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PERIODS OF CONTINUOUS MAPS ON CLOSED SURFACES

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ABSTRACT. The objective of the present work is to present information on the set of periodic points of a continuous self-map on a closed surface which can be obtained using the action of this map on homological groups of the closed surface.

1. Introduction. Periodic orbits play an important role in the description of the dynamics of a map; through their study, we can use topological information. Perhaps the best known result in this direction is that contained in the seminal paper entitled, "Period three implies chaos," for continuous self-maps on the interval, see [**8**], and of course, Sharkovskiĭ's famous theorem [**10**] describing the full set of periods of continuous self-maps on the interval. In addition, the set of periods of continuous self-maps on the circle has been characterized, see for instance, [**1**].

The interval and the circle are unique compact manifolds of dimension one. After studying the set of periods of self-continuous maps on these manifolds, the next natural step is to begin study of continuous self-maps on compact manifolds of dimension two, i.e., on closed surfaces. This is the main goal of this paper, and we shall use the homological information of these maps to provide information regarding the periods of their periodic orbits.

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Throughout, by a *closed surface*, we denote a connected compact surface, with or without boundary, orientable or not. More precisely, an *orientable, connected, compact surface without boundary of genus* $g \geq 0$, $\mathbb{M}_g$, is homeomorphic to the sphere if $g = 0$, to the torus if $g = 1$, or to the connected sum of g copies of the torus if $g \geq 2$. An *orientable, connected, compact surface with boundary of genus* $g \geq 0$, $\mathbb{M}_{g,b}$, is homeomorphic to $\mathbb{M}_g$, minus a finite number $b > 0$ of open discs with pairwise disjoint closure. In what follows, $\mathbb{M}_{g,0} = \mathbb{M}_g$.

A *non-orientable, connected, compact surface* without boundary of genus $g \geq 1$, $\mathbb{N}_g$, is homeomorphic to the real projective plane if $g = 1$, or to the connected sum of g copies of the real projective plane if $g > 1$. A *non-orientable, connected, compact surface* with boundary of genus $g \geq 1$, $\mathbb{N}_{g,b}$, is homeomorphic to $\mathbb{N}_g$, minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows, $\mathbb{N}_{g,0} = \mathbb{N}_g$.

Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map on a closed surface $\mathbb{X}$. A point $x \in \mathbb{X}$ is periodic of period n if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, \ldots, n-1$. We denote the set of periods of all periodic points of f by $\mathrm{Per}(f)$. The aim of the present paper is to provide information on $\mathrm{Per}(f)$.

Let A be an $n \times n$ complex matrix. A $k \times k$ *principal submatrix* of A is a submatrix lying in the same set of k rows and columns, and a $k \times k$ *principal minor* is the determinant of such a principal submatrix. There are $\binom{n}{k}$ different $k \times k$ principal minors of A, and the sum of these is denoted by $E_k(A)$. In particular, $E_1(A)$ is the trace of A, and $E_n(A)$ is the determinant of A, denoted by $\det(A)$.

It is well known that the characteristic polynomial of A is given by

$$\det(tI - A) = t^n - E_1(A)t^{n-1} + E_2(A)t^{n-2} - \cdots + (-1)^n E_n(A).$$

Our main result is stated in the next theorem.

Theorem 1.1. *Let $\mathbb{X}$ be a closed surface, let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map, and let A and (d) be the integral matrices of the endomorphisms $f_{*i} : H_i(\mathbb{X}, \mathbb{Q}) \to H_i(\mathbb{X}, \mathbb{Q})$ induced by f on the ith homology group of $\mathbb{X}$, $i = 1, 2$.*

If $\mathbb{X}$ is either $\mathbb{M}_{g,b}$ with $b > 0$, or $\mathbb{N}_{g,b}$ with $b \geq 0$, then the following statements hold:

(a) *If* $E_1(A) \neq 1$, *then* $1 \in \mathrm{Per}(f)$.
(b) *If* $E_1(A) = 1$ *and* $E_2(A) \neq 0$, *then* $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.

