ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 47, Number 7, 2017

THE IDEAL OF UNCONDITIONALLY p-COMPACT OPERATORS

JU MYUNG KIM

ABSTRACT. We investigate the ideal $\mathcal{K}_{\rm up}$, $1 \le p \le \infty$, of unconditionally p-compact operators. We obtain the isometric identities $\mathcal{K}_{\rm up} = \mathcal{K}_{\rm up} \circ \mathcal{K}_{\rm up}$, $\mathcal{K}_{\rm up}^{\max} = \mathcal{L}_{p^*}^{\rm sur}$, $\mathcal{K}_{\rm up}^{\min} = \widehat{\otimes}_{/w_{p^*}}$ and $\mathcal{K}_{\rm up} = \mathcal{N}_{\rm up}^{\rm Qdual}$ and prove that, if X^* has the approximation property or Y has the $\mathcal{K}_{\rm up}$-approximation property, then $\mathcal{K}_{\rm up}(X,Y)$ is isometrically equal to $\mathcal{K}_{\rm up}^{\min}(X,Y)$, and the dual space $\mathcal{K}_{\rm up}(X,Y)^*$ is isometric to $(\mathcal{L}_p^{\rm inj})^*(X^*,Y^*)$. As a consequence, for every Banach space X, we obtain the isometric identities $\mathcal{K}_{\rm up}^{\max}(\ell_1(\Gamma),X) = \mathcal{L}_{p^*}(\ell_1(\Gamma),X)$, $\mathcal{K}_{\rm up}^{\min}(\ell_1(\Gamma),X) = \ell_\infty(\Gamma)\widehat{\otimes}_{w_{p^*}}X$ and $\mathcal{K}_{\rm up}(\ell_1(\Gamma),X)^* = \mathcal{D}_{p^*}(\ell_\infty(\Gamma),X^*)$.

1. Introduction. The main notion of the paper stems from the criterion of compactness. Grothendieck **[7]** proved that a subset K of a Banach space X is relatively compact if and only if, for every $\varepsilon > 0$, there exists a null sequence (x_n) in X such that

$$K \subset \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n) \in B_{\ell_1} \right\}$$

and $\sup_n \|x_n\| \le \sup_{x\in K} \|x\| + \varepsilon$, where B_{ℓ_1} denotes the closed unit ball of ℓ_1 and, in general, B_Z denotes the closed unit ball of a Banach space Z. From this result, the operator norm of a compact operator

$$T : Y \longrightarrow X$$

2010 AMS *Mathematics subject classification.* Primary 46B28, 46B45, 46B50, 47L20.

Keywords and phrases. Unconditionally p-summable sequence, unconditionally p-compact operator, Banach operator ideal, tensor norm, approximation property.

The author is supported by the Korean government, grant No. NRF-2013R1A1A2A10058087.

Received by the editors on May 24, 2015, and in revised form on May 27, 2016.

DOI:10.1216/RMJ-2017-47-7-2277

can be determined via null sequences as follows:

$$(\dagger) \quad \|T\| = \inf\left\{ \sup_n \|x_n\| : \|x_n\| \longrightarrow 0, \right. \qquad \left. T(B_Y) \subset \left\{ \sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_1} \right\} \right\}.$$

That criterion of compactness was naturally extended by Sinha and Karn [**15**] as follows. For $1 \le p < \infty$ and a subset K of X, K is said to be relatively *p-compact* if there exists an $(x_n) \in \ell_p(X)$ such that

$$K \subset p\text{-co}(\{x_n\}) := \left\{ \sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_{p^*}} \right\},$$

where $1/p + 1/p^* = 1$, and $\ell_p(X)$ is the Banach space with the norm $\|\cdot\|_p$ of all X-valued absolutely p-summable sequences. A linear map

$$T : Y \longrightarrow X$$

is said to be *p-compact* if $T(B_Y)$ is a relatively p-compact subset of X. The collection of all p-compact operators from Y to X is denoted by $\mathcal{K}_p(Y, X)$. In view of ($\dagger$), the same method may be used for measuring p-compact operators. Similarly, Delgado, Piñeiro and Serrano [**4, 5**] introduced an operator ideal norm on $\mathcal{K}_p$. The norm $\|\cdot\|_{\mathcal{K}_p}$, $1 \le p < \infty$, on the space $\mathcal{K}_p(Y, X)$ is defined by

$$\|T\|_{\mathcal{K}_p} = \inf\{\|(x_n)\|_p : (x_n) \in \ell_p(X), T(B_Y) \subset p\text{-co}(\{x_n\})\}.$$

Then $[\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}]$ is a Banach operator ideal [**5**].

For $1 \le p \le \infty$, the closed subspace $\ell_p^{\mathrm{u}}(X)$ of $\ell_p^w(X)$, the Banach space with the norm $\|\cdot\|_p^w$ of all X-valued weakly p-summable sequences, consists of sequences (x_n) satisfying

$$\|(0, \ldots, 0, x_m, x_{m+1}, \ldots)\|_p^w \longrightarrow 0,$$

as $m \to \infty$. Elements in $\ell_p^u(X)$ are called *unconditionally p-summable sequences* [**8**]. We say that a subset K of X is relatively *unconditionally p-compact* (u-p-compact) if there exists an $(x_n) \in \ell_p^u(X)$ such that $K \subset p\text{-co}(\{x_n\})$. Also, a linear map $T : Y \to X$ is said to be *u-p-compact* if $T(B_Y)$ is a relatively u-p-compact subset of X. The collection of all u-p-compact operators from Y to X is denoted by

$\mathcal{K}_{\mathrm{up}}(Y,X)$, and the norm $\|\cdot\|_{\mathcal{K}_{\mathrm{up}}}$ on $\mathcal{K}_{\mathrm{up}}(Y,X)$ is defined by

$$\|T\|_{\mathcal{K}_{\mathrm{up}}} = \inf\{\|(x_n)\|_p^w : (x_n) \in \ell_p^u(X) \text{ and } T(B_Y) \subset p\text{-co}(\{x_n\})\}.$$

Then, the ideal $[\mathcal{K}, \|\cdot\|]$ of compact operators is isometrically equal to $[\mathcal{K}_{\mathrm{u}\infty}, \|\cdot\|_{\mathcal{K}_{\mathrm{u}\infty}}]$, and $[\mathcal{K}_{\mathrm{up}}, \|\cdot\|_{\mathcal{K}_{\mathrm{up}}}]$, $1 \le p < \infty$, is a Banach operator ideal [**8,** Theorem 2.1].

