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A GENERALIZATION OF THE CLASS OF PRINCIPALLY LIFTING MODULES

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ABSTRACT. Let R be an arbitrary ring with identity and M a right R-module. In this paper, we introduce a class of modules which is analogous to that of Goldie*-lifting and principally Goldie*-lifting modules. The module M is called *principally* $\mathcal{G}^*$-δ-*lifting* if, for any $m \in M$, there exists a direct summand N of M such that mR is β_δ^*-equivalent to N. We also introduce a generalization of Goldie*-supplemented modules, namely, a module M is said to be *principally* $\mathcal{G}^*$-δ-*supplemented* if, for any $m \in M$, there exists a δ-supplement N in M such that mR is β_δ^*-equivalent to N. We prove that some results of principally $\mathcal{G}^*$-lifting modules and Goldie*-lifting modules can be extended to principally $\mathcal{G}^*$-δ-lifting modules for this general setting. Several properties of these modules are given, and it is shown that the class of principally $\mathcal{G}^*$-δ-lifting modules lies between the classes of principally δ-lifting modules and principally $\mathcal{G}^*$-δ-supplemented modules.

1. Introduction. Throughout this paper, R denotes an associative ring with identity, and all modules are unital right R-modules. Let $N \leq M$ mean N is a submodule of a module M. A submodule N of a module M is called *small* in M if, for every submodule K of M, the equality $M = N + K$ implies $M = K$. A submodule P is a *supplement* of N in M if $M = P + N$ and $P \cap N$ is small in P, while M is called *supplemented* if every submodule of M has a supplement in M. In [**1**], M is said to be *principally supplemented* if every cyclic submodule of M has a supplement in M. Also, a module M is called (*principally*) *lifting* if, for every (cyclic) submodule N of M, there is a decomposition $M = D \oplus D'$ such that $D \subseteq N$ and $D' \cap N$ is small in M.

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In [**18**], Zhou introduced the concept of δ-small submodules as a generalization of the concept of small submodules. A submodule N of a module M is said to be *δ-small* in M if $N+K \neq M$ for any proper submodule K of M with M/K singular. Clearly, every small submodule N of M is δ-small in M. Let N be a submodule of M. A submodule P of M is called a *δ-supplement* of N in M if $M = N + P$ and $N \cap P$ is δ-small in P, and, in [**7**], M is called *principally δ-supplemented* in case every cyclic submodule of M has a δ-supplement in M. Recall that a module M is *$\oplus$-supplemented* if every submodule of M has a supplement which is a direct summand in M. Principally $\oplus$-supplemented modules are investigated in [**14**], and, in [**15**], principally $\oplus$-δ-supplemented modules generalizing principally δ-supplemented modules, principally $\oplus$-lifting modules and strengthening principally δ-supplemented modules are studied. A module M is said to be *principally $\oplus$-δ-supplemented* [**15**] if, for every cyclic submodule mR of M, M has a direct summand which is a δ-supplement of mR in M, that is, for any $m \in M$, there exists a direct summand A of M such that $M = mR + A$ and $mR \cap A$ is δ-small in A. A module M is called *principally δ-lifting* [**6**] if, for every cyclic submodule N of M, there is a decomposition $M = D \oplus D'$ such that $D \subseteq N$ and $D' \cap N$ is δ-small in M.

Let X and Y be submodules of a module M. In [**2**], X and Y are *β^*-equivalent*, $X\beta^*Y$ for short, if $(X+Y)/X$ is small in M/X and $(X+Y)/Y$ is small in M/Y. A module M is called *Goldie*-lifting* (or briefly, *$\mathcal{G}^*$-lifting*) [**2**] if, for each submodule X of M, there exists a direct summand D of M such that $X\beta^*D$. Also, M is called *$\mathcal{G}^*$-supplemented* [**2**] if, for each submodule X of M, there exists a supplement submodule Y of M such that $X\beta^*Y$. In [**5**], a module M is called *principally $\mathcal{G}^*$-lifting* if, for each cyclic submodule X of M, there exists a direct summand D of M such that $X\beta^*D$. Motivated by these principality of lifting modules, supplemented modules, Goldie*-lifting modules and Goldie*-supplemented modules, we introduce classes of modules, called principally $\mathcal{G}^*$-δ-lifting modules and principally $\mathcal{G}^*$-δ-supplemented modules, by using the δ-small concept generalizing smallness of submodules.

This paper is organized as follows. Section 2 is devoted to some properties of δ-small submodules that will be used in the sequel. In Section 3, we give a β^*_δ relation and some properties of this relation. In

Section 4, the notion of principally $\mathcal{G}^*$-δ-lifting modules is introduced. It is shown that every principally δ-lifting module is a principally $\mathcal{G}^*$-δ-lifting module. We give an example to show that the reverse implication need not hold in general. The conditions under which the reverse implication hold are investigated. Relations between principally $\mathcal{G}^*$-δ-lifting modules and some certain classes of modules are presented. Section 5 is devoted to the class of principally $\mathcal{G}^*$-δ-supplemented modules. In Section 6, we briefly mention principally semisimple modules in terms of the principally $\mathcal{G}^*$-δ-lifting modules.

In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$ we denote, respectively, integers, rational numbers, the ring of integers and the $\mathbb{Z}$-module of integers modulo n. Let $M_n(R)$, $\mathrm{Rad}(M)$ and $\mathrm{Soc}(M)$ denote the ring of $n \times n$ matrices over a ring R, the radical and the socle of a module M, respectively.

2. δ-small submodules. In this section, we collect basic properties of δ-small submodules of a module so that they may easily be referenced in the sequel of the paper. The next lemma is from **[18]**.

Lemma 2.1. *Let M be a module. Then the following hold.*

(1) *A submodule N of M is δ-small in M if and only if, for every submodule X of M, if $M = X + N$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$.*
(2) *If K is δ-small in M and $f\colon M \to N$ is a homomorphism, then $f(K)$ is δ-small in N. In particular, if K is δ-small in $N \subseteq M$, then K is δ-small in M.*
(3) *Let $M = M_1 \oplus M_2$. Then, K_1 is δ-small in M_1, and K_2 is δ-small in M_2 if and only if $K_1 \oplus K_2$ is δ-small in M.*
(4) *Let N and K be submodules of M with K δ-small in M and $N \subseteq K$. Then, N is also δ-small in M.*

Let $\mathcal{P}$ be the class of all singular simple modules. For a module M, in **[18]**, the submodule

$$\delta(M) = \bigcap \left\{ N \leq M \mid \frac{M}{N} \in \mathcal{P} \right\}$$

is defined.

Lemma 2.2. [**18,** Lemma 1.5]. *Let M be a module. Then,*

$$\delta(M) = \sum \{L \leq M \mid L \text{ is a } \delta\text{-small submodule of } M\}.$$

Since every small submodule of a module M is δ-small in M, $\mathrm{Rad}(M) \subseteq \delta(M)$, and there are modules in which this inclusion is strict.

The next lemma is an immediate consequence of Lemma 2.1 (1).

