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EQUIVARIANT PICARD GROUPS OF C^*-ALGEBRAS WITH FINITE DIMENSIONAL C^*-HOPF ALGEBRA COACTIONS

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ABSTRACT. Let A be a C^*-algebra and H a finite dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. Let (ρ, u) be a twisted coaction of H^0 on A. We shall define the (ρ, u, H)-equivariant Picard group of A, which is denoted by $\mathrm{Pic}_H^{\rho,u}(A)$, and discuss the basic properties of $\mathrm{Pic}_H^{\rho,u}(A)$. Also, we suppose that (ρ, u) is the coaction of H^0 on the unital C^*-algebra A, that is, $u = 1 \otimes 1^0$. We investigate the relation between $\mathrm{Pic}(A^s)$, the ordinary Picard group of A^s, and $\mathrm{Pic}_H^{\rho^s}(A^s)$, where A^s is the stable C^*-algebra of A and ρ^s is the coaction of H^0 on A^s induced by ρ. Furthermore, we shall show that $\mathrm{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho,u} H)$ is isomorphic to $\mathrm{Pic}_H^{\rho,u}(A)$, where $\widehat{\rho}$ is the dual coaction of H on the twisted crossed product $A \rtimes_{\rho,u} H$ of A by the twisted coaction (ρ, u) of H^0 on A.

1. Introduction. Let A be a C^*-algebra and H a finite dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. Let (ρ, u) be a twisted coaction of H^0 on A. We shall define the (ρ, u, H)-equivariant Picard group of A, which is denoted $\mathrm{Pic}_H^{\rho,u}(A)$. Also, we shall give a similar result to the ordinary Picard group as follows: let $\mathrm{Aut}_H^{\rho,u}(A)$ be the group of all automorphisms α of A satisfying that $(\alpha\otimes\mathrm{id})\circ\rho = \rho\circ\alpha$ and $(\underline{\alpha} \otimes \mathrm{id} \otimes \mathrm{id})(u) = u$, and let $\mathrm{Int}_H^{\rho,u}(A)$ be the normal subgroup of $\mathrm{Aut}_H^{\rho,u}(A)$ consisting of all generalized inner automorphisms $\mathrm{Ad}(v)$ of A satisfying that $\underline{\rho}(v) = v\otimes 1^0$ and $(v\otimes 1^0\otimes 1^0)u = u(v\otimes 1^0\otimes 1^0)$, where v is a unitary element in the multiplier algebra $M(A)$ of A. Then, we have the following exact sequence:

$$1 \longrightarrow \mathrm{Int}_H^{\rho,u}(A) \longrightarrow \mathrm{Aut}_H^{\rho,u}(A) \longrightarrow \mathrm{Pic}_H^{\rho,u}(A).$$

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In particular, let A^s be a stable C^*-algebra of a unital C^*-algebra A and ρ a coaction of H^0 on A. Also, let ρ^s be the coaction of H^0 on A^s induced by a coaction ρ of H^0 on A. Then, under a certain condition, we can obtain the exact sequence

$$1 \longrightarrow \mathrm{Int}_H^{\rho^s}(A^s) \longrightarrow \mathrm{Aut}_H^{\rho^s}(A^s) \longrightarrow \mathrm{Pic}_H^{\rho^s}(A^s) \longrightarrow 1.$$

In order to do this, we shall extend the definitions and results in the case of unital C^*-algebras to those in the case of non unital C^*-algebras in Section 2. Using this result, we shall investigate the relation between $\mathrm{Pic}(A^s)$, the ordinary Picard group of A^s, and $\mathrm{Pic}_H^{\rho^s}(A^s)$, the (ρ^s, H)-equivariant Picard group of A^s. Furthermore, we shall show that $\mathrm{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho,u} H)$ is isomorphic to $\mathrm{Pic}_H^{\rho,u}(A)$, where $\widehat{\rho}$ is the dual coaction of H on the twisted crossed product $A \rtimes_{\rho,u} H$ of A by the twisted coaction (ρ, u).

2. Preliminaries. Let H be a finite dimensional C^*-Hopf algebra. We denote its comultiplication, counit and antipode by Δ, ϵ and S, respectively. Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ is used for any $h \in H$ which suppresses a possible summation when comultiplications are written. The dimension of H is denoted by N. Let H^0 be the dual C^*-Hopf algebra of H. We denote its comultiplication, counit and antipode by Δ^0, ϵ^0 and S^0, respectively. There is a distinguished projection e in H. Note that e is the Haar trace on H^0. Also, there is a distinguished projection τ in H^0 which is the Haar trace on H. Since H is finite dimensional,

$$H \cong \bigoplus_{k=1}^{L} M_{f_k}(\mathbf{C}) \quad \text{and} \quad H^0 \cong \bigoplus_{k=1}^{K} M_{d_k}(\mathbf{C})$$

hold as C^*-algebras. Let

$$\{v_{ij}^k \mid k = 1, 2, \ldots, L, \ i, j = 1, 2, \ldots, f_k\}$$

be a system of matrix units of H. Let

$$\{w_{ij}^k \mid k = 1, 2, \ldots, K, \ i, j = 1, 2, \ldots, d_k\}$$

be a basis of H satisfying [**25,** Theorem 2.2,2], which is called a system of *comatrix* units of H, that is, the dual basis of a system of matrix units of H^0. Also, let

$$\{\phi_{ij}^k \mid k = 1, 2, \ldots, K,\ i, j = 1, 2, \ldots, d_k\}$$

and

$$\{\omega_{ij}^k \mid k = 1, 2, \ldots, L,\ i, j = 1, 2, \ldots, f_k\}$$

be systems of matrix and comatrix units of H^0, respectively.

Let A be a C^*-algebra and $M(A)$ its multiplier algebra. Let p, q be projections in A. If p and q are Murray-von Neumann equivalent, then we denote them by $p \sim q$ in A. We denote by id_A and 1_A the identity map on A and the unit element in A, respectively. They are simply denoted by id and 1, if no confusion arises. Modifying **[3,** Definition 2.1], we shall define a weak coaction of H^0 on A.

Definition 2.1. By a weak coaction of H^0 on A we mean a $*$-homomorphism $\rho : A \to A \otimes H^0$ satisfying the following conditions:

(1) $\overline{\rho(A)(A \otimes H^0)} = A \otimes H^0$,

(2) $(\mathrm{id} \otimes \epsilon^0)(\rho(x)) = x$ for any $x \in A$.

By a coaction of H^0 on A, we mean a weak coaction ρ such that

(3) $(\rho \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \Delta^0) \circ \rho$.

By Definition 2.1 (1), for any approximate unit $\{u_\alpha\}$ of A and $x \in A \otimes H^0$, $\rho(u_\alpha)x \to x$ $(\alpha \to \infty)$. Hence, $\rho(1) = 1 \otimes 1^0$ when A is unital. Since H^0 is finite dimensional, $M(A \otimes H^0) \cong M(A) \otimes H^0$. We identify $M(A \otimes H^0)$ with $M(A) \otimes H^0$. We also identify $M(A \otimes H^0 \otimes H^0)$ with $M(A) \otimes H^0 \otimes H^0$. Let ρ be a weak coaction of H^0 on A. By **[12,** Corollary 1.1.15], there is a unique strictly continuous homomorphism $\underline{\rho} : M(A) \to M(A) \otimes H^0$ extending ρ.

Lemma 2.2. *Using the above notation, $\underline{\rho}$ is a weak coaction of H^0 on $M(A)$.*

Proof. Clearly, $\underline{\rho}$ is a $*$-homomorphism of $M(A)$ to $M(A) \otimes H^0$. Let $\{u_\alpha\}$ be an approximate unit of A. Then, by Definition 2.1 (1), $\{\rho(u_\alpha)\}$ is an approximate unit of $A \otimes H^0$. Hence, $\underline{\rho}(1) = 1 \otimes 1^0$. Since H^0 is finite dimensional, $\mathrm{id} \otimes \epsilon^0$ is strictly continuous. Therefore, $\underline{\rho}$ satisfies Definition 2.1 (2). □

Let ρ be a weak coaction of H^0 on A and u a unitary element in $M(A) \otimes H^0 \otimes H^0$. Following [**19,** Section 3], we shall define a twisted coaction of H^0 on A.

Definition 2.3. The pair (ρ, u) is a *twisted coaction* of H^0 on A if the following conditions hold:

(1) $(\rho \otimes \mathrm{id}) \circ \rho = \mathrm{Ad}(u) \circ (\mathrm{id} \otimes \Delta^0) \circ \rho$,
(2) $(u \otimes 1^0)(\mathrm{id} \otimes \Delta^0 \otimes \mathrm{id})(u) = (\underline{\rho} \otimes \mathrm{id} \otimes \mathrm{id})(u)(\mathrm{id} \otimes \mathrm{id} \otimes \Delta^0)(u)$,
(3) $(\mathrm{id} \otimes \mathrm{id} \otimes \epsilon^0)(u) = (\mathrm{id} \otimes \epsilon^0 \otimes \mathrm{id})(u) = 1 \otimes 1^0$.

Remark 2.4. Let (ρ, u) be a twisted coaction of H^0 on A. Since H^0 is finite dimensional, $\mathrm{id}_{M(A)} \otimes \Delta^0$ is strictly continuous. Thus, by Lemma 2.2, $(\underline{\rho}, u)$ satisfies Definition 2.3. Therefore, $(\underline{\rho}, u)$ is a twisted coaction of H^0 on $M(A)$. Hence, if ρ is a coaction of H^0 on A, $\underline{\rho}$ is a coaction of H^0 on $M(A)$.

Let $\mathrm{Hom}(H, M(A))$ be the linear space of all linear maps from H to $M(A)$. Then, by [**24,** pages 69–70], it becomes a unital convolution *-algebra. Similarly, we define $\mathrm{Hom}(H \times H, M(A))$. Note that ϵ and $\epsilon \otimes \epsilon$ are the unit elements in $\mathrm{Hom}(H, M(A))$ and $\mathrm{Hom}(H \times H, M(A))$, respectively.

Modifying [**3,** Definition 1.1], we shall define a weak action of H on A.

Definition 2.5. By a weak action of H on A we mean a bilinear map $(h, x) \mapsto h \cdot x$ of $H \times A$ to A satisfying the following conditions:

(1) $h \cdot (xy) = [h_{(1)} \cdot x][h_{(2)} \cdot y]$ for any $h \in H$, $x, y \in A$,

(2) $[h \cdot u_\alpha]x \to \epsilon(h)x$ for any approximate unit $\{u_\alpha\}$ of A and $x \in A$,

(3) $1 \cdot x = x$ for any $x \in A$,

(4) $[h \cdot x]^* = S(h)^* \cdot x^*$ for any $h \in H$, $x \in A$.

By an action of H on A, we mean a *weak action* of H on A such that

(5) $h \cdot [l \cdot x] = (hl) \cdot x$ for any $x \in A$ and $h, l \in H$.

Since H is finite dimensional, as mentioned in [**3,** page 163], there is an isomorphism $\imath$ of $M(A) \otimes H^0$ onto $\mathrm{Hom}(H, M(A))$ defined by $\imath(x \otimes \phi)(h) = \phi(h)x$ for any $x \in M(A)$, $h \in H$, $\phi \in H^0$. Also, we can define an isomorphism $\jmath$ of $M(A) \otimes H^0 \otimes H^0$ onto $\mathrm{Hom}(H \times H, M(A))$ in a similar manner to the above. We note that

$$\imath(A \otimes H^0) = \mathrm{Hom}(H, A) \quad \text{and} \quad \jmath(A \otimes H^0 \otimes H^0) = \mathrm{Hom}(H \otimes H, A).$$

For any $x \in M(A) \otimes H^0$ and $y \in M(A) \otimes H^0 \otimes H^0$, we denote $\imath(x)$ and $\jmath(y)$ by $\widehat{x}$ and $\widehat{y}$, respectively.

Let a bilinear map $(h, x) \mapsto h \cdot x$ from $H \times A$ to A be a weak action. For any $x \in A$, let f_x be the linear map from H to A defined by $f_x(h) = h \cdot x$ for any $h \in H$. Let ρ be the linear map from A to $A \otimes H^0$ defined by $\rho(x) = \imath^{-1}(f_x)$ for any $x \in A$.

Lemma 2.6. *Using the above notation, ρ is a weak coaction of H^0 on A.*

Proof. By definition, ρ is a $*$-homomorphism of A to $A \otimes H^0$ satisfying Definition 2.1 (2). Thus, we only have to show that ρ satisfies Definition 2.1 (1). Let $\{u_\alpha\}$ be an approximate unit of A. We write that $\rho(u_\alpha) = \sum_j u_{\alpha j} \otimes \phi_j$, where $u_{\alpha j} \in A$, and $\{\phi_j\}$ is a basis of H^0 with

$$\sum_j \phi_j = 1^0.$$

Let $\{h_j\}$ be the dual basis of H corresponding to $\{\phi_j\}$. Then, for any $x \in A$ and j,

$$[h_j \cdot u_\alpha]x \longrightarrow \epsilon(h_j)x,$$

by Definition 2.5. Since $[h_j \cdot u_\alpha]x = (\mathrm{id} \otimes h_j)(\rho(u_\alpha))x = u_{\alpha j}x$,

$$u_{\alpha j}x \longrightarrow \epsilon(h_j)x \quad \text{for any } j.$$

Also, since $\sum_j \phi_j = 1^0$,

$$1 = \phi_j(h_j) = \sum_i \phi_i(h_j) = 1^0(h_j) = \epsilon(h_j)$$

for any j. Hence, $u_{\alpha j}x \to x$ for any j. Therefore, for any $x \in A$ and

$\phi \in H^0$,

$$\rho(u_\alpha)(x \otimes \phi) = \sum_j u_{\alpha j} x \otimes \phi_j \phi \longrightarrow \sum_j x \otimes \phi_j \phi = x \otimes \phi.$$

Thus, $\overline{\rho(A)(A \otimes H^0)} = A \otimes H^0$. □

For any weak coaction ρ of H^0 on A, we define the bilinear map $(h, x) \mapsto h \cdot_\rho x$ from $H \times A$ to A by

$$h \cdot_\rho x = (\mathrm{id} \otimes h)(\rho(x)) = \widehat{\rho(x)}(h).$$

We shall prove that the above map is a weak action of H on A.

Lemma 2.7. *With the above notation, the linear map $(h, x) \mapsto h \cdot_\rho x$ from $H \times A$ to A is a weak action of H on A.*

Proof. We only have to show that the above linear map satisfies Definition 2.5 (2). Let $\{u_\alpha\}$ be an approximate unit of A. Then, for any $x \in A \otimes H^0$, $\rho(u_\alpha)x \to x$ by the proof of Lemma 2.2. We write that

$$\rho(u_\alpha) = \sum_j u_{\alpha j} \otimes \phi_j,$$

where $u_{\alpha j} \in A$ and $\{\phi_j\}$ is a basis of H^0. Then, for any $a \in A$,

$$\begin{aligned}[h \cdot_\rho u_\alpha]a &= (\mathrm{id} \otimes h)(\rho(u_\alpha))a = \sum_j u_{\alpha j}\phi_j(h)a \\ &= (\mathrm{id} \otimes h)(\rho(u_\alpha)(a \otimes 1^0)) \longrightarrow \epsilon(h)a\end{aligned}$$

since $\mathrm{id} \otimes h$ is a bounded operator from $A \otimes H^0$ to A. □

Remark 2.8. By the proofs of Lemmas 2.6 and 2.7, Definition 2.5 (2) is equivalent to the following:

(2)′ $[h \cdot u_\alpha]x \to \epsilon(h)x$ for some approximate unit of A and any $x \in A$. Also, if A is unital, Definition 2.5 (2) means that $h \cdot 1 = \epsilon(h)$ for any $h \in H$.

Let ρ be a weak coaction of H^0 on A. Then, by Lemma 2.7, there is a weak action of H on A. We call it the *weak action* of H on A *induced* by ρ. Also, by Lemma 2.2, the weak coaction $\underline{\rho}$ of H^0 exists on $M(A)$, which is an extension of ρ to $M(A)$. Hence, we can obtain the action of H on $M(A)$ induced by $\underline{\rho}$. We see that this action is an extension of the action induced by ρ to $M(A)$.

Definition 2.9. Let $\sigma : H \times H \to M(A)$ be a bilinear map. σ is a *unitary cocycle* for a weak action of H on A if σ satisfies the following conditions:

(1) σ is a unitary element in $\mathrm{Hom}(H \times H, M(A))$;

(2) σ is normal, that is, for any $h \in H$, $\sigma(h, 1) = \sigma(1, h) = \epsilon(h)1$;

(3) (Cocycle condition). For any $h, l, m \in H$, $[h_{(1)} \cdot \sigma(l_{(1)}, m_{(1)})]\sigma(h_{(2)}, l_{(2)}m_{(2)}) = \sigma(h_{(1)}, l_{(1)})\sigma(h_{(2)}l_{(2)}, m)$;

(4) (Twisted modular condition). For any $h, l \in H$, $x \in A$, $[h_{(1)} \cdot [l_{(1)} \cdot x]]\sigma(h_{(2)}, l_{(2)}) = \sigma(h_{(1)}, l_{(1)})[(h_{(2)}l_{(2)}) \cdot x]$ where, if necessary, we consider the extension of the weak action to $M(A)$.

We call a pair which consists of a weak action of H on A and its unitary cocycle a *twisted action* of H on A.

Let (ρ, u) be a twisted coaction of H^0 on A. Then, we consider the twisted action of H on A and its unitary cocycle $\widehat{u}$, defined by

$$h \cdot_{\rho,u} x = \rho(x)\widehat{\ }(h) = (\mathrm{id} \otimes h)(\rho(x))$$

for any $x \in A$ and $h \in H$. We call it the *twisted action induced* by (ρ, u).

Further, we consider the twisted coaction $(\underline{\rho}, u)$ of H^0 on $M(A)$ and the twisted action of H on $M(A)$ induced by $(\underline{\rho}, u)$. Let $M(A) \rtimes_{\underline{\rho},u} H$ be the twisted crossed product by the twisted action of H on $M(A)$ induced by $(\underline{\rho}, u)$. Let $x \rtimes_{\underline{\rho},u} h$ be the element in $M(A) \rtimes_{\underline{\rho},u} H$ induced by elements $x \in M(A)$, $h \in H$. Let $A \rtimes_{\rho,u} H$ be the set of all finite sums of elements in the form $x \rtimes_{\underline{\rho},u} h$, where $x \in A$, $h \in H$. Simple computation shows that $A \rtimes_{\rho,u} H$ is a closed two-sided ideal of $M(A) \rtimes_{\underline{\rho},u} H$. We call it the *twisted crossed product* by (ρ, u), and its element is denoted by $x \rtimes_{\rho,u} h$, where $x \in A$ and $h \in H$. Let $E_1^{\underline{\rho},u}$ be the canonical conditional expectation from $M(A) \rtimes_{\underline{\rho},u} H$ onto $M(A)$, defined by

$$E_1^{\underline{\rho},u}(x \rtimes_{\underline{\rho},u} h) = \tau(h)x$$

for any $x \in M(A)$ and $h \in H$. Let Λ be the set of all triplets (i,j,k), where $i,j = 1,2,\ldots,d_k$ and $k = 1,2,\ldots,K$ with

$$\sum_{k=1}^{K} d_k^2 = N.$$

Let $W_I = \sqrt{d_k} \rtimes_{\underline{\rho},u} w_{ij}^k$ for any $I = (i,j,k) \in \Lambda$. By [**15,** Proposition 3.18], $\{(W_I^*, W_I)\}_{I\in\Lambda}$ is a quasi-basis for $E_1^{\rho,u}$. We assume that A faithfully and nondegenerately acts on a Hilbert space.

Lemma 2.10. *With the above notation,* $M(A) \rtimes_{\underline{\rho},u} H = M(A \rtimes_{\rho,u} H)$.