The main purpose of this paper is to establish some relationships among the ideals $[\mathcal{K}_{\mathrm{up}}, \|\cdot\|_{\mathcal{K}_{\mathrm{up}}}]$, some well-known operator ideals and tensor norms based on the investigation of the ideal $[\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}]$ of Galicer, Lassalle and Turco [**6, 11**].

2. A factorization of $\mathcal{K}_{\mathrm{up}}$. The following lemma may be verified from a standard argument.

Lemma 2.1. *Let K be a collection of sequences of positive numbers. If*

$$\sup_{(k_j)\in K} \sum_{j=1}^{\infty} k_j < \infty \quad \textit{and} \quad \lim_{l\to\infty} \sup_{(k_j)\in K} \sum_{j\ge l} k_j = 0,$$

then, for every $\varepsilon > 0$, there exists a sequence (b_j) of real numbers with $b_j \nearrow \infty$ and $b_j > 1$ for all j such that

$$\sup_{(k_j)\in K} \sum_{j=1}^{\infty} k_j b_j \le (1+\varepsilon) \sup_{(k_j)\in K} \sum_{j=1}^{\infty} k_j \quad \textit{and} \quad \lim_{l\to\infty} \sup_{(k_j)\in K} \sum_{j\ge l} k_j b_j = 0.$$

Theorem 2.2. *Let $1 \le p \le \infty$. Then, $T \in \mathcal{K}_{\mathrm{up}}(X,Y)$ if and only if there exist a quotient space Z of ℓ_{p^*} (c_0 if $p = 1$), $R \in \mathcal{K}_{\mathrm{up}}(X,Z)$ and $S \in \mathcal{K}_{\mathrm{up}}(Z,Y)$ such that $T = SR$. In this case, $\|T\|_{\mathcal{K}_{\mathrm{up}}} = \inf \|S\|_{\mathcal{K}_{\mathrm{up}}} \|R\|_{\mathcal{K}_{\mathrm{up}}}$, where the infimum is taken over all such factorizations.*

Proof. The "if" part is clear and, in this case,

$$\|T\|_{\mathcal{K}_{\mathrm{up}}} \le \inf \|S\|_{\mathcal{K}_{\mathrm{up}}} \|R\|_{\mathcal{K}_{\mathrm{up}}}.$$

Let $T \in \mathcal{K}_{\mathrm{up}}(X,Y)$, and let $\varepsilon > 0$ be given. The following proof is essentially due to [**15,** Theorem 3.2], [**2,** Theorem 3.1] and [**6,** Proposition 2.9]. Choose $(y_n) \in \ell_p^u(Y)$ such that $T(B_X) \subset p\text{-co}(\{y_n\})$

and $\|(y_n)\|_p^w \leq \|T\|_{\mathcal{K}_{\rm up}}(1+\varepsilon)$. Define the operators

$$E_y : \ell_{p^*} \longrightarrow Y$$

by

$$E_y\alpha = \sum_n \alpha_n y_n,$$

and

$$\widehat{E_y} : \ell_{p^*}/\ker(E_y) \longrightarrow Y$$

by $\widehat{E_y}[\alpha] = E_y\alpha$. Now, for each $x \in X$, there exists an $\alpha \in \ell_{p^*}$ such that

$$Tx = \sum_n \alpha_n y_n.$$

Define the map

$$T_y : X \longrightarrow \ell_{p^*}/\ker(E_y) \quad \text{by } T_y x = [\alpha].$$

Then, it is easily seen that T_y is well defined, linear and $\|T_y x\| \leq \|x\|$ for every $x \in X$. It follows that $T = \widehat{E_y}T_y$.

Now, by an application of Lemma 2.1, there exists a sequence (β_n) of positive numbers with $\lim_{n\to\infty} \beta_n = 0$ and $\beta_n < 1$ such that $(z_n) := (y_n/\beta_n) \in \ell_p^u(Y)$ and $\|(y_n/\beta_n)\|_p^w \leq \|(y_n)\|_p^w(1+\varepsilon)$. Define the operators

$$D_\beta : \ell_{p^*} \longrightarrow \ell_{p^*} \qquad \text{and} \qquad E_z : \ell_{p^*} \longrightarrow Y$$

by

$$D_\beta\alpha = (\alpha_n\beta_n) \qquad \text{and} \qquad E_z\alpha = \sum_n \alpha_n z_n,$$

respectively, and the map

$$\widehat{D_\beta} : \ell_{p^*}/\ker(E_y) \longrightarrow \ell_{p^*}/\ker(E_z) \quad \text{by } \widehat{D_\beta}([\alpha]) = [(\beta_n\alpha_n)].$$

Then, we see that $\widehat{D_\beta}$ is well defined and linear. Consider

$$[x_1] := [\beta_1 e_1], \ldots, [x_n] := [\beta_n e_n], \ldots \in \ell_{p^*}/\ker(E_z).$$

Then, it is easily verified that $([x_n])_{n=1}^{\infty} \in \ell_p^u(\ell_{p^*}/\ker(E_z))$,$\|([x_n])_n\|_p^w \leq 1$ and

$$\widehat{D_\beta}(B_{\ell_{p^*}/\ker(E_y)}) \subset \left\{ \sum_n \alpha_n [x_n] : (\alpha_n) \in B_{\ell_{p^*}} \right\}.$$

Thus, $\widehat{D_\beta}$ is u-p-compact, and $\|\widehat{D_\beta}\|_{\mathcal{K}_{\rm up}} \leq 1$. Define the map

$$\widehat{E_z} : \ell_{p^*}/\ker(E_z) \longrightarrow Y \quad \text{by } \widehat{E_z}([\alpha]) = E_z \alpha.$$

Recall that $(z_n) \in \ell_p^u(Y)$. Then,

$$\widehat{E_z}(B_{\ell_{p^*}/\ker(E_z)}) \subset \left\{ \sum_n \alpha_n z_n : (\alpha_n) \in B_{\ell_{p^*}} \right\}.$$

Therefore, $\widehat{E_z}$ is u-p-compact and $\|\widehat{E_z}\|_{\mathcal{K}_{\rm up}} \leq \|(y_n)\|_p^w (1+\varepsilon)$. It follows that $\widehat{E_y} = \widehat{E_z}\widehat{D_\beta}$.