Lemma 2.3. *Let M be a module. Then, M is δ-small in M if and only if M is semisimple projective.*

3. β^*_δ relation. In [**2**], a relation β^* on the set of submodules of a module M is defined to investigate so called $\mathcal{G}^*$-lifting modules and $\mathcal{G}^*$-supplemented modules. Also, in [**5**], that relation is used to study principally $\mathcal{G}^*$-lifting modules. In this section, we introduce a relation generalizing β^* on the set of submodules of a module M.

Definition 3.1. Let M be a module, X and Y submodules of M. We say X is β^*_δ *equivalent* to Y and write $X\beta^*_\delta Y$ for short, if $(X+Y)/X$ is δ-small in M/X and $(X+Y)/Y$ is δ-small in M/Y.

Lemma 3.2. *β^*_δ is an equivalence relation.*

Proof. The reflexive and symmetric properties are clear. For transitivity, assume that X is β^*_δ-equivalent to Y and Y is β^*_δ-equivalent to Z. Then $(X+Y)/X$ is δ-small in M/X, $(X+Y)/Y$ and $(Y+Z)/Y$ are δ-small in M/Y, and $(Y+Z)/Z$ is δ-small in M/Z. We prove that $(X+Z)/X$ is δ-small in M/X and $(X+Z)/Z$ is δ-small in M/Z. Let A/Z be a submodule of M/Z with $(X+Z)/Z+A/Z = M/Z$ and M/A singular. Note that every homomorphic image of a singular module is singular. Then, $(X+Z+A)/Z = M/Z$, thus $X+A = M$. Hence, $M/Y = (X+A)/Y = (X+Y)/Y + (Y+A)/Y$. Since $(X+Y)/Y$ is δ-small in M/Y and $M/(Y+A)$ singular, $(Y+A)/Y = M/Y$. Thus, $M = Y+A$. Then, $M/Z = (Y+A)/Z = (Z+Y)/Z + A/Z$. Since $(Y+Z)/Z$ is δ-small in M/Z and M/A is singular, $A/Z = M/Z$. Thus, $M = A$, so $(X+Z)/Z$ is δ-small in M/Z. Similarly, $(X+Z)/X$ is δ-small in M/X. Therefore, X is β^*_δ-equivalent to Z. $\square$

Since any small submodule is δ-small, for any submodules X and Y of a module, $X\beta^*Y$ implies $X\beta^*_\delta Y$. There are modules with submodules X and Y with $X\beta^*_\delta Y$ but not $X\beta^*Y$. We cite an example of Nicholson [**9,** Example 2.15] and Zhou [**18,** Example 4.3] in the following.

Example 3.3. Let F be a field and

$$I = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix},$$

$$R = \{(x_1, \ldots, x_n, x, x, \ldots) \mid n \in \mathbb{N},\ x_i \in M_2(F),\ x \in I\}.$$

Then, R is a ring with componentwise operations. Let M denote the right R-module R. It is seen that

$$\mathrm{Soc}(M) = \{(x_1, \ldots, x_n, 0, 0, \ldots) \mid n \in \mathbb{N},\ x_i \in M_2(F)\}$$

and

$$\delta(M) = \{(x_1, \ldots, x_n, x, x, \ldots) \mid n \in \mathbb{N},\ x_i \in M_2(F),\ x \in J\},$$

where

$$J = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}.$$

Consider the submodules

$$\begin{aligned} X &= \{(x_1, 0, x_3, \ldots, x_{2n+1}, 0, 0, \ldots) \mid n \in \mathbb{N},\ x_i \in M_2(F)\}, \\ Y &= \{(0, x_2, 0, x_4, \ldots, x_{2n}, 0, 0, \ldots) \mid n \in \mathbb{N},\ x_i \in M_2(F)\}, \\ Z &= \{(0, x_2, 0, x_4, \ldots, x_{2n}, x, x, \ldots) \mid n \in \mathbb{N},\ x_i \in M_2(F),\ x \in I\} \end{aligned}$$

of M. Then $(X+Y)/Y$ is δ-small but not small in M/Y, and $(X+Y)/X$ is δ-small but not small in M/X since $(X+Y)/Y + Z/Y = M/Y$ and X and Y are δ-small in M.

In the rest of this section, we investigate some basic properties of the β^*_δ relation.

Lemma 3.4. *Let $M = A \oplus B$ be a decomposition of a module M, and let U and V be any submodules of A. Then, the following hold.*

(1) *U is β^*_δ-equivalent to V in A if and only if U is β^*_δ-equivalent to V in M.*

(2) *Let* $A \subseteq X \leq M$. *Then* X *is* β^*_δ*-equivalent to* A *if and only if* $X \cap B$ *is* δ*-small in* B.

(3) *Let* $X \subseteq A$. *If* A *is singular, then* X *is* β^*_δ*-equivalent to* A *if and only if* $X = A$.

Proof.

(1) Let U and V be any submodules of A. Assume that U is β^*_δ-equivalent to V in A. Thus, $(U+V)/V$ is δ-small in A/V and $(U+V)/U$ is δ-small in A/U. Let X be a submodule of M with $(U+V)/V+X/V = M/V$ and M/X singular. Then, $(U+V)/V+(A\cap X)/V = A/V$. Since singularity is preserved under isomorphisms, taking submodules and $A/(A\cap X) \cong (A+X)/X \subseteq M/X$ and M/X is singular, $A/(A\cap X)$ is singular. By assumption, $(A \cap X)/V = A/V$. Hence, $A \subseteq X$. Moreover, $X/V = M/V$. A similar proof gives rise to the δ-smallness of $(U + V)/U$ in M/U.

Conversely, suppose that U is β^*_δ-equivalent to V in M. Let X be a submodule of A such that $(U + V)/V + X/V = A/V$ and A/X is singular. Then $M/X = A/X + (B + X)/X$, $A/X \cong M/(B + X)$ is singular and $M/V = (U + V)/V + (B + X)/V$. By supposition, $(B + X)/V = M/V$. Hence, $B + X = M$. So $A = X$. Therefore, $(U + V)/V$ is δ-small in A/V. Similarly, we may prove $(U + V)/U$ is δ-small in A/U.

(2) Let X be a submodule of M with $A \subseteq X$ and X β^*_δ-equivalent to A. Then, $X = A \oplus (X \cap B)$ and $(X + A)/A = X/A$ is δ-small in M/A. Since $X/A \cong X \cap B$ and $M/A \cong B$, by Lemma 2.1 (2), $X \cap B$ is δ-small in B. The converse is clear.

(3) Let X be a submodule of M with $X \subseteq A$ and A singular. Assume that X is β^*_δ-equivalent to A. Then, $(X + A)/X = A/X$ is δ-small in M/X. Singularity of A implies that of $M/(B+X)$. Also, $M/X = A/X + (B + X)/X$ implies $M = B + X$. Hence, $X = A$. The converse is clear. □

Proposition 3.5. *Let* $f\colon M \to N$ *be an epimorphism. Then the following hold.*

(1) *If* $X, Y \subseteq M$ *such that* X *is* β^*_δ*-equivalent to* Y*, then* $f(X)$ *is* β^*_δ*-equivalent to* $f(Y)$.

(2) *If* $X, Y \subseteq N$ *such that* X *is* β^*_δ*-equivalent to* Y*, then* $f^{-1}(X)$ *is* β^*_δ*-equivalent to* $f^{-1}(Y)$*.*

Proof.

