

## ON THREE CONSECUTIVE PRIME-GAPS

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**ABSTRACT.** We prove that the sequence of gaps in the sequence of prime numbers contains infinitely many runs of three terms, with the middle term exceeding both the first and third, provided that there is at least one integer  $m$  exceeding 3, and at least one set  $A$  of  $2^{m-2}$  integers, with infinitely many translations of this set  $n + A$  such that they contain at least  $m$  primes.

**1. Introduction.** Let  $a, b, h, i, j$  and  $k$  denote integers, let  $m$  and  $n$  be positive integers, let  $x$  be a (sufficiently-large) real number, and let  $p$  and  $q$  signify primes. Denote the  $n$ th prime by  $p_n$  and the  $n$ th prime-gap by  $g_n = p_{n+1} - p_n$ . Paul Erdős and P. Turan [3] conjectured that, given  $k$  exceeding 1, there are infinitely many values of  $n$  for which the sequence of prime-gaps  $g_n, g_{n+1}, \dots, g_{n+k-1}$  is strictly increasing. They also asked whether one could find infinitely many instances of consecutive prime-gaps where the gaps were each greater than or less than the preceding gap, for any fixed finite length of gaps, where greater than or less than is specified in advance. As Granville noted in his talk at the 2014 Joint Mathematics Meetings [2], the proof of this result follows from the (independent) recent work of Maynard, Tao and Zhang [9] on primes in intervals of bounded length. In this same talk, he also noted that a very similar proof shows the existence of infinitely many integers  $n$  for which the sequence of  $k$  consecutive prime-gaps is strictly decreasing. Both of these results were established by Banks, Freiberg and Turnage-Butterbaugh [1], along with the theorem that, infinitely often, there are sequences of consecutive prime-gaps where each gap is a divisor of the next, for any pre-specified length. In addition, they have shown the existence of infinitely many sequences of consecutive prime-gaps of any pre-specified length, where each gap is a multiple of the next. However, Granville noted that it is an open

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problem whether there are infinitely many runs of four consecutive primes with the middle prime-gap of this run exceeding both the first and last. This is a generalization of the so-called *decade primes*, such as 11, 13, 17, 19. While the current numerical work on primes in intervals of bounded length is not sufficient for us to establish that there are infinitely many such integers  $n$ , we quantify the required necessary improvement as our main theorem. In order to state it, we require the following definition.

**Definition 1.1.** We call a subset  $A$  of the nonnegative integers *admissible* if, for every prime  $p$ , there is a value of  $n$  such that the polynomial

$$f(n) = \prod_{a \in A} (n + a)$$

is not divisible by  $p$ . This set  $A$  is called *admissible with respect to the congruence  $h \pmod k$*  if, for every prime  $p$ , there is a value of  $n \equiv h \pmod k$  where the polynomial  $f(n)$  is not divisible by  $p$ .

We note that, to verify that a set  $A$  is admissible, it is sufficient to find a translation  $n + A = \{n + a : a \in A\}$  for some fixed  $n$ , so that each element of  $n + A$  is prime, and the smallest element exceeds the cardinality of  $A$ .

In terms of admissible sets, the twin prime conjecture states that, for infinitely many positive integers  $n$ , we have

$$|\{a \in A : n + a \text{ prime}\}| = 2,$$

for the admissible set  $A = \{0, 2\}$ . The Prime  $k$ -tuples conjecture asserts that, if  $A$  is any admissible set, then there are infinitely many values of  $n$  satisfying

$$(1.1) \quad |\{a \in A : n + a \text{ prime}\}| = |A| = k.$$

The Schinzel-Sierpinski hypothesis implies that, for any set  $A$  that is admissible with respect to the congruence  $h \pmod k$ , there are arbitrarily large  $n$  for which (1.1) holds, and the Bateman-Horn conjecture gives an asymptotic expression for the counting function of such positive integers  $n$  in terms of the set  $A$  and the congruence  $h \pmod k$ .

The Selberg sieve has yielded upper bounds for the average gap in the sequence of  $n$  fulfilling (1.1), provided that this set is infinite, see [5, page 175], which the Bateman-Horn conjecture implies are of the right order of magnitude. However, until very recently, there were no lower bounds for this function which even tended to infinity, for at minimum one admissible set  $A$  with at least two elements. In 2014, Zhang [9] showed that there are infinitely-many primes with a gap of at most 70,000,000, and Maynard [6] and Tao [8], in independent work, lowered this bound on the gap to 600. In related work, there are now calculations that reduce this bound to 270 or lower. In order to prove results of this type, Maynard and Tao [6, 8] independently established the following theorem. Their numerical work is comparable but slightly different. For more details on these differences, refer to [2, 6, 7, 8].