If $\mathbb{X} = \mathbb{M}_{g,b}$ *with* $b = 0$, *then the following statements hold*:

(c) *If* $E_1(A) \neq 1 + d$, *then* $1 \in \mathrm{Per}(f)$.
(d) *If* $E_1(A) = 1+d$ *and* $E_2(A) \neq d^2+d+1$, *then* $\mathrm{Per}(f)(f) \cap \{1, 2\} \neq \emptyset$.

If $\mathbb{X} = \mathbb{M}_{g,b}$ *with* $b > 0$, *then the following statement holds*:

(e) *If* $2g + b - 1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ *and* k *is the smallest integer of the set* $\{3, 4, \ldots, 2g + b - 1\}$ *such that* $E_k(A) \neq 0$, *then* $\mathrm{Per}(f)$ *has a periodic point of period a divisor of* k.

If $\mathbb{X} = \mathbb{N}_{g,b}$ *with* $b \geq 0$, *then the following statement holds*:

(f) *If* $g + b - 1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ *and* k *is the smallest integer of the set* $\{3, 4, \ldots, g + b - 1\}$ *such that* $E_k(A) \neq 0$, *then* $\mathrm{Per}(f)$ *has a periodic point of period a divisor of* k.

Theorem 1.1 is proven in Section 2.

Similar results to those obtained in Theorem 1.1 but for homeomorphisms on closed surfaces were obtained by Franks and Llibre [**5**], as well as the authors in [**6**]. Other related results may be found in [**5**]; also see the references there.

2. Proof of Theorem 1.1.

Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map, and let $\mathbb{X}$ either be $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$. Then the *Lefschetz number* of f is defined by

$$L(f) = \mathrm{trace}(f_{*0}) - \mathrm{trace}(f_{*1}) + \mathrm{trace}(f_{*2}).$$

For continuous self-maps f defined on $\mathbb{X}$ the Lefschetz fixed point theorem states, see for instance, [**2**].

Theorem 2.1. *If* $L(f) \neq 0$, *then* f *has a fixed point.*

With the objective of studying the periodic points of f, we shall use the Lefschetz numbers of the iterates of f, i.e., $L(f^n)$. Note that, if $L(f^n) \neq 0$, then f^n has a fixed point, and consequently, f has a periodic point of period a divisor of n. In order to study the whole sequence $\{L(f^n)\}_{n\geq 1}$, the formal *Lefschetz zeta function* of f is defined as

$$Z_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n\right). \tag{2.1}$$

The Lefschetz zeta function is in fact a generating function for the sequence of Lefschetz numbers $L(f^n)$.

Let f be a continuous self-map defined on $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$, respectively. For a closed surface, the homological groups with coefficients in $\mathbb{Q}$ are linear vector spaces over $\mathbb{Q}$. We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.,

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$ and $n_2 = 1$ if $b = 0$ and $n_2 = 0$ if $b > 0$; and the induced linear maps $f_{*k} : H_k(\mathbb{M}_{g,b}, \mathbb{Q}) \to H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ by f on the homological group $H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ are $f_{*0} = (1)$ and $f_{*2} = (d)$, where d is the *degree* of map f if $b = 0$, $f_{*2} = (0)$ if $b > 0$ and $f_{*1} = A$, where A is an $n_1 \times n_1$ integral matrix, for additional details, see **[9, 11]**.

Recall that the homological groups of $\mathbb{N}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.,

$$H_k(\mathbb{N}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = g + b - 1$ and $n_2 = 0$; and the induced linear maps are $f_{*0} = (1)$ and $f_{*1} = A$ where A is an $n_1 \times n_1$ integral matrix, again, for additional details, see **[9, 11]**.