(1) Assume that X is β^*_δ-equivalent to Y. Then, $(X+Y)/X$ is δ-small in M/X and $(X+Y)/Y$ is δ-small in M/Y. Let $U/f(X)$ be a submodule of $N/f(X)$ such that $((f(X)+f(Y))/f(X))+(U/f(X)) = N/f(X)$ with N/U singular. Then $X+Y+f^{-1}(U) = M$. Hence,

$$\left(\frac{X+Y}{X}\right) + \left(\frac{f^{-1}(U)+X}{X}\right) = \frac{M}{X}.$$

On the other hand, $M/(f^{-1}(U)+X)$ is singular as $M/(f^{-1}(U)+X)$ is a homomorphic image of $M/f^{-1}(U)$, and $M/f^{-1}(U)$ is isomorphic to the singular module N/U. By hypothesis, we have $M = f^{-1}(U)+X$. Hence, $N = U + f(X)$. Since $f(X) \subseteq U$, $N = U$. The remainder is clear by symmetry.

(2) Let $X, Y \subseteq N$ be such that X is β^*_δ-equivalent to Y. In order to prove that $f^{-1}(X)$ is β^*_δ-equivalent to $f^{-1}(Y)$, we begin by proving that $(f^{-1}(X)+f^{-1}(Y))/f^{-1}(X)$ is δ-small in $M/f^{-1}(X)$, for, if $L \leq M$ with $((f^{-1}(X)+f^{-1}(Y))/f^{-1}(X)) + (L/f^{-1}(X)) = M/f^{-1}(X)$ and M/L singular, then $f^{-1}(Y)+L = M$ since $f^{-1}(X) \subseteq L$. Thus, $X+Y+f(L) = N$. It is easily verified that $M/(L+\ker(f))$ is isomorphic to $N/f(L)$ which is singular since it is a homomorphic image of the singular module M/L. By hypothesis, $N = f(L)$. Thus, $M = L+\ker(f)$ since $\ker(f) \subseteq f^{-1}(X) \subseteq L$, $M = L$. A similar proof reveals that

$$\frac{f^{-1}(X)+f^{-1}(Y)}{f^{-1}(Y)}$$

is δ-small in $M/f^{-1}(Y)$. This completes the proof. $\square$

Lemma 3.6. *Let* M *be a module and* K*,* N*,* T *submodules of* M*. If* K *is* β^*_δ*-equivalent to* N *and* T *is* δ*-small in* M*, then* K *is* β^*_δ*-equivalent to* $N+T$ *in* M*.*

Proof. Assume that K is β^*_δ-equivalent to N. Thus, $(K+N)/K$ is δ-small in M/K and $(K+N)/N$ is δ-small in M/N. Let L be a submodule of M with $((K+N+T)/K)+(L/K) = M/K$ and M/L

singular. Then,

$$\frac{K+N}{K}+\frac{T+L}{K}=\frac{M}{K},$$

and the singularity of M/L implies that of $M/(L+T)$. Since $(K+N)/K$ is δ-small in M/K and $M/(L+T)$ is singular, $M=L+T$. Moreover, M/L is singular and T is δ-small, and we have $M=L$. Hence, $(K+N+T)/K$ is δ-small in M/K. In order to prove that $(K+N+T)/(N+T)$ is δ-small in $M/(N+T)$, let L be a submodule of M with $N+T\subseteq L$, $((K+N+T)/(N+T))+(L/(N+T))=M/(N+T)$ and M/L singular. Then,

$$\frac{K+N}{N}+\frac{T+L}{N}=\frac{M}{N},\quad\text{and}\quad\frac{M}{T+L}$$

is singular as a homomorphic image of singular M/L. Since $(K+N)/N$ is δ-small in M/N, $M=T+L$. By hypothesis and the singularity of M/L, we have $M=L$. This completes the proof. $\square$

Theorem 3.7. *Let N and K be submodules of a module M with N β_δ^*-equivalent to K. Then, the following hold.*

(1) *N is δ-small in M if and only if K is δ-small in M.*
(2) *Let U be a submodule of M with M/U singular. If R is right non-singular, then U is a δ-supplement of N in M if and only if U is a δ-supplement of K in M.*

Proof.

(1) Assume that N is δ-small in M. Let $K+K'=M$ with M/K' singular. Then, $(N+K)/N+(N+K')/N=M/N$. The singularity of M/K' implies the singularity of $M/(N+K')$. By hypothesis, $M=N+K'$, and by assumption, $M=K'$. Thus, K is δ-small in M. Conversely, since N is β_δ^*-equivalent to K, the implication is that K is β_δ^*-equivalent to N. By replacing N by K in the preceding proof, we conclude that N is δ-small in M.

(2) Assume that U is a δ-supplement of N in M. Then, $M=N+U=N+K+K+U$ implies $M/K=((N+K)/K)+((K+U)/K)$. By hypothesis, $M/(K+U)$ is singular, and thus, $M=K+U$. In order to prove U is a δ-supplement of K, we show that $K\cap U$ is δ-small in U, for if $L\subseteq U$ and $U=(K\cap U)+L$ with U/L singular, then $M=K+U=K+L$ and $M/N=(K+N)/N+(L+N)/N$. On the

other hand, by [**4**], in the exact sequence,

$$0 \longrightarrow \frac{U}{L} \longrightarrow \frac{M}{L} \longrightarrow \frac{M}{U} \longrightarrow 0,$$

M/L is singular since M/U and U/L are singular; therefore, $M/(N+L)$ is singular. Thus, $M/N = (N+L)/N$ or $M = N+L$. Hence, $U = L + (N \cap U)$. Since U/L is singular and $N \cap U$ is δ-small in U, $U = L$.

Conversely, suppose that U is a δ-supplement of K in M. Since N is β_δ^*-equivalent to K, K is β_δ^*-equivalent to N. By replacing N by K and K by N in the preceding proof, we may conclude that U is a δ-supplement of N in M. This completes the proof. $\square$

4. Principally $\mathcal{G}^*$-δ-lifting modules. Motivated by the generalizations of lifting and supplemented modules, we introduce principally $\mathcal{G}^*$-δ-lifting modules. This section is devoted to investigating some properties of this class of modules.

We now introduce principally $\mathcal{G}^*$-δ-lifting modules with the next lemma by using the β_δ^* relation.

Lemma 4.1. *Let M be a module, $m \in M$ and D a direct summand of M. Then the following are equivalent.*

(1) *mR is β_δ^*-equivalent to D.*
(2) *If $M = mR + D + A$ and M/A is singular for any $A \leq M$, then $M = mR + A$ and $M = D + A$.*

Proof.

(1) $\Rightarrow$ (2). Let $M = mR + D + A$ with M/A singular for some submodule A of M. Then

$$\frac{M}{mR} = \left(\frac{mR+D}{mR}\right) + \left(\frac{mR+A}{mR}\right).$$

Also, $M/(mR+A)$ is singular as a homomorphic image of the singular module M/A. Since $(mR+D)/mR$ is δ-small in M/mR,

$$\frac{M}{mR} = \frac{mR+A}{mR}.$$

Thus, $M = mR + A$. Similarly, $M = D + A$.