Proof. By the definition of multiplier algebras $M(A)$ and $M(A \rtimes_{\rho,u} H)$, it is clear that

$$M(A) \rtimes_{\underline{\rho},u} H \subset M(A \rtimes_{\rho,u} H)$$

since $M(A) \rtimes_{\underline{\rho},u} H$ and $M(A \rtimes_{\rho,u} H)$ act on the same Hilbert space. We now show another inclusion. Let $x \in M(A \rtimes_{\rho,u} H)$. Then, there is a bounded net $\{x_\alpha\}_{\alpha\in\Gamma} \subset A \rtimes_{\rho,u} H$ such that $\{x_\alpha\}_{\alpha\in\Gamma}$ converges to x strictly. Since $x_\alpha \in A \rtimes_{\rho,u} H$,

$$x_\alpha = \sum_I E_1^{\rho,u}(x_\alpha W_I^*)W_I.$$

By the definition of $E_1^{\rho,u}$, $E_1^{\rho,u}(x_\alpha W_I^*) \in A$. Also, for any $a \in A$,

$$\begin{aligned}\lim_{\alpha\to\infty} E_1^{\rho,u}(x_\alpha W_I^*)a &= \lim_{\alpha\to\infty} E_1^{\rho,u}(x_\alpha W_I^* a)\\ &= E_1^{\rho,u}(xW_I^* a) = E_1^{\rho,u}(xW_I^*)a.\end{aligned}$$

Similarly, $\lim_{\alpha\to\infty} aE_1^{\rho,u}(x_\alpha W_I^*) = aE_1^{\rho,u}(xW_I^*)$. Hence, $E_1^{\rho,u}(xW_I^*) \in M(A)$. In addition, by the above discussion, we can see that $E_1^{\rho,u}(\,\cdot\, W_I^*)$ is strictly continuous for any $I \in \Lambda$. For any $a \in A$ and $h \in H$,

$$\begin{aligned}(a \rtimes_{\rho,u} h)E_1^{\rho,u}(x_\alpha W_I^*) &= a[h_{(1)} \cdot_{\rho,u} E_1^{\rho,u}(x_\alpha W_I^*)] \rtimes_{\rho,u} h_{(2)}\\ &= a((\mathrm{id}\otimes h_{(1)})\circ(\underline{\rho}\otimes\mathrm{id}))(E_1^{\rho,u}(x_\alpha W_I^*)) \rtimes_{\rho,u} h_{(2)}.\end{aligned}$$

Since $\mathrm{id}\otimes h_{(1)}$, $\underline{\rho}\otimes\mathrm{id}$ and $E_1^{\rho,u}(\,\cdot\, W_I)$ are strictly continuous for any $I \in \Lambda$, we see that

$$\lim_{\alpha\to\infty}(a \rtimes_{\rho,u} h)E_1^{\rho,u}(x_\alpha W_I^*) = (a \rtimes_{\rho,u} h)E_1^{\rho,u}(xW_I^*).$$

Similarly, we see that, for any $a \in A$, $h \in H$,

$$\lim_{\alpha \to \infty} E_1^{\rho,u}(x_\alpha W_I^*)(a \rtimes_{\rho,u} h) = E_1^{\rho,u}(xW_I^*)(a \rtimes_{\rho,u} h).$$

Thus, $E_1^{\rho,u}(x_\alpha W_I^*)$ strictly converges to $E_1^{\rho,u}(xW_I^*)$ in $M(A \rtimes_{\rho,u} H)$. Therefore,

$$x = \sum_I E_1^{\rho,u}(xW_I^*)W_I$$

since

$$x_\alpha = \sum_I E_1^{\rho,u}(x_\alpha W_I^*)W_I.$$

It follows that $x \in M(A) \rtimes_{\underline{\rho},u} H$. □

Remark 2.11. Let $(\underline{\rho})\widehat{\ }$ be the dual coaction of $\underline{\rho}$ of H on $M(A) \rtimes_{\underline{\rho},u} H$ and $\underline{(\widehat{\rho})}$ the coaction of H on $M(A \rtimes_{\rho,u} H)$ induced by the dual coaction $\widehat{\rho}$ of H on $A \rtimes_{\rho,u} H$. By Lemma 2.10, we can see that $(\underline{\rho})\widehat{\ } = \underline{(\widehat{\rho})}$. Indeed, by Lemma 2.10, it suffices to show that $\underline{(\widehat{\rho})}(x \rtimes_{\underline{\rho},u} h) = (\underline{\rho})\widehat{\ }(x \rtimes_{\underline{\rho},u} h)$ for any $x \in M(A)$ and $h \in H$. Since $x \in M(A)$, there is a bounded net $\{x_\alpha\} \subset A$ such that x_α strictly converges to x in $M(A)$. Then, since $x_\alpha \rtimes_{\rho,u} h$ strictly converges to $x \rtimes_{\underline{\rho},u} h$ in $M(A) \rtimes_{\underline{\rho},u} H$ and $\underline{(\widehat{\rho})}$ is strictly continuous,

$$\begin{aligned}\underline{(\widehat{\rho})}(x \rtimes_{\underline{\rho},u} h) &= \lim_{\alpha \to \infty} \widehat{\rho}(x_\alpha \rtimes_{\rho,u} h) \\ &= \lim_{\alpha \to \infty} (x_\alpha \rtimes_{\rho,u} h_{(1)}) \otimes h_{(2)} \\ &= (x \rtimes_{\underline{\rho},u} h_{(1)}) \otimes h_{(2)} = (\underline{\rho})\widehat{\ }(x \rtimes_{\underline{\rho},u} h),\end{aligned}$$

where the limits are taken under the strict topology. We denote this by $\widehat{\underline{\rho}}$.

Next, we extend [**16,** Theorem 3.3] to a twisted coaction of H^0 on a (non-unital) C^*-algebra A. Before doing so, we define the exterior equivalence for twisted coactions of a finite dimensional C^*-Hopf algebra H^0 on a C^*-algebra A.

Definition 2.12. Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A. We say that (ρ, u) is *exterior equivalent* to (σ, v) if there is a unitary element $w \in M(A) \otimes H^0$ satisfying the following conditions:

(1) $\sigma = \mathrm{Ad}(w) \circ \rho$,
(2) $v = (w \otimes 1^0)(\underline{\rho} \otimes \mathrm{id})(w)u(\mathrm{id} \otimes \Delta^0)(w^*)$.

Conditions (1) and (2) are equivalent to the following, respectively:

(1)′ $h \cdot_{\sigma,v} a = \widehat{w}(h_{(1)})[h_{(2)} \cdot_{\rho,u} a]\widehat{w}^*(h_{(3)})$ for any $a \in A$ and $h \in H$,
(2)′ $\widehat{v}(h,l) = \widehat{w}(h_{(1)})[h_{(2)} \cdot_{\underline{\rho},u} \widehat{w}(l_{(1)})]\widehat{u}(h_{(3)}, l_{(2)})\widehat{w}^*(h_{(4)}l_{(3)})$ for any $h, l \in H^0$.

If ρ and σ are coactions of H^0 on A, (1), (2) and (1)′, (2)′ are as follows:

(i) $\sigma = \mathrm{Ad}(w) \circ \rho$,
(ii) $(w \otimes 1^0)(\underline{\rho} \otimes \mathrm{id})(w) = (\mathrm{id} \otimes \Delta^0)(w)$,

(i)′ $h \cdot_\sigma a = \widehat{w}(h_{(1)})[h_{(2)} \cdot_\rho a]\widehat{w}^*(h_{(3)})$ for any $a \in A$, $h \in H^0$,
(ii)′ $\widehat{w}(h_{(1)})[h_{(2)} \cdot_{\underline{\rho}} \widehat{w}(l)] = \widehat{w}(hl)$ for any $h, l \in H^0$.

Furthermore, let (ρ, u) be a twisted coaction of H^0 on A, and let w be any unitary element in $M(A) \otimes H^0$ with $(\mathrm{id} \otimes \epsilon^0)(w) = 1^0$. Let

$$\sigma = \mathrm{Ad}(w) \circ \rho, \qquad v = (w \otimes 1^0)(\underline{\rho} \otimes \mathrm{id})(w)u(\mathrm{id} \otimes \Delta^0)(w^*).$$

Then (σ, v) is a twisted coaction of H^0 on A by simple computation.

In the case of twisted coactions on von Neumann algebras, Vaes and Vanierman **[26]** and, in the case of ordinary coactions on C^*-algebras, Baaj and Skandalis **[1]** have already obtained much more generalized results than the following. We give a proof related to Watatani index–finite-type inclusions of unital C^*-algebras.

Proposition 2.13. *Let A be a C^*-algebra and H a finite dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. Let (ρ, u) be a twisted coaction of H^0 on A. Then there is an isomorphism Ψ of $M(A) \otimes M_N(\mathbf{C})$ onto $M(A) \rtimes_{\underline{\rho},u} H \rtimes_{\underline{\hat{\rho}}} H^0$ and a unitary element $U \in (M(A) \rtimes_{\underline{\rho},u} H \rtimes_{\underline{\hat{\rho}}} H^0) \otimes H^0$ such that*

$$\mathrm{Ad}(U) \circ \widehat{\widehat{\rho}} = (\Psi \otimes \mathrm{id}_{H^0}) \circ (\rho \otimes \mathrm{id}_{M_N(\mathbf{C})}) \circ \Psi^{-1},$$
$$(\Psi \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(u \otimes I_N) = (U \otimes 1^0)(\underline{\widehat{\widehat{\rho}}} \otimes \mathrm{id}_{H^0})(U)(\mathrm{id} \otimes \Delta^0)(U^*),$$
$$\Psi(A \otimes M_N(\mathbf{C})) = A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0,$$

that is, the coaction $\widehat{\widehat{\rho}}$ of H^0 on $A \rtimes_{\rho,u} \rtimes_{\hat{\rho}} H^0$ is exterior equivalent to *the twisted coaction*

$$((\Psi\otimes \mathrm{id}_{H^0})\circ(\rho\otimes \mathrm{id}_{M_N(\mathbf{C})})\circ\Psi^{-1},\ (\Psi\otimes \mathrm{id}_{H^0}\otimes \mathrm{id}_{H^0})(u\otimes I_N)),$$

where we identify $A\otimes H^0\otimes H^0\otimes M_N(\mathbf{C})$ *with* $A\otimes M_N(\mathbf{C})\otimes H^0\otimes H^0$.

Proof. By [**16,** Theorem 3.3], there is an isomorphism Ψ of $M(A)\otimes M_N(\mathbf{C})$ onto $M(A)\rtimes_{\underline{\rho},u}H\rtimes_{\underline{\hat\rho}}H^0$ and a unitary element $U\in(M(A)\rtimes_{\underline{\rho},u}H\rtimes_{\underline{\hat\rho}}H^0)\otimes H^0$ satisfying the required conditions, except for the equation

$$\Psi(A\otimes M_N(\mathbf{C}))=A\rtimes_{\rho,u}H\rtimes_{\hat\rho}H^0.$$

Therefore, we show the equation. By [**16,** Section 3],

$$\Psi([a_{IJ}])=\sum_{I,J}V_I^*(a_{IJ}\rtimes_{\rho,u}1\rtimes_{\hat\rho}1^0)V_J$$

for any $[a_{IJ}]\in A\otimes M_N(\mathbf{C})$, where

$$V_I=(1\rtimes_{\underline{\hat\rho}}\tau)(W_I\rtimes_{\underline{\hat\rho}}1^0)$$

for any $I\in\Lambda$. Since $V_I\in M(A)\rtimes_{\underline{\rho},u}H\rtimes_{\underline{\hat\rho}}H^0$ for any $I\in\Lambda$,

$$\Psi(A\otimes M_N(\mathbf{C}))\subset A\rtimes_{\rho,u}\rtimes_{\hat\rho}H^0.$$

For any $z\in A\rtimes_{\rho,u}H\rtimes_{\hat\rho}H^0$, we write that

$$z=\sum_{i=1}^{n}(x_i\rtimes_{\hat\rho}1^0)(1\rtimes_{\hat\rho}\tau)(y_i\rtimes_{\hat\rho}1^0),$$

where $x_i,y_i\in M(A)\rtimes_{\underline{\rho},u}H$ for any i. Let $\{u_\alpha\}$ be an approximate unit of A. Then $(u_\alpha\rtimes_{\rho,u}1\rtimes_{\hat\rho}1^0)(x_i\rtimes_{\hat\rho}1^0)$ and $(y_i\rtimes_{\hat\rho}1^0)(u_\alpha\rtimes_{\rho,u}1\rtimes_{\hat\rho}1^0)$ are in $A\rtimes_{\rho,u}H\rtimes_{\hat\rho}H^0$ for any i and α. Hence,

$$(A\rtimes_{\rho,u}H\rtimes_{\hat\rho}1^0)(1\rtimes_{\hat\rho}\tau)(A\rtimes_{\rho,u}H\rtimes_{\hat\rho}1^0)$$

is dense in $A\rtimes_{\rho,u}H\rtimes_{\hat\rho}H^0$. On the other hand, for any $x,y\in A\rtimes_{\rho,u}H$,

$$\Psi([E_1^{\rho,u}(W_Ix)E_1^{\rho,u}(yW_J^*)]_{I,J})=(x\rtimes_{\hat\rho}1^0)(1\rtimes_{\hat\rho}\tau)(y\rtimes_{\hat\rho}1^0)$$

by the proof of [**16,** Theorem 3.3]. Since $E_1^{\rho,u}(A\rtimes_{\rho,u}H)=A$ and $E_1^{\rho,u}$ is continuous by definition,

$$A\rtimes_{\rho,u}H\rtimes_{\hat\rho}H^0\subset\Psi(A\otimes M_N(\mathbf{C})).$$ □

We extend [**15,** Theorem 6.4] to coactions of H^0 on a (non-unital) C^*-algebra. First, we recall a saturated coaction. We say that a coaction ρ of H^0 on a unital C^*-algebra A is *saturated* if the induced action from ρ of H on A is saturated in the sense of [**25,** Definition 4.2].

Let B be a C^*-algebra and σ a coaction of H^0 on B. Let

$$B^\sigma = \{b \in B \mid \sigma(b) = b \otimes 1^0\}$$

be the fixed point C^*-subalgebra of B for the coaction σ. We suppose that B acts non-degenerately and faithfully on a Hilbert space $\mathcal{H}$. Also, we suppose that $\underline{\sigma}$ is saturated. Then, the canonical conditional expectation $E^{\underline{\sigma}}$ from $M(B)$ onto $M(B)^{\underline{\sigma}}$ defined by $E^{\underline{\sigma}}(x) = e \cdot_{\underline{\sigma}} x$ for any $x \in M(B)$ is of Watatani index-finite type by [**25,** Theorem 4.3]. Thus, there is a quasi-basis $\{(u_i, u_i^*)\}_{i=1}^n$ of $E^{\underline{\sigma}}$. Let $\{v_\alpha\}$ be an approximate unit of B^σ. For any $x \in B$,

$$v_\alpha x = v_\alpha \sum_{i=1}^n E^{\underline{\sigma}}(xu_i)u_i^* \longrightarrow \sum_{i=1}^n E^{\underline{\sigma}}(xu_i)u_i^* = x, \quad \alpha \to \infty,$$

since $E^\sigma(xu_i) \in B^\sigma$. Similarly, $xv_\alpha \to x$, $\alpha \to \infty$, since

$$x = \sum_{i=1}^n u_i E^{\underline{\sigma}}(u_i^* x).$$

Thus, $\{v_\alpha\}$ is an approximate unit of B. Hence, B^σ acts non-degenerately and faithfully on $\mathcal{H}$.

Lemma 2.14. *With the above notation, we suppose that $\underline{\sigma}$ is saturated. Then, $M(B^\sigma) = M(B)^{\underline{\sigma}}$.*

Proof. By the above discussion, we may suppose that B and B^σ act non-degenerately and faithfully on a Hilbert space. Let $x \in M(B^\sigma)$. Then, there is a bounded net $\{a_\alpha\} \subset B^\sigma$ such that $a_\alpha \to x$, $\alpha \to \infty$, strictly in $M(B^\sigma)$. Since any approximate unit of B^σ is an approximate unit of B by the above discussion, for any $y \in B^\sigma$,

$$\begin{aligned}\underline{\sigma}(x)(y \otimes 1^0) = \sigma(xy) = \sigma(\lim_{\alpha\to\infty} a_\alpha y) &= \lim_{\alpha\to\infty} \sigma(a_\alpha y) \\ &= \lim_{\alpha\to\infty} a_\alpha y \otimes 1^0 = xy \otimes 1^0.\end{aligned}$$

Thus, $x \in M(B)^{\underline{\sigma}}$. Next, let $x \in M(B)^{\underline{\sigma}}$. Then, for any $b \in B^{\sigma}$, xb and bx are in B. Thus,

$$\sigma(xb) = \underline{\sigma}(x)\sigma(b) = (x \otimes 1^0)(b \otimes 1^0) = xb \otimes 1^0,$$
$$\sigma(bx) = \sigma(b)\underline{\sigma}(x) = (b \otimes 1^0)(x \otimes 1^0) = bx \otimes 1^0.$$

Hence, $x \in M(B^{\sigma})$. □

We suppose that $\widehat{\underline{\sigma}}(1 \rtimes_{\underline{\sigma}} e) \sim (1 \rtimes_{\underline{\sigma}} e) \otimes 1$ in $(M(B) \rtimes_{\underline{\sigma}} H) \otimes H$. As mentioned in [**16,** Section 2], without the assumption of saturation for an action, all the statements in [**15,** Sections 4, 5, 6] hold. Hence, by [**15,** Sections 4, 5], $\underline{\sigma}$ is saturated, and there is a unitary element $w^{\sigma} \in M(B) \otimes H$ satisfying

$$w^{\sigma *}((1 \rtimes_{\underline{\sigma}} e) \otimes 1)w^{\sigma} = \widehat{\underline{\sigma}}(1 \rtimes_{\underline{\sigma}} e).$$

Let $U^{\sigma} = w^{\sigma}(z^{\sigma *} \otimes 1)$, where $z^{\sigma} = (\mathrm{id}_{M(B)} \otimes \epsilon)(w^{\sigma}) \in M(B)^{\underline{\sigma}}$. Then, $U^{\sigma} \in M(B) \otimes H$ satisfies

$$\widehat{U}^{\sigma}(1^0) = 1, \qquad \widehat{U}^{\sigma}(\phi_{(1)})a\widehat{U}^{\sigma *}(\phi_{(2)}) \in M(B)^{\underline{\sigma}}$$

for any $a \in M(B)^{\underline{\sigma}}$, $\phi \in H^0$. Let $\widehat{u}^{\sigma}$ be a bilinear map from $H^0 \times H^0$ to $M(B)$, defined by

$$\widehat{u}^{\sigma}(\phi, \psi) = \widehat{U}^{\sigma}(\phi_{(1)})\widehat{U}^{\sigma}(\psi_{(1)})\widehat{U}^{\sigma *}(\phi_{(2)}\psi_{(2)})$$

for any $\phi, \psi \in H^0$. Then, by [**15,** Lemma 5.4], $\widehat{u}^{\sigma}(\phi, \psi) \in M(B)^{\underline{\sigma}}$ for any $\phi, \psi \in H^0$ and, by [**15,** Corollary 5.3], the map

$$H^0 \times M(B)^{\underline{\sigma}} \longrightarrow M(B)^{\underline{\sigma}} : (\phi, a) \longmapsto \widehat{U}^{\sigma}(\phi_{(1)})a\widehat{U}^{\sigma *}(\phi_{(2)})$$

is a weak action of H^0 on $M(B)^{\underline{\sigma}}$. Furthermore, by [**15,** Proposition 5.6], $\widehat{u}^{\sigma}$ is a unitary cocycle for the above weak action. Let u^{σ} be the unitary element in $M(B)^{\underline{\sigma}} \otimes H \otimes H$ induced by $\widehat{u}^{\sigma}$ and ρ' the weak coaction of H on $M(B)^{\underline{\sigma}}$ induced by the above weak action. Thus, we obtain a twisted coaction (ρ', u^{σ}) of H on $M(B)^{\underline{\sigma}}$. Let π' be the map from $M(B)^{\underline{\sigma}} \rtimes_{\rho', u^{\sigma}} H^0$ to $M(B)$, defined by

$$\pi'(a \rtimes_{\rho', u^{\sigma}} \phi) = a\widehat{U}^{\sigma}(\phi)$$

for any $a \in M(B)^{\underline{\sigma}}$, $\phi \in H^0$. Then, by [**15,** Proposition 6.1, Theorem 6.4], π' is an isomorphism of $M(B)^{\underline{\sigma}} \rtimes_{\rho', u^{\sigma}} H^0$ onto $M(B)$

satisfying

$$\underline{\sigma} \circ \pi' = (\pi' \otimes \mathrm{id}_{H^0}) \circ \widehat{\rho'}, \qquad E_1^{\rho', u^\sigma} = E^{\underline{\sigma}} \circ \pi',$$

where E_1^{ρ', u^σ} is the canonical conditional expectation from $M(B)^{\underline{\sigma}} \rtimes_{\rho', u^\sigma} H^0$ onto $M(B)^{\underline{\sigma}}$ and $E^{\underline{\sigma}}$ is the canonical conditional expectation from $M(B)$ onto $M(B)^{\underline{\sigma}}$. Let $\rho = \rho'|_{B^\sigma}$.

Lemma 2.15. *With the above notation, (ρ, u^σ) is a twisted coaction of H on B^σ and $\underline{\rho} = \rho'$.*

Proof. By the definition of ρ, for any $a \in B^\sigma$,

$$\rho(a) = U^\sigma (a \otimes 1) U^{\sigma *}.$$

Since $a \in B^\sigma \subset M(B^\sigma) = M(B)^{\underline{\sigma}}$, by Lemma 2.14, $\rho(a) \in M(B)^{\underline{\sigma}} \otimes H$. On the other hand, since $U^\sigma \in M(B) \otimes H$, $\rho(a) \in B \otimes H$. Thus, $\rho(a) \in (M(B)^{\underline{\sigma}} \otimes H) \cap (B \otimes H) = B^\sigma \otimes H$. Hence, ρ is a homomorphism of B^σ to $B^\sigma \otimes H$. Since $(\rho' \otimes \mathrm{id}) \circ \rho' = \mathrm{Ad}(u^\sigma) \circ (\mathrm{id} \otimes \Delta) \circ \rho'$ and $\rho(a) \in B^\sigma \otimes H$ for any $a \in B^\sigma$, we see that $(\rho \otimes \mathrm{id}) \circ \rho = \mathrm{Ad}(u^\sigma) \circ (\mathrm{id} \otimes \Delta) \circ \rho$. By the definition of ρ', ρ' is strictly continuous on $M(B)^{\underline{\sigma}}$. Hence, for any approximate unit $\{u_\alpha\}$ of B^σ,

$$1 \otimes 1 = \rho'(1) = \rho'(\lim_{\alpha \to \infty} u_\alpha) = \lim_{\alpha \to \infty} \rho'(u_\alpha) = \lim_{\alpha \to \infty} \rho(u_\alpha),$$

where the limits are taken under strict topologies in $M(B^\sigma)$ and $M(B^\sigma) \otimes H$, respectively. This means that

$$\overline{\rho(B^\sigma)(B^\sigma \otimes H)} = B^\sigma \otimes H.$$

It follows that (ρ, u^σ) is a twisted coaction of H on B^σ. Furthermore, since ρ' is strictly continuous, $\rho' = \underline{\rho}$ on $M(B^\sigma)$. $\square$

Let $\pi = \pi'|_{B^\sigma \rtimes_{\rho, u^\sigma} H^0}$.