**Theorem 1.2.** [6, 8]. *Let  $k \geq 2$ , and let  $m$  be sufficiently large, depending upon  $k$ . In addition, let  $A$  be any admissible set of cardinality  $m$ . Then, there is a positive effectively computable constant  $c$ , depending possibly on  $k$ , so that the number of subsets  $B$  of  $A$  of cardinality  $k$  such that*

$$(1.2) \quad |\{n : n + b \text{ is prime for all } b \in B\}| = \infty$$

*exceeds  $c$  times the number of subsets of  $A$  with cardinality  $k$ .*

There is active collaborative work, see [7, 8], to improve the constant  $c$ . Note that this theorem implies that a positive proportion of  $k$ -tuples satisfy the Prime  $k$ -tuples conjecture [2, 4]. Granville discussed applying these methods to obtain comparable results where the linear terms  $n + b$  in (1.2) are replaced by expressions of the form  $an + b$ . This is equivalent to requiring that the set  $A$  be admissible subject to a congruence  $h \pmod{k}$ . These kinds of sets have also been studied with the Selberg lower-bound sieve, see [5, page 174]. Elsewhere in his paper [6], Maynard applies this method to obtain the result that infinitely often, there are  $m$  or more primes in an interval of length  $cm^3e^{4m}$  for some positive constant  $c$ .

In order to state our main theorem we will need to assume the following property of a positive integer.

**Property 1.3.** *The integer  $m > 3$  is such that, for any set  $A$  which is admissible subject to the congruence  $h \pmod j$  and satisfies  $|A| \geq 2^{m-2}$ , there are arbitrarily large  $n$  with  $|\{a \in A : n + a \text{ prime}\}| \geq m$ .*

**Theorem 1.4.** *Assume that there exists at least one integer  $m$  exceeding 3 with Property 1.3. Then there are infinitely many runs of consecutive primes  $p, q, r, s$  such that  $r - q$  exceeds both  $q - p$  and  $s - r$ . Equivalently, the sequence of gaps of consecutive prime numbers contains infinitely many runs of three gaps for which the second exceeds both the first and the third.*

While the result of Theorem 1.4 depends upon the conjecture that there is at least one value of  $m > 3$  satisfying Property 1.3, we believe that, at some point, this conjecture will be established. However, at this point, we do not even know how to deduce it from the Elliott-Halberstam conjecture, see [2, 4, 6]. Maynard [6] deduced from the Elliott-Halberstam conjecture that a weaker theorem than Property 1.3 holds for all sufficiently large  $m$ , but there, instead of the base 2, he used a substantially larger base. We require the side condition that  $A$  be admissible subject to the congruence  $h \pmod j$ , to be able to guarantee that we have consecutive primes.

**2. Proof of the main theorem.** We assume the hypotheses of Theorem 1.4 and argue by induction on  $m$ . For  $m = 4$ , these conditions imply that, for any admissible set  $A$  of cardinality 4, there are infinitely many  $n$  such that (1.1) holds. For example, if we take  $A = \{11, 13, 17, 19\}$ , then we immediately deduce from (1.1) that there are infinitely many  $n$  such that the sequence  $[n+11, n+13, n+17, n+19]$  gives four consecutive primes, in that  $n + 15$  must be divisible by 3. Now, assume that Theorem 1.4 holds for all values of  $m$  not exceeding some integer  $M \geq 4$ . We begin by recursively constructing a sequence of admissible sets  $A_i$ , where  $|A_i| = 2^i$ , for  $i \geq 2$ .

**Lemma 2.1.** *If we recursively define the sets  $A_i$ , for  $i \geq 2$ , by*

$$A_2 = \{2, 4, 8, 10\};$$

$$(2.1) \quad A_{i+1} = A_i \cup \{(2^{i+1})! + a : a \in A_i\}, \quad \text{if } i > 2,$$

then  $A_i$  is admissible and has cardinality  $2^i$ . The union in (2.1) is disjoint. In addition, all elements of  $A_i$  lie between 2 and  $10 + \sum_{t=3}^i 2^{t!}$ , inclusive.