(2) $\Rightarrow$ (1). Let A be a submodule of M such that $mR \subseteq A$ and

$$\frac{M}{mR} = \left(\frac{mR + D}{mR}\right) + \left(\frac{A}{mR}\right)$$

with M/A singular. Then, $M = mR + D + A$. By (2), $M = mR + A$, and thus, $M = A$. Therefore, $(mR + D)/mR$ is δ-small in M/mR. Similarly, it is shown that $(mR + D)/D$ is δ-small in M/D by using $M = D + A$. $\square$

We call a module M *principally $\mathcal{G}^*$-δ-lifting* if, for every cyclic submodule mR of M, there exists a direct summand D satisfying the equivalent conditions of Lemma 4.1. Clearly, every (principally) $\mathcal{G}^*$-lifting module is principally $\mathcal{G}^*$-δ-lifting. However, the converse does not hold in general, as the following example shows.

Example 4.2. Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Every cyclic submodule of $\mathbb{Q}$ is δ-small in $\mathbb{Q}$. By [**2,** Example 2.15], $\mathbb{Q}$ is principally $\mathcal{G}^*$-lifting, and thus, principally $\mathcal{G}^*$-δ-lifting. However, the $\mathbb{Z}$-module $\mathbb{Q}$ is not supplemented; thus, it is not $\mathcal{G}^*$-lifting.

Recall that a module M is called *principally δ-hollow* if every proper cyclic submodule of M is δ-small in M, while M is said to be a *δ-radical module* if $\delta(M) = M$. Since every submodule of a δ-small submodule is again δ-small, every δ-radical module is principally δ-hollow. The $\mathbb{Z}$-module $\mathbb{Q}$ is δ-radical since every cyclic submodule of $\mathbb{Q}$ is small, thus, δ-small in $\mathbb{Q}$.

Proposition 4.3. *Every principally δ-hollow module is principally $\mathcal{G}^*$-δ-lifting.*

Proof. Let M be a principally δ-hollow module. Then, for each $m \in M$, mR is δ-small in M. Thus, mR is β^*_δ-equivalent to a zero submodule which is a direct summand of M. Therefore, M is principally $\mathcal{G}^*$-δ-lifting. $\square$

According to the next result, the class of principally $\mathcal{G}^*$-δ-lifting modules is a generalization of that of principally δ-lifting modules.

Theorem 4.4. *If M is a principally δ-lifting module, then it is principally $\mathcal{G}^*$-δ-lifting.*

Proof. Let $m \in M$. Then, there is a decomposition $M = D \oplus D'$ with $D \leq mR$ and $mR \cap D'$ δ-small in D' and therefore in M also. Since $D \leq mR$, $(mR + D)/mR$ is δ-small in M/mR. By modularity, $mR = D \oplus (mR \cap D')$. Then, $mR/D \cong mR \cap D'$ and $M/D \cong D'$. The fact that the submodule $mR \cap D'$ is δ-small in D' implies that $(mR + D)/D$ is δ-small in M/D. Hence, mR is β^*_δ-equivalent to D. Thus, M is principally $\mathcal{G}^*$-δ-lifting. $\square$

The next example shows that a principally $\mathcal{G}^*$-δ-lifting module need not be principally δ-lifting; thus, the converse of Theorem 4.4 is not true in general.

Example 4.5. Let M denote the $\mathbb{Z}$-module $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$. By [**16,** Example 3.7], M is H-supplemented, and thus, $\mathcal{G}^*$-lifting. Therefore, M is principally $\mathcal{G}^*$-δ-lifting. On the other hand, M is not principally δ-lifting from [**6,** Example 3.11].

Our next endeavor is to find conditions under which a principally $\mathcal{G}^*$-δ-lifting module is principally δ-lifting. In [**12**], Talebi and Vanaja defined the *cosingular submodule* of a module M as

$$\overline{Z}(M) = \bigcap \{\operatorname{Ker} g \mid g\colon M \longrightarrow N,\ N \text{ is a small module}\}.$$

If $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$), then M is called a *cosingular* (*non-cosingular*) module. In [**10**], inspired by this definition, Özcan defines the submodule $\overline{Z}_\delta(M)$ of M as

$$\overline{Z}_\delta(M) = \bigcap \{\operatorname{Ker} g \mid g\colon M \longrightarrow N,\ N \text{ is a } \delta\text{-small module}\}.$$

Clearly, $\overline{Z}_\delta(M) \subseteq \overline{Z}(M)$. A module M is called *δ-cosingular* (*non-δ-cosingular*) if $\overline{Z}_\delta(M) = 0$ ($\overline{Z}_\delta(M) = M$). Every cosingular module is δ-cosingular and every non-δ-cosingular module is non-cosingular.

Proposition 4.6. *Let M be a principally $\mathcal{G}^*$-δ-lifting and non-δ-cosingular module. Then, M is principally δ-lifting.*

Proof. Let $m \in M$. Since M is principally $\mathcal{G}^*$-δ-lifting, there exists a direct summand D of M such that $(mR + D)/D$ is δ-small in M/D and $(mR+D)/mR$ is δ-small in M/mR. Hence, $(mR+D)/mR$ is both non-δ-cosingular and δ-cosingular, and thus, $mR + D = mR$. Hence, $D \leq mR$ and mR/D is δ-small in M/D. Therefore, M is principally δ-lifting. □

The following example shows that there are principally $\mathcal{G}^*$-δ-lifting modules but not principally δ-lifting. Hence, the condition of the module being non-δ-cosingular is not irrelevant in Proposition 4.6.

Example 4.7. Let M denote the $\mathbb{Z}$-module $\mathbb{Q} \oplus (\mathbb{Z}/2\mathbb{Z})$. By [**15,** Example 3.1], M is principally $\oplus$-δ-supplemented module but not principally δ-lifting. We claim that M is a principally $\mathcal{G}^*$-δ-lifting $\mathbb{Z}$-module. For, if $(u,\overline{v}) \in M$, as in the proof of [**15,** Example 3.1], $M = (u,\overline{1})\mathbb{Z} + (\mathbb{Q} \oplus (\overline{0}))$ and $(u,\overline{0})\mathbb{Z}$ is small in M. Note that the only direct summands of M are $\mathbb{Q} \oplus (\overline{0})$ and $(0,\overline{1})\mathbb{Z}$. We claim that $(u,\overline{1})\mathbb{Z}$ is β_δ^*-equivalent to $(0,\overline{1})\mathbb{Z}$, which is a direct summand of M. Let

$$\frac{M}{(u,\overline{1})\mathbb{Z}} = \frac{(u,\overline{1})\mathbb{Z} + (0,\overline{1})\mathbb{Z}}{(u,\overline{1})\mathbb{Z}} + \frac{L}{(u,\overline{1})\mathbb{Z}}$$