Lemma 2.16. *With the above notation, π is an isomorphism of $B^\sigma \rtimes_{\rho, u^\sigma} H^0$ onto B, satisfying*

$$\sigma \circ \pi = (\pi \otimes \mathrm{id}_{H^0}) \circ \widehat{\rho}, \qquad E_1^{\rho, u^\sigma} = E^\sigma \circ \pi,$$

where E_1^{ρ, u^σ} is the canonical conditional expectation from $B^\sigma \rtimes_{\rho, u^\sigma} H^0$ onto B^σ, and E^σ is the canonical conditional expectation from B onto B^σ. Furthermore, $\pi' = \underline{\pi}$.

Proof. Let $E^{\underline{\sigma}}$ be the canonical conditional expectation from $M(B)$ onto $M(B)^{\underline{\sigma}}$. By [**15,** Proposition 4.3, Remark 4.9],

$$\{(\sqrt{f_k}\widehat{U}^{\sigma}(\omega_{ij}^k)^*,\ \sqrt{f_k}\widehat{U}^{\sigma}(\omega_{ij}^k))\}_{i,j,k}$$

is a quasi-basis for $E^{\underline{\sigma}}$. Hence, for any $b \in B$,

$$b = \sum_{i,j,k} f_k E^{\underline{\sigma}}(b\widehat{U}^{\sigma}(\omega_{ij}^k)^*)\widehat{U}^{\sigma}(\omega_{ij}^k).$$

Since $\widehat{U}^{\sigma}(\omega_{ij}^k) \in M(B)$ for any i, j, k,

$$E^{\underline{\sigma}}(b\widehat{U}^{\sigma}(\omega_{ij}^k)^*) \in B^{\sigma}$$

for any i, j, k and $b \in B$. Let

$$a = \sum_{i,j,k} f_k E^{\underline{\sigma}}(b\widehat{U}^{\sigma}(\omega_{ij}^k)^*) \rtimes_{\rho,u^{\sigma}} \omega_{ij}^k.$$

Then $a \in B^{\sigma} \rtimes_{\rho,u^{\sigma}} H^0$ and $\pi(a) = b$. Thus, π is surjective. Since π' is an isomorphism of $M(B)^{\underline{\sigma}} \rtimes_{\underline{\rho},u^{\sigma}} H^0$ onto $M(B)$, we see that π is an isomorphism of $B^{\sigma} \rtimes_{\rho,u^{\sigma}} H^0$ onto B. Also, since $\underline{\sigma}\circ\pi' = (\pi'\otimes\mathrm{id})\circ\underline{\widehat{\rho}}$ and $E_1^{\underline{\rho},u^{\sigma}} = E^{\underline{\sigma}} \circ \pi'$, we see that

$$\sigma \circ \pi = (\pi \otimes \mathrm{id}) \circ \widehat{\rho}, \qquad E_1^{\rho,u^{\sigma}} = E^{\sigma} \circ \pi.$$

Furthermore, by the definition of π', π' is strictly continuous. Thus, $\pi' = \underline{\pi}$. □

Combining Lemmas 2.14, 2.15 and 2.16, we obtain the next proposition.

Proposition 2.17. *Let B be a C^*-algebra and σ a coaction of H^0 on B. We suppose that $\widehat{\underline{\sigma}}(1 \rtimes_{\underline{\sigma}} e) \sim (1 \rtimes_{\underline{\sigma}} e)\otimes 1$ in $(M(B) \rtimes_{\underline{\sigma}} H) \otimes H$. Then, there are a twisted coaction (ρ, u^{σ}) of H on B^{σ} and an isomorphism π of $B^{\sigma} \rtimes_{\rho,u^{\sigma}} H^0$ onto B satisfying*

$$\sigma \circ \pi = (\pi \otimes \mathrm{id}_H) \circ \widehat{\rho}, \qquad E_1^{\rho,u^{\sigma}} = E^{\sigma} \circ \pi,$$

where B^{σ} is the fixed point C^*-subalgebra of B for σ, and $E_1^{\rho,u^{\sigma}}$ and E^{σ} are the canonical conditional expectations from B and $B^{\sigma} \rtimes_{\rho,u^{\sigma}} H^0$ onto B^{σ}, respectively.

3. Twisted coactions on a Hilbert C^*-bimodule and strong Morita equivalence for twisted coactions. First, we shall define crossed products of Hilbert C^*-bimodules in the sense of Brown, Mingo and Shen [**6**] and show their duality theorem, which is similar to [**17,** Theorem 5.7]. The definition of a Hilbert C^*-bimodule is as follows: Let A and B be C^*-algebras. Let X be a left pre-Hilbert A-bimodule and a right pre-Hilbert B-module. Its left A-valued inner and right B-valued inner products are denoted by ${}_A\langle \cdot\, , \, \cdot \rangle$ and $\langle \cdot\, , \, \cdot \rangle_B$, respectively.

Definition 3.1. We call X a *pre-Hilbert* $A-B$*-bimodule* if X satisfies the condition

$$ {}_A\langle x, \, y\rangle z = x\langle y, \, z\rangle_B $$

for any $x, y, z \in X$. We call X a *Hilbert* $A-B$*-bimodule* if X is complete with the norms.

Remark 3.2. We suppose that X is a pre-Hilbert $A-B$-bimodule. Then, by [**6,** Remark 1.9], we have the following:

(1) for any $x \in x$, $\|{}_A\langle x, \, x\rangle\| = \|\langle x, \, x\rangle_B\|$;
(2) for any $a \in A$, $b \in B$ and $x, y \in X$,

$$ {}_A\langle x, \, yb\rangle = {}_A\langle xb^*, \, y\rangle, \qquad \langle ax, \, y\rangle_B = \langle x, \, a^* y\rangle_B; $$

(3) if X is complete with the norm and full with both-sided inner products, then X is an $A-B$-equivalence bimodule.

In this paper, by "pre-Hilbert C^*-bimodules" and "Hilbert C^*-bimodules," we mean pre-Hilbert C^*-bimodules and Hilbert C^*-bimodules in the sense of [**6**], respectively.

Let A and B be C^*-algebras. Let X be a Hilbert $A-B$-bimodule, and let $\mathbf{B}_B(X)$ be the C^*-algebra of all right B-linear operators on X for which there is a right adjoint B-linear operator on X. We note that a right B-linear operator on X is bounded. For each $x, y \in X$, let $\theta_{x,y}$ be a rank 1 operator on X defined by $\theta_{x,y}(z) = x\langle y, z\rangle_B$ for any $z \in X$. Then, $\theta_{x,y}$ is a right B-linear operator on X. Let $\mathbf{K}_B(X)$ be the closure of all linear spans of such $\theta_{x,y}$. Then, $\mathbf{K}_B(X)$ is a closed two-sided ideal of $\mathbf{B}_B(X)$.

Similarly, we define ${}_A\mathbf{B}(X)$ and ${}_A\mathbf{K}(X)$. If X is an $A-B$-equivalence bimodule, we identify A and $M(A)$ with $\mathbf{K}_B(X)$ and

$\mathbf{B}_B(X)$, respectively, and B and $M(B)$ with ${}_A\mathbf{B}(X)$ and ${}_A\mathbf{K}(X)$, respectively. For any $a \in M(A)$, we regard $a \in M(A)$ as an element in $\mathbf{B}_B(X)$ as follows: for any $b \in A$, $x \in X$,

$$a(bx) = (ab)x.$$

Since $X = \overline{AX}$, by [**6,** Proposition 1.7], we obtain an element in $\mathbf{B}_B(X)$ induced by $a \in M(A)$. Similarly, we can obtain an element in ${}_A\mathbf{B}(X)$ induced by any $b \in M(B)$.

Lemma 3.3. *With the above notation, we suppose that X is a Hilbert $A-B$-bimodule. For any $a \in M(A)$, there is a bounded net $\{a_\alpha\}_{\alpha\in\Gamma} \subset A$ such that $ax = \lim_{\alpha\to\infty} a_\alpha x$ for any $x \in X$.*

Proof. Since $a \in M(A)$, there is a bounded net $\{a_\alpha\}_{\alpha\in\Gamma} \subset A$ such that $\{a_\alpha\}_{\alpha\in\Gamma}$ converges to a strictly. We can prove that $ax = \lim_{\alpha\to\infty} a_\alpha x$ for any $x \in X$ in a routine manner since $X = \overline{AX}$ by [**6,** Proposition 1.7]. □

Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A and B, respectively.

Definition 3.4. Let λ be a linear map from a Hilbert $A-B$-bimodule X to $X \otimes H^0$. Then we say that λ is a *twisted coaction* of H^0 on X with respect to $(A, B, \rho, u, \sigma, v)$ if the following conditions hold:

(1) $\lambda(ax) = \rho(a)\lambda(x)$ for any $a \in A$, $x \in X$;
(2) $\lambda(xb) = \lambda(x)\sigma(b)$ for any $b \in B$, $x \in X$;
(3) $\rho({}_A\langle x, y\rangle) = {}_{A\otimes H^0}\langle\lambda(x), \lambda(y)\rangle$ for any $x, y \in X$;
(4) $\sigma(\langle x, y\rangle_B) = \langle\lambda(x), \lambda(y)\rangle_{B\otimes H^0}$ for any $x, y \in X$;
(5) $(\mathrm{id}_X \otimes \epsilon^0) \circ \lambda = \mathrm{id}_X$;
(6) $(\lambda \otimes \mathrm{id})(\lambda(x)) = u(\mathrm{id} \otimes \Delta^0)(\lambda(x))v^*$ for any $x \in X$; where u and v are regarded as elements in $\mathbf{B}_B(X)$ and ${}_A\mathbf{B}(X)$, respectively.

Note that the twisted coaction λ of H^0 on the Hilbert $A-B$-bimodule X with respect to $(A, B, \rho, u, \sigma, v)$ is isometric. Indeed, for any $x \in X$,

$$\|\lambda(x)\|^2 = \|{}_{A\otimes H^0}\langle\lambda(x), \lambda(x)\rangle\| = \|\rho({}_A\langle x, y\rangle)\| = \|{}_A\langle x, y\rangle\| = \|x\|^2.$$

Let λ be a twisted coaction of H^0 on a Hilbert $A-B$-bimodule X with respect to $(A, B, \rho, u, \sigma, v)$. We define the *twisted action* of H on X induced by λ as follows: for any $x \in X$, $h \in H$,

$$h \cdot_\lambda x = (\mathrm{id} \otimes h)(\lambda(x)) = \lambda(x)\widehat{\ }(h),$$

where $\lambda(x)\widehat{\ }$ is the element in $\mathrm{Hom}(H, X)$ induced by $\lambda(x)$ in $X \otimes H^0$. Then, we obtain the following conditions which are equivalent to Definition 3.4 (1)–(6), respectively:

(1)′ $h \cdot_\lambda ax = [h_{(1)} \cdot_{\rho,u} a][h_{(2)} \cdot_\lambda x]$ for any $a \in A$, $x \in X$;

(2)′ $h \cdot_\lambda xb = [h_{(1)} \cdot_\lambda x][h_{(2)} \cdot_{\sigma,v} b]$ for any $b \in B$, $x \in X$;

(3)′ $h \cdot_\rho {}_A\langle x, y\rangle = {}_A\langle [h_{(1)} \cdot_\lambda x],\ [S(h^*_{(2)}) \cdot_\lambda y]\rangle$ for any $x, y \in X$;

(4)′ $h \cdot_\sigma \langle x, y\rangle_B = \langle [S(h^*_{(1)}) \cdot_\lambda x],\ [h_{(2)} \cdot_\lambda y]\rangle_B$ for any $x, y \in X$;

(5)′ $1_H \cdot_\lambda x = x$ for any $x \in X$;

(6)′ $h \cdot_\lambda [l \cdot_\lambda x] = \widehat{u}(h_{(1)}, l_{(1)})[h_{(2)}l_{(2)} \cdot_\lambda x]\widehat{v}^*(h_{(3)}, l_{(3)})$ for any $x \in X$, $h, l \in H$; where $\widehat{u}$ and $\widehat{v}$ are elements in $\mathrm{Hom}(H \times H, M(A))$ and $\mathrm{Hom}(H \times H, M(B))$ induced by $u \in M(A) \otimes H^0 \otimes H^0$ and $v \in M(B) \otimes H^0 \otimes H^0$, respectively.

Remark 3.5. In Definition 3.4, if ρ and σ are coactions of H^0 on A and B, respectively, then Definition 3.4 (6) and its equivalent (6)′ are the following, respectively:

(6) $(\lambda \otimes \mathrm{id}) \circ \lambda = (\mathrm{id} \otimes \Delta^0) \circ \lambda$;

(6)′ $h \cdot_\lambda [l \cdot_\lambda x] = hl \cdot_\lambda x$ for any $x \in X$.

In this case, we call λ a *coaction* of H^0 on X with respect to (A, B, ρ, σ).

Next, we shall define crossed products of Hilbert C^*-bimodules by twisted coactions in the same way as in [**17,** Section 4] and give a duality theorem for them.

Let (ρ, u) and (σ, v) be twisted coactions of H^0 on C^*-algebras A and B, respectively. Let λ be a twisted coaction of H^0 on a Hilbert $A-B$-bimodule X with respect to $(A, B, \rho, u, \sigma, v)$. We define $X \rtimes_\lambda H$, a Hilbert $A \rtimes_{\rho,u} H - B \rtimes_{\sigma,v} H$-bimodule as follows: let $(X \rtimes_\lambda H)_0$ merely be $X \otimes H$ (the algebraic tensor product) as vector spaces. Its left and right actions are given by

$$(a \rtimes_{\rho,u} h)(x \rtimes_\lambda l) = a[h_{(1)} \cdot_\lambda x]\widehat{v}(h_{(2)}, l_{(1)}) \rtimes_\lambda h_{(3)}l_{(2)},$$
$$(x \rtimes_\lambda l)(b \rtimes_{\sigma,v} m) = x[l_{(1)} \cdot_{\sigma,v} b]\widehat{v}(l_{(2)}, m_{(1)}) \rtimes_\lambda l_{(3)}m_{(2)}$$

for any $a \in A$, $b \in B$, $x \in X$ and $h, l, m \in H$. Also, its left $A \rtimes_{\rho,u} H$-valued and right $B \rtimes_{\sigma,v} H$-valued inner products are given by

$$\begin{aligned}{}_{A\rtimes_{\rho,u}H}\langle x \rtimes_\lambda h,\, y \rtimes_\lambda l\rangle &= {}_A\langle x,\, [S(h_{(2)}l^*_{(3)})^* \cdot_\lambda y]\widehat{v}(S(h_{(1)}l^*_{(2)})^*,\, l_{(1)})\rangle \\ &\quad \rtimes_{\rho,u} h_{(3)}l^*_{(4)}, \\ \langle x \rtimes_\lambda h,\, y \rtimes_\lambda l\rangle_{B\rtimes_{\sigma,v}H} &= \widehat{v}^*(h^*_{(2)},\, S(h_{(1)})^*)[h^*_{(3)} \cdot_{\sigma,v} \langle x, y\rangle_B]\widehat{v}(h^*_{(4)}, l_{(1)}) \\ &\quad \rtimes_{\sigma,v} h^*_{(5)}l_{(2)}\end{aligned}$$

for any $x, y \in X$ and $h, l \in H$. In the same manner as in **[17,** Section 4], we see that $(X \rtimes_\lambda H)_0$ is a pre-Hilbert $A \rtimes_{\rho,u} H - B \rtimes_{\sigma,v} H$-bimodule. Let $X \rtimes_\lambda H$ be the completion of $(X \rtimes_\lambda H)_0$. It is a Hilbert $A \rtimes_{\rho,u} H - B \rtimes_{\sigma,v} H$-bimodule. Let $\widehat{\lambda}$ be a linear map from $(X \rtimes_\lambda H)_0$ to $(X \rtimes_\lambda H)_0 \otimes H$, defined by

$$\widehat{\lambda}(x \rtimes_\lambda h) = (x \rtimes_\lambda h_{(1)}) \otimes h_{(2)}$$

for any $x \in X$, $h \in H$. By simple computation, we can see that $\widehat{\lambda}$ is a linear map from H to $(X \rtimes_\lambda H)_0 \otimes H$ satisfying in Definition 3.4 (1)–(6). Thus, for any $x \in (X \rtimes_\lambda H)_0$,

$$\begin{aligned}\|\widehat{\lambda}(x)\|^2 &= \|{}_{(A\rtimes_{\rho.u}H)\otimes H}\langle\widehat{\lambda}(x),\, \widehat{\lambda}(x)\rangle\| = \|\widehat{\rho}({}_A\langle x,\, x\rangle)\| \\ &= \|{}_A\langle x, x\rangle\| = \|x\|^2.\end{aligned}$$

Hence, $\widehat{\lambda}$ is an isometry. We extend $\widehat{\lambda}$ to $X \rtimes_\lambda H$. We see that the extension of $\widehat{\lambda}$ is a coaction of H on $X \rtimes_\lambda H$ with respect to $(A \rtimes_{\rho,u} H, B \rtimes_{\sigma,v} H, \widehat{\rho}, \widehat{\sigma})$. We also denote it by the same symbol $\widehat{\lambda}$ and call it the *dual coaction* of λ.

Similarly, we define the second dual coaction of λ, which is a coaction of H^0 on $X \rtimes_\lambda H \rtimes_{\hat{\lambda}} H^0$. Let Λ be as in Section 2. For any $I = (i, j, k) \in \Lambda$, let W^ρ_I and V^ρ_I be elements in $M(A) \rtimes_{\underline{\rho},u} H \rtimes_{\underline{\widehat{\rho}}} H^0$, defined by

$$W^\rho_I = \sqrt{d_k} \rtimes_{\underline{\rho},u} w^k_{ij}, \qquad V^\rho_I = (1 \rtimes_{\underline{\rho},u} 1 \rtimes_{\underline{\hat{\rho}}} \tau)(W^\rho_I \rtimes_{\underline{\hat{\rho}}} 1^0).$$

Similarly, for any $I = (i, j, k) \in \Lambda$, we define the elements

$$W_I^\sigma = \sqrt{d_k} \rtimes_{\underline{\sigma}, v} w_{ij}^k, \qquad V_I^\sigma = (1 \rtimes_{\underline{\sigma}, v} 1 \rtimes_{\underline{\hat{\sigma}}} \tau)(W_I^\sigma \rtimes_{\underline{\hat{\sigma}}} 1^0)$$

in $M(B) \rtimes_{\underline{\sigma}, v} H \rtimes_{\underline{\hat{\sigma}}} H^0$. We regard $M_N(\mathbf{C})$ as an equivalence $M_N(\mathbf{C}) - M_N(\mathbf{C})$-bimodule in the usual way. Let $X \otimes M_N(\mathbf{C})$ be the exterior tensor product of X and $M_N(\mathbf{C})$, which is a Hilbert $A \otimes M_N(\mathbf{C}) - B \otimes M_N(\mathbf{C})$-bimodule. Let $\{f_{IJ}\}_{I,J \in \Lambda}$ be a system of matrix units of $M_N(\mathbf{C})$. Let Ψ_X be a linear map from $X \otimes M_N(\mathbf{C})$ to $X \rtimes_\lambda H \rtimes_{\hat{\lambda}} H^0$, defined by

$$\Psi_X\left(\sum_{I,J} x_{IJ} \otimes f_{IJ}\right) = \sum_{I,J} V_I^{\rho *}(x_{IJ} \rtimes_\lambda 1 \rtimes_{\hat{\lambda}} 1^0) V_J^\sigma.$$

Let Ψ_A and Ψ_B be the isomorphisms of $A \otimes M_N(\mathbf{C})$ and $B \otimes M_N(\mathbf{C})$ onto $A \rtimes_{\rho, u} H \rtimes_{\hat{\rho}} H^0$ and $B \rtimes_{\sigma, v} H \rtimes_{\hat{\sigma}} H^0$ defined in Proposition 2.13, respectively. Then, we have the same lemmas as [**17,** Lemmas 5.1, 5.5]. Hence, Ψ_X is an isometry from $X \otimes M_N(\mathbf{C})$ to $X \rtimes_\lambda H \rtimes_{\hat{\lambda}} H^0$ whose image is $(X \rtimes_\lambda H)_0 \rtimes_{\hat{\lambda}} H^0$, the linear span of the set

$$\{x \rtimes_\lambda h \rtimes_{\hat{\lambda}} \phi \mid x \in X, h \in H, \phi \in H^0\}.$$

Since $X \otimes M_N(\mathbf{C})$ is complete, so is $(X \rtimes_\lambda H)_0 \rtimes_{\hat{\lambda}} H^0$. Furthermore, we claim that $(X \rtimes_\lambda H)_0$ is also complete. In order to show this, we need the following lemma: let E_1^λ be a linear map from $(X \rtimes_\lambda H)_0$ onto X defined by

$$E_1^\lambda(x \rtimes_\lambda h) = \tau(h)x$$

for any $x \in X$, $h \in H$.