Now, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y \\ {\scriptstyle T_y}\downarrow & \overset{\widehat{E_y}}{\nearrow} & \uparrow{\scriptstyle \widehat{E_z}} \\ \ell_{p^*}/\ker(E_y) & \xrightarrow[\widehat{D_\beta}]{} & \ell_{p^*}/\ker(E_z), \end{array}$$

and we have

$$\begin{aligned} \inf \|S\|_{\mathcal{K}_{\rm up}} \|R\|_{\mathcal{K}_{\rm up}} &\leq \|\widehat{E_z}\|_{\mathcal{K}_{\rm up}} \|\widehat{D_\beta} T_y\|_{\mathcal{K}_{\rm up}} \\ &\leq \|(y_n)\|_p^w (1+\varepsilon) \leq \|T\|_{\mathcal{K}_{\rm up}} (1+\varepsilon)^2. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\inf \|S\|_{\mathcal{K}_{\rm up}} \|R\|_{\mathcal{K}_{\rm up}} \leq \|T\|_{\mathcal{K}_{\rm up}}$. □

From the proof of Theorem 2.2, we also obtain a factorization of $\mathcal{K}_p$ via $\mathcal{K}_{\rm up}$.

Theorem 2.3. *Let $1 \leq p \leq \infty$. Then,*

$$T \in \mathcal{K}_p(X, Y)$$

if and only if there exist a quotient space Z of ℓ_{p^*} (c_0 if $p = 1$), $R \in \mathcal{K}_{\rm up}(X, Z)$ and $S \in \mathcal{K}_p(Z, Y)$ such that $T = SR$. In this case,

$\|T\|_{\mathcal{K}_p} = \inf \|S\|_{\mathcal{K}_p}\|R\|_{\mathcal{K}_{\rm up}}$, *where the infimum is taken over all such factorizations.*

Corollary 2.4. *Let* $1 \le p \le \infty$. *Then* $[\mathcal{K}_{\rm up}, \|\cdot\|_{\mathcal{K}_{\rm up}}] = [\mathcal{K}_{\rm up} \circ \mathcal{K}_{\rm up}, \|\cdot\|_{\mathcal{K}_{\rm up}} \circ \|\cdot\|_{\mathcal{K}_{\rm up}}]$ *and* $[\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}] = [\mathcal{K}_p \circ \mathcal{K}_{\rm up}, \|\cdot\|_{\mathcal{K}_p} \circ \|\cdot\|_{\mathcal{K}_{\rm up}}]$.

3. The maximal hull and minimal kernel of $[\mathcal{K}_{\rm up}, \|\cdot\|_{\mathcal{K}_{\rm up}}]$.

Given a Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$, we denote by $[\mathcal{A}^{\rm max}, \|\cdot\|_{\mathcal{A}^{\rm max}}]$, $[\mathcal{A}^{\rm min}, \|\cdot\|_{\mathcal{A}^{\rm min}}]$, $[\mathcal{A}^{\rm sur}, \|\cdot\|_{\mathcal{A}^{\rm sur}}]$, $[\mathcal{A}^{\rm inj}, \|\cdot\|_{\mathcal{A}^{\rm inj}}]$, $[\mathcal{A}^{*}, \|\cdot\|_{\mathcal{A}^{*}}]$ and $[\mathcal{A}^{\rm dual}, \|\cdot\|_{\mathcal{A}^{\rm dual}}]$, the *maximal hull*, *minimal kernel*, *surjective hull*, *injective hull*, *adjoint ideal* and *dual ideal*, respectively. The definitions may be found in **[3, 14]**.

A *classical p-compact* operator $T \in \mathfrak{K}_p(X,Y)$, $1 \le p \le \infty$, from X to Y, is represented as

$$T = \sum_n x_n^* \otimes y_n, \qquad (x_n^*) \in \ell_p^u(X^*), \qquad (y_n) \in \ell_{p^*}^u(Y),$$

and its norm is

$$\|T\|_{\mathfrak{K}_p} := \inf \|(x_n^*)\|_p^w \|(y_n)\|_{p^*}^w,$$

where the infimum is taken over all such representations of T. Then, $[\mathfrak{K}_p, \|\cdot\|_{\mathfrak{K}_p}]$ is a Banach operator ideal, cf., **[3,** subsection 22.3] and **[14,** subsection 18.3].

From **[3,** Proposition 9.8] and **[8,** Lemma 3.2], we have:

Proposition 3.1. *Let* $1 \le p \le \infty$. *Then,* $[\mathcal{K}_{\rm up}, \|\cdot\|_{\mathcal{K}_{\rm up}}] = [\mathfrak{K}_{p^*}^{\rm sur}, \|\cdot\|_{\mathfrak{K}_{p^*}^{\rm sur}}]$.

From Proposition 3.1 and **[3,** Corollary 9.8], we have:

Corollary 3.2. *Let* $1 \le p \le \infty$, *and let* Γ *be a set. Then, for every Banach space* X, $\mathcal{K}_{\rm up}(\ell_1(\Gamma), X)$ *is isometrically equal to* $\mathfrak{K}_{p^*}(\ell_1(\Gamma), X)$.

A Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is said to be *associated* to a tensor norm α if the canonical map

$$(\mathcal{A}(M,N), \|\cdot\|_{\mathcal{A}}) \longrightarrow M^* \otimes_\alpha N$$

is an isometry for all finite-dimensional normed spaces M and N. We denote by $/\alpha$, $\alpha\backslash$, $\backslash\alpha$ and $\alpha/$, the *left-injective associate*, *right-injective associate*, *left-projective associate* and *right-projective associate*, respectively, of α. See [**3,** subsections 20.6, 20.7] for the corresponding definitions.

The following is a crucial tensor norm in this paper. Let $u \in X \otimes Y$. For $1 \leq p \leq \infty$, define

$$w_p(u) = \inf \left\{ \|(x_j)\|_p^w \|(y_j)\|_{p^*}^w : u = \sum_{j=1}^n x_j \otimes y_j,\ n \in \mathbb{N} \right\}.$$

Then, w_p is a finitely generated tensor norm, cf., [**3,** Section 12]. For the definition of *accessibility* of tensor norms, see [**3,** subsection 21.1].

Proposition 3.3. *Let $1 \leq p \leq \infty$. Then, the ideal $[\mathcal{K}_{\mathrm{up}}, \|\cdot\|_{\mathcal{K}_{\mathrm{up}}}]$ is associated to the totally accessible tensor norm $/w_{p^*}$.*

Proof. Since w_{p^*} is accessible, cf., [**3,** Theorem 21.5 (1)], by the symmetric version of [**3,** Proposition 21.1 (2)] $/w_{p^*}$ is totally accessible.