with M/L singular. Then $M = (0,\overline{1})\mathbb{Z} + L$. Let $(u/2,\overline{0}) \in M$. There exist $(0,\overline{a}) \in (0,\overline{1})\mathbb{Z}$ and $(x,\overline{y}) \in L$ such that $(u/2,\overline{0}) = (0,\overline{a}) + (x,\overline{y})$. Then, $u/2 = x$ and $\overline{a} + \overline{y} = \overline{0}$. Thus, $\overline{a} = \overline{y} = \overline{0}$ or $\overline{a} = \overline{y} = \overline{1}$. Hence, $(u/2,\overline{0}) \in L$ or $(u/2,\overline{1}) \in L$. Multiplying the latter cases by 2, we have $(u,\overline{0}) \in L$. This and $(u,\overline{1}) \in L$ imply $(0,\overline{1}) \in L$. Hence, $(0,\overline{1})\mathbb{Z} \subseteq L$. Thus, $M = L$, and further, $((u,\overline{1})\mathbb{Z} + (0,\overline{1})\mathbb{Z})/(u,\overline{1})\mathbb{Z}$ is δ-small in $M/(u,\overline{1})\mathbb{Z}$. Also, clearly, $((u,\overline{1})\mathbb{Z}+(0,\overline{1})\mathbb{Z})/(0,\overline{1})\mathbb{Z}$ is δ-small in $M/(0,\overline{1})\mathbb{Z}$. By [**10**], $\overline{Z}_\delta(\mathbb{Q}) = \mathbb{Q}$. Since $\mathbb{Z}/2\mathbb{Z}$ is a small $\mathbb{Z}$-module, $\overline{Z}_\delta(\mathbb{Z}/2\mathbb{Z}) = 0$ by [**12**]. Hence, $\overline{Z}_\delta(M) = \mathbb{Q} \oplus (\overline{0})$, and thus, M is not non-δ-cosingular.

Recall that a module M is called *π-projective* if, for any U and V with $M = U + V$, there exists an endomorphism of M such that $\mathrm{Im} f \subseteq U$ and $\mathrm{Im}(1 - f) \subseteq V$. For π-projective singular modules, the concepts of principally $\mathcal{G}^*$-δ-lifting modules and principally $\oplus$-δ-supplemented modules are the same as shown next.

Proposition 4.8. *Let M be a π-projective module. Consider the following conditions.*

(1) *M is principally $\oplus$-δ-supplemented.*
(2) *M is principally $\mathcal{G}^*$-δ-lifting.*

Then $(1) \Rightarrow (2)$. *If, in addition, M is singular, then* $(2) \Rightarrow (1)$.

Proof.

$(1) \Rightarrow (2)$. Let $m \in M$. From (1), mR has a δ-supplement D which is a direct summand of M, that is, $M = mR + D = D \oplus D'$ for some submodule D' of M and $mR \cap D$ is δ-small in D. By hypothesis, there exists a submodule $N \subseteq mR$ with $M = N \oplus D$. Then, $(mR + N)/mR$ is δ-small in M/mR, and $(mR + N)/N = mR/N$ is δ-small in M/N since $D \cong M/N$ and $mR/N \cong mR \cap D$ is δ-small in D.

$(2) \Rightarrow (1)$. Assume that M is singular and $m \in M$. By (2), there exists a direct summand D of M such that $M = D \oplus D'$ for some submodule D' of M and mR is β_δ^*-equivalent to D. Then, $(mR+D)/D$ is δ-small in M/D and $(mR + D)/mR$ is δ-small in M/mR. Since $((mR + D)/mR) + ((mR + D')/mR) = M/mR$ and $(mR + D)/mR$ is δ-small in M/mR and M is singular, $M = mR + D'$.

Next, we prove that $mR \cap D'$ is δ-small in D'. For, if $(mR \cap D') + L = D'$, then $M = (mR \cap D') + L + D = mR + L + D$ implies $M/D = ((mR + D)/D) + ((L + D)/D)$. Since $(mR + D)/D$ is δ-small in M/D and M is singular, $M = L + D$. Hence, $L = D'$, and thus, $mR \cap D'$ is δ-small in D'. This completes the proof. $\square$

Note that the notions of principally δ-lifting modules and principally $\mathcal{G}^*$-δ-lifting modules are not equivalent. As we shall see in the next proposition, we need projectivity and singularity conditions.

Proposition 4.9. *Let M be a π-projective and singular module. Then, the following are equivalent*:

(1) *M is principally δ-lifting.*
(2) *M is principally $\mathcal{G}^*$-δ-lifting.*
(3) *M is principally $\oplus$-δ-supplemented.*

Proof.

$(1) \Rightarrow (2)$. By Theorem 4.4.

(2) $\Leftrightarrow$ (3). Follows from Proposition 4.8.

(3) $\Rightarrow$ (1). By [**15,** Theorem 3.1]. $\square$

A relation between principally $\mathcal{G}^*$-δ-lifting modules and principally δ-supplemented modules is presented in the next result.

Theorem 4.10. *If M is a singular principally $\mathcal{G}^*$-δ-lifting module, then it is principally δ-supplemented.*

Proof. Let $m \in M$. There exists a direct summand K such that $M = K \oplus L$ and mR is β^*_δ-equivalent to K. Then, $M/mR = (mR + K)/mR + (mR + L)/mR$ and $M/(mR + L)$ is singular. By hypothesis, $M = mR + L$ since $(mR + K)/mR$ is δ-small in M/mR and every homomorphic image of a singular module is singular.

Next, we prove that $(mR) \cap L$ is δ-small in L. For, if $(mR) \cap L + U = L$ and L/U is singular (already given), then $M = K + L = K + (mR) \cap L + U = K + mR + U$. Thus, $M/K = (mR + K)/K + (U + K)/K$. Since $(mR + K)/K$ is δ-small in M/K and $M/(U + K)$ is singular, $M = U + K$. By modularity, $L = U$. This completes the proof. $\square$

We now give some characterizations of being a principally $\mathcal{G}^*$-δ-lifting module for π-projective singular modules and indecomposable modules.

Proposition 4.11. *Let M be a π-projective and singular module. Then, M is principally $\mathcal{G}^*$-δ-lifting if and only if every cyclic submodule X of M can be written as $X = D \oplus A$ such that D is a direct summand of M and A is δ-small in M.*

Proof. Let M be a principally $\mathcal{G}^*$-δ-lifting module. By Proposition 4.9, M is principally δ-lifting. Then, for any cyclic submodule X of M, there exists a decomposition $M = D \oplus D'$ such that $D \leq X$ and $X \cap D'$ is δ-small M. By modularity, we conclude that $X = D \oplus (X \cap D')$. For the converse, by assumption and [**6,** Theorem 3.6], M is principally δ-lifting. Hence, from Theorem 4.4, M is principally $\mathcal{G}^*$-δ-lifting. $\square$

Theorem 4.12. *Let M be an indecomposable module. Then, the following are equivalent*:

(1) *M is principally δ-lifting.*
(2) *M is principally δ-hollow.*
(3) *M is principally $\mathcal{G}^*$-δ-lifting.*

Proof.

(1) $\Rightarrow$ (3). By Theorem 4.4.

(3) $\Rightarrow$ (1). Let $m \in M$. By (3), there exists a direct summand D of M such that $M = D \oplus D'$ and mR is β_δ^*-equivalent to D. Then, $D = 0$ or $D = M$. Assume that $D = 0$. Hence, mR is δ-small in M. If $D = M$, this implies that mR is β_δ^*-equivalent to M, or equivalently, $(mR+M)/mR = M/mR$ is δ-small in M/mR. By Lemma 2.3, M/mR is semisimple projective. Thus, mR is a direct summand of M, and hence, $mR = 0$ or $mR = M$. In both cases, the rest is clear. Therefore, M is principally δ-lifting.