*Proof.* For  $i = 2$ , we observe that

$$\{a + 9 : a \in A_2\} = \{11, 13, 17, 19\}$$

is a set of four primes exceeding the cardinality of our set; thus, it is admissible and has the asserted cardinality. By inspection, all the elements in  $A_2$  lie in the required range. Now, assume the lemma for all values of  $i$  not exceeding some integer  $I \geq 2$ . By the induction hypothesis, the largest element of  $A_{I+1}$  is at most

$$10 + \sum_{t=3}^I 2^{t!} + 2^{(I+1)!} = 10 + \sum_{t=3}^{I+1} 2^{t!},$$

and this completes the proof of the last statement of Lemma 2.1. By this last statement, the union in (2.1) is disjoint, and we conclude that  $|A_{I+1}| = 2|A_I|$ , from which it follows that  $|A_i| = 2^i$  for all  $i$ . It remains to show that the set  $A_{I+1}$  is admissible. Thus, we need to show that, given a prime  $p$ , there is a positive integer  $n$  such that  $\prod_{a \in A} n + a$  is relatively prime to  $p$ . If  $p$  exceeds  $|A_{I+1}| = 2^{I+1}$ , the result is clear. Since the union in (2.1) is disjoint, the set  $A_I$  is admissible by the induction hypotheses, and we have

$$(2.2) \quad (2^{I+1})! + a \equiv a \pmod p \quad \text{for } p \leq 2^{I+1}.$$

We conclude that there is some  $n$  with  $\prod_{a \in A} (n + a)$  is relatively prime to  $p$ . This leaves the case where  $p$  lies between  $2^I$  and  $2^{I+1}$ . By (2.2), the elements of the set on the right-hand side of (2.1) cover at most  $2^I$  residue classes modulo  $p$ , and this verifies that our set  $A_{I+1}$  is, indeed, admissible. This completes the induction.  $\square$

Next, we show that, if we have any  $i + 2$  elements of the set  $A_i$ , and we arrange them in ascending order, then we can find a run of four of them where the gap between the second and third exceeds the gap between the first and second, as well as the gap between the third and fourth. We may view this as a combinatorial lemma.

**Lemma 2.2.** *Assume that  $m \geq 2$ . In the notation of Lemma 2.1, for any subset  $B$  of  $A_m$  with  $|B| \geq m + 2$ , there are elements  $b_1, b_2, b_3, b_4$  of  $B$  so that*

$$b_1 < b_2 < b_3 < b_4,$$

$$b_3 - b_2 > \max(b_4 - b_3, b_2 - b_1),$$

*and the interval  $[b_1, b_4]$  contains no other element of  $B$  except for these four elements.*

*Proof.* Again, we reason by induction on the integer  $m > 1$ . For  $m = 2$ , the assumption implies that  $B = A_2 = \{2, 4, 8, 10\}$ , which satisfies the conclusion of our lemma. Now, assume that Lemma 2.2 holds for all  $m$  less than some integer  $M \geq 3$ . Consider the decomposition in (2.1) of  $A_M$  into two disjoint sets. If there are at least  $M + 1$  elements in  $A_{M-1}$ , then we can deduce the conclusion of the lemma from the induction hypothesis. Similarly, since the other set is a linear translation of  $A_{M-1}$ , we can apply the same reasoning. Thus, we are left with the case in which there are at least two elements lying in each of these sets. Take the four elements of  $B$  to be the largest two elements in  $A_{M-1}$  together with the smallest two elements in the other set, and label them in ascending order. Then by Lemma 2.1, we have

$$(2.3) \quad b_3 - b_2 \geq 2^{M!} + 2 - \left(10 + \sum_{t=3}^{M-1} 2^{t!}\right),$$

$$(2.4) \quad \max(b_2 - b_1, b_4 - b_3) < 10 + \sum_{t=3}^{M-1} 2^{t!}.$$

The left side of (2.3) exceeds the right side of (2.4) due to the growth rate of the function  $2^t$ .  $\square$

In order to establish Theorem 1.2, we require that we have consecutive primes, and the next lemma allows us to do that.