Lemma 3.6. *With the above notation, E_1^λ is continuous.*

Proof. In the same manner as in the proof of [**17,** Lemma 5.6], we see that

$$E_1^\lambda(x \rtimes_\lambda h) = \tau \cdot_{\hat{\lambda}} (x \rtimes_\lambda h) = \widehat{V}^{\hat{\rho}}(\tau_{(1)})(x \rtimes_\lambda h \rtimes_{\hat{\lambda}} 1^0)\widehat{V}^{\hat{\sigma}*}(\tau_{(2)}),$$

where we identify $X \rtimes_\lambda H \rtimes_{\hat{\lambda}} 1^0$ with $X \rtimes_\lambda H$ and

$$\widehat{V}^{\hat{\rho}}(\phi) = 1 \rtimes_{\underline{\rho}, u} 1 \rtimes_{\underline{\hat{\rho}}} \phi, \quad \widehat{V}^{\hat{\sigma}}(\phi) = 1 \rtimes_{\underline{\sigma}, v} 1 \rtimes_{\underline{\hat{\sigma}}} \phi$$

for any $\phi \in H^0$. Hence, E_1^λ is continuous. ☐

Let E_2^λ be a linear map from $(X \rtimes_\lambda H \rtimes_{\hat{\lambda}} H^0)_0$ to $X \rtimes_\lambda H$, defined by

$$E_2^\lambda(x \rtimes_{\hat{\lambda}} \phi) = \phi(e)x$$

for any $x \in X \rtimes_\lambda H$, $\phi \in H^0$.

Lemma 3.7. *With the above notation, $(X \rtimes_\lambda H)_0$ is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $(X \rtimes_\lambda H)_0$. Using Lemma 3.6 and the linear map E_2^λ, we can see that $\{x_n\}$ is convergent in $(X \rtimes_\lambda H)_0$. □

By Lemma 3.7, $X \rtimes_\lambda H = (X \rtimes_\lambda H)_0$. In the same way as in the proof of [**17,** Theorem 5.7], we obtain the following proposition using Lemma 3.7.

Proposition 3.8. *Let A and B be C^*-algebras and H a finite dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A and B, respectively. Let λ be a twisted coaction of H^0 on a Hilbert $A-B$-bimodule X with respect to $(A, B, \rho, u, \sigma, v)$. Then, there is an isomorphism Ψ_X from $X \otimes M_N(\mathbf{C})$ onto $X \rtimes_\lambda H \rtimes_{\hat{\lambda}} H^0$ satisfying that*

(1) $$\Psi_X\left(\left(\sum_{I,J} a_{IJ} \otimes f_{IJ}\right)\left(\sum_{I,J} x_{IJ} \otimes f_{IJ}\right)\right) = \Psi_A\left(\sum_{I,J} a_{IJ} \otimes f_{IJ}\right)\Psi_X\left(\sum_{I,J} x_{IJ} \otimes f_{IJ}\right),$$

(2) $$\Psi_X\left(\left(\sum_{I,J} x_{IJ} \otimes f_{IJ}\right)\left(\sum_{I,J} b_{IJ} \otimes f_{IJ}\right)\right) = \Psi_X\left(\sum_{I,J} x_{IJ} \otimes f_{IJ}\right)\Psi_B\left(\sum_{I,J} b_{IJ} \otimes f_{IJ}\right),$$

(3) $${}_{A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0}\left\langle \Psi_X\left(\sum_{I,J} x_{IJ} \otimes f_{IJ}\right), \Psi_X\left(\sum_{I,J} y_{IJ} \otimes f_{IJ}\right)\right\rangle = \Psi_A\left({}_{A \otimes M_N(\mathbf{C})}\left\langle \sum_{I,J} x_{IJ} \otimes f_{IJ}, \sum_{I,J} y_{IJ} \otimes f_{IJ}\right\rangle\right),$$

$$(4) \qquad \Big\langle \Psi_X\Big(\sum_{I,J} x_{IJ}\otimes f_{IJ}\Big), \Psi_X\Big(\sum_{I,J} y_{IJ}\otimes f_{IJ}\Big)\Big\rangle_{B\rtimes_{\sigma,v}H\rtimes_{\hat{\sigma}}H^0}$$
$$= \Psi_B\Big(\Big\langle \sum_{I,J} x_{IJ}\otimes f_{IJ}, \sum_{I,J} y_{IJ}\otimes f_{IJ}\Big\rangle_{B\otimes M_N(\mathbf{C})}\Big)$$

for any $a_{IJ}\in A$, $b_{IJ}\in B$, x_{IJ}, $y_{IJ}\in X$, I and $J\in\Lambda$, where $X\rtimes_\lambda H\rtimes_{\hat{\lambda}} H^0$ is a Hilbert $A\rtimes_{\rho,u}H\rtimes_{\hat{\rho}}H^0 - B\rtimes_{\sigma,v}H\rtimes_{\hat{\sigma}}H^0$- bimodule, $X\otimes M_N(\mathbf{C})$ is an exterior tensor product of X and the Hilbert $M_N(\mathbf{C})-M_N(\mathbf{C})$-bimodule $M_N(\mathbf{C})$. Furthermore, there are unitary elements $U\in(M(A)\rtimes_{\underline{\rho},u}H\rtimes_{\underline{\hat{\rho}}}H^0)\otimes H^0$ and $V\in(M(B)\rtimes_{\underline{\sigma},v}H\rtimes_{\underline{\hat{\sigma}}}H^0)\otimes H^0$ such that

$$U\hat{\hat{\lambda}}(x)V = ((\Psi_X\otimes \mathrm{id})\circ(\lambda\otimes \mathrm{id}_{M_N(\mathbf{C})})\circ\Psi_X^{-1})(x)$$

for any $x\in X\otimes M_N(\mathbf{C})$.

Proposition 3.8 has already been obtained in the case of Kac systems by Guo and Zhang [**10**], which is a generalization of the above result. Also, we have the following lemmas:

Lemma 3.9. *With the above notation, if X is full with both-sided inner products, then so is $X\rtimes_\lambda H$.*

Proof. Modifying the proof of [**17,** Lemma 4.5], yields the proof of Lemma 3.9. □

Lemma 3.10. *With the above notation, if $X\rtimes_\lambda H$ is full with both-sided inner products, then so is X.*

Proof. Since $X\rtimes_\lambda H$ is full with both-sided inner products, so is $X\rtimes_\lambda H\rtimes_{\hat{\lambda}} H^0$ by Lemma 3.9. Thus, $X\otimes M_N(\mathbf{C})$ is full with both-sided inner products by Proposition 3.8. Let f be a minimal projection in $M_N(\mathbf{C})$. Then,

$$\begin{aligned} A\otimes f &= (1_{M(A)}\otimes f)(A\otimes M_N(\mathbf{C}))(1_{M(A)}\otimes f) \\ &= (1\otimes f)\,\overline{{}_{A\otimes M_N(\mathbf{C})}\langle X\otimes M_N(\mathbf{C}),\, X\otimes M_N(\mathbf{C})\rangle}(1\otimes f) \\ &= \overline{{}_A\langle X, X\rangle\otimes fM_N(\mathbf{C})f} = \overline{{}_A\langle X, X\rangle}\otimes f. \end{aligned}$$

Hence, X is full with the left-sided inner product. Similarly, we can see that X is full with the right-sided inner product. Therefore, we obtain the conclusion. □

Definition 3.11. Let (ρ, u) and (σ, v) be twisted coactions of H^0 on C^*-algebras A and B, respectively. Then, (ρ, u) is *strongly Morita equivalent* to (σ, v) if there are an $A-B$-equivalence bimodule X and a twisted coaction λ of H^0 on X with respect to $(A, B, \rho, u, \sigma, v)$.

In the same manner as in [**17,** Section 3], we see that the strong Morita equivalence for twisted coactions of H^0 on C^*-algebras is an equivalence relation. Also, we obtain the following lemma in a similar manner to [**17,** Lemma 3.12] using approximate units in a C^*-algebra. It is given without its proof.

Lemma 3.12. *Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A. Then, the following conditions are equivalent*:

(1) *the twisted coactions (ρ, u) and (σ, v) are exterior equivalent*;
(2) *the twisted coactions (ρ, u) and (σ, v) are strongly Morita equivalent by a twisted coaction λ of H^0 on ${}_A A_A$, which is a linear map from ${}_A A_A$ to ${}_{A\otimes H^0} A \otimes H^0_{A\otimes H^0}$ where ${}_A A_A$ and ${}_{A\otimes H^0} A \otimes H^0_{A\otimes H^0}$ are regarded as an $A-A$-equivalence bimodule and an $A\otimes H^0 - A\otimes H^0$-equivalence bimodule in the usual way.*

Remark 3.13. Let A and B be C^*-algebras and σ a coaction of H^0 on B. Let X be an $A-B$-equivalence bimodule and λ a linear map from X to $X \otimes H^0$ satisfying

(1) $\lambda(xb) = \lambda(x)\sigma(b)$ for any $b \in B$, $x \in X$;
(2) $\sigma(\langle x, y\rangle_B) = \langle \lambda(x), \lambda(y)\rangle_{B\otimes H^0}$ for any $x, y \in X$;
(3) $(\mathrm{id}_X \otimes \epsilon^0) \circ \lambda = \mathrm{id}_X$;
(4) $(\lambda \otimes \mathrm{id}) \circ \lambda = (\mathrm{id} \otimes \Delta^0) \circ \lambda$.

We call $(B, X, \sigma, \lambda, H^0)$ a *right covariant system*, see [**17,** Definition 3.4]. Then, we construct an action "$\cdot$" of H on $\mathbf{K}_B(X)$ as follows. For any $a \in \mathbf{B}_B(X)$, $h \in H$ and $x \in X$,

$$[h \cdot a]x = h_{(1)} \cdot_\lambda a[S(h_{(2)}) \cdot_\lambda x].$$

If $a \in \mathbf{K}_B(X)$, we see that $h \cdot a \in \mathbf{K}_B(X)$. Thus, identifying A with $\mathbf{K}_B(X)$, we obtain an action of H on A.

4. Linking C^*-algebras and coactions on C^*-algebras. Let (ρ, u) and (σ, v) be twisted coactions of H^0 on C^*-algebras A and B, respectively. Suppose that there are a Hilbert $A-B$- bimodule X and a twisted coaction λ of H^0 on X with respect to $(A, B, \rho, u, \sigma, v)$. Let C be the linking C^*-algebra for X defined in [**6**]. By [**6,** Proposition 2.3], C is the C^*-algebra consisting of all 2×2-matrices

$$\begin{bmatrix} a & x \\ \widetilde{y} & b \end{bmatrix}, \quad a \in A,\ b \in B,\ x, y \in X,$$

where $\widetilde{y}$ denotes y as an element in $\widetilde{X}$, the dual Hilbert C^*-bimodule of X. Before defining the coaction of H^0 on C induced by the twisted coaction λ of H^0 on X, with respect to $(A, B, \rho, u, \sigma, v)$, we give the next remark.

Remark 4.1. We identify the $H^0 - H^0$-equivalence bimodule $\widetilde{H^0}$ with H^0 as the $H^0 - H^0$-equivalence bimodule by the map

$$\widetilde{H^0} \longrightarrow H^0 : \widetilde{\phi} \longmapsto \phi^*.$$

Also, we identify the Hilbert $B \otimes H^0 - A \otimes H^0$-bimodule $\widetilde{X \otimes H^0}$ with $\widetilde{X} \otimes H^0$ by the map

$$\widetilde{X \otimes H^0} \longrightarrow \widetilde{X} \otimes H^0 : \widetilde{x \otimes \phi} \longmapsto \widetilde{x} \otimes \phi^*.$$

Furthermore, we identify the linking C^*-algebra for $X \otimes H^0$, the Hilbert $A \otimes H^0 - B \otimes H^0$-bimodule with $C \otimes H^0$ by the isomorphism defined by

$$\Phi\left(\begin{bmatrix} a \otimes \phi_{11} & x \otimes \phi_{12} \\ \widetilde{y \otimes \phi_{21}} & b \otimes \phi_{22} \end{bmatrix} \right) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \otimes \phi_{11} + \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \otimes \phi_{12} + \begin{bmatrix} 0 & 0 \\ \widetilde{y} & 0 \end{bmatrix} \otimes \phi_{21}^* + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \otimes \phi_{22},$$

where $a \in A$, $b \in B$, $x, y \in X$ and $\phi_{ij} \in H^0$, $i, j = 1, 2$.

Let γ be the homomorphism of C to $C \otimes H^0$ defined by, for any $a \in A$, $b \in B$, $x, y \in X$,

$$\gamma\left(\begin{bmatrix} a & x \\ \widetilde{y} & b \end{bmatrix}\right) = \begin{bmatrix} \rho(a) & \lambda(x) \\ \widetilde{\lambda(y)} & \sigma(b) \end{bmatrix}.$$

Let w be the unitary element in $M(C)$ defined by $w = \left[\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right]$. By routine computation, (γ, w) is a twisted coaction of H^0 on C.

Remark 4.2.

(1) We note the twisted action of H on C induced by (γ, w) as follows: for any $a \in A, b \in B, x, y \in X$ and $h \in H$,

$$h \cdot_\gamma \begin{bmatrix} a & x \\ \widetilde{y} & b \end{bmatrix} = \begin{bmatrix} h \cdot_{\rho,u} a & h \cdot_\lambda x \\ \widetilde{S(h)^* \cdot_\lambda y} & h \cdot_{\sigma,v} b \end{bmatrix}.$$

(2) Let $\widetilde{\lambda}$ be a linear map from $\widetilde{X}$ to $\widetilde{X} \otimes H^0$ defined by, for any $x \in X$,

$$\widetilde{\lambda}(\widetilde{x}) = \widetilde{\lambda(x)}.$$

Then, $\widetilde{\lambda}$ is the twisted coaction of H^0 on $\widetilde{X}$ induced by λ. Also, the twisted action of H on $\widetilde{X}$ induced by $\widetilde{\lambda}$ is as follows: for any $x \in X, h \in H$,

$$h \cdot_{\widetilde{\lambda}} \widetilde{x} = \widetilde{S(h^*) \cdot_\lambda x}.$$

Let C_1 be the linking C^*-algebra for the Hilbert $A \rtimes_\rho H - B \rtimes_\sigma H$-bimodule $X \rtimes_\lambda H$. Then, we obtain the next lemma by Remarks 4.1 and 4.2.

Lemma 4.3. *With the above notation, there is an isomorphism π_1 of $C \rtimes_{\gamma,w} H$ onto C_1.*

Proof. Let π_1 be the map from $C \rtimes_{\gamma,w} H$ to C_1, defined by

$$\pi_1\left(\begin{bmatrix} a & x \\ \widetilde{y} & b \end{bmatrix} \rtimes_{\gamma,w} h\right) = \begin{bmatrix} a \rtimes_{\rho,u} h & x \rtimes_\lambda h \\ \{\widehat{u}(S(h_{(2)}), h_{(1)})^* [h_{(3)}^* \cdot_\lambda y] \rtimes_\lambda h_{(4)}^*\}^{\sim} & b \rtimes_{\sigma,v} h \end{bmatrix}$$

for any $a \in A$, $b \in B$, $x, y \in X$ and $h \in H$. Let θ_1 be the map from C_1 to $C \rtimes_{\gamma,w} H$, defined by

$$\theta_1\left(\begin{bmatrix} a \rtimes_{\rho,u} h & x \rtimes_\lambda l \\ \widetilde{y \rtimes_\lambda k} & b \rtimes_{\sigma,v} m \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \rtimes_{\gamma,w} h + \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \rtimes_{\gamma,w} l$$
$$+ \begin{bmatrix} 0 & 0 \\ \{[S(k_{(3)} \cdot_\lambda y)]\widehat{v}(S(k_{(2)}), k_{(1)})\}^{\widetilde{}} & 0 \end{bmatrix} \rtimes_{\gamma,w} k_{(4)}^* + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \rtimes_{\gamma,w} m$$

for any $a \in A$, $b \in B$, $x, y \in X$ and $h, k, l, m \in H$. Then, by routine computation, π_1 is a homomorphism of $C \rtimes_{\gamma,w} H$ to C_1, and θ_1 is a homomorphism of C_1 to $C \rtimes_{\gamma,w} H$, Moreover, we see that θ_1 is the inverse map of π_1. Therefore, we obtain the conclusion. □

By the proof of Lemma 4.3, we obtain the following corollary.

Corollary 4.4. *With the above notation, there is a Hilbert $B \rtimes_\sigma H - A \rtimes_\rho H$-bimodule isomorphism π of $\widetilde{X \rtimes_\lambda H}$ onto $\widetilde{X} \rtimes_{\widetilde{\lambda}} H$.*

Remark 4.5. Let γ_1 be a coaction of H on C_1, defined by

$$\gamma_1 = (\pi_1 \otimes \mathrm{id}_H) \circ \widehat{\gamma} \circ \pi_1^{-1}.$$

Then, by routine computation, for any $a \in A$, $b \in B$, $x, y \in X$ and $h, l, k, m \in H$,

$$\gamma_1\left(\begin{bmatrix} a \rtimes_{\rho,u} h & x \rtimes_\lambda l \\ \widetilde{y \rtimes_\lambda k} & b \rtimes_{\sigma,v} m \end{bmatrix}\right) = \begin{bmatrix} a \rtimes_{\rho,u} h_{(1)} & 0 \\ 0 & 0 \end{bmatrix} \otimes h_{(2)} + \begin{bmatrix} 0 & x \rtimes_\lambda l_{(1)} \\ 0 & 0 \end{bmatrix}$$
$$\otimes l_{(2)} + \begin{bmatrix} 0 & 0 \\ (y \rtimes_\lambda k_{(1)})^{\widetilde{}} & 0 \end{bmatrix} \otimes k_{(2)}^* + \begin{bmatrix} 0 & 0 \\ 0 & b \rtimes_{\sigma,v} m_{(1)} \end{bmatrix} \otimes m_{(2)}.$$

We give a result similar to **[15,** Theorem 6.4] for coactions of H^0 on a Hilbert C^*-bimodule, applying Proposition 2.17 to a linking C^*-algebra. Let ρ and σ be coactions of H^0 on A and B, respectively, and let X be a Hilbert $A - B$-bimodule. Let λ be a coaction of H^0 on X with respect to (A, B, ρ, σ). Let C be the linking C^*-algebra for X and γ the coaction of H^0 on C induced by ρ, σ and λ. As defined in Section 3, let

$$X^\lambda = \{x \in X \mid \lambda(x) = x \otimes 1^0\}.$$

Then, by Lemma 3.10, X^λ is a Hilbert $A^\rho - B^\sigma$- bimodule. Let C_0 be the linking C^*-algebra for X^λ.

We prove the next lemma in a straightforward way. Therefore, we give it with no proof.