(2) $\Rightarrow$ (3). Let M be principally δ-hollow and $m \in M$. If mR is a proper submodule of M, then mR is δ-small in M. This implies that mR is β_δ^*-equivalent to (0). If $mR = M$, then there is nothing to show. Thus, M is principally $\mathcal{G}^*$-δ-lifting.

(3) $\Rightarrow$ (2). Let mR be a proper cyclic submodule of M. By (3), there exists a decomposition $M = D \oplus D'$ such that mR is β_δ^*-equivalent to D. Since M is indecomposable, $D = M$ or $D = 0$. Assume that $D = M$. Then, by Lemma 2.3, M/mR is semisimple projective, and thus, mR is a direct summand of M. Since mR is proper, we have $mR = 0$, and it is δ-small in M. If $D = 0$, then mR is β_δ^*-equivalent to (0); thus, mR is δ-small in M, that is, M is principally δ-hollow. $\square$

As a converse of Theorem 4.10, one may expect that every singular principally δ-supplemented module is principally $\mathcal{G}^*$-δ-lifting. However, in light of Theorem 4.12, the next example illustrates that this is not the case (see [**15,** Example 3.3]).

Example 4.13. Let F be a field. Consider the polynomial ring $R = F[x, y]$ with x and y commuting indeterminates, and the ring $S = R/(x^2, y^2)$. Let $\overline{x}$ and $\overline{y}$ denote the canonical images of x and y from R onto S, and let $M = \overline{x}S + \overline{y}S$. Then, M is an indecomposable and singular S-module. By [**15,** Example 3.3], M is principally δ-supplemented but not principally $\oplus$-δ-supplemented. Further, M is not principally δ-lifting by [**15,** Theorem 3.1]. Thus, it is not principally $\mathcal{G}^*$-δ-lifting by Theorem 4.12.

Note that a direct summand D of a module M is said to be *relatively projective* if, whenever M has a decomposition $M = D \oplus D'$, then D and D' are relatively projective.

Proposition 4.14. *Let M be a module, and let any cyclic submodule of M have a δ-supplement which is a relatively projective direct summand of M. Then, M is principally $\mathcal{G}^*$-δ-lifting.*

Proof. Let $m \in M$. By hypothesis, there exists a decomposition $M = D \oplus D'$ such that $M = mR + D$ and $mR \cap D$ is δ-small in D where D and D' are relatively projective. Since D' is D-projective, $M = A \oplus D$ for some submodule A of mR by [**8,** Lemma 4.47]. Thus, M is principally δ-lifting. It follows from Theorem 4.4 that M is principally $\mathcal{G}^*$-δ-lifting. $\square$

Direct summands of principally $\mathcal{G}^*$-δ-lifting modules need not be principally $\mathcal{G}^*$-δ-lifting. Under some conditions, the property of being a principally $\mathcal{G}^*$-δ-lifting module may be inherited by direct summands of principally $\mathcal{G}^*$-δ-lifting modules.

Proposition 4.15. *Let M_0 be a direct summand of a module M such that, for every decomposition $M = N \oplus K$ of M, there exist submodules N' of N and K' of K such that $M = M_0 \oplus N' \oplus K'$. If M is principally $\mathcal{G}^*$-δ-lifting, then M/M_0 is principally $\mathcal{G}^*$-δ-lifting.*

Proof. Let mR/M_0 be a submodule of M/M_0. Since M is principally $\mathcal{G}^*$-δ-lifting, there exists a decomposition $M = N \oplus K$ such that mR is β_δ^*-equivalent to N. Then, $(mR + N)/N$ is δ-small in M/N and $(mR + N)/mR$ is δ-small in M/mR. By hypothesis, $M = M_0 \oplus N' \oplus K'$ for some submodules N' of N and K' of K. Clearly, $(mR + M_0 + N')/mR$ is δ-small in M/mR. On the other hand, since $M/N \cong K$ and $M/(M_0 + K') \cong K'$, it can be shown via Lemma 2.1 (2) that $(mR + M_0 + N')/(M_0 + N')$ is δ-small in $M/(M_0 + N')$. Therefore,

mR/M_0 is β^*_δ-equivalent to $(M_0 \oplus N')/M_0$. This implies that M/M_0 is principally $\mathcal{G}^*$-δ-lifting. $\square$

Recall that a module M is said to be *distributive* if, for all submodules U, V and W,

$$U \cap (V + W) = (U \cap V) + (U \cap W)$$

or

$$U + (V \cap W) = (U + V) \cap (U + W).$$

A submodule N of a module of M is called *fully invariant* if $f(N) \subseteq N$ for all endomorphisms f of M. Also, M is said to be a *duo* (or *weak-duo*) *module* if every submodule (or direct summand) of M is fully invariant in M (see **[11]** for details).

Theorem 4.16. *Let $M = M_1 \oplus M_2$ be a duo* (*or distributive*) *module. Then, M is principally $\mathcal{G}^*$-δ-lifting if and only if M_1 and M_2 are principally $\mathcal{G}^*$-δ-lifting.*

Proof. Let $m \in M_1$. Since M is principally $\mathcal{G}^*$-δ-lifting, there is a decomposition $M = D \oplus D'$ such that mR is β^*_δ-equivalent to D in M. Since M is a duo module,

$$\begin{aligned} M_1 &= (M_1 \cap D) \oplus (M_1 \cap D'), \\ mR &= (mR \cap D) \oplus (mR \cap D'), \\ D' &= (M_1 \cap D') \oplus (M_2 \cap D'). \end{aligned}$$

We claim that mR is β^*_δ-equivalent to $M_1 \cap D$ in M_1. Consider the isomorphisms

$$mR \cap D' \cong \frac{mR}{mR \cap D} \cong \frac{mR + D}{D}$$

and

$$\frac{mR + (M_1 \cap D)}{M_1 \cap D} \cong \frac{mR}{mR \cap (M_1 \cap D)} = \frac{mR}{mR \cap D} \cong mR \cap D'.$$

Then, $(mR + D)/D$ is δ-small in $M/D \cong D'$. By Lemma 2.1 (2), $mR \cap D'$ is δ-small in D'. Again, by Lemma 2.1 (3), we have that $mR \cap D'$ is δ-small in $M_1 \cap D' \cong M/(M_1 \cap D)$. Thus, $(mR + (M_1 \cap D))/(M_1 \cap D)$ is δ-small in $M_1/(M_1 \cap D)$. In order to complete the proof, we need to show that $(mR + (M_1 \cap D))/mR$

is δ-small in M_1/mR. Since $(mR+D)/mR$ is δ-small in M/mR, by Lemma 2.1 (4), $(mR+(M_1\cap D))/mR$ is δ-small in M/mR.