Now, let α be a finitely generated tensor norm associated to $[\mathcal{K}_{\mathrm{up}}, \|\cdot\|_{\mathcal{K}_{\mathrm{up}}}]$. Then by Corollary 3.2, for every $n \in \mathbb{N}$ and every finite-dimensional normed space N, we have the following isometries:

$$\ell_\infty^n \otimes_{w_{p^*}} N \longrightarrow \mathfrak{K}_{p^*}(\ell_1^n, N) \longrightarrow \mathcal{K}_{\mathrm{up}}(\ell_1^n, N) \longrightarrow \ell_\infty^n \otimes_\alpha N.$$

Then, using the proof of [**6,** Theorem 3.3], the proof is complete. $\square$

Corollary 3.4. *Let $1 \leq p \leq \infty$. Then $[\mathcal{K}_{\mathrm{up}}^{\max}, \|\cdot\|_{\mathcal{K}_{\mathrm{up}}^{\max}}]$, $[\mathcal{K}_{\mathrm{up}}, \|\cdot\|_{\mathcal{K}_{\mathrm{up}}}]$ and $[\mathcal{K}_{\mathrm{up}}^{\min}, \|\cdot\|_{\mathcal{K}_{\mathrm{up}}^{\min}}]$ are all totally accessible.*

Proof. By Proposition 3.3, $[\mathcal{K}_{\mathrm{up}}^{\max}, \|\cdot\|_{\mathcal{K}_{\mathrm{up}}^{\max}}]$ is associated to $/w_{p^*}$. Hence, by Proposition 3.3 and [**3,** Proposition 21.3], $[\mathcal{K}_{\mathrm{up}}^{\max}, \|\cdot\|_{\mathcal{K}_{\mathrm{up}}^{\max}}]$ is totally accessible. The other parts follow from [**3,** Exercise 21.2 (b)]. $\square$

We denote the ideal of *p-factorable operators* by $[\mathcal{L}_p, \|\cdot\|_{\mathcal{L}_p}]$, cf., [**3,** Section 18] and [**14,** subsection 19.3].

Theorem 3.5. *Let* $1 \le p \le \infty$. *Then* $[\mathcal{K}_{\rm up}^{\max}, \|\cdot\|_{\mathcal{K}_{\rm up}^{\max}}] = [\mathcal{L}_{p^*}^{\rm sur}, \|\cdot\|_{\mathcal{L}_{p^*}^{\rm sur}}]$ *and* $\mathcal{K}_{\rm up}^{\min}(X,Y)$ *is isometric to* $X^*\widehat{\otimes}_{/w_{p^*}}Y$ *for all Banach spaces* X *and* Y.

Proof. Since $[\mathcal{L}_{p^*}, \|\cdot\|_{\mathcal{L}_{p^*}}]$ is associated to w_{p^*}, see [**3,** subsection 17.12], by [**3,** Theorem 20.11 (2)], $[\mathcal{L}_{p^*}^{\rm sur}, \|\cdot\|_{\mathcal{L}_{p^*}^{\rm sur}}]$ is associated to $/w_{p^*}$. By Proposition 3.3, we obtain the first part since the maximal ideal associated to a finitely generated tensor norm is unique. Due to the fact that $/w_{p^*}$ is totally accessible, the second part follows from [**3,** Corollary 22.2]. □

From [**3,** Corollary 9.8 and the symmetric version of Corollary 20.7] and Theorem 3.5, we have:

Corollary 3.6. *Let* $1 \le p \le \infty$, *and let* Γ *be a set. Then, for every Banach space* X, $\mathcal{K}_{\rm up}^{\max}(\ell_1(\Gamma), X)$ *is isometrically equal to* $\mathcal{L}_{p^*}(\ell_1(\Gamma), X)$ *and* $\mathcal{K}_{\rm up}^{\min}(\ell_1(\Gamma), X)$ *is isometric to* $\ell_\infty(\Gamma)\widehat{\otimes}_{w_{p^*}}X$.

A Banach space X is said to have the *approximation property* (AP) if, for every compact subset K of X and every $\varepsilon > 0$, there exists a finite rank operator S on X such that $\sup_{x\in K}\|Sx-x\| \le \varepsilon$. Grothendieck [**7**] proved that X has the AP if and only if, for every Banach space Y,

$$\mathcal{K}(Y,X) = \overline{\mathcal{F}(Y,X)}^{\|\cdot\|}.$$

Based on this criterion, Oja [**12**] and Lassalle and Turco [**10**] introduced the notion of approximation property related to a Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$, where the norm ideal $\|\cdot\|_{\mathcal{A}}$ is taken into account, namely, given a Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$, a Banach space X is said to have the $\mathcal{A}$-AP if, for every Banach space Y,

$$\mathcal{A}(Y,X) = \overline{\mathcal{F}(Y,X)}^{\|\cdot\|_{\mathcal{A}}}.$$

The $\mathcal{K}_{\rm up}$-AP was investigated in [**9**], and it was shown that, if X has the AP, then, for every $1 \le p \le \infty$, X has the $\mathcal{K}_{\rm up}$-AP.

For a definition and further information on accessibility of Banach operator ideals, see [**3,** subsection 21.2]. The next lemma follows immediately.

Lemma 3.7. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a totally accessible Banach operator ideal. Then, for all Banach spaces Y and X, $\|T\|_{\mathcal{A}} = \|T\|_{\mathcal{A}^{\min}}$ for every $T \in \mathcal{F}(Y, X)$.*

Proposition 3.8. *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a totally accessible Banach operator ideal. Then, for all Banach spaces Y and X, $\mathcal{A}(Y, X) = \overline{\mathcal{F}(Y, X)}^{\|\cdot\|_{\mathcal{A}}}$ if and only if $\mathcal{A}(Y, X)$ is isometrically equal to $\mathcal{A}^{\min}(Y, X)$.*

Proof. Suppose that $\mathcal{A}(Y, X) = \overline{\mathcal{F}(Y, X)}^{\|\cdot\|_{\mathcal{A}}}$, and let $T \in \mathcal{A}(Y, X)$. Then, there exists a sequence (T_n) in $\mathcal{F}(Y, X)$ such that $\lim_{n\to\infty} \|T_n - T\|_{\mathcal{A}} = 0$. Then, by Lemma 3.7, (T_n) is a Cauchy sequence in $(\mathcal{A}^{\min}(Y, X), \|\cdot\|_{\mathcal{A}^{\min}})$. Thus, there exists an $R \in \mathcal{A}^{\min}(Y, X)$ such that $\lim_{n\to\infty} \|T_n - R\|_{\mathcal{A}^{\min}} = 0$. Hence, $T = R \in \mathcal{A}^{\min}(Y, X)$ and

$$\|T\|_{\mathcal{A}^{\min}} = \lim_{n\to\infty} \|T_n\|_{\mathcal{A}^{\min}} = \lim_{n\to\infty} \|T_n\|_{\mathcal{A}} = \|T\|_{\mathcal{A}}.$$

From [**3,** Proposition 22.1 (2)], the converse is always true. □

Consequently, for a totally accessible Banach operator ideal $\mathcal{A}$, X has the $\mathcal{A}$-AP if and only if, for every Banach space Y, $\mathcal{A}(Y, X)$ is isometrically equal to $\mathcal{A}^{\min}(Y, X)$. Hence, from Corollary 3.4 and [**3,** Proposition 22.1 (3)], we have the following result which should be compared with [**11,** Proposition 3.4].