Lemma 4.6. *With the above notation and assumptions, $C^\gamma = C_0$, where C^γ is the fixed point C^*-subalgebra of C for γ.*

Lemma 4.7. *With the above notation, if $\widehat{\underline{\rho}}(1 \rtimes_{\underline{\rho}} e) \sim (1 \rtimes_{\underline{\rho}} e) \otimes 1$ in $(M(A) \rtimes_{\underline{\rho}} H) \otimes H$ and $\widehat{\underline{\sigma}}(1 \rtimes_{\underline{\rho}} e) \sim (1 \rtimes_{\underline{\sigma}} e) \otimes 1$ in $(M(B) \rtimes_{\underline{\sigma}} H) \otimes H$, then $\widehat{\underline{\gamma}}(1_{M(C)} \rtimes_{\underline{\gamma}} e) \sim (1_{M(C)} \rtimes_{\underline{\gamma}} e) \otimes 1$ in $(M(C) \rtimes_{\underline{\gamma}} H) \otimes H$.*

Proof. By Remark 4.3, we identify $C \rtimes_\gamma H$ with C_1, the linking C^*-algebra for the Hilbert $A \rtimes_\rho H - A \rtimes_\rho H$-bimodule $X \rtimes_\lambda H$. Also, we identify $\widehat{\gamma}$ with γ_1, the coaction of H on C_1 defined in Remark 4.5. Hence,

$$\widehat{\underline{\gamma}}(1 \rtimes_{\underline{\gamma}} e) = \begin{bmatrix} 1 \rtimes_{\underline{\rho}} e_{(1)} & 0 \\ 0 & 0 \end{bmatrix} \otimes e_{(2)} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \rtimes_{\underline{\sigma}} e_{(1)} \end{bmatrix} \otimes e_{(2)}.$$

By the assumptions,

$$\begin{bmatrix} 1 \rtimes_{\underline{\rho}} e_{(1)} & 0 \\ 0 & 0 \end{bmatrix} \otimes e_{(2)} \sim \begin{bmatrix} 1 \rtimes_{\underline{\rho}} e & 0 \\ 0 & 0 \end{bmatrix} \otimes 1 \quad \text{in} \begin{bmatrix} M(A) \rtimes_{\underline{\rho}} H & 0 \\ 0 & 0 \end{bmatrix} \otimes H,$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \rtimes_{\underline{\sigma}} e_{(1)} \end{bmatrix} \otimes e_{(2)} \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \rtimes_{\underline{\sigma}} e \end{bmatrix} \otimes 1 \quad \text{in} \begin{bmatrix} 0 & 0 \\ 0 & M(A) \rtimes_{\underline{\sigma}} H \end{bmatrix} \otimes H.$$

Since $\left[\begin{smallmatrix} M(A) \rtimes_{\underline{\rho}} H & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 0 \\ 0 & M(A) \rtimes_{\underline{\sigma}} H \end{smallmatrix}\right]$ are C^*-subalgebras of $M(C_1)$ by the proof of Echterhoff and Raeburn [**9,** Proposition A.1],

$$\begin{bmatrix} 1 \rtimes_{\underline{\rho}} e_{(1)} & 0 \\ 0 & 0 \end{bmatrix} \otimes e_{(2)} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \rtimes_{\underline{\sigma}} e_{(1)} \end{bmatrix} \otimes e_{(2)} \sim \begin{bmatrix} 1 \rtimes_{\underline{\rho}} e & 0 \\ 0 & 1 \rtimes_{\underline{\sigma}} e \end{bmatrix} \otimes 1$$

in $M(C_1) \otimes H$. Therefore, we obtain the conclusion since $M(C_1) \otimes H$ is identified with $(M(C) \rtimes_{\underline{\gamma}} H) \otimes H$. □

By **[15,** Section 4], there is a unitary element $w^\rho \in M(A) \otimes H$ satisfying

$$w^{\rho *}((1 \rtimes_{\underline{\rho}} e) \otimes 1) w^\rho = \widehat{\underline{\rho}}(1 \rtimes_{\underline{\rho}} e),$$
$$U^\rho = w^\rho (z^{\rho *} \otimes 1),$$
$$z^\rho = (\mathrm{id}_{M(A)} \otimes \epsilon)(w^\rho) \in M(A)^{\underline{\rho}}.$$

Also, there is a unitary element $w^\sigma \in M(B) \otimes H$ satisfying

$$w^{\sigma *}((1 \rtimes_{\underline{\sigma}} e) \otimes 1) w^\sigma = \widehat{\underline{\sigma}}(1 \rtimes_{\underline{\sigma}} e),$$
$$U^\sigma = w^\sigma (z^{\sigma *} \otimes 1),$$
$$z^\sigma = (\mathrm{id}_{M(A)} \otimes \epsilon)(w^\sigma) \in M(A)^{\underline{\sigma}}.$$

Let $w^\gamma = \begin{bmatrix} w^\rho & 0 \\ 0 & w^\sigma \end{bmatrix} \in M(C) \otimes H$. Then, w^γ is a unitary element satisfying $w^{\gamma *}((1 \rtimes_{\underline{\gamma}} e) \otimes) w^\gamma = \widehat{\underline{\gamma}}(1 \rtimes_{\underline{\gamma}} e)$. Let $U^\gamma = w^\gamma (z^{\gamma *} \otimes 1)$, where $z^\gamma = (\mathrm{id}_{M(C)} \otimes \epsilon)(w^\gamma) \in M(C)^{\underline{\gamma}}$. Then, by Section 2, U^γ satisfies

$$\widehat{U}^\gamma(1^0) = 1, \qquad \widehat{U}^\gamma(\phi_{(1)}) c \widehat{U}^{\gamma *}(\phi_{(2)}) \in M(C)^{\underline{\gamma}}$$

for any $c \in M(C)^{\underline{\gamma}}$, $\phi \in H^0$. Let (η, u^γ) be a twisted coaction of H on C^γ induced by U^γ, which is defined in Section 2. Then, by the proof of Proposition 2.17, there is an isomorphism π_C of $C^\gamma \rtimes_{\eta, u^\gamma} H^0$ onto C, defined by

$$\pi_C(c \rtimes_{\eta, u^\gamma} \phi) = c \widehat{U}^\gamma(\phi)$$

for any $c \in C^\gamma, \phi \in H^0$, which satisfies

$$\gamma \circ \pi_C = (\pi_C \otimes \mathrm{id}_H) \circ \widehat{\eta}, \qquad E^{\eta . u^\gamma} = E^\gamma \circ \pi_C,$$

where E^{η, u^γ} and E^γ are the canonical conditional expectations from $C^\gamma \rtimes_{\eta, u^\gamma} H^0$ and C onto C^γ, respectively. Let $p = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}$, $q = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}$. Then p and q are projections in $M(C^\gamma)$. We note that $M(C^\gamma) = M(C)^{\underline{\gamma}}$ by Lemma 2.14.

Lemma 4.8. *With the above notation and assumptions,*

$$\pi_C(p \rtimes_{\underline{\eta}, u^\gamma} 1^0) = p, \qquad u^\gamma(p \otimes 1 \otimes 1) = (p \otimes 1 \otimes 1) u^\gamma,$$
$$\pi_C(q \rtimes_{\underline{\eta}, u^\gamma} 1^0) = q, \qquad u^\gamma(q \otimes 1 \otimes 1) = (q \otimes 1 \otimes 1) u^\gamma,$$

Proof. Note that C^γ is identified with the C^*-subalgebra $C^\gamma \rtimes_{\eta,u^\gamma} 1^0$ of $C^\gamma \rtimes_{\eta,u^\gamma} H^0$. Then, by [**25,** Proposition 2.12],

$$\begin{aligned} p &= E_1^{\eta,u^\gamma}(p \rtimes_{\underline{\eta},u^\gamma} 1^0) = E^\gamma(\pi_C(p \rtimes_{\underline{\eta},u^\gamma} 1^0)) \\ &= e \cdot_\gamma \pi_C(p \rtimes_{\underline{\eta},u^\gamma} 1^0) = \pi_C(e \cdot_{\widehat{\eta}} (p \rtimes_{\underline{\eta},u^\gamma} 1^0)) \\ &= \pi_C(p) = \pi_C(p \rtimes_{\underline{\eta},u^\gamma} 1^0) \end{aligned}$$

since $\gamma \circ \pi_C = (\pi_C \otimes \mathrm{id}_H) \circ \widehat{\eta}$. Similarly, we obtain that $\pi_C(q \rtimes_{\underline{\eta},u^\gamma} 1^0) = q$. Furthermore, by the definition of U^γ, $U^\gamma = \begin{bmatrix} U^\rho & 0 \\ 0 & U^\sigma \end{bmatrix} \in M(C) \otimes H$. Hence, $U^\gamma(p \otimes 1) = (p \otimes 1)U^\gamma$. Since

$$\widehat{u}^\gamma(\phi, \psi) = \widehat{U}^\gamma(\phi_{(1)})\widehat{U}^\gamma(\psi_{(1)})\widehat{U}^{\gamma *}(\phi_{(2)}\psi_{(2)})$$

for any $\phi, \psi \in H^0$, we see that $u^\gamma(p \otimes 1 \otimes 1) = (p \otimes 1 \otimes 1)u^\gamma$. Similarly, $u^\gamma(q \otimes 1 \otimes 1) = (q \otimes 1 \otimes 1)u^\gamma$. □

Let $\alpha = \eta|_{A^\rho}$, $\beta = \eta|_{B^\sigma}$ and $\mu = \eta|_{X^\lambda}$. Let $u^\rho = u^\gamma(p \otimes 1 \otimes 1)$ and $u^\sigma = u^\gamma(q \otimes 1 \otimes 1)$. Furthermore, let $\pi_A = \pi_C|_A$, $\pi_B = \pi_C|_B$ and $\pi_X = \pi_C|_X$. Then, (α, u^ρ) and (β, u^σ) are twisted coactions H^0 on A^ρ and B^σ, respectively, and μ is a twisted coaction of H^0 on X^λ with respect to $(A, B, \alpha, u^\rho, \beta, u^\sigma)$. Also, π_A and π_B are isomorphisms of $A^\rho \rtimes_{\alpha,u^\rho} H^0$ and $B^\sigma \rtimes_{\beta,u^\sigma} H^0$ onto A and B satisfying the results in Proposition 2.17, respectively. Furthermore, we obtain the following.

Theorem 4.9. *Let A and B be C^*-algebras and H a finite dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. Let ρ and σ be coactions of H^0 on A and B, respectively. Let λ be a coaction of H^0 on a Hilbert $A-B$-bimodule X with respect to (A, B, ρ, σ). We suppose that $\widehat{\underline{\rho}}(1 \rtimes_{\underline{\rho}} e) \sim (1 \rtimes_{\underline{\rho}} e) \otimes 1$ in $M(A) \rtimes_{\underline{\rho}} H$ and that $\widehat{\underline{\sigma}}(1 \rtimes_{\underline{\sigma}} e) \sim (1 \rtimes_{\underline{\sigma}} e) \otimes 1$ in $M(B) \rtimes_{\underline{\sigma}} H$. Then, there are a twisted coaction μ of H^0 on X^λ and a bijective linear map π_X from $X^\lambda \rtimes_\mu H^0$ onto X satisfying the following conditions*:

(1) $\pi_X((a \rtimes_{\alpha,u^\rho} \phi)(x \rtimes_\mu \psi)) = \pi_A(a \rtimes_{\alpha,u^\rho} \phi)\pi_X(x \rtimes_\mu \psi)$;
(2) $\pi_X((x \rtimes_\mu \phi)(b \rtimes_{\beta,u^\sigma} \psi)) = \pi_X(x \rtimes_\mu \phi)\pi_B(b \rtimes_{\beta,u^\sigma} \psi)$;
(3) $\pi_A({}_{A^\rho \rtimes_{\alpha,u^\rho} H^0}\langle x \rtimes_\mu \phi,\, y \rtimes_\mu \psi\rangle) = {}_A\langle \pi_X(x \rtimes_\mu \phi),\, \pi_X(y \rtimes_\mu \psi)\rangle$;
(4) $\pi_B(\langle x \rtimes_\mu \phi,\, y \rtimes_\mu \psi\rangle_{B^\sigma \rtimes_{\beta,u^\sigma} H^0}) = \langle \pi_X(x \rtimes_\mu \phi),\, \pi_X(y \rtimes_\mu \psi)\rangle_B$;
(5) $h \cdot_\lambda \pi_X(x \rtimes_\mu \phi) = \pi_X(h \cdot_{\widehat{\mu}} (x \rtimes_\mu \phi))$ *for any* $x, y \in X^\lambda$, $a \in A^\rho$, $b \in B^\sigma$, $h \in H$, $\phi, \psi \in H^0$.

Proof. Using the above discussion, we can prove the theorem in a straightforward manner. □

Let A be a unital C^*-algebra and ρ a coaction of H^0 on A. Let $\mathbf{K}$ be the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space. Let $A^s = A \otimes \mathbf{K}$ and $\rho^s = \rho \otimes \mathrm{id}$. We identify $H^0 \otimes \mathbf{K}$ with $\mathbf{K} \otimes H^0$. Then, ρ^s is a coaction of H^0 on A^s.

Lemma 4.10. *With the above notation, ρ and ρ^s are strongly Morita equivalent.*

Proof. Immediate by routine computation. □

Let A and B be unital C^*-algebras. Let ρ and σ be coactions of H^0 on A and B, respectively. Suppose that ρ and σ are strongly Morita equivalent. Also, suppose that there are an $A-B$-equivalence bimodule X and a coaction λ of H^0 on X with respect to (A, B, ρ, σ). Let C be the linking C^*-algebra for X and γ the coaction of H^0 on C induced by ρ, σ and λ, which is defined above. Let $A^s = A \otimes \mathbf{K}$, $B^s = B \otimes \mathbf{K}$ and $C^s = C \otimes \mathbf{K}$. Let $X^s = X \otimes \mathbf{K}$ be the exterior tensor product of X and $\mathbf{K}$, which is an $A^s - B^s$-equivalence bimodule in the usual way. Let $\rho^s = \rho \otimes \mathrm{id}$, $\sigma^s = \sigma \otimes \mathrm{id}$ and $\gamma^s = \gamma \otimes \mathrm{id}$. Let $\lambda^s = \lambda \otimes \mathrm{id}$, which is a coaction of H^0 on X^s. Let

$$p = \begin{bmatrix} 1_A \otimes 1_{M(\mathbf{K})} & 0 \\ 0 & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \otimes 1_{M(\mathbf{K})} \end{bmatrix}.$$

Then, p and q are full projections in $M(C^s)$ and $A^s \cong pC^s p$, $B^s \cong qC^s q$. We identify A^s and B^s with $pC^s p$ and $qC^s q$, respectively. By [**4,** Lemma 2.5], there is a partial isometry $w \in M(C^s)$ such that $w^* w = p$, $ww^* = q$. Let θ be a map from A^s to C^s, defined by

$$\theta(a) = waw^* = w \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} w^*$$

for any $a \in A$. Since $w^* w = p$ and $ww^* = q$, by easy computation, we see that θ is an isomorphism of A^s onto B^s.

Proposition 4.11. *With the above notation, there is a unitary element* $u \in M(B^s) \otimes H^0$ *such that*

$$(\theta \otimes \mathrm{id}_{H^0}) \circ \rho^s \circ \theta^{-1} = \mathrm{Ad}(u) \circ \sigma^s,$$
$$(u \otimes 1^0)(\underline{\sigma^s} \otimes \mathrm{id}_{H^0})(u) = (\mathrm{id}_{M(B^s)} \otimes \Delta^0)(u),$$

where $\underline{\sigma^s}$ *is the strictly continuous coaction of* H^0 *on* $M(B^s)$ *extending the coaction* σ^s *of* H^0 *on* B^s.

Proof. We note that $\theta = \mathrm{Ad}(w)$. Since $\rho^s = \gamma^s|_{A^s}$ and $\sigma^s = \gamma^s|_{B^s}$, we obtain that

$$(\theta \otimes \mathrm{id}_{H^0}) \circ \rho^s \circ \theta^{-1} = \mathrm{Ad}((w \otimes 1^0)\underline{\gamma^s}(w^*)) \circ \sigma^s,$$

where $\underline{\gamma^s}$ is the strictly continuous coaction of H^0 on $M(C^s)$ extending the coaction γ^s of H^0 on C^s. Let $u = (w \otimes 1^0)\underline{\gamma^s}(w^*)$. By routine computation, we can show that u is a desired unitary element in $M(B^s) \otimes H^0$. □

5. Equivariant Picard groups. Following [**11**], we shall define the equivariant Picard group of a C^*-algebra.

Let A be a C^*-algebra and H a finite dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. Let (ρ, u) be a twisted coaction of H^0 on A. We denote by (X, λ) a pair of an $A-A$-equivalence bimodule X and a twisted coaction λ of H^0 on X with respect to (A, A, ρ, u, ρ, u). Let $\mathrm{Equi}_H^{\rho,u}(A)$ be the set of all such pairs (X, λ) as above. We define an equivalence relation $\sim$ in $\mathrm{Equi}_H^{\rho,u}(A)$ as follows: for $(X, \lambda), (Y, \mu) \in \mathrm{Equi}_H^{\rho,u}(A)$, $(X, \lambda) \sim (Y, \mu)$ if and only if there is an $A-A$-equivalence bimodule isomorphism π of X onto Y such that $\mu \circ \pi = (\pi \otimes \mathrm{id}_{H^0}) \circ \lambda$, that is, for any $x \in X$ and $h \in H$, $\pi(h \cdot_\lambda x) = h \cdot_\mu \pi(x)$. We denote by $[X, \lambda]$ the equivalence class of (X, λ) in $\mathrm{Equi}_H^{\rho,u}(A)$. Let $\mathrm{Pic}_H^{\rho,u}(A) = \mathrm{Equi}_H^{\rho,u}(A)/\sim$. We define the product in $\mathrm{Pic}_H^{\rho,u}(A)$ as follows: for $(X, \lambda), (Y, \mu) \in \mathrm{Equi}_H^{\rho,u}(A)$,

$$[X, \lambda][Y, \mu] = [X \otimes_A Y, \lambda \otimes \mu],$$

where $\lambda \otimes \mu$ is the twisted coaction of H^0 on $X \otimes_A Y$ induced by the action "$\cdot_{\lambda\otimes\mu}$" of H on $X \otimes_A Y$ defined in [**17,** Proposition 3.1]. By simple computation, we see that the above product is well defined. We regard A as an $A-A$-equivalence bimodule in the usual way. Sometimes it is denoted by ${}_AA_A$. Also, we can regard a twisted coaction ρ of H^0

on C^*-algebra A as a twisted coaction of H^0 on the $A-A$-equivalence bimodule ${}_AA_A$ with respect to (A, A, ρ, u, ρ, u). Then, $[{}_AA_A, \rho]$ is the unit element in $\mathrm{Pic}_H^{\rho,u}(A)$. Let $\widetilde{\lambda}$ be the coaction of H^0 on $\widetilde{X}$ defined by $\widetilde{\lambda}(\widetilde{x}) = \widetilde{\lambda(x)}$ for any $x \in X$, which is also defined in Remark 4.2 (2). Then, we see that $[\widetilde{X}, \widetilde{\lambda}]$ is the inverse element of $[X, \lambda]$ in $\mathrm{Pic}_H^{\rho,u}(A)$. By the above product, $\mathrm{Pic}_H^{\rho,u}(A)$ is a group. We call it the (ρ, u, H)-*equivariant Picard group* of A.

Let $\mathrm{Aut}_H^{\rho,u}(A)$ be the group of all automorphisms α of A satisfying that $(\alpha\otimes\mathrm{id}_{H^0})\circ\rho = \rho\circ\alpha$, $(\underline{\alpha}\otimes\mathrm{id}\otimes\mathrm{id})(u) = u$ and let $\mathrm{Int}_H^{\rho,u}(A)$ be the set of all generalized inner automorphisms $\mathrm{Ad}(v)$ of A satisfying that $\underline{\rho}(v) = v\otimes 1^0$, $(v\otimes 1^0\otimes 1^0)u = u(v\otimes 1^0\otimes 1^0)$, where v is a unitary element in $M(A)$. By easy computation, $\mathrm{Int}_H^{\rho,u}(A)$ is a normal subgroup of $\mathrm{Aut}_H^{\rho,u}(A)$. Modifying [**5**], for each $\alpha \in \mathrm{Aut}_H^{\rho,u}(A)$, we construct the element $(X_\alpha, \lambda_\alpha) \in \mathrm{Equi}_H^{\rho,u}(A)$ as follows: let $\alpha \in \mathrm{Aut}_H^{\rho,u}(A)$. Let X_α be the vector space A with the obvious left action of A on X_α and the obvious left A-valued inner product, but define the right action of A on X_α by $x\cdot a = x\alpha(a)$ for any $x \in X_\alpha$, $a \in A$ and the right A-valued inner product by $\langle x, y\rangle_A = \alpha^{-1}(x^*y)$ for any $x, y \in X_\alpha$. Then, by [**5**], X_α is an $A-A$-equivalence bimodule. Also, ρ may be regarded as a linear map from X_α to an $A\otimes H^0 - A\otimes H^0$-equivalence bimodule $X_\alpha\otimes H^0$. We denote it by λ_α. By simple computation, λ_α is a twisted coaction of H^0 on X_α with respect to (A, A, ρ, u, ρ, u). Thus, we obtain the map Φ,

$$\Phi : \mathrm{Aut}_H^{\rho,u}(A) \longrightarrow \mathrm{Pic}_H^{\rho,u}(A) : \alpha \longmapsto [X_\alpha, \lambda_\alpha].$$

Modifying [**5**], we see that the map Φ is a homomorphism of $\mathrm{Aut}_H^{\rho,u}(A)$ to $\mathrm{Pic}_H^{\rho,u}(A)$. This yields a similar result to [**5,** Proposition 3.1].

Proposition 5.1. *With the above notation, we have the exact sequence*

$$1 \longrightarrow \mathrm{Int}_H^{\rho,u}(A) \xrightarrow{\imath} \mathrm{Aut}_H^{\rho,u}(A) \xrightarrow{\Phi} \mathrm{Pic}_H^{\rho,u}(A),$$

where $\imath$ is the inclusion map of $\mathrm{Int}_H^{\rho,u}(A)$ *to* $\mathrm{Aut}_H^{\rho,u}(A)$.