On the other hand,

$$\frac{mR+D}{mR}=\frac{mR+(M_1\cap D)}{mR}\oplus\frac{mR+(M_2\cap D)}{mR}.$$

Then, $(mR+(M_1\cap D))/mR$ is δ-small in M_1/mR since

$$\frac{M}{mR}\cong\frac{M_1}{mR}\oplus\frac{M_2+mR}{mR},$$

and the natural projection π from M/mR onto M_1/mR with kernel $(M_2+mR)/mR$ maps

$$\pi\left(\frac{mR+D}{mR}\right)=\frac{mR+(M_1\cap D)}{mR}$$

This completes the proof. As a trivial note, similarly, M_2 is principally $\mathcal{G}^*$-δ-lifting.

Conversely, assume that M_1 and M_2 are principally $\mathcal{G}^*$-δ-lifting. Note that, in either case, M is duo or distributive, for any submodule N of M, $N=(N\cap M_1)\oplus(N\cap M_2)$. Let $m\in M$, and thus, $m=m_1+m_2$ for some $m_1\in M_1$ and $m_2\in M_2$. Then, $mR=(mR\cap M_1)\oplus(mR\cap M_2)$. Hence, $m_1R=mR\cap M_1$ and $m_2R=mR\cap M_2$ are cyclic submodules of M_1 and M_2, respectively. By hypothesis, there exist direct summands D_1 of M_1 and D_2 of M_2 such that $(m_1R+D_1)/m_1R$ is δ-small in M_1/m_1R and $(m_1R+D_1)/D_1$ is δ-small in M_1/D_1, similarly, $(m_2R+D_2)/m_2R$ is δ-small in M_2/m_2R and $(m_2R+D_2)/D_2$ is δ-small in M_2/D_2. Let $D=D_1\oplus D_2$. Then, it is easily verified that $((m_1R+D_1)/m_1R)\times((m_2R+D_2)/m_2R)$ is isomorphic to $(mR+D)/mR$, and $(M_1/m_1)\times(M_2/m_2R)$ is isomorphic to M/mR. Hence, $(mR+D)/mR$ is δ-small in M/mR. Similarly, $(mR+D)/D$ is δ-small in M/D. Thus, M is principally $\mathcal{G}^*$-δ-lifting. $\square$

We close this section by observing some results related to homomorphic images of principally $\mathcal{G}^*$-δ-lifting modules.

Lemma 4.17. [**15,** Lemma 3.3]. *Let M be a module and N a fully invariant submodule of M. If $M=M_1\oplus M_2$ for some submodules M_1*

and M_2 of M, then

$$\frac{M}{N} = \left(\frac{M_1 + N}{N}\right) \oplus \left(\frac{M_2 + N}{N}\right).$$

Proposition 4.18. *Let M be a principally $\mathcal{G}^*$-δ-lifting module. Then, M/N is principally $\mathcal{G}^*$-δ-lifting for every fully invariant submodule N of M.*

Proof. Let $m \in M$ and $\overline{L} = (L + N)/N$ denote the image of $L \subseteq M$ under the natural map from M onto $\overline{M} = M/N$. By hypothesis, there exists a direct summand D of M such that $M = D \oplus D'$ and mR is β^*_δ-equivalent to D. We prove that $\overline{m}R$ is β^*_δ-equivalent to $\overline{D}$. Note that $\overline{m}R = (mR + N)/N$ and $\overline{D} = (D + N)/N$. Therefore, we must show that $(\overline{m}R + \overline{D})/\overline{D}$ is δ-small in $\overline{M}/\overline{D}$ and $(\overline{m}R + \overline{D})/\overline{m}R$ is δ-small in $\overline{M}/\overline{m}R$. Thus, let $D + N \subseteq L \leq M$ be such that

$$\left(\frac{mR + D + N}{D + N}\right) + \left(\frac{L}{D + N}\right) = \frac{M}{D + N}$$

with M/L singular. Then $mR + D + L = M$ and $((mR + D)/D) + (L/D) = M/D$. By hypothesis, $M = L$.

Similarly, to prove that $(\overline{m}R + \overline{D})/\overline{m}R$ is δ-small in $\overline{M}/\overline{m}R$, let $mR + N \subseteq L \leq M$, and assume that

$$\left(\frac{mR + N + D}{mR + N}\right) + \left(\frac{L}{mR + N}\right) = \frac{M}{mR + N}$$

and M/L singular. Then, $M = mR + L + D$. Therefore, $M/mR = ((mR + D)/mR) + (L/mR)$. By hypothesis, $M = L$. This completes the proof. $\square$

As an immediate consequence of Proposition 4.18, we deduce that, if M is principally $\mathcal{G}^*$-δ-lifting, then so are $M/\operatorname{Rad}(M)$, $M/\operatorname{Soc}(M)$ and $M/\delta(M)$.

Corollary 4.19. *Let M be a weak-duo and principally $\mathcal{G}^*$-δ-lifting module. Then, every direct summand of M is principally $\mathcal{G}^*$-δ-lifting.*

5. Principally $\mathcal{G}^*$-δ-supplemented modules. In this section, we introduce a class of modules, the so-called principally $\mathcal{G}^*$-δ-supplemented modules, as a generalization of that of the principally $\mathcal{G}^*$-δ-lifting modules.

Definition 5.1. A module M is called *principally $\mathcal{G}^*$-δ-supplemented* if, for all $m \in M$, each cyclic submodule mR of M, there exists a δ-supplement submodule N of M such that mR is β^*_δ-equivalent to N.

The next theorem shows that the class of principally $\mathcal{G}^*$-δ-lifting modules lies between the classes of principally δ-lifting modules and principally $\mathcal{G}^*$-δ-supplemented modules.

Theorem 5.2. *Let M be a module. Consider the following conditions.*

(1) *M is principally δ-lifting.*
(2) *M is principally $\mathcal{G}^*$-δ-lifting.*
(3) *M is principally $\mathcal{G}^*$-δ-supplemented.*

Then $(1) \Rightarrow (2) \Rightarrow (3)$.

Proof.

$(1) \Rightarrow (2)$. By Theorem 4.4.

$(2) \Rightarrow (3)$. Let $m \in M$. By (2), there exists a direct summand D of M such that mR is β^*_δ-equivalent to D. Since $M = D \oplus D'$ for some submodule D' of M, D is a δ-supplement of D' in M. Therefore, M is principally $\mathcal{G}^*$-δ-supplemented. $\square$

Theorem 5.3. *If M is a π-projective principally $\mathcal{G}^*$-δ-supplemented module, then it is principally $\mathcal{G}^*$-δ-lifting.*

Proof. Let $m \in M$. There exists a δ-supplement K in M such that mR is β^*_δ-equivalent to K and $M = K + K'$ with $K \cap K'$ δ-small in K, for some submodule K' of M. By π-projectivity of M, there exists a $D \subseteq K$ such that $M = D \oplus K'$. In order to complete the proof, we prove that mR is β^*_δ-equivalent to D. Due to the fact that $(mR + K)/mR$ is δ-small in M/mR, $(mR + D)/mR$ is δ-small in M/mR.