Corollary 3.9. *Let $1 \le p \le \infty$. Then, a Banach space X has the $\mathcal{K}_{\mathrm{up}}$-AP(respectively, $\mathcal{K}_{\mathrm{up}}^{\max}$-AP) if and only if, for every Banach space Y, $\mathcal{K}_{\mathrm{up}}(Y, X)$ (respectively, $\mathcal{K}_{\mathrm{up}}^{\max}(Y, X)$) is isometrically equal to $\mathcal{K}_{\mathrm{up}}^{\min}(Y, X)$.*

4. The dual space of $(\mathcal{K}_{\mathrm{up}}(X, Y), \|\cdot\|_{\mathcal{K}_{\mathrm{up}}})$.

Proposition 4.1. *Let $1 \le p \le \infty$. If X^* has the AP, then, for every Banach space Y, $\mathcal{K}_{\mathrm{up}}(X, Y)$ is isometrically equal to $\mathcal{K}_{\mathrm{up}}^{\min}(X, Y)$.*

Proof. Let Y be a Banach space, and let $T \in \mathcal{K}_{\mathrm{up}}(X, Y)$. Then, by Corollary 2.4, there exist Banach space Z, $R \in \mathcal{K}_{\mathrm{up}}(X, Z)$ and $S \in \mathcal{K}_{\mathrm{up}}(Z, Y)$ such that $T = SR$. It is well known that X^* has the

AP if and only if, for every Banach space Y,

$$\mathcal{K}(X,Y) = \overline{\mathcal{F}(X,Y)}^{\|\cdot\|}.$$

Thus, we see that $T \in (\mathcal{K}_{\rm up} \circ \overline{\mathcal{F}})(X,Y)$. By [**3,** Proposition 25.2 (2)] and Corollary 3.4, $T \in \mathcal{K}_{\rm up}^{\rm min}(X,Y)$. Also, by [**3,** Proposition 22.1 (3) and Corollary 22.5], we have

$$\|T\|_{\mathcal{K}_{\rm up}^{\rm max}} \le \|T\|_{\mathcal{K}_{\rm up}} \le \|T\|_{\mathcal{K}_{\rm up}^{\rm min}} = \|T\|_{(\mathcal{K}_{\rm up}^{\rm max})^{\rm min}} = \|T\|_{\mathcal{K}_{\rm up}^{\rm max}}. \qquad \Box$$

We denote by α^t and α', respectively, the *transposed* and *dual* tensor norm of a tensor norm α, see [**3,** subsection 15.2]. The *adjoint* tensor norm is defined by $\alpha^* := (\alpha^t)' = (\alpha')^t$.

Theorem 4.2. *Let $1 \le p \le \infty$. If X^* has the* AP *or Y has the* $\mathcal{K}_{\rm up}$-AP*, then the dual space $(\mathcal{K}_{\rm up}(X,Y), \|\cdot\|_{\mathcal{K}_{\rm up}})^*$ is isometric to $((\mathcal{L}_p^{\rm inj})^*(X^*,Y^*), \|\cdot\|_{(\mathcal{L}_p^{\rm inj})^*})$. Moreover, we have*:

(a) *if X^* has the* AP*, then, for every $T \in (\mathcal{L}_p^{\rm inj})^*(X^*,Y^*)$ and every $S \in \mathcal{K}_{\rm up}(X,Y)$,*

$$\langle T, S\rangle = {\rm tr}_{X^*}(S^*T).$$

(b) *If Y^* has the* AP*, then, for every $T \in (\mathcal{L}_p^{\rm inj})^*(X^*,Y^*)$ and every $S \in \mathcal{K}_{\rm up}(X,Y)$,*

$$\langle T, S\rangle = {\rm tr}_{Y^*}(TS^*).$$

Proof. By Corollary 3.9 and Proposition 4.1, $\mathcal{K}_{\rm up}(X,Y)$ is isometrically equal to $\mathcal{K}_{\rm up}^{\rm min}(X,Y)$. In view of Theorem 3.5, the canonical map

$$J_{/w_{p^*}} : X^* \widehat{\otimes}_{/w_{p^*}} Y \longrightarrow \mathcal{K}_{\rm up}^{\rm min}(X,Y)$$

in [**3,** Theorem 22.2] is an isometry.

Now, by [**3,** Proposition 20.10] and its symmetric version,

$$(/w_{p^*})' = \backslash w'_{p^*} = \backslash w_p^* = (w_p\backslash)^*.$$

Since $\mathcal{L}_p$ is associated to w_p, by [**3,** Theorem 20.11 (1)] $\mathcal{L}_p^{\rm inj}$ is associated to $w_p\backslash$. Thus, $(\mathcal{L}_p^{\rm inj})^*$ is associated to $(w_p\backslash)^* = (/w_{p^*})'$. Hence, by [**3,** Theorem 17.5], we have the following isometries.

$$((\mathcal{L}_p^{\text{inj}})^*(X^*,Y^*), \|\cdot\|_{(\mathcal{L}_p^{\text{inj}})^*}) \longrightarrow (X^*\widehat{\otimes}_{/w_{p^*}}Y)^*$$
$$\longrightarrow (\mathcal{K}_{\text{up}}(X,Y), \|\cdot\|_{\mathcal{K}_{\text{up}}})^*.$$

