ is compact if and only if

$$\lim_{n\to\infty} \|A_n\|_1 = \lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{nk}| = 0. \tag{2.9}$$

**Remark 2.1.** From (2.9) and [**10,** (21.1)], an operator $L \in \mathcal{B}(c_0)$ is compact if and only if $A \in (\ell_\infty, c_0)$ for the matrix $A$ which represents $L$.

**Theorem 2.2.** *Let $T$ be a multiplication operator given by the sequence $\lambda$. Then, we have*:

(i) $T \in \mathcal{B}(c_0)$ *if and only if* $\lambda \in \ell_\infty$; *in addition,* $\|T\| = \|\lambda\|_\infty$ [**11,** Lemma 3.2].

(ii) *If* $T \in \mathcal{B}(c_0)$, *then*

$$\|T\|_\chi = \limsup_{n\to\infty} |\lambda_n|, \tag{2.10}$$

*and $T$ is compact if and only if $\lambda \in c_0$.*

*Proof.*

(i) As mentioned above, $T \in \mathcal{B}(c_0)$ if and only if $D(\lambda) \in (c_0)$, which is the case if and only if $D(\lambda)$ satisfies the conditions in (2.3) and (2.4). The condition in (2.3) is equivalent to

$$\|D(\lambda)\|_{(c_0)} = \sup_n \|D_n(\lambda)\|_1 = \sup_n |\lambda_n| = \|\lambda\|_\infty < \infty,$$

that is, $\lambda \in \ell_\infty$. Furthermore, since $D(\lambda)e^{(k)} = \lambda_k e^{(k)}$ for each $k$, the condition in (2.4) is automatically satisfied. We also have $\|T\| = \|D(\lambda)\|_{(c_0)} = \|\lambda\|_\infty$ by (2.5).

(ii) If $T \in \mathcal{B}(c_0)$, then (2.10) is an immediate consequence of (2.8), and from (2.9), $T$ is compact if and only if $\lim_{n\to\infty} |\lambda_n| = 0$, that is, $\lambda \in c_0$. $\square$

**Remark 2.3.** Theorem 2.2 (ii) establishes the equivalence on conditions (1) and (4) in [**11,** Theorem 3.4].

**Theorem 2.4.**

(i) ([**11,** Lemma 3.5]). *An operator $T$ is a composition operator in $\mathcal{B}(c_0)$ if and only if $\sigma$ is proper*; *in this case,* $\|T\| = 1$.

(ii) ([**11,** Proposition 3.6]). *A composition operator in $\mathcal{B}(c_0)$ is never compact.*

*Proof.*

(i) If $T$ is a composition operator, then, from (2.1), $T$ is given by the matrix $A$ with the rows $A_n = e^{(\sigma(n))}$ for $n = 0, 1, \dots$, and thus, $\|A_n\|_1 = 1$ for all $n$ and $\|A\|_{(c_0)} = 1$, that is, $A$ satisfies the condition in (2.3). Hence, we have $T \in \mathcal{B}(c_0)$ if and only if (2.4) is satisfied. Thus, we must show that (2.4) is satisfied if and only if $\sigma$ is proper.

First, we assume that $\sigma$ is proper. Let $k \in \mathbf{N}_0$ be given. Since

$$\sigma(n) \longrightarrow \infty, \quad n \to \infty,$$

there exists an $n_0 \in \mathbf{N}_0$ such that $\sigma(n) > k$ for all $n \geq n_0$; hence, $a_{nk} = 0$ for all $n \geq n_0$, that is, (2.4) is satisfied.

Conversely, we assume that $\sigma$ is not proper. Then, there exists a subsequence $(\sigma(n(l)))_{l=0}^{\infty}$ of the sequence $(\sigma_n)_{n=0}^{\infty}$ of nonnegative integers such that $\sigma(n(l)) = m$, $l = 0, 1, 2, \dots$, for some $m \in \mathbf{N}_0$. Hence, it follows that $a_{n(l),m} = 1$ for all $l$, and thus, (2.4) is not satisfied.

Finally, if $T \in \mathcal{B}(c_0)$, then $\|T\| = \|A\|_{(c_0)} = 1$.