Proof. Modifying the proof of [**5,** Proposition 3.1], we shall prove this proposition. Let v be a unitary element in $M(A)$ with $\underline{\rho}(v) = v\otimes 1^0$, $(v\otimes 1^0\otimes 1^0)u = u(v\otimes 1^0\otimes 1^0)$. We show that $[X_{\mathrm{Ad}(v)}, \lambda_{\mathrm{Ad}(v)}] = [{}_AA_A, \rho]$ in $\mathrm{Pic}_H^{\rho,u}(A)$. Let π be the map from ${}_AA_A$ to $X_{\mathrm{Ad}(v)}$ defined by $\pi(a) = av^*$ for any $a \in {}_AA_A$. Then π is an $A-A$-equivalence bimodule isomorphism. Also, for any $a \in {}_A A_A$ and $h \in H$,

$$\begin{aligned} h \cdot_{\lambda_{\mathrm{Ad}(v)}} \pi(a) &= h \cdot_{\lambda_{\mathrm{Ad}(v)}} (av^*) \\ &= [h_{(1)} \cdot_\rho a][h_{(2)} \cdot_{\underline{\rho}} v^*] \\ &= [h \cdot_\rho a]v^* = \pi(h \cdot_\rho a). \end{aligned}$$

Thus, $[X_{\mathrm{Ad}(v)}, \lambda_{\mathrm{Ad}(v)}] = [{}_A A_A, \rho]$ in $\mathrm{Pic}_H^{\rho,u}(A)$. Conversely, let $\alpha \in \mathrm{Aut}_H^{\rho,u}(A)$ with $[X_\alpha, \lambda_\alpha] = [{}_A A_A, \rho]$ in $\mathrm{Pic}_H^{\rho,u}(A)$. Then, there is an $A-A$-equivalence bimodule isomorphism π of ${}_A A_A$ onto X_α such that

$$\lambda_\alpha \circ \pi = (\pi \otimes \mathrm{id}) \circ \rho.$$

By the proof of [**5,** Proposition 3.1], $(\pi \circ \alpha^{-1}, \pi)$ is a double centralizer of A. Hence, $(\pi \circ \alpha^{-1}, \pi) \in M(A)$. Let $v = (\pi \circ \alpha^{-1}, \pi)$. Then, v is a unitary element in $M(A)$ such that $\alpha = \mathrm{Ad}(v^*)$. Furthermore, since $\lambda_\alpha \circ \pi = (\pi \otimes \mathrm{id}) \circ \rho$, for any $a \in A$, $\lambda_\alpha(\pi(a)) = (\pi \otimes \mathrm{id})(\rho(a))$. It follows that $\rho(av^*) = \rho(a)(v \otimes 1^0)^*$ for any $a \in A$, that is, $\underline{\rho}(v) = v \otimes 1^0$. Also, since $(\rho \otimes \mathrm{id}) \circ \rho = \mathrm{Ad}(u) \circ (\mathrm{id} \otimes \Delta^0) \circ \rho$, $(v \otimes 1^0 \otimes 1^0)u = u(v \otimes 1^0 \otimes 1^0)$. Therefore, we obtain the conclusion. □

Next, we shall show a similar result to [**5,** Corollary 3.5]. Let A be a C^*-algebra and X an $A-A$-equivalence bimodule. Let ρ be a coaction of H^0 on A and λ a coaction of H^0 on X with respect to (A, A, ρ, ρ). Let C be the linking C^*-algebra for X and γ the coaction of H^0 on C induced by ρ and λ which is defined in Section 4. Furthermore, suppose that A is unital and that $\widehat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(A \rtimes_\rho H) \otimes H$. Then ρ is saturated by [**15,** Section 4]. Let $(\widehat{\rho})^s$ be the coaction of H on $(A \rtimes_\rho H)^s \otimes H$ induced by the dual coaction $\widehat{\rho}$ of H on $A \rtimes_\rho H$. Also, let $(\rho^s)\widehat{}$ be the dual coaction of ρ^s which is a coaction of H on $A^s \rtimes_{\rho^s} H$. By their definitions, we can see that $(\widehat{\rho})^s = (\rho^s)\widehat{}$, where we identify $(A \rtimes_\rho H)^s$ with $A^s \rtimes_{\rho^s} H$. We denote them by $\widehat{\rho^s}$.

Lemma 5.2. *With the above notation, if* $\widehat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ *in* $(A \rtimes_\rho H) \otimes H$, *then* $\underline{\widehat{\rho}}^s(1 \rtimes_{\underline{\rho^s}} e) \sim (1 \rtimes_{\underline{\rho^s}} e) \otimes 1$ *in* $(M(A^s) \rtimes_{\underline{\rho^s}} H) \otimes H$.

Proof. Immediate by straightforward computation. □

Let C be the linking C^*-algebra for an $A^s - A^s$-equivalence bimodule X^s and γ the coaction of H on C induced by ρ^s and λ^s.

Lemma 5.3. *With the above notation, if* $\widehat{\rho}(1\rtimes_\rho e)\sim(1\rtimes_\rho e)\otimes 1$ *in* $(A\rtimes_\rho H)\otimes H$*, then* $\widehat{\gamma}(1_{M(C)}\rtimes_{\underline{\gamma}} e)\sim(1_{M(C)}\rtimes_{\underline{\gamma}} e)\otimes 1$ *in* $(M(C)\rtimes_{\underline{\gamma}} H)\otimes H$*.*

Proof. Immediate by Lemmas 4.7 and 5.2. □

Lemma 5.4. *With the above notation, we suppose that* $\widehat{\rho}(1\rtimes_\rho e)\sim(1\rtimes_\rho e)\otimes 1$ *in* $(A\rtimes_\rho H)\otimes H$*. Let* Φ *be the homomorphism of* $\mathrm{Aut}_H^{\rho^s}(A^s)$ *to* $\mathrm{Pic}_H^{\rho^s}(A^s)$ *defined by* $\Phi(\alpha)=[X_\alpha,\,\lambda_\alpha]$ *for any* $\alpha\in\mathrm{Aut}_H^{\rho^s}(A^s)$*. Then,* Φ *is surjective.*

Proof. Let $[X,\lambda]$ be any element in $\mathrm{Pic}_H^{\rho^s}(A^s)$. Let

$$X^\lambda=\{x\in X\mid\lambda(x)=x\otimes 1^0\}.$$

Since $\widehat{\rho}(1\rtimes_\rho e)\sim(1\rtimes_\rho e)\otimes 1$ in $(A\rtimes_\rho H)\otimes H$, by Lemma 5.2, $\widehat{\underline{\rho}}^s(1\rtimes_{\underline{\rho^s}} e)\sim(1\rtimes_{\underline{\rho^s}} e)\otimes 1$ in $(M(A^s)\rtimes_{\underline{\rho^s}} H)\otimes H$. Since X is an A^s-A^s-equivalence bimodule, by Lemma 3.10 and Theorem 4.9, X^λ is an $(A^s)^{\rho^s}-(A^s)^{\rho^s}$-equivalence bimodule, where $(A^s)^{\rho^s}$ is the fixed point C^*-subalgebra of A^s for the coaction ρ^s. Let C be the linking C^*-algebra for X and γ the coaction of H^0 on C induced by ρ^s and λ. Let C^γ be the fixed point C^*-algebra of C for γ. Then, by Lemma 4.6, C^γ is isomorphic to C_0, the linking C^*-algebra for X^λ. We identify C^γ with C_0. Let

$$p=\begin{bmatrix}1_A\otimes 1_{M(\mathbf{K})} & 0\\ 0 & 0\end{bmatrix},\qquad q=\begin{bmatrix}0 & 0\\ 0 & 1_A\otimes 1_{M(\mathbf{K})}\end{bmatrix}.$$

Then p and q are projections in $M(C)^{\underline{\gamma}}$. Since $M(C)^{\underline{\gamma}}=M(C^\gamma)$ by Lemmas 2.14 and 4.7, p and q are full for C^γ. By the proof of [**5,** Theorem 3.4], there is a partial isometry $w\in M(C)^{\underline{\gamma}}$ such that

$$w^*w=p,\qquad q=ww^*.$$

Hence, $w\in M(C)$. Let α be the map on A^s, defined by

$$\alpha(a)=w^*aw=w^*\begin{bmatrix}0&0\\0&a\end{bmatrix}w$$

for any $a\in A^s$. By routine computation, α is an automorphism of A^s. Let π be a linear map from X to X_α, defined by

$$\pi(x)=\begin{bmatrix}0&x\\0&0\end{bmatrix}w=p\begin{bmatrix}0&x\\0&0\end{bmatrix}wp$$

for any $x \in X$. In the same manner as in the proof of [**5,** Lemma 3.3], we can see that π is an $A^s - A^s$-equivalence bimodule isomorphism of X onto X_α. For any $a \in A^s$,

$$\begin{aligned}(\rho^s \circ \alpha)(a) &= \rho^s(w^*aw) = \gamma\left(w^* \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} w\right) \\ &= \underline{\gamma}(w^*) \begin{bmatrix} 0 & 0 \\ 0 & \rho^s(a) \end{bmatrix} \underline{\gamma}(w) \\ &= (\alpha \otimes \mathrm{id}_{H^0})(\rho^s(a))\end{aligned}$$

since $w \in M(C)^{\underline{\gamma}}$. Hence, $\alpha \in \mathrm{Aut}_H^{\rho^s}(A^s)$. Furthermore, for any $x \in X$,

$$\begin{aligned}(\lambda_\alpha \circ \pi)(x) &= \lambda_\alpha\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} w\right) = \rho^s\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} w\right) \\ &= \gamma\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} w\right) = \begin{bmatrix} 0 & \lambda(x) \\ 0 & 0 \end{bmatrix} (w \otimes 1^0) \\ &= (\pi \otimes \mathrm{id}_{H^0})(\lambda(x)),\end{aligned}$$

where we identify $\mathbf{K} \otimes H^0$ with $H^0 \otimes \mathbf{K}$. Thus, $\Phi(\alpha) = [X, \lambda]$. Therefore, we obtain the conclusion. □

Theorem 5.5. *Let A be a unital C^*-algebra and ρ a coaction of H^0 on A. We suppose that $\widehat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(A \rtimes_\rho H) \otimes H$. Then, we have the following exact sequence:*

$$1 \longrightarrow \mathrm{Int}_H^{\rho^s}(A^s) \xrightarrow{\imath} \mathrm{Aut}_H^{\rho^s}(A^s) \xrightarrow{\Phi} \mathrm{Pic}_H^{\rho^s}(A^s) \longrightarrow 1,$$

where $\imath$ is the inclusion map of $\mathrm{Int}_H^{\rho^s}(A^s)$ to $\mathrm{Aut}_H^{\rho^s}(A^s)$.

Proof. Immediate by Proposition 5.1 and Lemma 5.4. □

Since the following lemma is obtained in a straightforward manner, we omit its proof.

Lemma 5.6. *Let (ρ, u) and (σ, v) be twisted coactions on C^*-algebras A and B, respectively. We suppose that (ρ, u) is strongly Morita equivalent to (σ, v). Then, $\mathrm{Pic}_H^{\rho,u}(A) \cong \mathrm{Pic}_H^{\sigma,v}(B)$.*

6. Ordinary and equivariant Picard groups. In this section, we shall investigate the relation between ordinary and equivariant Picard groups. Let ρ be a coaction of H^0 on a C^*-algebra A, and let f_ρ be the map from $\mathrm{Pic}_H^\rho(A)$ to $\mathrm{Pic}(A)$, defined by

$$f_\rho : \mathrm{Pic}_H^\rho(A) \longrightarrow \mathrm{Pic}(A) : [X, \lambda] \longmapsto [X],$$

where $\mathrm{Pic}(A)$ is the ordinary Picard group of A. Clearly, f_ρ is a homomorphism of $\mathrm{Pic}_H^\rho(A)$ to $\mathrm{Pic}(A)$. Let $\mathrm{Aut}(A)$ be the group of all automorphisms of A, and let $\alpha \in \mathrm{Aut}(A)$. Let X_α be the $A-A$-equivalence bimodule induced by α defined in Section 5. Let λ be a coaction of H^0 on X_α with respect to (A, A, ρ, ρ). Then, for any $a \in A$ and $x, y \in X_\alpha$,

(1) $\lambda(ax) = \lambda(a \cdot x) = \rho(a) \cdot \lambda(x) = \rho(a)\lambda(x)$;
(2) $\lambda(x\alpha(a)) = \lambda(x \cdot a) = \lambda(x) \cdot \rho(a) = \lambda(x)(\alpha \otimes \mathrm{id})(\rho(a))$;
(3) $\rho(xy^*) = \rho({}_A\langle x, y\rangle) = {}_{A\otimes H^0}\langle \lambda(x), \lambda(y)\rangle = \lambda(x)\lambda(y)^*$;
(4) $\rho(\alpha^{-1}(x^*y))=\rho(\langle x, y\rangle_A)=\langle \lambda(x), \lambda(y)\rangle_{A\otimes H^0}=(\alpha^{-1}\otimes\mathrm{id})(\lambda(x)^*\lambda(y))$;
(5) $(\mathrm{id} \otimes \epsilon^0)(\lambda(x)) = x$;
(6) $(\lambda \otimes \mathrm{id})(\lambda(x)) = (\mathrm{id} \otimes \Delta^0)(\lambda(x))$.

Let $\{u_\gamma\}$ be an approximate unit of A. Then, $\lambda(u_\gamma) \in X_\alpha \otimes H^0$. Since $X_\alpha = A$ as vector spaces, we regard $\lambda(u_\gamma)$ as an element in $A \otimes H^0$.

Lemma 6.1. *With the above notation, we regard $\lambda(u_\gamma)$ as an element in $A \otimes H^0$. Then, $\{\lambda(u_\gamma)\}$ strictly converges to a unitary element in $M(A \otimes H^0)$, and the unitary element does not depend upon the choice of an approximate unit of A.*

Proof. Let $a \in A$ and $x \in A \otimes H^0$. Then, by equation (2),

$$\begin{aligned} \|(\lambda(u_\gamma) - \lambda(u_{\gamma'}))(\alpha \otimes \mathrm{id})(\rho(a)x)\| &= \|\lambda((u_\gamma - u_{\gamma'})\alpha(a))(\alpha \otimes \mathrm{id})(x)\| \\ &\le \|\lambda((u_\gamma - u_{\gamma'})\alpha(a))\|\|x\| \\ &= \|(u_\gamma - u_{\gamma'})\alpha(a)\|\|x\| \end{aligned}$$

since λ is isometric. Since $\rho(A)(A\otimes H^0)$ is dense in $A\otimes H^0$, $\{\lambda(u_\gamma)y\}$ is a Cauchy net for any $y \in A \otimes H^0$. Similarly, by equation (1), $\{y\lambda(u_\gamma)\}$ is also a Cauchy net for any $y \in A \otimes H^0$. Thus, $\{\lambda(u_\gamma)\}$ converges to some element $u \in M(A \otimes H^0)$ strictly. We note

$$\lim_{\gamma\to\infty} \rho(u_\gamma) = \lim_{\gamma\to\infty} \underline{\rho}\,(u_\gamma) = \underline{\rho}(\lim_{\gamma\to\infty} u_\gamma) = \underline{\rho}(1) = 1,$$
$$\lim_{\gamma\to\infty} \alpha^{-1}(u_\gamma) = \lim_{\gamma\to\infty} \underline{\alpha}^{-1}(u_\gamma) = \underline{\alpha}^{-1}(\lim_{\gamma\to\infty} u_\gamma) = \underline{\alpha}^{-1}(1) = 1,$$

where the limits are taken under the strict topologies in $M(A\otimes H^0)$ and $M(A)$, respectively, and $\underline{\alpha}^{-1}$ is an automorphism of $M(A)$ extending α^{-1} to $M(A)$, which is strictly continuous on $M(A)$. Hence, by equations (3) and (4), we can see that u is a unitary element in $M(A \otimes H^0)$. Let $\{v_\beta\}$ be another approximate unit of A, and let v be the limit of $\lambda(v_\beta)$ under the strict topology in $M(A \otimes H^0)$. Then, by the above discussion, we have that

$$\|(\lambda(u_\gamma) - \lambda(v_\beta))(\alpha \otimes \mathrm{id})(\rho(a)x)\| \le \|(u_\gamma - v_\beta)\alpha(a)\|\|x\|$$

for any $a \in A$ and $x \in A\otimes H^0$. Since $\rho(A)(A\otimes H^0)$ is dense in $A\otimes H^0$, $u = v$. □

Lemma 6.2. *Let u be as in the proof of Lemma* 6.1. *Then, u satisfies $\lambda(x) = \rho(x)u$ for any $x \in X_\alpha$, $\rho(\alpha(a)) = u(\alpha \otimes \mathrm{id})(\rho(a))u^*$ for any $a \in A$ and $(\underline{\rho} \otimes \mathrm{id})(u)(u \otimes 1^0) = (\mathrm{id} \otimes \Delta^0)(u)$.*

Proof. Let $\{u_\gamma\}$ be an approximate unit of A. By equation (1), for any $x \in X_\alpha$, $\lambda(xu_\gamma) = \rho(x)\lambda(u_\gamma)$. Thus, $\lambda(x) = \rho(x)u$. Also, by equation (2) for any $a \in A$,

$$\lambda(u_\gamma\alpha(a)) = \lambda(u_\gamma)(\alpha \otimes \mathrm{id})(\rho(a)).$$

Hence, $\lambda(\alpha(a)) = u(\alpha \otimes \mathrm{id})(\rho(a))$. Since $\lambda(\alpha(a)) = \rho(\alpha(a))u$ for any $a \in A$ by the above discussion, for any $a \in A$,

$$\rho(\alpha(a))u = u(\alpha \otimes \mathrm{id})(\rho(a))$$

for any $a \in A$. Since u is a unitary element in $M(A \otimes H^0)$,

$$\rho(\alpha(a)) = u(\alpha \otimes \mathrm{id})(\rho(a))u^*$$

for any $a \in A$. Furthermore, for any $a \in A$,

$$\begin{aligned}(\lambda \otimes \mathrm{id})(\lambda(u_\gamma a)) &= (\lambda \otimes \mathrm{id})(\lambda(u_\gamma)(\alpha \otimes \mathrm{id})(\rho(\alpha^{-1}(a))) \\ &= (\rho \otimes \mathrm{id})(\lambda(u_\gamma))((\lambda \otimes \mathrm{id}) \circ (\alpha \otimes \mathrm{id}) \circ \rho \circ \alpha^{-1})(a),\end{aligned}$$

by equations (1) and (2). Thus, equation (2) yields

$$
\begin{aligned}
(\lambda\otimes\mathrm{id})(\lambda(a)) &= (\underline{\rho}\otimes\mathrm{id})(u)((\lambda\otimes\mathrm{id})\circ(\alpha\otimes\mathrm{id})\circ\rho\circ\alpha^{-1})(a) \\
&= \lim_{\gamma\to\infty}(\underline{\rho}\otimes\mathrm{id})(u) \\
&\quad\cdot(\lambda\otimes\mathrm{id})((u_\gamma\otimes 1^0)((\alpha\otimes\mathrm{id})\circ\rho\circ\alpha^{-1})(a)) \\
&= \lim_{\gamma\to\infty}(\underline{\rho}\otimes\mathrm{id})(u)(\lambda(u_\gamma)\otimes 1^0) \\
&\quad\cdot((\alpha\otimes\mathrm{id}\otimes\mathrm{id})\circ(\rho\otimes\mathrm{id})\circ\rho\circ\alpha^{-1})(a) \\
&= (\underline{\rho}\otimes\mathrm{id})(u)(u\otimes 1^0) \\
&\quad\cdot((\alpha\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta^0)\circ\rho\circ\alpha^{-1})(a) \\
&= (\underline{\rho}\otimes\mathrm{id})(u)(u\otimes 1^0)((\mathrm{id}\otimes\Delta^0\circ(\alpha\otimes\mathrm{id})\circ\rho\circ\alpha^{-1})(a).
\end{aligned}
$$

Also, by equation (2),

$$
\begin{aligned}
(\mathrm{id}\otimes\Delta^0)(\lambda(u_\gamma a)) &= (\mathrm{id}\otimes\Delta^0)(\lambda(u_\gamma)((\alpha\otimes\mathrm{id})\circ\rho\circ\alpha^{-1})(a)) \\
&= (\mathrm{id}\otimes\Delta^0)(\lambda(u_\gamma)) \\
&\quad\cdot((\mathrm{id}\otimes\Delta^0)\circ(\alpha\otimes\mathrm{id})\circ\rho\circ\alpha^{-1})(a).
\end{aligned}
$$

Thus,

$$
(\mathrm{id}\otimes\Delta^0)(\lambda(a)) = (\mathrm{id}\otimes\Delta^0)(u)((\mathrm{id}\otimes\Delta^0)\circ(\alpha\otimes\mathrm{id})\circ\rho\circ\alpha^{-1})(a).
$$

By equation (6),

$$
[(\rho\otimes\mathrm{id})(u)(u\otimes 1^0)-((\mathrm{id}\otimes\Delta^0)(u)]((\mathrm{id}\otimes\Delta^0)\circ(\alpha\otimes\mathrm{id})\circ\rho\circ\alpha^{-1})(a)=0
$$

for any $a\in A$. Therefore,

$$
(\rho\otimes\mathrm{id})(u)(u\otimes 1^0) = (\mathrm{id}\otimes\Delta^0)(u). \qquad \square
$$

Remark 6.3. By Lemma 6.2, we can see that the coaction $(\alpha\otimes\mathrm{id})\circ\rho\circ\alpha^{-1}$ of H^0 on A is exterior equivalent to ρ.