From the work of Franks **[3]** we have, for a continuous self-map of a closed surface, that its Lefschetz zeta function is the rational function:

$$Z_f(t) = \frac{\det(I - tf_{*1})}{\det(I - tf_{*0})\det(I - tf_{*2})},$$

where, in $I - tf_{*k}$, I denotes the $n_k \times n_k$ identity matrix and $\det(I - tf_{*2}) = 1$ if $f_{*2} = (0)$. Then, for a continuous map $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$, we have

$$Z_f(t) = \begin{cases} \dfrac{\det(I-tA)}{(1-t)(1-dt)} & \text{if } b=0, \\ \dfrac{\det(I-tA)}{1-t} & \text{if } b>0, \end{cases} \tag{2.2}$$

and, for a continuous map $f : \mathbb{N}_{g,b} \to \mathbb{N}_{g,b}$, we have

$$Z_f(t) = \frac{\det(I-tA)}{1-t}. \tag{2.3}$$

Proof of Theorem 1.1. Combining expressions (2.1) and (2.2), if $\mathbb{X} = \mathbb{M}_{g,b}$, $b>0$, and expressions (2.1) and (2.3), if $\mathbb{X} = \mathbb{N}_{g,b}$, with $b \geq 0$, we obtain the following equalities:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log(Z_f(t)) \\ &= \log\left(\frac{\det(I-tA)}{1-t}\right) \\ &= \log\left(\frac{1 - E_1(A)t + E_2(A)t^2 - \cdots + (-1)^m E_m(A)t^m}{1-t}\right) \\ &= \log(1 - E_1(A)t + E_2(A)t^2 - \cdots) - \log(1-t) \\ &= \left(- E_1(A)t + \left(E_2(A) - \frac{E_1(A)^2}{2}\right)t^2 - \cdots \right) \\ &\quad - \left(-t - \frac{t^2}{2} - \cdots \right) \\ &= (1 - E_1(A))t + \left(\frac{1}{2} - \frac{E_1(A)^2}{2} + E_2(A)\right)t^2 + O(t^3). \end{aligned}$$

Here, $n_1 = 2g+b-1$, if $\mathbb{X} = \mathbb{M}_{g,b}$, with $b>0$ or $n_1 = g+b-1$, if $\mathbb{X} = \mathbb{N}_{g,b}$, with $b \geq 0$. Therefore, we have

$$L(f) = 1 - E_1(A) \quad \text{and} \quad L(f^2) = 1 - E_1(A)^2 + 2E_2(A).$$

Hence, if $E_1(A) \neq 1$, then $L(f) \neq 0$, and by Theorem 2.1, statement (a) follows.

If $E_1(A) = 1$ and $E_2(A) \neq 0$, then $L(f^2) = 2E_2(A) \neq 0$, and again by Theorem 2.1, we obtain that $\operatorname{Per}(f) \cap \{1,2\} \neq \emptyset$. Thus, statement (b) has been proved.

Let $\mathbb{X} = \mathbb{M}_{g,b}$ with $b = 0$. By (2.1) and (2.2) with $b = 0$, we obtain the following equalities:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log(Z_f(t)) \\
&= \log\left(\frac{\det(I - tA)}{(1-t)(1-dt)}\right) \\
&= \log\left(\frac{1 - E_1(A)t + E_2(A)t^2 - \cdots + (-1)^m E_m(A)t^m}{(1-t)(1-dt)}\right) \\
&= \log(1 - E_1(A)t + E_2(A)t^2 - \cdots) - \log((1-t)(1-dt)) \\
&= \left(- E_1(A)t + \left(E_2(A) - \frac{E_1(A)^2}{2}\right)t^2 - \cdots\right) \\
&\quad - \left(- (1+d)t - \left(\frac{d^2+1}{2}\right)t^2 - \cdots\right) \\
&= (1 + d - E_1(A))t \\
&\quad + \left(E_2(A) - \frac{E_1(A)^2}{2} - \frac{d^2+1}{2}\right)t^2 + O(t^3).
\end{aligned}$$

Here, $n_1 = 2g$; therefore, we have

$$L(f) = 1 + d - E_1(A),$$

and

$$L(f^2) = 2E_2(A) - E_1(A)^2 - (d^2+1).$$

Hence, if $E_1(A) \neq 1 + d$, then $L(f) \neq 0$, and, by Theorem 2.1, statement (c) follows.