(ii) Applying (2.8), we have

$$\|T\|_\chi = \limsup_{n\to\infty} \|A_n\|_1 = 1 \neq 0,$$

and thus, the composition operator $T$ cannot be compact by (2.9). □

**Theorem 2.5.**

(i) ([**11,** Lemma 3.7]). *An operator $T$ is a weighted composition operator in $\mathcal{B}(c_0)$ if and only if $\sigma$ is proper and $\lambda \in \ell_\infty$*; *in this case,* $\|T\| = \|\lambda\|_\infty$.

(ii) ([**11,** Theorem 3.8]). *A weighted composition operator in $\mathcal{B}(c_0)$ is compact if and only if $\lambda \in c_0$.*

*Proof.*

(i) If $T$ is a weighted composition operator, from (2.2), $T$ is given by the matrix $A$ with the rows $A_n = \lambda_n e^{(\sigma(n))}$, $n = 0, 1, \ldots$. Since $\|A_n\|_1 = |\lambda_n|$ for all $n$, we have $T \in \mathcal{B}(c_0)$ if and only if the conditions in (2.3) and (2.4) are satisfied; however, (2.3) is equivalent to $\sup_n \|A_n\|_1 = \|\lambda\|_\infty < \infty$, that is, $\lambda \in \ell_\infty$. Also, the condition in (2.4) is satisfied by Theorem 2.4 (i) if and only if $\sigma$ is proper and $\lambda \in \ell_\infty$.

(ii) The proof is identical to the proof of Theorem 2.2 (ii). □

**2.3. The space $c$.** In the case of the $BK$ space $c$, which does not have $AK$, we need the following results concerning the representations of operators $L \in \mathcal{B}(c)$ and an estimate for $\|L\|_\chi$. If $X$ and $Y$ are subsets of $\omega$ and

$$L : X \longrightarrow Y,$$

then we write

$$L_n(x) = (L(x))_n \ (x \in X) \quad \text{for } n = 0, 1, \ldots.$$

**Theorem 2.6** ([**2,** Theorem 3.19])**.** *Every operator $L \in \mathcal{B}(c)$ can be represented by a matrix $B = (b_{nk})_{n=0,k=-1}^\infty$ such that the following conditions hold*:

$$(2.11) \qquad L(x) = \left(b_{n,-1}\xi + \sum_{k=0}^\infty b_{nk}x_k\right)_{n=0}^\infty \quad \textit{where } \xi = \lim_{k\to\infty} x_k,$$
$$b_{nk} = L_n(e^{(k)}) \ (k \geq 0),$$
$$b_{n,-1} = L_n(e) - \sum_{k=0}^\infty L_n(e^{(k)}) \quad \textit{for } n = 0, 1, \ldots,$$

$$(2.12) \qquad \lim_{n\to\infty} b_{nk} = \beta_k \quad \textit{exists for each } k = 0, 1, \ldots,$$

$$(2.13) \qquad \lim_{n\to\infty} \sum_{k=-1}^\infty b_{nk} = \beta$$

*and*

$$\|L\| = \sup_n \sum_{k=-1}^{\infty} |b_{nk}| < \infty. \tag{2.14}$$

**Theorem 2.7** ([**1,** Theorem 1], [**2,** Theorem 3.21]). *Let $L \in \mathcal{B}(c)$. Then, we have, using the notation of Theorem* 2.6*,*

$$\frac{1}{2} \limsup_{n\to\infty} \left( \left| b_{n,-1} - \beta + \sum_{k=0}^{\infty} \beta_k \right| + \sum_{k=0}^{\infty} |b_{nk} - \beta_k| \right) \le \|L\|_\chi \le \limsup_{n\to\infty} \left( \left| b_{n,-1} - \beta + \sum_{k=0}^{\infty} \beta_k \right| + \sum_{k=0}^{\infty} |b_{nk} - \beta_k| \right). \tag{2.15}$$

In [**11**], the authors considered the multiplication operator in $\mathcal{B}(c)$.