It remains to prove that $(mR + D)/D$ is δ-small in M/D. For, if $(mR + D)/D + L/D = M/D$ with M/L singular, then $M = mR + D +$

$L = mR + K + L$. By hypothesis, $M/K = (mR + K)/K + (L + K)/K$ and $M/(L + K)$ singular implies that $M = L + K$. By modularity, $K = D \oplus (K \cap K')$; thus, $M = L + D + (K \cap K')$. Since $K \cap K'$ is δ-small in K, by Lemma 2.1 (2), $K \cap K'$ is δ-small in M. Thus, $M = L + D$, and $M = L$ since $D \subseteq L$. This completes the proof. $\square$

Theorem 5.4. *Let M be an $\oplus$-δ-supplemented module. If M is distributive and singular, then it is principally δ-lifting.*

Proof. Let $m \in M$. By hypothesis, there exists a direct summand δ-supplement K in M such that $M = mR + K = L \oplus K$ and $mR \cap K$ is δ-small in K for some submodule L of M.

Next, we prove $L \subseteq mR$. By distributivity, we have $mR = (mR \cap L) \oplus (mR \cap K)$. Then,

$$M = mR + K = (mR \cap L) + (mR \cap K) + K = (mR \cap L) + K.$$

Hence, $L = mR \cap L \subseteq mR$. Therefore, M is principally δ-lifting. $\square$

The following example shows that there are principally $\mathcal{G}^*$-δ-lifting and $\oplus$-δ-supplemented but not principally δ-lifting modules. It also shows that, in Theorem 5.4, the conditions of the module being distributive and singular are not irrelevant.

Example 5.5. Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus (\mathbb{Z}/2\mathbb{Z})$. In Example 4.7, it is shown that M is a principally $\mathcal{G}^*$-δ-lifting $\mathbb{Z}$-module. Since $Z(\mathbb{Q}) = 0$, and thus, $Z(M) = \mathbb{Z}/2\mathbb{Z}$, the module M is not singular. Now,

$$\begin{aligned}(1, \overline{1})\mathbb{Z} &\neq ([(1, \overline{1})\mathbb{Z}] \cap [\mathbb{Q} \oplus (\overline{0})]) \oplus ([(1, \overline{1})\mathbb{Z}] \cap [(0 \oplus \mathbb{Z}/2\mathbb{Z})]) \\ &= \{(2n, \overline{0}) \mid n \in \mathbb{Z}\}\end{aligned}$$

shows that M is not distributive.

Theorem 5.6. *Let M be a singular principally $\mathcal{G}^*$-δ-supplemented module. Then it is principally δ-supplemented.*

Proof. Assume that M is singular. Let $m \in M$. Suppose that mR is β_δ^*-equivalent to N and there exists a δ-supplement K of N in M

such that $M = N + K$ and $N \cap K$ is δ-small in K. Then,

$$\frac{M}{mR} = \frac{mR+N}{mR} + \frac{mR+K}{mR}.$$

The singularity of $M/(mR+K)$ implies $M = mR + K$ since $(mR+N)/mR$ is δ-small in M/mR.

Next, we prove $(mR) \cap K$ is δ-small in K. Let $L \subseteq K$ be such that $K = (mR) \cap K + L$. Then, $M = L + mR + N$. Therefore, $M/N = (L+N)/N + (mR+N)/N$. By the singularity of $M/(L+N)$, we have $M = L + N$. Hence, $K = L + (K \cap N)$. Then, $K = L$ since $N \cap K$ is δ-small in K and K/L is singular as a submodule of singular module M/L. Therefore, K is a δ-supplement of mR. □

Recall from **[15]** that a module M is said to be *principally δ-semiperfect* if every factor module of M by a cyclic submodule has a projective δ-cover, that is, for any $m \in M$, there exists a projective module P and an epimorphism $f\colon P \to M/mR$ such that $\operatorname{Ker} f$ is δ-small in P. A ring R is called *principally δ-semiperfect* in the case where the right R-module R is principally δ-semiperfect. Every δ-semiperfect module is principally δ-semiperfect.

Next, we deal with projective principally δ-semiperfect modules in terms of the notion of principally $\oplus$-δ-supplemented.

Proposition 5.7. *Let M be a projective module, and consider the following statements.*

(1) *M is principally δ-semiperfect.*
(2) *M is principally $\mathcal{G}^*$-δ-lifting.*
(3) *M is principally $\mathcal{G}^*$-δ-supplemented.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). *If M is a singular module, then all of them are equivalent.*

Proof.

(1) $\Leftrightarrow$ (2). This is clear by **[15,** Theorem 3.11] and Proposition 4.8.

(2) $\Leftrightarrow$ (3). By Theorems 5.2 and 5.3. □

If R is δ-semiperfect, then by Proposition 5.7, the right R-module R is principally $\mathcal{G}^*$-δ-lifting.

Example 5.8. Let

$$Q = \prod_{i=1}^{\infty} F_i,$$

where $F_i = \mathbb{Z}_2$ for all i, and R denote the subring of Q generated by

$$\bigoplus_{i=1}^{\infty} F_i$$

and 1_Q. Then, R is δ-semiperfect by [**18,** Example 4.1], and thus, principally $\mathcal{G}^*$-δ-lifting by Proposition 5.7, but not semiperfect (i.e., not lifting). However, it is principally lifting since it is von Neumann regular.

6. On principally semisimple modules. A module M is said to be *principally semisimple* if every cyclic submodule of M is a direct summand of M. Tuganbaev calls a principally semisimple module a *regular module* in [**13**]. Obviously, every semisimple module is principally semisimple. Every principally semisimple module is principally δ-lifting, and thus, principally $\oplus$-δ-supplemented. A ring R is called *principally semisimple* if the right R-module R is principally semisimple. It is clear that every principally semisimple ring is von Neumann regular and vice versa. In this section, we briefly mention principally semisimple modules in terms of the notion of principally $\mathcal{G}^*$-δ-lifting.

Proposition 6.1. *Every principally semisimple module is principally $\mathcal{G}^*$-δ-lifting.*

Proof. Let M be a principally semisimple module and $m \in M$. Then, mR is a direct summand of M. Also, mR is β^*-equivalent to mR. Therefore, M is principally $\mathcal{G}^*$-δ-lifting. □

Proposition 6.2. *Let M be a principally $\mathcal{G}^*$-δ-lifting module. If M is singular and distributive, then $M/\delta(M)$ is a principally semisimple module.*

Proof. Let $m \in M$. There exists a decomposition $M = D \oplus D'$ such that mR is β_δ^*-equivalent to D. We show that D' is a δ-supplement of

mR. Then

$$\frac{M}{mR} = \left(\frac{mR + D}{mR}\right) + \left(\frac{mR + D'}{mR}\right).$$

The singularity of M and the δ-smallness of $(mR + D)/mR$ in M/mR implies $M = mR + D'$. Let $(mR \cap D') + L = D'$ where L is a submodule of D'. Similarly,

$$\frac{M}{D} = \left(\frac{mR + D}{D}\right) + \left(\frac{D + L}{D}\right)$$

implies $M = D \oplus L$. Hence, $L = D'$. Thus, $mR \cap D'$ is δ-small in D', and therefore, $mR \cap D' \subseteq \delta(M)$. By the distributivity of M, we have

$$(mR + \delta(M)) \cap (D' + \delta(M)) = (mR \cap D') + \delta(M) = \delta(M).$$

Thus, $(mR + \delta(M))/\delta(M)$ is a direct summand of $M/\delta(M)$. Therefore, $M/\delta(M)$ is a principally semisimple module. $\square$

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