Conversely, let u be a unitary element in $M(A\otimes H^0)$ satisfying

$$
\begin{aligned}
\rho &= \mathrm{Ad}(u)\circ(\alpha\otimes\mathrm{id})\circ\rho\circ\alpha^{-1}, \\
(\underline{\rho}\otimes\mathrm{id})(u)(u\otimes 1^0) &= (\mathrm{id}\otimes\Delta^0)(u).
\end{aligned}
$$

Let λ_u be the linear map from X_α to $X_\alpha \otimes H^0$, defined by

$$\lambda_u(x) = \rho(x)u$$

for any $x \in X_\alpha$. Then, by routine computation, we can see that λ_u is a coaction of H^0 on X_α with respect to (A, A, ρ, ρ).

Proposition 6.4. *With the above notation, the following conditions are equivalent*:

(1) $[X_\alpha] \in \mathrm{Im} f_\rho$;
(2) *there is a unitary element* $u \in M(A \otimes H^0)$ *such that*

$$\rho = \mathrm{Ad}(u) \circ (\alpha \otimes \mathrm{id}) \circ \rho \circ \alpha^{-1},$$
$$(\underline{\rho} \otimes \mathrm{id})(u)(u \otimes 1^0) = (\mathrm{id} \otimes \Delta^0)(u).$$

Proof. Immediate from Lemma 6.2 and the above discussion. □

Let u be a unitary element in $M(A \otimes H^0)$ satisfying Proposition 6.4 (2). Let λ_u be as above. We call λ_u the coaction of H^0 on X_α with respect to (A, A, ρ, ρ) *induced* by u.

Let $\alpha, \beta \in \mathrm{Aut}(A)$ satisfy that there are unitary elements $u, v \in M(A \otimes H^0)$ such that

$$\rho = \mathrm{Ad}(u) \circ (\alpha \otimes \mathrm{id}) \circ \rho \circ \alpha^{-1},$$
$$(\underline{\rho} \otimes \mathrm{id})(u)(u \otimes 1^0) = (\mathrm{id} \otimes \Delta^0)(u),$$
$$\rho = \mathrm{Ad}(v) \circ (\beta \otimes \mathrm{id}) \circ \rho \circ \beta^{-1},$$
$$(\underline{\rho} \otimes \mathrm{id})(v)(v \otimes 1^0) = (\mathrm{id} \otimes \Delta^0)(v).$$

Lemma 6.5. *With the above notation, we have the following*:

$$(\underline{\rho} \otimes \mathrm{id})(u(\underline{\alpha} \otimes \mathrm{id})(v))(u(\underline{\alpha} \otimes \mathrm{id})(v) \otimes 1^0) = (\mathrm{id} \otimes \Delta^0)(u(\underline{\alpha} \otimes \mathrm{id})(v)).$$

Proof. By routine computation, we see that

$$((\alpha \circ \beta) \otimes \mathrm{id}) \circ \rho \circ (\alpha \circ \beta)^{-1} = \mathrm{Ad}((\underline{\alpha} \otimes \mathrm{id})(v^*)) \circ \mathrm{Ad}(u^*) \circ \rho.$$

Thus, we obtain

$$\rho = \mathrm{Ad}(u(\underline{\alpha} \otimes \mathrm{id})(v)) \circ ((\alpha \circ \beta) \otimes \mathrm{id}) \circ \rho \circ (\alpha \circ \beta)^{-1}.$$

Since $\rho\circ\alpha = \mathrm{Ad}(u)\circ(\alpha\otimes\mathrm{id})\circ\rho$,

$$(\underline{\rho}\otimes\mathrm{id})((\underline{\alpha}\otimes\mathrm{id})(v)) = (u\otimes 1^0)(\underline{\alpha}\otimes\mathrm{id}\otimes\mathrm{id})((\underline{\rho}\otimes\mathrm{id})(v))(u\otimes 1^0)^*.$$

Hence, by routine computation, we see that

$$(\underline{\rho}\otimes\mathrm{id})(u(\underline{\alpha}\otimes\mathrm{id})(v))(u(\underline{\alpha}\otimes\mathrm{id})(v)\otimes 1^0) = (\mathrm{id}\otimes\Delta^0)(u(\underline{\alpha}\otimes\mathrm{id})(v)). \quad \square$$

Let α, β and u, v be as above. Let λ_u and λ_v be coactions of H^0 on X_α and X_β with respect to (A, A, ρ, ρ) induced by u and v, respectively. Let $u\,\sharp\,v = u(\underline{\alpha}\otimes\mathrm{id})(v)\in M(A\otimes H^0)$. By Lemma 6.5, we can define the coaction $\lambda_{u\sharp v}$ of H^0 on $X_{\alpha\circ\beta}$ with respect to (A, A, ρ, ρ), induced by $u\,\sharp\,v$. By simple computation, we see that $X_\alpha\otimes X_\beta$ is isomorphic to $X_{\alpha\circ\beta}$ by an $A-A$-equivalence bimodule isomorphism π, as follows:

$$\pi : X_\alpha\otimes_A X_\beta \longrightarrow X_{\alpha\circ\beta} : x\otimes y\longmapsto x\alpha(y).$$

We identify $X_\alpha\otimes_A X_\beta$ with $X_{\alpha\circ\beta}$ by the above $A-A$-equivalence bimodule isomorphism π.

Lemma 6.6. *With the above notation, for* $[X_\alpha, \lambda_u]$, $[X_\beta, \lambda_v]\in\mathrm{Pic}_H^\rho(A)$,

$$[X_\alpha, \lambda_u][X_\beta, \lambda_v] = [X_{\alpha\circ\beta}, \lambda_{u\sharp v}]\in\mathrm{Pic}_H^\rho(A),$$

where $u\,\sharp\,v = u(\underline{\alpha}\otimes\mathrm{id})(v)\in M(A\otimes H^0)$.

Proof. By the definition of the product in $\mathrm{Pic}_H^\rho(A)$,

$$[X_\alpha, \lambda_u][X_\beta, \lambda_v] = [X_\alpha\otimes_A X_\beta, \lambda_u\otimes\lambda_v].$$

Hence, it suffices to show that

$$\pi(h\cdot_{\lambda_u\otimes\lambda_v} x\otimes y) = h\cdot_{\lambda_{u\sharp v}}\pi(x\otimes y)$$

for any $x\in X_\alpha$, $y\in X_\beta$ and $h\in H$. For any $x\in X_\alpha$, $y\in X_\beta$ and $h\in H$,

$$\begin{aligned}\pi(h\cdot_{\lambda_u\otimes\lambda_v} x\otimes y) &= \pi([h_{(1)}\cdot_{\lambda_u} x]\otimes[h_{(2)}\cdot_{\lambda_v} y]) \\ &= \pi([h_{(1)}\cdot_\rho x]\widehat{u}(h_{(2)})\otimes[h_{(3)}\cdot_\rho y]\widehat{v}(h_{(4)})) \\ &= [h_{(1)}\cdot_\rho x]\widehat{u}(h_{(2)})\alpha([h_{(3)}\cdot_\rho y]\widehat{v}(h_{(4)})).\end{aligned}$$

Since $\rho\circ\alpha=\mathrm{Ad}(u)\circ(\alpha\otimes\mathrm{id})\circ\rho$,

$$\begin{aligned}\pi(h\cdot_{\lambda_u\otimes\lambda_v}x\otimes y)&=[h_{(1)}\cdot_\rho x][h_{(2)}\cdot_\rho\alpha(y)]\widehat{u}(h_{(3)})\alpha(\widehat{v}(h_{(4)}))\\&=[h_{(1)}\cdot_\rho x\alpha(y)](u(\underline{\alpha}\otimes\mathrm{id})(v))\widehat{\ }(h_{(2)})\\&=h\cdot_{\lambda_{u\sharp v}}x\alpha(y).\end{aligned}$$

Therefore, we obtain the conclusions. □

Corollary 6.7. *With the above notation, for any* $[X_\alpha,\lambda_u]\in\mathrm{Pic}_H^\rho(A)$,

$$[X_\alpha,\lambda_u]^{-1}=[X_{\alpha^{-1}},\lambda_{(\underline{\alpha}^{-1}\otimes\mathrm{id})(u^*)}]\in\mathrm{Pic}_H^\rho(A).$$

Proof. Immediate by Lemma 6.6 and routine computation. □

For any $\alpha\in\mathrm{Aut}(A)$, let $\mathrm{U}_\alpha^\rho(M(A\otimes H^0))$ be the set of all unitary elements $u\in M(A\otimes H^0)$ satisfying

$$\rho=\mathrm{Ad}(u)\circ(\alpha\otimes\mathrm{id})\circ\rho\circ\alpha^{-1},$$
$$(\underline{\rho}\otimes\mathrm{id})(u)(u\otimes 1^0)=(\mathrm{id}\otimes\Delta^0)(u).$$

Lemma 6.8. *With the above notation, for any* $\alpha\in\mathrm{Aut}(A)$, *we have the following*:

(1) *for any* $u\in\mathrm{U}_{\mathrm{id}}^\rho(M(A\otimes H^0))$ *and* $v\in\mathrm{U}_\alpha^\rho(M(A\otimes H^0))$, $uv\in\mathrm{U}_\alpha^\rho(M(A\otimes H^0))$;

(2) *for any* $u,v\in\mathrm{U}_\alpha^\rho(M(A\otimes H^0))$, $uv^*\in\mathrm{U}_{\mathrm{id}}^\rho(M(A\otimes H^0))$.

Proof.

(1) This is immediate by Lemma 6.6;

(2) By Corollary 6.7, $(\alpha^{-1}\otimes\mathrm{id})(v^*)\in\mathrm{U}_{\alpha^{-1}}^\rho(M(A\otimes H^0))$. Hence, $uv^*\in\mathrm{U}_{\mathrm{id}}^\rho(M(A\otimes H^0))$. □

Lemma 6.9. *Let* $u\in\mathrm{U}_{\mathrm{id}}^\rho(M(A\otimes H^0))$. *Then, the following conditions are equivalent*:

(1) $[{}_AA_A,\lambda_u]=[{}_AA_A,\rho]$ *in* $\mathrm{Pic}_H^\rho(A)$;

(2) *there is a unitary element* $w\in M(A)\cap A'$ *such that* $u=(w^*\otimes 1^0)\underline{\rho}(w)$.

Proof. Suppose condition (1). Then, there is an $A-A$-equivalence bimodule automorphism π of ${}_AA_A$ such that

$$\rho(\pi(x)) = (\pi\otimes \mathrm{id})(\lambda_u(x)) = (\pi\otimes\mathrm{id})(\rho(x)u)$$

for any $x\in {}_AA_A$. We note that $\pi\in {}_A\mathbf{B}_A({}_AA_A)$ and

$${}_A\mathbf{B}_A({}_AA_A)\cong A'\cap \mathbf{B}_A(A_A)\cong A'\cap M(A).$$

Hence, there is a unitary element $w\in A'\cap M(A)$ such that $\pi(x)=wx$ for any $x\in A$. Thus, for any $x\in A$,

$$\rho(wx) = (w\otimes 1^0)\rho(x)u.$$

Therefore, $u=(w^*\otimes 1^0)\underline{\rho}(w)$.

Next, we suppose condition (2). Let π be the $A-A$-equivalence bimodule automorphism of ${}_AA_A$ defined by $\pi(x)=wx$ for any $x\in {}_AA_A$. Then, for any $x\in {}_AA_A$,

$$\begin{aligned}\rho(\pi(x)) &= \rho(wx)=\rho(xw)\\ &= \rho(x)\underline{\rho}(w) = \rho(x)(w\otimes 1^0)u\\ &= (w\otimes 1^0)\rho(x)u = (\pi\otimes\mathrm{id})(\lambda_u(x)).\end{aligned}$$

Thus, we obtain condition (1). □

Corollary 6.10. *Let* $\alpha\in \mathrm{Aut}(A)$ *and* $u,v\in \mathrm{U}_\alpha^\rho(M(A\otimes H^0))$. *Then, the following conditions are equivalent*:

(1) $[X_\alpha,\lambda_u]=[X_\alpha,\lambda_v]$ *in* $\mathrm{Pic}_H^\rho(A)$;
(2) *there is a unitary element* $w\in M(A)\cap A'$ *such that* $u=(w^*\otimes 1^0)\underline{\rho}(w)v$.

Proof. Suppose condition (1). By Lemma 6.6 and Corollary 6.7, we see that $[{}_AA_A,\lambda_{uv^*}]=[{}_AA_A,\rho]$ in $\mathrm{Pic}_H^\rho(A)$. Thus, by Lemma 6.9, there is a unitary element in $w\in M(A)\cap A'$ such that $uv^*=(w^*\otimes 1^0)\underline{\rho}(w)$. Hence, we obtain condition (2).

Conversely, suppose condition (2). Then, there is a unitary element $w\in M(A)\cap A'$ such that $uv^*=(w^*\otimes 1^0)\underline{\rho}(w)$. Hence, $[{}_AA_A,\lambda_{uv^*}]=[{}_AA_A,\rho]$ in $\mathrm{Pic}_H^\rho(A)$. Since $[X_\alpha,\lambda_u][X_\alpha,\lambda_v]^{-1}=[{}_AA_A,\lambda_{uv^*}]$ in $\mathrm{Pic}_H^\rho(A)$ by Lemma 6.6 and Corollary 6.7, $[X_\alpha,\lambda_u]=[X_\alpha,\lambda_v]$ in $\mathrm{Pic}_H^\rho(A)$. □

We shall compute $\mathrm{Ker} f_\rho$, the kernel of f_ρ. Let $[X, \lambda] \in \mathrm{Pic}_H^\rho(A)$. Then, by Proposition 6.4, we see that $[X] = [{}_A A_A]$ in $\mathrm{Pic}(A)$ if and only if there is a unitary element $u \in \mathrm{U}_{\mathrm{id}}^\rho(M(A \otimes H^0))$ such that $[X, \lambda] = [{}_A A_A, \lambda_u]$ in $\mathrm{Pic}_H^\rho(A)$. Furthermore, by Corollary 6.10, $[{}_A A_A, \lambda_u] = [{}_A A_A, \lambda_v]$ in $\mathrm{Pic}_H^\rho(A)$ if and only if there is a unitary element $w \in M(A) \cap A'$ such that $u = (w^* \otimes 1^0)\underline{\rho}(w)v$, where $u, v \in \mathrm{U}_{\mathrm{id}}^\rho(M(A \otimes H^0))$. We define an equivalence relation in $\mathrm{U}_{\mathrm{id}}^\rho(M(A \otimes H^0))$ as follows: let $u, v \in \mathrm{U}_{\mathrm{id}}^\rho(M(A \otimes H^0))$, written $u \sim v$ if there is a unitary element $w \in M(A) \cap A'$ such that

$$u = (w^* \otimes 1^0)\underline{\rho}(w)v.$$

Let $\mathrm{U}_{\mathrm{id}}^\rho(M(A \otimes H^0))/\!\sim$ be the set of all equivalence classes in $\mathrm{U}_{\mathrm{id}}^\rho(M(A \otimes H^0))$. We denote by $[u]$ the equivalence class of $u \in \mathrm{U}_{\mathrm{id}}^\rho(M(A \otimes H^0))$. By Lemma 6.8, $\mathrm{U}_{\mathrm{id}}^\rho(M(A \otimes H^0))$ is a group. Hence, $\mathrm{U}_{\mathrm{id}}^\rho(M(A \otimes H^0))/\!\sim$ is a group by simple computation.

Proposition 6.11. *With the above notation,* $\mathrm{Ker} f_\rho \cong \mathrm{U}_{\mathrm{id}}^\rho(M(A \otimes H^0))/\!\sim$ *as groups.*

Proof. Let π be a map from $\mathrm{U}_{\mathrm{id}}^\rho(M(A \otimes H^0))/\!\sim$ to $\mathrm{Ker} f_\rho$, defined by

$$\pi([u]) = [{}_A A_A, \lambda_u]$$

for any $u \in \mathrm{U}_{\mathrm{id}}^\rho(M(A \otimes H^0))$. By the above discussion, we see that π is well defined and bijective. For any $u, v \in \mathrm{U}_{\mathrm{id}}^\rho(M(A \otimes H^0))$,

$$\pi([u])\pi([v]) = [{}_A A_A, \lambda_u][{}_A A_A, \lambda_v] = [{}_A A_A, \lambda_{uv}] = \pi([uv]),$$

by Lemma 6.6. Therefore, we obtain the conclusion. □

We recall that there is a homomorphism Φ of $\mathrm{Aut}_H^{\rho^s}(A^s)$ to $\mathrm{Pic}_H^{\rho^s}(A^s)$, defined by

$$\Phi(\alpha) = [X_\alpha, \lambda_\alpha]$$

for any $\alpha \in \mathrm{Aut}_H^{\rho^s}(A^s)$, where λ_α is a coaction of H^0 on X_α induced by ρ^s, see Section 5. Then the following results hold:

Lemma 6.12. *With the above notation, for any* $\alpha \in \mathrm{Aut}_H^{\rho^s}(A^s)$,

$$(f_{\rho^s} \circ \Phi)(\alpha) = [X_\alpha]$$

in $\mathrm{Pic}(A^s)$. *Furthermore, if* $\widehat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ *in* $(A \rtimes_\rho H) \otimes H$, *then*

$$\mathrm{Im} f_{\rho^s} = \{[X_\alpha] \in \mathrm{Pic}(A^s) \mid \alpha \in \mathrm{Aut}_H^{\rho^s}(A^s)\}.$$

Proof. Immediate by simple computation. □

Let G be a subgroup of $\mathrm{Pic}(A^s)$, defined by

$$G = \{[X_\alpha] \in \mathrm{Pic}(A^s) \mid \alpha \in \mathrm{Aut}_H^{\rho^s}(A^s)\}.$$

Theorem 6.13. *Let H be a finite dimensional C^*-Hopf algebra with its dual C^*-algebra H^0. Let A be a unital C^*-algebra and ρ a coaction of H^0 on A with $\widehat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(A \rtimes_\rho H) \otimes H$. Let $A^s = A \otimes \mathbf{K}$, and let ρ^s be the coaction of H^0 on A^s induced by ρ. Let $\mathrm{U}_{\mathrm{id}}^{\rho}(M(A^s \otimes H^0))$ be the group of all unitary elements $u \in M(A^s \otimes H^0)$ satisfying*

$$\rho^s = \mathrm{Ad}(u) \circ \rho^s, \qquad (\underline{\rho^s} \otimes \mathrm{id})(u)(u \otimes 1^0) = (\mathrm{id} \otimes \Delta^0)(u).$$

Then, we have the following exact sequence:

$$1 \longrightarrow \mathrm{U}_{\mathrm{id}}^{\rho^s}(M(A^s \otimes H^0))/\sim \longrightarrow \mathrm{Pic}_H^{\rho^s}(A^s) \longrightarrow G \longrightarrow 1,$$

where "$\sim$" is the equivalence relation in $\mathrm{U}_{\mathrm{id}}^{\rho^s}(M(A^s \otimes H^0))$ defined in this section.

Proof. Immediate by Proposition 6.11 and Lemma 6.12. □

Let A be a UHF-algebra of type N^∞, where $N = \dim H$. Let ρ be the coaction of H^0 on A defined in [**16,** Section 7], which has the Rohlin property. Note that

$$\widehat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1 \quad \text{in } (A \rtimes_\rho H) \otimes H,$$

by [**16,** Definition 5.1].

Corollary 6.14. *With the above notation, we have the following exact sequence*:

$$1 \longrightarrow \mathrm{U}_{\mathrm{id}}^{\rho^s}(M(A^s \otimes H^0)) \longrightarrow \mathrm{Pic}_H^{\rho^s}(A^s) \longrightarrow G \longrightarrow 1.$$

Proof. Since A^s is simple, $M(A^s) \cap (A^s)' = \mathbf{C}1$ by [**21,** Corollary 4.4.8]. Therefore, by Theorem 6.13, we obtain the conclusion. □

7. Equivariant Picard groups and crossed products. Let (ρ, u) be a twisted coaction of H^0 on a unital C^*-algebra A. Let f be a map from $\mathrm{Pic}_H^{\rho,u}(A)$ to $\mathrm{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho,u} H)$, defined by

$$f([X,\lambda]) = [X \rtimes_\lambda H, \widehat{\lambda}]$$

for any $[X,\lambda] \in \mathrm{Pic}_H^{\rho,u}(A)$. In this section, we shall show that f is an isomorphism of $\mathrm{Pic}_H^{\rho,u}(A)$ onto $\mathrm{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho,u} H)$. We see that f is well defined in a straightforward way. We show that f is a homomorphism of $\mathrm{Pic}_H^{\rho,u}(A)$ to $\mathrm{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho,u} H)$. Let A, B and C be unital C^*-algebras and (ρ, u), (σ, v) and (γ, w) be twisted coactions of H^0 on A, B and C, respectively. Let λ be a twisted coaction of H^0 on an $A-B$-equivalence bimodule X with respect to $(A, B, \rho, u, \sigma, v)$. Also, let μ be a twisted coaction of H^0 on a $B-C$-equivalence bimodule Y with respect to $(B, C, \sigma, v, \gamma, w)$. Let Φ be a linear map from $(X \otimes_B Y) \rtimes_{\lambda\otimes\mu} H$ to $(X \rtimes_\lambda H) \otimes_{B\rtimes_{\sigma,v}H} (Y \rtimes_\mu H)$, defined by

$$\Phi(x \otimes y \rtimes_{\lambda\otimes\mu} h) = (x \rtimes_\lambda 1) \otimes (y \rtimes_\mu h)$$

for any $x \in X$, $y \in Y$ and $h \in H$. By routine computation, Φ is well defined. We note that $(X \rtimes_\lambda H) \otimes_{B\rtimes_{\sigma,v}H} (Y \rtimes_\mu H)$ consists of finite sums of elements in the form $(x \rtimes_\lambda 1) \otimes (y \rtimes_\mu h)$ by the definition of $(X \rtimes_\lambda H) \otimes_{B\rtimes_{\sigma,v}H} (Y \rtimes_\mu H)$, where $x \in X$, $y \in Y$ and $h \in H$. Hence, we can see that Φ is bijective and its inverse map Φ^{-1} is:

$$\begin{aligned}(X \rtimes_\lambda H) \otimes_{B\rtimes_{\sigma,u}H} (Y \rtimes_\mu H) &\longrightarrow (X \otimes_B Y) \rtimes_{\lambda\otimes\mu} H : \\ (x \rtimes_\lambda 1) \otimes (y \rtimes_\mu h) &\longmapsto x \otimes y \rtimes_{\lambda\otimes\mu} h.\end{aligned}$$

Furthermore, we have the following lemmas.