(a) Let $T \in (\mathcal{L}_p^{\text{inj}})^*(X^*,Y^*)$. Then, by an application of [**3,** Theorem 17.15] it may be verified that the map

$$\text{id}_{X^*} \otimes T^* : X^* \otimes_{(w_p\backslash)^t} Y^{**} \longrightarrow X^* \otimes_\pi X^{**}$$

is continuous and $(w_p\backslash)^t = /w_{p^*}$, where π is the projective tensor norm. Let

$$\text{id}_{X^*}\widehat{\otimes}T^* : X^*\widehat{\otimes}_{(w_p\backslash)^t}Y^{**} \longrightarrow X^*\widehat{\otimes}_\pi X^{**}$$

be the continuous extension of $\text{id}_{X^*} \otimes T^*$. Let

$$\Phi : (\mathcal{K}_{\text{up}}(X,Y), \|\cdot\|_{\mathcal{K}_{\text{up}}}) \longrightarrow \mathcal{N}(X^*,X^*)$$

be the composition of the following maps, where $[\mathcal{N}, \|\cdot\|_{\mathcal{N}}]$ is the ideal of nuclear operators:

$$\mathcal{K}_{\text{up}}(X,Y) = \mathcal{K}_{\text{up}}^{\min}(X,Y) \overset{J^{-1}_{/w_{p^*}}}{\longrightarrow} X^*\widehat{\otimes}_{/w_{p^*}}Y$$
$$= X^*\widehat{\otimes}_{(w_p\backslash)^t}Y \overset{\text{id}_{X^*}\hat{\otimes} i_Y}{\longrightarrow} X^*\widehat{\otimes}_{(w_p\backslash)^t}Y^{**} \overset{\text{id}_{X^*}\hat{\otimes} T^*}{\longrightarrow}$$
$$X^*\widehat{\otimes}_\pi X^{**} \overset{i_t}{\longrightarrow} X^{**}\widehat{\otimes}_\pi X^* \overset{J_\pi}{\longrightarrow} \mathcal{N}(X^*,X^*).$$

Here, $i_Y : Y \to Y^{**}$ is the canonical isometry and i_t is the transposed map. Since X^* has the AP, it is well known that the trace map

$$\text{tr}_{X^*} : (\mathcal{N}(X^*,X^*), \|\cdot\|_{\mathcal{N}}) \longrightarrow \mathbb{C}$$

is well defined and continuous. It may easily be verified that, for every $R \in \mathcal{F}(X,Y)$,

$$\Phi(R) = R^*T$$

and $\langle T,R\rangle = \text{tr}_{X^*}(R^*T)$, and then it follows that $\langle T,S\rangle = \text{tr}_{X^*}(S^*T)$ for every $S \in \mathcal{K}_{\text{up}}(X,Y)$.

(b) Let $T \in (\mathcal{L}_p^{\text{inj}})^*(X^*,Y^*)$. Then, by [**3,** Theorem 17.15], the map

$$\text{id}_{Y^{**}} \otimes T : Y^{**} \otimes_{w_p\backslash} X^* \longrightarrow Y^{**} \otimes_\pi Y^*$$

is continuous. Let

$$\mathrm{id}_{Y^{**}}\widehat{\otimes}T : Y^{**}\widehat{\otimes}_{w_p\backslash}X^* \longrightarrow Y^{**}\widehat{\otimes}_{\pi}Y^*$$

be the continuous extension of $\mathrm{id}_{Y^{**}} \otimes T$. Let

$$\Phi : (\mathcal{K}_{\mathrm{up}}(X,Y), \|\cdot\|_{\mathcal{K}_{\mathrm{up}}}) \longrightarrow \mathcal{N}(Y^*,Y^*)$$

be the composition of the following maps:

$$\begin{aligned}\mathcal{K}_{\mathrm{up}}(X,Y) = \mathcal{K}_{\mathrm{up}}^{\min}(X,Y) &\overset{J^{-1}_{/w_{p^*}}}{\longrightarrow} X^*\widehat{\otimes}_{/w_{p^*}}Y \overset{i_t}{\longrightarrow} Y\widehat{\otimes}_{w_p\backslash}X^* \overset{i_Y\hat{\otimes}\mathrm{id}_{X^*}}{\longrightarrow} \\ & Y^{**}\widehat{\otimes}_{w_p\backslash}X^* \overset{\mathrm{id}_{Y^{**}}\hat{\otimes}T}{\longrightarrow} Y^{**}\widehat{\otimes}_{\pi}Y^* \overset{J_\pi}{\longrightarrow} \mathcal{N}(Y^*,Y^*).\end{aligned}$$

Since Y^* has the AP, the trace map

$$\mathrm{tr}_{Y^*} : (\mathcal{N}(Y^*,Y^*), \|\cdot\|_{\mathcal{N}}) \longrightarrow \mathbb{C}$$

is well defined and continuous. As in the proof of (a), the proof is complete. □

We denote the ideal of *p-dominated operators* by $\mathcal{D}_p$, cf., [**3,** Section 19] and [**14,** subsection 17.4].

Corollary 4.3. *Let $1 \le p \le \infty$, and let Γ be a set. Then, for every Banach space X, the dual space $\mathcal{K}_{\mathrm{up}}(\ell_1(\Gamma),X)^*$ is isometric to $\mathcal{D}_{p^*}(\ell_\infty(\Gamma),X^*)$ and, for every $T \in \mathcal{D}_{p^*}(\ell_\infty(\Gamma),X^*)$ and every $S \in \mathcal{K}_{\mathrm{up}}(\ell_1(\Gamma),X)$, $\langle T,S\rangle = \mathrm{tr}_{\ell_\infty(\Gamma)}(S^*T)$.*

Proof. Recall that $\ell_\infty(\Gamma)$ has the AP. From the symmetric version of [**3,** Corollary 20.7] and the fact that $w'_{p^*} = w^*_p$, in the proof of Theorem 4.2, we have the following isometries.

$$\begin{aligned}(\mathcal{L}^*_p(\ell_\infty(\Gamma),X^*), \|\cdot\|_{\mathcal{L}^*_p}) &\longrightarrow (\ell_\infty(\Gamma)\widehat{\otimes}_{w_{p^*}}X)^* \\ &\longrightarrow (\mathcal{K}_{\mathrm{up}}(\ell_1(\Gamma),X), \|\cdot\|_{\mathcal{K}_{\mathrm{up}}})^*.\end{aligned}$$

Since $\mathcal{L}^*_p$ is isometrically equal to $\mathcal{D}_{p^*}$, cf., [**3,** subsection 17.12], we complete the proof. □

5. The ideal of $\mathcal{A}$-compact operators. Carl and Stephani [**1**] introduced the notion of compactness determined by operator ideals. Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal. A subset K of a Banach space X is said to be relatively $\mathcal{A}$-*compact* if there exist a Banach space Z, $U \in \mathcal{A}(Z,X)$ and a relatively compact subset C of Z such that $K \subset U(C)$. A linear map

$$R : Y \longrightarrow X$$

is said to be $\mathcal{A}$-*compact* if $R(B_Y)$ is a relatively $\mathcal{A}$-compact subset of X. We denote by $\mathcal{K}_{\mathcal{A}}(Y,X)$ the space of all $\mathcal{A}$-compact operators from Y to X.