**Theorem 2.8.** *Let $T$ be a multiplication operator given by the sequence $\lambda$. Then, we have*

(i) *$T \in \mathcal{B}(c)$ if and only if $\lambda \in c$; in addition, $\|T\| = \|\lambda\|_\infty$* [**11,** Theorem 3.9].

(ii) *If $T \in \mathcal{B}(c)$, then*

$$\lim_{n\to\infty} |\lambda_n| \le \|T\|_\chi \le 2 \cdot \lim_{n\to\infty} |\lambda_n|, \tag{2.16}$$

*and $T$ is compact if and only if $\lambda \in c_0$* [**11,** Theorem 3.10].

*Proof.*

(i) We have $D(\lambda) \in (c)$ if and only if the conditions in (2.3), (2.6) and (2.7) are satisfied. Now, (2.7) is equivalent to $\lambda \in c$. Since $\|D(\lambda)\|_{(c_0)} = \|\lambda\|_\infty$, the condition in (2.3) is redundant. As in the proof of Theorem 2.2 (i), the condition in (2.4) is also redundant.

(ii) Since $T(x) = D(\lambda)x$ for all $x$, we obtain in Theorem 2.6,

$$b_{n,-1} = 0 \quad \text{for all } n, \qquad (b_{nk})_{n,k=0}^{\infty} = D(\lambda),$$
$$\beta_k = 0 \quad \text{for all } k, \qquad \beta = \lim_{n\to\infty} \lambda_n.$$

Hence, (2.15) yields

$$\tfrac{1}{2} \limsup_{n\to\infty} (|\beta| + |\lambda_n|) = \lim_{n\to\infty} |\lambda_n| \le \|T\|_\chi \le 2 \cdot \lim_{n\to\infty} |\lambda_n|,$$

that is, (2.16) holds.

Finally, $T \in \mathcal{B}(c)$ is compact if and only if $\lim_{n\to\infty} |\lambda_n| = 0$, that is, $\lambda \in c_0$. □

Let $R, S, T \in \mathcal{B}(X)$ be multiplication, composition and weighted composition operators, respectively, with $T = R \circ S$. It is known [**6,** Corollary 2.26 (2.61)] that

$$0 \le \|T\|_\chi \le \|R\|_\chi \cdot \|S\|_\chi.$$

It is clear that, if $R$ is compact, then $T$ is a compact operator. Now, it is clear that some results in [**11**] on weighted composition operators follow directly from the results on multiplication operators.

The question arises whether the converse implication is also true; that is, if $X$ is one of the classical sequence spaces $c$ or $c_0$, and $T$ is a compact operator in $\mathcal{B}(X)$, is the operator $R$ a compact operator in $\mathcal{B}(X)$?

Let $B_1$ be the unit ball in $X$. We have $\|R\|_\chi = \chi(R(B_1))$. However, if $\sigma$ is a proper permutation, the ball $B_1$ is the image of some unit ball $B_2$ by $S$, that is, $B_1 = S(B_2)$, and thus, we obtain from Theorem 1.3,

$$\|R\|_\chi = \chi(R(B_1)) = \chi(R(S(B_2))) = \chi(T(B_2)) = \|T\|_\chi.$$

Hence, we conclude that the weighted composition operator $T$ is compact on $c$ (or $c_0$) if and only if the multiplication operator $R$ is compact on $c$ (or $c_0$) and $\sigma$ is a proper permutation. Thus, we have obtained [**11,** Theorem 6.1], but in a different way.

**2.4. The space $\ell_1$.** Since $\ell_1$ is a $BK$ space with $AK$, it follows from Lemma 1.1 that $\mathcal{B}(\ell_1) = (\ell_1)$. The characterization of the class $(\ell_1)$ and the conditions for compactness are known.