**Lemma 2.3.** *In the notation of Lemma 2.1, for every integer  $m > 1$ , there is a congruence  $h \pmod{j}$  such that  $A_m$  is admissible with respect to the congruence  $h \pmod{j}$ , and such that, if  $n$  is a positive integer, then the only primes between the minimal and maximal element of  $\{n + a : a \in A_m\}$ , inclusive, are elements of  $\{n + a : a \in A_m\}$ .*

*Proof.* Fix  $m$ , and let the minimal element of  $A_m$  be  $\alpha$  and the maximal element of  $A_m$  be  $\beta$ . Denote the set of integers in  $[\alpha, \beta]$  that are not in  $A_m$  by  $S$ . Then, select the first  $|S|$  primes exceeding  $2 + |A_m|$ , and call them  $p(s)_{s \in S}$ . For each  $s \in S$ , consider the congruence

$$(2.5) \quad n \equiv -s \pmod{p(s)}.$$

By the Chinese remainder theorem, the set of congruences in (2.5) is equivalent to the single congruence

$$(2.6) \quad n \equiv h \pmod{\prod_{s \in S} p(s)}.$$

We claim that the set  $A_m$  is admissible, subject to congruence (2.6). Indeed, if  $p$  is any prime not in  $S$ , then the Chinese remainder theorem and the definition of admissibility still imply that there is some integer  $n$  for which  $p$  does not divide  $\prod_{a \in A_m} (n + a)$ . If  $p \in S$ , then  $n = 1$  will have this property by construction, and this establishes our claim. In addition, we have arranged it so that whenever (2.6) holds, then so does (2.5), and this guarantees that, for any  $n$  satisfying (2.6), the prime elements of the set

$$\{n + a : a \in A_m\}$$

are consecutive. □

We are now ready to establish the main theorem.

*Proof.* Assume that there is an integer  $m > 2$  fulfilling the hypotheses. Let  $n$  be an integer so that the set  $\{n + a : a \in A_m\}$  contains at least  $2^m + 2$  primes. By Lemma 2.2, there is a subset  $B$  of

$$\{a \in A_m : n + a \text{ is prime}\}$$

of cardinality 4, so that the elements  $\{b_i\}_{i=1, \dots, 4}$  satisfy its conclusions. In the above argument, there was no restriction on the value of  $n$  other than at least  $2^m + 2$  primes in the set  $\{n + a : a \in A_m\}$ , and hence, we can apply Lemma 2.3 to satisfy this for some integer  $n$  satisfying (2.6). It then follows from Lemma 2.3 that the primes in

$$\{n + b : b \in B\}$$

are consecutive. □

**3. Final remarks.** The first three admissible sets in our construction are as follows:

$$A_2 = \{2, 4, 8, 10\},$$

$$A_3 = \{2, 4, 8, 10, 40322, 40324, 40328, 40330\},$$

$$A_4 = \{2, 4, 8, 10, 40322, 40324, 40328, 40330, 20922789888002, \\ 20922789888004, 20922789888008, 20922789888010, \\ 20922789928322, 20922789928324, \\ 20922789928328, 20922789928320\}$$

With care, we can find admissible sets with the same cardinality and with much smaller elements that would suffice, if the purpose were more computational in nature. For example, the set

$$A'_2 = \{2, 4, 8, 10, 212, 214, 218, 220\},$$

used in place of the set  $A_2$  that we have used, would still work to give an admissible set of the correct cardinality so that the lemmas and the main theorem would work for that case (and, of course, the bound on the elements in our sets would need to be adjusted accordingly).

The configuration *high, low, high* of consecutive prime-gaps appears to be a more difficult problem. Certainly, if there are infinitely many values of  $n$  such that all of the elements of the set  $\{n + 7, n + 11, n + 13, n + 17\}$  are prime, then we will have infinitely many instances of high, low, high gaps, but construction of additional admissible sets  $A_i$  along the lines of the argument of this paper appears to yield a requirement for significantly more prime elements of the corresponding sets  $\{n + a : a \in A_i\}$  than we require in Theorem 1.2. Indeed, for all other examples of gaps where no theorem is established (other than the *low, high, low* run), and where there are at least four terms, our methods give a linear relationship between the number of prime elements of the set  $\{a \in A : n + a\}$  (for the best case) and the cardinality of the set  $\{a \in A : n + a\}$ . The fact that we have an exponential relationship in Property 1.3, sets the *low, high, low* case apart from others.



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