Lassalle and Turco [**11**] introduced a method of measuring the size of relatively $\mathcal{A}$-compact sets. For a relatively $\mathcal{A}$-compact subset K of X, let

$$m_{\mathcal{A}}(K;X) := \inf\{\|U\|_{\mathcal{A}} : U \in \mathcal{A}(Z,X), \text{ relatively compact } C \subset B_Z, K \subset U(C)\},$$

and let $\|R\|_{\mathcal{K}_{\mathcal{A}}} := m_{\mathcal{A}}(R(B_Y);X)$ for $R \in \mathcal{K}_{\mathcal{A}}(Y,X)$. Then, $[\mathcal{K}_{\mathcal{A}}, \|\cdot\|_{\mathcal{K}_{\mathcal{A}}}]$ is a Banach operator ideal, see [**11,** Section 2].

The following lemma combines, with standard modifications, [**11,** Remark 1.3] and the proof of [**6,** Proposition 2.9]. We give a proof for the sake of completeness.

Lemma 5.1. *Let* $1 \le p \le \infty$. *If* $(x_n) \in \ell_p^u(X)$, *then, for every* $\varepsilon > 0$, *there exist a relatively compact subset* M *of* $B_{\ell_{p^*}}$ *and* $T \in \mathfrak{K}_{p^*}(\ell_{p^*}, X)$ *with*

$$\|T\|_{\mathfrak{K}_{p^*}} \le \|(x_n)\|_p^w + \varepsilon$$

such that

$$p\text{-co}(\{x_n\}) = T(M).$$

Proof. Let $\varepsilon > 0$ be given. By an application of Lemma 2.1, there exists a sequence (β_n) with $\lim_{n\to\infty} \beta_n = \infty$ and $\beta_n > 1$ such that $(\beta_n x_n) \in \ell_p^u(X)$ and $\|(\beta_n x_n)\|_p^w \le \|(x_n)\|_p^w + \varepsilon$. We see that the set

$$M := \left\{ \left(\frac{\alpha_n}{\beta_n}\right)_{n=1}^{\infty} : (\alpha_n) \in B_{p^*} \right\}$$

is a relatively compact subset of B_{p^*} and

$$T := \sum_{n=1}^{\infty} e_n \otimes \beta_n x_n \in \mathfrak{K}_{p^*}(\ell_{p^*}, X),$$

where (e_n) is the sequence of canonical unit vectors in ℓ_p. This yields

$$p\text{-co}(\{x_n\}) = \left\{ \sum_{n=1}^{\infty} \gamma_n \beta_n x_n : (\gamma_n) \in M \right\} = T(M)$$

and

$$\|T\|_{\mathfrak{K}_{p^*}} \leq \|(e_n)\|_{p^*}^w \|(\beta_n x_n)\|_p^w \leq \|(x_n)\|_p^w + \varepsilon. \qquad \square$$

Proposition 5.2. *Let $K \subset X$, and let $1 \leq p \leq \infty$. The following statements are equivalent.*

(a) *K is relatively u-p-compact.*

(b) *K is relatively $\mathfrak{K}_{p^*}$-compact.*

(c) *K is relatively $\mathcal{K}_{\mathrm{up}}$-compact. In this case,*

$$\begin{aligned} m_{\mathcal{K}_{\mathrm{up}}}(K;X) &= m_{\mathfrak{K}_{p^*}}(K;X) \\ &= \inf\{\|(x_n)\|_p^w : K \subset p\text{-co}(\{x_n\}), (x_n) \in \ell_p^u(X)\}. \end{aligned}$$

Proof.

(a) $\Rightarrow$ (b). Let $(x_n) \in \ell_p^u(X)$ be arbitrary such that $K \subset p$-co$(\{x_n\})$. Let $\varepsilon > 0$ be given. By Lemma 5.1, there exist a relatively compact subset M of $B_{\ell_{p^*}}$ and $T \in \mathfrak{K}_{p^*}(\ell_{p^*}, X)$ with $\|T\|_{\mathfrak{K}_{p^*}} \leq \|(x_n)\|_p^w + \varepsilon$ such that

$$K \subset p\text{-co}(\{x_n\}) = T(M).$$

Thus, K is relatively $\mathfrak{K}_{p^*}$-compact and

$$m_{\mathfrak{K}_{p^*}}(K;X) \leq \|T\|_{\mathfrak{K}_{p^*}} \leq \|(x_n)\|_p^w + \varepsilon.$$

Since $\varepsilon > 0$ and $(x_n) \in \ell_p^u(X)$ are arbitrary, we have

$$m_{\mathfrak{K}_{p^*}}(K;X) \leq \inf\{\|(x_n)\|_p^w : K \subset p\text{-co}(\{x_n\}), (x_n) \in \ell_p^u(X)\}.$$

(b) $\Rightarrow$ (c). Simple verification shows that $\mathfrak{K}_{p^*} \subset \mathcal{K}_{\mathrm{up}}$ and $\|\cdot\|_{\mathcal{K}_{\mathrm{up}}} \leq \|\cdot\|_{\mathfrak{K}_{p^*}}$. Hence, (b) $\Rightarrow$ (c) follows and $m_{\mathcal{K}_{\mathrm{up}}}(K;X) \leq m_{\mathfrak{K}_{p^*}}(K;X)$.