**Theorem 2.9** ([**6,** Theorem 2.27]). *We have $L \in \mathcal{B}(\ell_1)$ if and only if*

$$\|A\|_{(\ell_1)} = \sup_k \|A^{(k)}\|_1 = \sup_k \sum_{n=0}^{\infty} |a_{nk}| < \infty, \tag{2.17}$$

*where $A \in (\ell_1)$ is the matrix that represents $L$ and $A^{(k)} = (a_{nk})_{n=0}^{\infty}$ denotes the sequence in the $k$th column of $A$; if $L \in \mathcal{B}(\ell_1)$, then we have*

$$\|L\| = \|A\|_{(\ell_1)}. \tag{2.18}$$

*The operator* $L \in \mathcal{B}(\ell_1)$ *is compact if and only if* [**6,** Theorem 2.29],

$$\lim_{m\to\infty} \sup_k \sum_{n=m}^{\infty} |a_{nk}| = 0.$$

Now, as in the previous cases for $c_0$ and $c$, we consider the multiplication operator in $\mathcal{B}(\ell_1)$ represented by the diagonal matrix $D(\lambda)$ as a special case of an operator in $\mathcal{B}(\ell_1)$. The following results hold.

**Theorem 2.10.** *Let* $T$ *be a multiplication operator given by the sequence* $\lambda$*. Then, we have*

(i) $T \in \mathcal{B}(\ell_1)$ *if and only if* $\lambda \in \ell_\infty$*; in addition,* $\|T\| = \|\lambda\|_\infty$ [**11,** Theorem 4.2].

(ii) *If* $T \in \mathcal{B}(\ell_1)$*, then we have*

$$\|T\|_\chi = \limsup_{n\to\infty} |\lambda_n|, \tag{2.19}$$

*and* $T$ *is compact if and only if* $\lambda \in c_0$ [**11,** Theorem 4.3].

(iii) *The identity operator* $I_{\ell_1}$ *is not compact.*

*Proof.* Since $\ell_1$ is a $BK$ space with $AK$, the multiplication operator $T$ is given by the diagonal matrix $D(\lambda)$.

(i) For $A = D(\lambda)$, the condition in (2.17) reduces to $\sup_k |\lambda_k| < \infty$; hence, $\lambda \in \ell_\infty$ and $\|T\| = \sup_k |\lambda_k| = \|\lambda\|_\infty$.

(ii) We obtain, for $A = D(\lambda)$ from [**6,** Theorem 2.28 (2.68)],

$$\begin{aligned}\|T\|_\chi &= \lim_{m\to\infty} \left( \sup_k \sum_{n=m}^{\infty} |d_{nk}(\lambda)| \right) = \lim_{m\to\infty} \left( \sup_{n\geq m} |\lambda_n| \right) \\ &= \inf_m \left( \sup_{n\geq m} |\lambda_n| \right) = \limsup_{m\to\infty} |\lambda_m|.\end{aligned}$$

From this and (1.2), we obtain that $T \in \mathcal{B}(\ell_1)$ is compact if and only if $\lambda \in c_0$.

(iii) Since the identity operator $I_{\ell_1}$ on $\ell_1$ is given by the sequence $\lambda = e \notin c_0$, it cannot be compact by (ii). □

**2.5. The space $\ell_p$, $1 < p < \infty$.** Again, since $\ell_p$ is a $BK$ space with $AK$, $1 < p < \infty$, it follows from Lemma 1.1 that $\mathcal{B}(\ell_p) = (\ell_p)$. No characterization, however, seems to be known for the classes $(\ell_p)$, $1 < p < \infty$, except for the case $p = 2$. Thus, we cannot proceed as in the previous cases. In view of this, we consider the compactness of the multiplication operator $T \in \mathcal{B}(\ell_p)$, $1 < p < \infty$, directly by applying the Hausdorff measure of noncompactness.

**Theorem 2.11.** *Let $T \in \mathcal{B}(\ell_p)$, $1 < p < \infty$, be a multiplication operator given by the sequence $\lambda$. Then, $\|T\|_\chi$ satisfies* (2.19), *and $T \in \mathcal{B}(\ell_p)$ is compact if and only if $\lambda \in c_0$* [**11,** Theorem 5.3].