If $E_1(A) = 1 + d$ and $E_2(A) \neq d^2 + d + 1$, then $L(f^2) = 2E_2(A) - 2(d^2 + d + 1) \neq 0$, and again, by Theorem 2.1, we obtain that $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$. Thus, statement (d) has been proved.

Now, assume, that $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, $2g + b - 1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$, and k is the smallest integer of the set $\{3, 4, \ldots, 2g + b - 1\}$ such that $E_k(A) \neq 0$. Therefore,

$$\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n = \log\left(\tfrac{1-t+(-1)^k E_k(A)t^k}{1-t} + \cdots + \tfrac{(-1)^{b-1} E_{2g+b-1}(A)t^{2g+b-1}}{1-t}\right)$$

$$= \log\left(1 + \tfrac{(-1)^k E_k(A)t^k}{1-t} + \cdots + \tfrac{(-1)^{b-1}E_{2g+b-1}(A)t^{2g+b-1}}{1-t}\right)$$
$$= (-1)^k E_k(A)t^k + O(t^{k+1}).$$

Hence, $L(f) = \cdots = L(f^{k-1}) = 0$ and $L(f^k) = (-1)^k k E_k(A) \neq 0$. Thus, statement (e) follows from Theorem 2.1.

Suppose that $\mathbb{X} = \mathbb{N}_{g,b}$, with $b \geq 0$, $g+b-1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$, and k is the smallest integer of the set $\{3, 4, \ldots, g+b-1\}$ such that $E_k(A) \neq 0$. Therefore,

$$\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n = \log\left(\tfrac{1-t+(-1)^k E_k(A)t^k}{1-t} + \cdots + \tfrac{(-1)^{g+b-1}E_{g+b-1}(A)t^{g+b-1}}{1-t}\right)$$
$$= \log\left(1 + \tfrac{(-1)^k E_k(A)t^k}{1-t} + \cdots + \tfrac{(-1)^{g+b-1}E_{g+b-1}(A)t^{g+b-1}}{1-t}\right)$$
$$= (-1)^k E_k(A)t^k + O(t^{k+1}).$$

Again, $L(f) = \cdots = L(f^{k-1}) = 0$ and $L(f^k) = (-1)^k k E_k(A) \neq 0$. Therefore, statement (f) follows from Theorem 2.1. □

REFERENCES

1. L. Alsedà, J. Llibre and M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, Adv. Ser. Nonlin. Dynam. **5**, World Scientific, Singapore, 2000.

2. R.F. Brown, *The Lefschetz fixed point theorem*, Scott Foresman and Company, Glenview, IL, 1971.

3. J. Franks, *Homology and dynamical systems*, CBMS Reg. Conf. Ser. **49**, American Mathematical Society, Providence, RI, 1982.

4. J. Franks and J. Llibre, *Periods of surface homeomorphisms*, Contemp. Math. **117** (1991), 63–77.

5. J.M. Gambaudo and J. Llibre, *A note on the set of periods of surface homeomorphisms*, J. Math. Anal. Appl. **177** (1993), 627–632.

6. J.L. García Guirao and J. Llibre, *Periods of homeomorphisms on surfaces*, in *Difference equations*, L. Alsedà i Soler, et al., eds., Discr. Dynam. Syst. Appl. **180** (2016), 171–178.

7. B. Halpern, *Fixed point for iterates*, Pacific J. Math. **25** (1968), 255–275.

8. T.Y. Li and J. Yorke, *Period three implies chaos*, Amer. Math. Month. **82** (1975), 985–992.

9. J.R. Munkres, *Elements of algebraic topology*, Addison-Wesley, Boston, 1984.

10. A.N. Sharkovskiĭ, *Coexistence of cycles of a continuous map of the line into itself*, Ukraine Math. Z. **16** (1964), 61–71.

11. J.W. Vicks, *Homology theory*, *An introduction to algebraic topology*, Springer-Verlag, New York, 1994.

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