(c) $\Rightarrow$ (a). Let Y be a Banach space, and let $T \in \mathcal{K}_{\mathrm{up}}(Y,X)$ be such that M is a relatively compact subset of B_Y and $K \subset T(M)$. Let $\varepsilon > 0$ be given. Choose $(z_n) \in \ell_p^u(X)$ with $\|(z_n)\|_p^w \leq \|T\|_{\mathcal{K}_{\mathrm{up}}} + \varepsilon$ such that

$$K \subset T(M) \subset T(B_Y) \subset p\text{-co}(\{z_n\}).$$

Then, K is relatively u-p-compact, and we have

$$\begin{aligned}\inf\{\|(x_n)\|_p^w : K \subset p\text{-co}(\{x_n\}), (x_n) \in \ell_p^u(X)\} &\le \|(z_n)\|_p^w \\ &\le \|T\|_{\mathcal{K}_{\text{up}}} + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\inf\{\|(x_n)\|_p^w : K \subset p\text{-co}(\{x_n\}), (x_n) \in \ell_p^u(X)\} \le m_{\mathcal{K}_{\text{up}}}(K; X). \qquad \square$$

Corollary 5.3. *Let $1 \le p \le \infty$. Then, we have*

$$[\mathcal{K}_{\text{up}}, \|\cdot\|_{\mathcal{K}_{\text{up}}}] = [\mathcal{K}_{\mathfrak{K}_{p^*}}, \|\cdot\|_{\mathcal{K}_{\mathfrak{K}_{p^*}}}] = [\mathcal{K}_{\mathcal{K}_{\text{up}}}, \|\cdot\|_{\mathcal{K}_{\mathcal{K}_{\text{up}}}}].$$

Proposition 5.2 and [**11,** Corollary 2.3] improve [**8,** Corollary 2.9] by the following proposition.

Proposition 5.4. *Let $K \subset X$, and let $1 \le p \le \infty$. Then, K is relatively u-p-compact if and only if $i_X(K)$ is relatively u-p-compact in X^{**}. In this case,*

$$\begin{aligned}&\inf\{\|(x_n)\|_p^w : K \subset p\text{-co}(\{x_n\}), (x_n) \in \ell_p^u(X)\} \\ &\qquad = \inf\{\|(x_n^{**})\|_p^w : i_X(K) \subset p\text{-co}(\{x_n^{**}\}), (x_n^{**}) \in \ell_p^u(X^{**})\}.\end{aligned}$$

Also, Proposition 5.2 and [**11,** Corollary 2.4] give the following:

Corollary 5.5. *Let $1 \le p \le \infty$. Then, $T \in \mathcal{K}_{\text{up}}(X, Y)$ if and only if $T^{**} \in \mathcal{K}_{\text{up}}(X^{**}, Y^{**})$. In this case, $\|T\|_{\mathcal{K}_{\text{up}}} = \|T^{**}\|_{\mathcal{K}_{\text{up}}}$.*

For $1 \le p \le \infty$, a linear map

$$T : X \longrightarrow Y$$

is called *quasi unconditionally p-nuclear* (quasi u-p-nuclear) [**8**] if there exists an $(x_n^*) \in \ell_p^u(X^*)$ such that $\|Tx\| \le \|(x_n^*(x))\|_p$ for every $x \in X$. This notion originates from [**13**], namely, when the space $\ell_p^u(X^*)$ is replaced by the space $\ell_p(X^*)$, the linear map T is called the *quasi p-nuclear* operator. We denote by $\mathcal{N}_{\text{up}}^Q(X, Y)$ the collection of all quasi u-p-nuclear operators from X to Y. For $T \in \mathcal{N}_{\text{up}}^Q(X, Y)$, let

$$\|T\|_{\mathcal{N}_{\text{up}}^Q} := \inf \|(x_n^*)\|_p^w,$$

where the infimum is taken over all such inequalities. As in the proof of [**13,** Lemma 4], it may be shown that $[\mathcal{N}_{\rm up}^Q, \|\cdot\|_{\mathcal{N}_{\rm up}^Q}]$ is a Banach operator ideal.

Theorem 5.6. *Let* $1 \le p \le \infty$. *Then,* $T \in \mathcal{K}_{\rm up}(X,Y)$ *if and only if* $T^* \in \mathcal{N}_{\rm up}^Q(Y^*, X^*)$. *In this case,* $\|T\|_{\mathcal{K}_{\rm up}} = \|T^*\|_{\mathcal{N}_{\rm up}^Q}$.

Proof. According to [**8,** Theorem 2.4], we need only to check that $\|T\|_{\mathcal{K}_{\rm up}} \le \|T^*\|_{\mathcal{N}_{\rm up}^Q}$. If $T \in \mathcal{K}_{\rm up}(X,Y)$, then, by Corollary 5.5 and [**8,** Theorem 2.3], we have

$$\|T\|_{\mathcal{K}_{\rm up}} = \|T^{**}\|_{\mathcal{K}_{\rm up}} = \|T^*\|_{\mathcal{N}_{\rm up}^Q}. \qquad \square$$

Acknowledgments. The author would like to express sincere gratitude to the referee for valuable comments.

REFERENCES

1. B. Carl and I. Stephani, *On A-compact operators, generalized entropy numbers and entropy ideals*, Math. Nachr. **199** (1984), 77–95.

2. Y.S. Choi and J.M. Kim, *The dual space of* $(\mathcal{L}(X,Y),\tau_p)$ *and the p-approximation property*, J. Funct. Anal. **259** (2010), 2437–2454.

3. A. Defant and K. Floret, *Tensor norms and operator ideals*, Elsevier, North-Holland, 1993.

4. J.M. Delgado, C. Piñeiro and E. Serrano, *Density of finite rank operators in the Banach space of p-compact operators*, J. Math. Anal. Appl. **370** (2010), 498–505.

5. ______, *Operators whose adjoints are quasi p-nuclear*, Stud. Math. **197** (2010), 291–304.

6. D. Galicer, S. Lassalle and P. Turco, *The ideal of p-compact operators: A tensor product approach*, Stud. Math. **211** (2012), 269–286.

7. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955).

8. J.M. Kim, *Unconditionally p-null sequences and unconditionally p-compact operators*, Stud. Math. **224** (2014), 133–142.

9. ______, *The* $\mathcal{K}_{\rm up}$*-approximation property and its duality*, J. Australian Math. Soc. **98** (2015), 364–374.

10. S. Lassalle and P. Turco, *On p-compact mappings and the p-approximation property*, J. Math. Anal. Appl. **389** (2012), 1204–1221.

11. ______, *The Banach ideal of* $\mathcal{A}$*-compact operators and related approximation properties*, J. Funct. Anal. **265** (2013), 2452–2464.

12. E. Oja, *A remark on the approximation of p-compact operators by finite-rank operators*, J. Math. Anal. Appl. **387** (2012), 949–952.

13. A. Persson and A. Pietsch, *p-nukleare und p-integrale Abbildungen in Banachräumen*, Stud. Math. **33** (1969), 19–62.

14. A. Pietsch, *Operator ideals*, North-Holland, Amsterdam, 1980.

15. D.P. Sinha and A.K. Karn, *Compact operators whose adjoints factor through subspaces of* ℓ_p, Stud. Math. **150** (2002), 17–33.

Sejong University, Department of Mathematics, Seoul, 143-747 Korea
Email address: **kjm21@kaist.ac.kr**