*Proof.* Let $T \in (\ell_p)$ be given by the diagonal matrix $D(\lambda)$. Then, it follows that $D(\lambda) \in (\ell_1, \ell_\infty)$, which is the case by [**12,** 8.4.1A] if and only if

$$\sup_{n,k} |d_{nk}(\lambda)| = \sup_k |\lambda_k| < \infty,$$

that is, $\lambda \in \ell_\infty$. Now, let $B$ be the unit ball in $\ell_p$. Applying Theorem 1.5 and the definition of the Hausdorff measure of noncompactness of an operator, we have

$$\begin{aligned} \|T\|_\chi = \chi(T_{D(\lambda)}(B)) &= \lim_{n\to\infty}\Big(\sup_{x\in B}\|\mathcal{R}_n(D(\lambda)x)\|_p\Big) \\ &= \lim_{n\to\infty}\left(\sup_{x\in B}\sum_{k=n+1}^{\infty}|\lambda_k x_k|^p\right)^{1/p}. \end{aligned}$$

We fix $n \in \mathbf{N}_0$. Then, we obtain

$$\sup_{x\in B}\left(\sum_{k=n+1}^{\infty}|\lambda_k x_k|^p\right)^{1/p} \le \sup_{k\ge n+1}|\lambda_k| \cdot \sup_{x\in B}\|x\|_p = \sup_{k\ge n+1}|\lambda_k|.$$

Conversely, we have, for $x = e^{(n+1)} \in B$,

$$\sup_{k\ge n+1}|\lambda_k| = \left(\sum_{k=n+1}^{\infty}|\lambda_k x_k|^p\right)^{1/p} \le \sup_{x\in B}\left(\sum_{k=n+1}^{\infty}|\lambda_k x_k|^p\right)^{1/p}.$$

Thus, we have shown

$$\sup_{x\in B}\left(\sum_{k=n+1}^{\infty}|\lambda_k x_k|^p\right)^{1/p} = \sup_{k\ge n+1}|\lambda_k|;$$

therefore, as in the proof of Theorem 2.10,

$$\|T\|_\chi = \lim_{n\to\infty}\left(\sup_{k\geq n}|\lambda_k|\right) = \limsup_{n\to\infty}|\lambda_n|,$$

that is, (2.19) holds.

Finally, we conclude from (2.19) that $T \in \mathcal{B}(\ell_1)$ is compact if and only if $\lambda \in c_0$. □

The above argument could also be used to show that $D(\lambda) \in (\ell_p)$ if and only if $\lambda \in \ell_\infty$ and $\|D(\lambda)\|_{(\ell_p)} = \|\lambda\|_\infty$.

**3. Main results: Compactness of some classes of operators II.** Here, we present another approach to obtain the results of the previous section by using multiplier spaces.

We need the following well-known result.

**Remark 3.1.**

(a) Let $X$ and $Y$ be subspaces of $\omega$, and let the multiplication operator $T$ be given by the diagonal matrix $D(\lambda)$ with the sequence $\lambda = (\lambda_n)$ on the diagonal. Then, we have $T : X \to Y$ if and only if $\lambda \in \mathcal{M}(X,Y)$, where

$$\mathcal{M}(X,Y) = \{a \in \omega : a \cdot x = (a_k x_k) \in Y \text{ for all } x \in X\}$$

is the *multiplier space* of $X$ and $Y$; we write $\mathcal{M}(X) = \mathcal{M}(X,X)$ for short.

(b) ([**12,** Theorem 4.3.15]). Let $X$ and $Y$ be $BK$ spaces. Then, $\mathcal{M}(X,Y)$ is a $BK$ space with respect to the norm $\|\cdot\|_{\mathcal{M}}$ defined by

$$\|a\|_{\mathcal{M}} = \sup\left\{\frac{\|a\cdot x\|}{\|x\|} : \|x\| = 1\right\} \quad \text{for all } a \in \mathcal{M}(X,Y).$$

Since the topology of a $BK$ space is unique (up to homeomorphisms) [**12,** Corollary 4.2.4], the characterizations of multiplication operators and compact multiplication operators in $\mathcal{B}(c_0)$ and $\mathcal{B}(\ell_p)$, $1 \leq p < \infty$, can easily be obtained by applying Remark 3.1. The evaluation of the operator norms needs an additional argument.