Lemma 7.1. *With the above notation,*

$$\begin{aligned}&{}_{A\rtimes_{\rho,u}H}\langle \Phi(x \otimes y \rtimes_{\lambda\otimes\mu} h),\, \Phi(z \otimes r \rtimes_{\lambda\otimes\mu} l)\rangle \\ &\qquad = {}_{A\rtimes_{\rho,u}H}\langle x \otimes y \rtimes_{\lambda\otimes\mu} h,\, z \otimes r \rtimes_{\lambda\otimes\mu} l\rangle, \\ &\langle \Phi(x \otimes y \rtimes_{\lambda\otimes\mu} h),\, \Phi(z \otimes r \rtimes_{\lambda\otimes\mu} l)\rangle_{C\rtimes_{\gamma,w}H} \\ &\qquad = \langle x \otimes y \rtimes_{\lambda\otimes\mu} h,\, z \otimes r \rtimes_{\lambda\otimes\mu} l\rangle_{C\rtimes_{\gamma,w}H}\end{aligned}$$

for any $x, z \in X$, $y, r \in Y$ and $h, l \in H$.

Proof. We can prove this lemma by routine computation. Indeed,

$${}_{A\rtimes_{\rho,u}H}\langle \Phi(x \otimes y \rtimes_{\lambda\otimes\mu} h),\, \Phi(z \otimes r \rtimes_{\lambda\otimes\mu} l)\rangle$$

$$\begin{aligned}
&= {}_{A\rtimes_{\rho,u}H}\langle (x\rtimes_\lambda 1)\otimes(y\rtimes_\mu h),\, (z\rtimes_\lambda 1)\otimes(r\rtimes_\mu l)\rangle \\
&= {}_{A\rtimes_{\rho,u}H}\langle (x\rtimes_\lambda 1)\,{}_{B\rtimes_{\sigma,y}H}\langle y\rtimes_\mu h,\, r\rtimes_\mu l\rangle,\, z\rtimes_\lambda 1\rangle \\
&= {}_{A\rtimes_{\rho,u}H}\langle (x\rtimes_\lambda 1)\,({}_B\langle y,\, [S(h_{(2)}l^*_{(3)})^*\cdot_\mu r]\widehat{w}(S(h_{(1)}l^*_{(2)})^*,\, l_{(1)})\rangle \\
&\qquad\qquad \rtimes_{\sigma,v} h_{(3)}l^*_{(4)}),\, z\rtimes_\lambda 1\rangle \\
&= {}_{A\rtimes_{\rho,u}H}\langle x_B\langle y,\, [S(h_{(2)}l^*_{(3)})^*\cdot_\mu r]\widehat{w}(S(h_{(1)}l^*_{(2)})^*,\, l_{(1)})\rangle \\
&\qquad\qquad \rtimes_\lambda h_{(3)}l^*_{(4)},\, z\rtimes_\lambda 1\rangle \\
&= {}_A\langle x\,{}_B\langle y,\, [S(h_{(2)}l^*_{(3)})^*\cdot_\mu r]\widehat{w}(S(h_{(1)}l^*_{(2)})^*, l_{(1)})\rangle,\, [S(h_{(3)}l^*_{(4)})^*\cdot_\lambda z]\rangle \\
&\quad \rtimes_{\rho,u} h_{(4)}l^*_{(5)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&{}_{A\rtimes_{\rho,u}H}\langle x\otimes y\rtimes_{\lambda\otimes\mu} h,\, z\otimes r\rtimes_{\lambda\otimes\mu} l\rangle \\
&\quad = {}_A\langle x\otimes y,\, [S(h_{(2)}l^*_{(3)})^*\cdot_{\lambda\otimes\mu} z\otimes r]\widehat{w}(S(h_{(1)}l^*_{(2)})^*,\, l_{(1)})\rangle \\
&\qquad \rtimes_{\rho,u} h_{(3)}l^*_{(4)} \\
&\quad = {}_A\langle x\otimes y,\, [S(h_{(3)}l^*_{(4)})^*\cdot_\lambda z]\otimes[S(h_{(2)}l^*_{(3)})^*\cdot_\mu r]\widehat{w}(S(h_{(1)}l^*_{(2)})^*,\, l_{(1)})\rangle \\
&\qquad \rtimes_{\rho,u} h_{(4)}l^*_{(5)} \\
&\quad = {}_A\langle x\,{}_B\langle y,\, [S(h_{(2)}l^*_{(3)})^*\cdot_\mu r]\widehat{w}(S(h_{(1)}l^*_{(2)})^*, l_{(1)})\rangle,\, [S(h_{(3)}l^*_{(4)})^*\cdot_\lambda z]\rangle \\
&\qquad \rtimes_{\rho,u} h_{(4)}l^*_{(5)}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&{}_{A\rtimes_{\rho,u}H}\langle \Phi(x\otimes y\rtimes_{\lambda\otimes\mu} h), \Phi(z\otimes r\rtimes_{\lambda\otimes\mu} l)\rangle \\
&\qquad\qquad\qquad = {}_{A\rtimes_{\rho,u}H}\langle x\otimes y\rtimes_{\lambda\otimes\mu} h,\, z\otimes r\rtimes_{\lambda\otimes\mu} l\rangle.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\langle \Phi(x\otimes y\rtimes_{\lambda\otimes\mu} h),\, \Phi(z\otimes r\rtimes_{\lambda\otimes\mu} l)\rangle_{C\rtimes_{\gamma,w}H} \\
&\qquad\qquad\qquad = \langle x\otimes y\rtimes_{\lambda\otimes\mu} h,\, z\otimes r\rtimes_{\lambda\otimes\mu} l\rangle_{C\rtimes_{\gamma,w}H}. \quad \square
\end{aligned}$$

Lemma 7.2. *With the above notation, Φ is an $A\rtimes_{\rho,u} H - C\rtimes_{\gamma,w} H$-equivalence bimodule isomorphism of $(X\otimes_B Y)\rtimes_{\lambda\otimes\mu} H$ onto $(X\rtimes_\lambda H)\otimes_{B\rtimes_{\sigma,v}H}(Y\rtimes_\mu H)$, satisfying*

$$\Phi(\phi\cdot_{\widehat{\lambda\otimes\mu}}(x\otimes y\rtimes_{\lambda\otimes\mu} h)) = \phi\cdot_{\hat{\lambda}\otimes\hat{\mu}}\Phi(x\otimes y\rtimes_{\lambda\otimes\mu} h)$$

for any $x\in X$, $y\in Y$, $h\in H$ and $\phi\in H^0$.

Proof. From Lemma 7.1 and the remark after [**12,** Definition 1.1.18], we see that Φ is an $A \rtimes_{\rho,u} H - C \rtimes_{\gamma,w} H$-equivalence bimodule isomorphism of $(X \otimes_B Y) \rtimes_{\lambda\otimes\mu} H$ onto $(X \rtimes_\lambda H) \otimes_{B\rtimes_{\sigma,v}H} (Y \rtimes_\mu H)$. Furthermore, for any $x \in X$, $y \in Y$, $h \in H$ and $\phi \in H^0$,

$$\begin{aligned}\Phi(\phi \cdot_{\widehat{\lambda\otimes\mu}} (x \otimes y \rtimes_{\lambda\otimes\mu} h)) &= \Phi(x \otimes y \rtimes_{\lambda\otimes\mu} h_{(1)}\phi(h_{(2)})) \\ &= (x \rtimes_\lambda 1) \otimes (y \rtimes_\mu h_{(1)})\phi(h_{(2)}) \\ &= [\phi_{(1)} \cdot_{\hat{\lambda}} (x \rtimes_\lambda 1)] \otimes [\phi_{(2)} \cdot_{\hat{\mu}} (y \rtimes_\mu h)] \\ &= \phi \cdot_{\hat{\lambda}\otimes\hat{\mu}} \Phi(x \otimes y \rtimes_{\lambda\otimes\mu} h).\end{aligned}$$

Therefore, we obtain the conclusion. □

Corollary 7.3. *Let f be a map from $\mathrm{Pic}_H^{\rho,u}(A)$ to $\mathrm{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho,u} H)$, defined by $f([X,\lambda]) = [X \rtimes_\lambda H, \widehat{\lambda}]$ for any $[X,\lambda] \in \mathrm{Pic}_H^{\rho,u}(A)$. Then, f is a homomorphism of $\mathrm{Pic}_H^{\rho,u}(A)$ to $\mathrm{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho,u} H)$.*

Proof. Immediate by Lemma 7.2. □

Next, we construct the inverse homomorphism of f of $\mathrm{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_\rho H)$ to $\mathrm{Pic}_H^{\rho,u}(A)$. First, note the following: let (α, v) and (β, z) be twisted coactions of H^0 on unital C^*-algebras A and B, respectively. Suppose that there is an isomorphism Φ of B onto A such that $(\Phi\otimes\mathrm{id})\circ\beta = \alpha\circ\Phi$ and $v = (\Phi\otimes\mathrm{id})(z)$. Let $(X,\lambda) \in \mathrm{Equi}_H^{\alpha,v}(A)$. We construct an element (X_Φ, λ_Φ) in $\mathrm{Equi}_H^{\beta,z}(B)$ from $(X,\lambda) \in \mathrm{Equi}_H^{\alpha,v}(A)$ and Φ as follows: let $X_\Phi = X$ as vector spaces. For any $x, y \in X_\Phi$ and $b \in B$,

$$b \cdot x = \Phi(b)x, \qquad x \cdot b = x\Phi(b)$$
$${}_B\langle x, y\rangle = \Phi^{-1}({}_A\langle x, y\rangle), \qquad \langle x, y\rangle_B = \Phi^{-1}(\langle x, y\rangle_A).$$

We regard λ as a linear map from X_Φ to $X_\Phi \otimes H^0$. We denote it by λ_Φ. Then, (X_Φ, λ_Φ) is an element in $\mathrm{Equi}_H^{\beta,z}(B)$. By simple computation, the map

$$\mathrm{Pic}_H^{\alpha,v}(A) \longrightarrow \mathrm{Pic}_H^{\beta,z}(B) : [X,\lambda] \longmapsto [X_\Phi, \lambda_\Phi]$$

is well defined, and it is an isomorphism of $\mathrm{Pic}_H^{\alpha,v}(A)$ onto $\mathrm{Pic}_H^{\beta,z}(B)$. By Corollary 7.3, there is a homomorphism $\widehat{f}$ of $\mathrm{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho,u} H)$ to

$\mathrm{Pic}_H^{\hat{\hat{\rho}}}(A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0)$, defined by

$$\widehat{f}([Y, \mu]) = [Y \rtimes_\mu H^0, \widehat{\mu}]$$

for any $[Y, \mu] \in \mathrm{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho,u} H)$. By Proposition 2.13, there are an isomorphism Ψ_A of $A \otimes M_N(\mathbf{C})$ onto $A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0$ and a unitary element $U \in (A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0) \otimes H^0$ such that

$$\mathrm{Ad}(U) \circ \widehat{\widehat{\rho}} = (\Psi_A \otimes \mathrm{id}_{H^0}) \circ (\rho \otimes \mathrm{id}_{M_N(\mathbf{C})}) \circ \Psi_A^{-1},$$
$$(\Psi_A \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0})(u \otimes I_N) = (U \otimes 1^0)(\widehat{\widehat{\rho}} \otimes \mathrm{id}_{H^0})(U)(\mathrm{id} \otimes \Delta^0)(U^*).$$

Let $\overline{\rho} = (\Psi_A^{-1} \otimes \mathrm{id}_{H^0}) \circ \widehat{\widehat{\rho}} \circ \Psi_A$. By the above discussion, there is an isomorphism g_1 of $\mathrm{Pic}_H^{\hat{\hat{\rho}}}(A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0)$ onto $\mathrm{Pic}_H^{\bar{\rho}}(A \otimes M_N(\mathbf{C}))$, defined by

$$g_1([X, \lambda]) = [X_{\Psi_A}, \lambda_{\Psi_A}]$$

for any $[X, \lambda] \in \mathrm{Pic}_H^{\hat{\hat{\rho}}}(A \rtimes_{\rho,u} H \rtimes_{\hat{\rho}} H^0)$. Furthermore, the coaction $\overline{\rho}$ of H^0 on $A \otimes M_N(\mathbf{C})$ is exterior equivalent to the twisted coaction $(\rho \otimes \mathrm{id}, u \otimes I_N)$. Indeed,

$$\rho \otimes \mathrm{id}_{M_N(\mathbf{C})} = (\Psi_A^{-1} \otimes \mathrm{id}_{H^0}) \circ \mathrm{Ad}(U) \circ \widehat{\widehat{\rho}} \circ \Psi_A = \mathrm{Ad}(U_1) \circ \overline{\rho},$$

where $U_1 = (\Psi_A^{-1} \otimes \mathrm{id}_{H^0})(U)$. Since $(\Psi_A^{-1} \otimes \mathrm{id}_{H^0} \otimes \mathrm{id}_{H^0}) \circ (\mathrm{id} \otimes \Delta^0) = (\mathrm{id} \otimes \Delta^0) \circ (\Psi_A^{-1} \otimes \mathrm{id}_{H^0})$,

$$u \otimes I_N = (U_1 \otimes 1^0)(\overline{\rho} \otimes \mathrm{id})(U_1)(\mathrm{id} \otimes \Delta^0)(U_1^*).$$

We also note the following: consider twisted coactions (α, v) and (β, z) of H^0 on a unital C^*-algebra A. We suppose that (α, v) and (β, z) are exterior equivalent. Then, there is a unitary element w in $A \otimes H^0$ such that

$$\beta = \mathrm{Ad}(w) \circ \alpha,$$
$$z = (w \otimes 1^0)(\rho \otimes \mathrm{id})(w)v(\mathrm{id} \otimes \Delta^0)(w^*).$$

From Lemmas 3.12, 5.6 and their proofs, there is an isomorphism g_2 of $\mathrm{Pic}_H^{\alpha,v}(A)$ onto $\mathrm{Pic}_H^{\beta,z}(A)$, defined by $g_2([X, \lambda]) = [X, \mathrm{Ad}(w) \circ \lambda]$ for any $[X, \lambda] \in \mathrm{Pic}_H^{\alpha,v}(A)$, where $\mathrm{Ad}(w) \circ \lambda$ means a linear map from X to $X \otimes H^0$, defined by $(\mathrm{Ad}(w) \circ \lambda)(x) = w\lambda(x)w^*$ for any $x \in X$, which is a coaction of H^0 on $X \otimes H^0$ with respect to $(A, A, \beta, z, \beta, z)$. Since $\overline{\rho}$ and $(\rho \otimes \mathrm{id}, u \otimes I_N)$ are exterior equivalent, by the above discussion, there is an isomorphism g_2 of $\mathrm{Pic}_H^{\bar{\rho}}(A\otimes M_N(\mathbf{C}))$ onto $\mathrm{Pic}_H^{\rho\otimes\mathrm{id}_{M_N(\mathbf{C})},\, u\otimes I_N}(A\otimes M_N(\mathbf{C}))$, defined by

$$g_2([X,\lambda])=[X,\,\mathrm{Ad}(U_1)\circ\lambda]$$

for any $[X,\lambda]\in\mathrm{Pic}_H^{\bar{\rho}}(A\otimes M_N(\mathbf{C}))$. By simple computation, (ρ,u) is strongly Morita equivalent to $(\rho\otimes\mathrm{id}_{M_N(\mathbf{C})},\,u\otimes I_N)$. Hence, by Lemma 5.6 and its proof, there is an isomorphism g_3 of $\mathrm{Pic}_H^{\rho,u}(A)$ onto $\mathrm{Pic}_H^{\rho\otimes\mathrm{id}_{M_N(\mathbf{C})},\, u\otimes I_N}(A\otimes M_N(\mathbf{C}))$, defined by

$$g_3([X,\lambda])=[X\otimes M_N(\mathbf{C}),\,\lambda\otimes\mathrm{id}_{M_N(\mathbf{C})}]$$

for any $[X,\lambda]\in\mathrm{Pic}_H^{\rho,u}(A)$. Let $g=g_3^{-1}\circ g_2\circ g_1\circ\widehat{f}$. Then, g is a homomorphism of $\mathrm{Pic}_{H^0}^{\hat{\rho}}(A\rtimes_{\rho,u}H)$ to $\mathrm{Pic}_H^{\rho,u}(A)$.

Proposition 7.4. *With the above notation,* $g\circ f=\mathrm{id}$ *on* $\mathrm{Pic}_H^{\rho,u}(A)$.

Proof. Let $[X,\lambda]\in\mathrm{Pic}_H^{\rho,u}(A)$. By the definitions of $f,\widehat{f},g_1$ and g_2,

$$(g_2\circ g_1\circ\widehat{f}\circ f)([X,\lambda])=[(X\rtimes_\lambda H\rtimes_{\hat{\lambda}}H^0)_{\Psi_A},\,\mathrm{Ad}(U_1)\circ(\widehat{\widehat{\lambda}})_{\Psi_A}].$$

Let Ψ_X be the linear map from $X\otimes M_N(\mathbf{C})$ to $X\rtimes_\lambda H\rtimes_{\hat{\lambda}}H^0$ defined in Proposition 3.8, and regard Ψ_X as an $A\otimes M_N(\mathbf{C})-A\otimes M_N(\mathbf{C})$-equivalence bimodule isomorphism of $X\otimes M_N(\mathbf{C})$ onto $(X\rtimes_\lambda H\rtimes_{\hat{\lambda}}H^0)_{\Psi_A}$. Also, since

$$\mathrm{Ad}(U)\circ\widehat{\widehat{\lambda}}=(\Psi_X\otimes\mathrm{id})\circ(\lambda\otimes\mathrm{id})\circ\Psi_X^{-1}$$

by Proposition 3.8, for any $x\in A\otimes M_N(\mathbf{C})$,

$$\begin{aligned}(\mathrm{Ad}(U_1)\circ(\widehat{\widehat{\lambda}})_{\Psi_A})(x)&=U_1\cdot(\widehat{\widehat{\lambda}})_{\Psi_A}(x)\cdot U_1^*=U\widehat{\widehat{\lambda}}(x)U^*\\&=((\Psi_X\otimes\mathrm{id})\circ(\lambda\otimes\mathrm{id})\circ\Psi_X^{-1})(x).\end{aligned}$$

Thus,

$$[(X\rtimes_\lambda H\rtimes_{\hat{\lambda}}H^0)_{\Psi_A},\,\mathrm{Ad}(U_1)\circ(\widehat{\widehat{\lambda}})_{\Psi_A}]=[X\otimes M_N(\mathbf{C}),\,\lambda\otimes\mathrm{id}]$$

in $\mathrm{Pic}_H^{\rho\otimes\mathrm{id}_{M_N(\mathbf{C})},\, u\otimes I_N}(A\otimes M_N(\mathbf{C}))$. Since $g_3([X,\lambda])=[X\otimes M_N(\mathbf{C}),\,\lambda\otimes\mathrm{id}_{M_N(\mathbf{C})}]$, we obtain the conclusion. □

Theorem 7.5. *Let* (ρ, u) *be a twisted coaction of* H^0 *on a unital* C^**-algebra* A*. Then* $\mathrm{Pic}_H^{\rho,u}(A) \cong \mathrm{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho,u} H)$*.*

Proof. Let $f, \widehat{f}, g_i$, $i = 1, 2, 3$, and g be as in the proof of Proposition 7.4. By Proposition 7.4, $g \circ f = \mathrm{id}$ on $\mathrm{Pic}_H^{\rho,u}(A)$. Hence, f is injective and g is surjective. Furthermore, we can see that $\widehat{f}$ is injective by Proposition 7.4. Since $g = g_3^{-1} \circ g_2 \circ g_1 \circ \widehat{f}$ and g_i, $i = 1, 2, 3$, are bijective, g is injective. It follows that g is bijective. Therefore, f is an isomorphism of $\mathrm{Pic}_H^{\rho,u}(A)$ onto $\mathrm{Pic}_{H^0}^{\hat{\rho}}(A \rtimes_{\rho,u} H)$. □

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