**Proposition 3.2.** *Let $X$ be any of the spaces $c_0$ or $\ell_p$, $1 \le p < \infty$. Then, we have $\mathcal{M}(X) = \ell_\infty$ and*

$$\|\lambda\|_{\mathcal{M}} = \|\lambda\|_\infty \quad \text{for all } \lambda \in \mathcal{M}(X); \tag{3.1}$$

*in addition, $\mathcal{M}(c) = c$, and* (3.1) *holds for all $\lambda \in \mathcal{M}(c)$.*

*Proof.* We have $\mathcal{M}(c_0) = \ell_\infty$ and $\mathcal{M}(c) = c$ from [**5,** Lemma 3.1 (a), (b)]; in addition,

$$\mathcal{M}(\ell_p) \subset \mathcal{M}(\ell_p, \ell_\infty) = \ell_\infty, \quad 1 \le p < \infty,$$

from [**5,** Lemma 3.1 (e)]. Furthermore, if $\lambda \in \ell_\infty$, then it follows that, for all $x \in \ell_p$, $1 \le p < \infty$,

$$\begin{aligned}\|\lambda \cdot x\|_p = \left(\sum_{k=0}^\infty |\lambda_k x_k|^p\right)^{1/p} &\le \sup_k |\lambda_k| \left(\sum_{k=0}^\infty |x_k|^p\right)^{1/p} \\ &= \|x\|_p \cdot \|\lambda\|_\infty < \infty,\end{aligned}$$

that is, $\lambda \in \mathcal{M}(\ell_p)$ and

$$\|\lambda\|_{\mathcal{M}} \le \|\lambda\|_\infty. \tag{3.2}$$

Let $X = c_0$ or $X = c$, and let $\lambda \in \mathcal{M}(X)$ be given. Then, we have, for all $x \in X$,

$$|\lambda_k x_k| \le \|x\|_\infty \cdot \|\lambda\|_\infty \quad \text{for } k = 0, 1, \ldots;$$

hence,

$$\|\lambda\|_{\mathcal{M}} \le \|\lambda\|_\infty. \tag{3.3}$$

In order to prove the converse inequalities of those in (3.2) and (3.3), let $m \in \mathbf{N}_0$ and $\lambda \in \ell_\infty$ be given. We set $x = e^{(m)}$. Then, we have $x \in X$ for $X = c$, $c_0$, $\ell_p$, $1 \le p < \infty$, and

$$|\lambda_m| = |\lambda_m| \cdot \|e^{(m)}\|_X = \|\lambda \cdot x\|_X \le \|\lambda\|_{\mathcal{M}} \cdot \|x\|_X = \|\lambda\|_{\mathcal{M}}.$$

Since $m \in \mathbf{N}_0$ is arbitrary, we have

$$\|\lambda\|_\infty \le \|\lambda\|_{\mathcal{M}}. \tag{3.4}$$

Now, (3.1) follows from (3.2), (3.3) and (3.4). □

Next, we apply Proposition 3.2.

**Theorem 3.3.** *Let* $X = c_0$ *or* $X = \ell_p$, $1 \leq p < \infty$, *and the multiplication operator be given by the sequence* $\lambda$. *Then, we have*:

(i) $T \in \mathcal{B}(X)$ *if and only if* $\lambda \in \ell_\infty$; *in addition,* $\|T\|_{(X)} = \|\lambda\|_\infty$. *Furthermore,* $T \in \mathcal{B}(c)$ *if and only if* $\lambda \in c$. *In addition,* $\|T\|_{(c)} = \|\lambda\|_\infty$.

(ii) *If* $T \in \mathcal{B}(X)$, *then*

$$\|T\|_\chi = \limsup_{n\to\infty} |\lambda_n|, \tag{3.5}$$

*and* $T$ *is compact if and only if* $\lambda \in c_0$.

(iii) *If* $T \in \mathcal{B}(c)$, *then*

$$\lim_{n\to\infty} |\lambda_n| \leq \|T\|_\chi \leq 2 \cdot \lim_{n\to\infty} |\lambda_n|, \tag{3.6}$$

*and* $T$ *is compact if and only if* $\lambda \in c_0$.

*Proof.* We denote the unit ball in $X$ by $B$.

(i) This is an immediate consequence of Proposition 3.2.

(ii) For each $n \in \mathbf{N}_0$, we write $\lambda^{\langle n\rangle} = \lambda - \lambda^{[n]}$. Then, it follows from Theorems 1.3, 1.5 and (3.1)

$$\begin{aligned}\|T\|_\chi = \chi(T(B)) &= \lim_{n\to\infty}\left(\sup_{x\in B}\|\lambda^{\langle n\rangle}\cdot x\|_\infty\right)\\ &= \lim_{n\to\infty}\left(\sup_{k\geq n+1}|\lambda_k|\right) = \limsup_{n\to\infty}|\lambda_n|,\end{aligned}$$

that is, (3.5) holds. Finally, (3.5) and (1.2) imply that $T \in \mathcal{B}(X)$ is compact if and only if $\lim_{n\to\infty}|\lambda_n| = 0$, that is, $\lambda \in c_0$.

(iii) We assume that $T \in \mathcal{B}(c)$. Let $x \in c$ be given and $\xi = \lim_{k\to\infty} x_k$. It follows from part (i) that $\lambda \in c$; hence, $\mu = \lim_{k\to\infty}\lambda_k$ exists, and thus, $\lim_{k\to\infty}\lambda_k x_k = \mu\xi$.

Let $n \in \mathbf{N}_0$ be given. Then, we have $\mathcal{R}_n(\lambda\cdot x) = (\lambda\cdot x - \mu\xi e)^{\langle n\rangle}$, and, for all $k \geq n+1$,

$$|\lambda_k x_k - \mu\xi| = (\mathcal{R}_n(\lambda\cdot x))_k \leq |\lambda_k|\cdot|x_k| + |\mu|\cdot|\xi| \leq (|\lambda_k| + |\mu|)\cdot\|x\|_\infty;$$

hence,

$$\sup_{x\in B}\|\mathcal{R}_n(\lambda\cdot x)\|_\infty \leq \sup_{k\geq n+1}(|\lambda_k| + |\mu|). \tag{3.7}$$

Let $m \geq n+1$ be given. Now, we consider the sequence $x = \operatorname{sgn}(\lambda_m)e^{(m)} - \operatorname{sgn}(\mu)e^{\langle m\rangle}$, where $\operatorname{sgn}(z) = |z|/z$ for $z \in \mathbf{C} \setminus \{0\}$ and $\operatorname{sgn}(0) = 0$. Then, we have $x \in c$, $\lim_{k\to\infty} x_k = -\operatorname{sgn}(\mu)$, $\|x\|_\infty \leq 1$ and

$$|\lambda_m x_m - \mu\xi| = |\lambda_m| + |\mu| \leq \sup_{x\in B} \|\mathcal{R}_n(\lambda \cdot x)\|_\infty.$$

Since $m \geq n+1$ is arbitrary, it follows that

$$\sup_{k\geq n+1} (|\lambda_k| + |\mu|) \leq \sup_{x\in B} \|\mathcal{R}_n(\lambda \cdot x)\|_\infty,$$

and this, together with (3.7), imply

$$\sup_{x\in B} \|\mathcal{R}_n(\lambda \cdot x)\|_\infty = \sup_{k\geq n+1} (|\lambda_k| + |\mu|) \quad \text{for all } n. \tag{3.8}$$

Now, it follows from (3.8), the estimate in Theorem 1.4 and the facts $\lambda \in c$ and $a = 2$ for $X = c$, that

$$\begin{aligned}\frac{1}{2} \cdot \limsup_{n\to\infty} \|\mathcal{R}_n(\lambda \cdot x)\|_\infty &= \frac{1}{2} \cdot \limsup_{n\to\infty} \Big( \sup_{k\geq n+1} (|\lambda_k| + |\mu|)\Big)\\ &= \lim_{n\to\infty} \Big( \sup_{k\geq n+1} (|\lambda_k|)\Big)\\ &= \lim_{n\to\infty} |\lambda_n| \leq \|T\|_\chi \leq 2 \cdot \lim_{n\to\infty} |\lambda_n|.\end{aligned}$$

Thus, we have shown (3.6).

Finally, (3.6) and (1.2) imply that $T \in \mathcal{B}(c)$ is compact if and only if $\lim_{n\to\infty} |\lambda_n| = 0$, that is, $\lambda \in c_0$. □

**4. Further applications of results.** Applying the Hausdorff measure of noncompactness, we find sufficient conditions for multiplication operator $T \in \mathcal{B}(X)$ to be a Fredholm operator when $X$ is one of the sequence spaces $c_0$, $c$ or $\ell_p$, $1 \leq p < \infty$.

First, let us recall the definition of a Fredholm operator. If $X$ and $Y$ are Banach spaces, $T \in \mathcal{B}(X, Y)$ and $N(T)$ and $R(T)$ are the null and range spaces of $T$, respectively, then an operator $T$ is a Fredholm operator if $N(T)$ is finite-dimensional, $R(T)$ is closed in $Y$ and $Y/R(T)$ is finite-dimensional. The set of Fredholm operators from $X$ to $Y$ is denoted by $\Phi(X, Y)$.

The next result is useful for our purposes [**9,** page 106]: if $X = Y$ and $T$ is compact, then $I - T$ is a Fredholm operator where $I$ is the identity operator on $X$.

**Theorem 4.1.** *Let the multiplication operator $T$ be given by the sequence $\lambda$. Then, we have*

(i) $T \in \Phi(X)$ *if*

$$\limsup_{n\to\infty} |1 - \lambda_n| = 0, \tag{4.1}$$

*where $X$ is one of the spaces $c_0$ or $\ell_p$, $1 \le p < \infty$.*

(ii) $T \in \Phi(c)$ *if*

$$\lim_{n\to\infty} |1 - \lambda_n| = 0. \tag{4.2}$$

*Proof.* From Section 2, we know that the multiplication operator $T$ is associated with the diagonal matrix $D(\lambda)$. Similarly, the operator $I - T$ is the multiplication operator on $X$ defined by the infinite matrix $I - D(\lambda)$. Now, it is clear that $I - T$ is compact on $X$ if and only if $\limsup_{n\to\infty} |1 - \lambda_n| = 0$ when $X$ is one of the spaces $c_0$ or $\ell_p$, $1 \le p < \infty$, and $I - T$ is compact on $c$ if and only if $\lim_{n\to\infty} |1 - \lambda_n| = 0$. This means that $T$ is a Fredholm operator, that is, these conditions are sufficient for the multiplication operator $T$ to be Fredholm on $X$. $\square$

The question naturally arises as to what the conditions should be for the composition operator $T$ on $c_0$ to be Fredholm.

If $T \in \mathcal{B}(c_0)$ is a composition operator, the appropriate associated infinite matrix $A$ is given in (2.1). When we want to find conditions for $T$ to be Fredholm, we must consider at the same time the operator $I - T$. This operator is also defined by the infinite matrix $B = (b_{nk})_{n,k=0}^{\infty}$, with

$$b_{nk} = \begin{cases} 0 & \text{if } n = k = \sigma(n) \\ 1 & \text{if } n = k \neq \sigma(n) \\ -1 & \text{if } n \neq k \text{ and } k = \sigma(n) \\ 0 & \text{if } n \neq k \text{ and } k \neq \sigma(n), \end{cases} \tag{4.3}$$

$n = 0, 1, \ldots$. From the results of the previous sections, we obtain

$$\|I - T\|_\chi = \lim_{n\to\infty} \sum_k |b_{nk}|.$$

It would also be interesting to study when the operator $I - T$ is compact, or, whether $\lim_{n\to\infty} \sum_k |b_{nk}|$ can be equal to zero.

We have, for each $n$,

$$\sum_k |b_{nk}| = |b_{nn}| + \sum_{k\neq n} |b_{nk}|.$$

If $n = \sigma(n)$, then

$$\sum_k |b_{nk}| = 0 + 0 = 0.$$

If $n \neq \sigma(n)$, then

$$\sum_k |b_{nk}| = 1 + \sum_{k\neq n} |b_{nk}| \neq 0.$$

Hence, we conclude that $\|I - T\|_\chi = 0$ if and only if $n = \sigma(n)$, which implies that $T$ is the identity operator.

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