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PROPER RESOLUTIONS AND GORENSTEINNESS IN TRIANGULATED CATEGORIES

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ABSTRACT. Let $\mathcal{T}$ be a triangulated category with triangulation Δ, $\xi \subseteq \Delta$ a proper class of triangles and $\mathcal{C}$ an additive full subcategory of $\mathcal{T}$. We provide a method for constructing a proper $\mathcal{C}(\xi)$-resolution (respectively, coproper $\mathcal{C}(\xi)$-coresolution) of one term in a triangle in ξ from those of the other two terms. By using this construction, we show the stability of the Gorenstein category $\mathcal{GC}(\xi)$ in triangulated categories. Some applications are given.

Introduction. Triangulated categories were introduced by Grothendieck and Verdier in the early 1960s as the proper framework for homological algebra in an abelian category. Since then, triangulated categories have found important applications in algebraic geometry, stable homotopy theory and representation theory. Examples for this may be found in duality theory, Hartshorne [**9**] and Iversen [**12**], or in the fundamental work on perverse sheaves, Beilinson, Bernstein and Deligne [**3**].

Relative homological algebra has been formulated by Hochschild in categories of modules and later by Heller and Butler and Horrocks in more general categories with a relative abelian structure. Let $\mathcal{T}$ be a triangulated category with triangulation Δ. Beligiannis [**4**] developed a homological algebra in $\mathcal{T}$ which parallels the homological algebra in an exact category in the sense of Quillen. To develop the homology, a class of triangles $\xi \subseteq \Delta$, called proper class of triangles, is specified. This class is closed under translations and satisfies the analogous formal

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properties of a proper class of short exact sequences. Moreover, ξ-projective objects, ξ-projective resolution, ξ-projective dimension and their duals are introduced [**4,** Section 4].

In the category of modules, there is a natural generalization of the class of finitely generated projective modules over a commutative Noetherian ring, due to Auslander and Bridger [**2**], that is the notion of modules in Auslander's G-class (modules of Gorenstein dimension 0). To complete the analogy, Enochs and Jenda [**7**] introduced Gorenstein projective modules that generalize the notion of modules of Gorenstein dimension 0 to the class of not necessarily finitely generated modules. Using this class, they developed a relative homological algebra in the category of modules. Motivated by this, Beligiannis [**5**] defined the concept of an $\mathcal{X}$-Gorenstein object induced by a pair $(\mathcal{A}, \mathcal{X})$ consisting of an additive category $\mathcal{A}$ and a contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{A}$, assuming that any $\mathcal{X}$-epic has a kernel in $\mathcal{A}$. This notion is a natural generalization of a module of Gorenstein dimension 0 in the sense of Auslander and Bridger [**2**]. Based on the works of Auslander and Bridger [**2**], Enochs and Jenda [**7**] and Beligiannis [**5**], Asadollahi and Salarian [**1**] developed the above-mentioned relative homological algebra in triangulated categories with enough ξ-projectives. They introduced and studied ξ-Gorenstein projective objects and ξ-Gorenstein projective dimensions with respect to a proper class of triangles ξ.

Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ an additive full subcategory of $\mathcal{A}$. Sather-Wagstaff, Sharif and White [**15**] introduced the Gorenstein category $\mathcal{G}(\mathcal{C})$ which unifies the notions: modules of Gorenstein dimension 0 [**2**], Gorenstein projective modules, Gorenstein injective modules [**7**], V-Gorenstein projective modules, V-Gorenstein injective modules [**8**], and so on. Huang [**10**] provided a method for constructing a proper $\mathcal{C}$-resolution (respectively, coproper $\mathcal{C}$-coresolution) of one term in a short exact sequence in $\mathcal{A}$ from those of the other two terms. By using these, he affirmatively answered an open question on the stability of the Gorenstein category $\mathcal{G}(\mathcal{C})$ posed by Sather-Wagstaff, Sharif and White [**15**] and also proved that $\mathcal{G}(\mathcal{C})$ is closed under direct summands.

Let $\mathcal{T} = (\mathcal{T}, \Sigma, \Delta)$ be a triangulated category with Σ the suspension functor and Δ the triangulation, $\xi \subseteq \Delta$ a proper class of triangles and $\mathcal{C}$ an additive full subcategory of $\mathcal{T}$ closed under isomorphisms and Σ-stable, i.e., $\Sigma(\mathcal{C}) = \mathcal{C}$.

In this paper, we make a general study of relative homological algebra on triangulated categories which may not have enough ξ-projectives or enough ξ-injectives. Section 1 gives some notions and basic consequences of the proper class ξ. Section 2 provides a method for constructing a proper $\mathcal{C}(\xi)$-resolution (respectively, coproper $\mathcal{C}(\xi)$-coresolution) of one term in a triangle in ξ from those of the other two terms. Section 3 is devoted to establishing the stability of the Gorenstein category $\mathcal{GC}(\xi)$ in triangulated categories, and some applications are given.

1. Definitions and basic facts. This section is devoted to discussing the axioms of a proper class of triangles and drawing some basic consequences for use throughout this paper. The basic reference for triangulated categories is the monograph of Neeman [**14**]. For terminology, we shall follow [**4, 16**].

Triangulated categories. Let $\mathcal{T}$ be an additive category and $\Sigma : \mathcal{T} \to \mathcal{T}$ an additive functor. Let $\mathrm{Diag}(\mathcal{T}, \Sigma)$ denote the category whose objects are diagrams in $\mathcal{T}$ of the form $X \overset{\mu}{\to} Y \overset{\nu}{\to} Z \overset{\omega}{\to} \Sigma X$, and morphisms between two objects $X_i \overset{\mu_i}{\to} Y_i \overset{\nu_i}{\to} Z_i \overset{\omega_i}{\to} \Sigma X_i$, $i = 1, 2$, are triples of morphisms $f : X_1 \to X_2$, $g : Y_1 \to Y_2$ and $h : Z_1 \to Z_2$, such that the following diagram commutes:

$$\begin{tikzcd} X_1 \arrow[r, "\mu_1"] \arrow[d, "f"] & Y_1 \arrow[r, "\nu_1"] \arrow[d, "g"] & Z_1 \arrow[r, "\omega_1"] \arrow[d, "h"] & \Sigma X_1 \arrow[d, "\Sigma f"] \\ X_2 \arrow[r, "\mu_2"] & Y_2 \arrow[r, "\nu_2"] & Z_2 \arrow[r, "\omega_2"] & \Sigma X_2. \end{tikzcd}$$

Such a morphism is called an *isomorphism* if f, g, h are isomorphisms in $\mathcal{T}$.

A triple $(\mathcal{T}, \Sigma, \Delta)$ is called a *triangulated category*, where $\mathcal{T}$ is an additive category, Σ is an autoequivalence of $\mathcal{T}$ and Δ is a full subcategory of $\mathrm{Diag}(\mathcal{T}, \Sigma)$ which satisfies the following axioms. The elements of Δ are then called *triangles*.

(TR1) Every diagram isomorphic to a triangle is a triangle. For every object X in $\mathcal{T}$, the diagram $X \overset{1}{\to} X \to 0 \to \Sigma X$ is a triangle. Every morphism $\mu : X \to Y$ in $\mathcal{T}$ can be embedded into a triangle $X \overset{\mu}{\to} Y \to Z \to \Sigma X$.

(TR2) $X \overset{\mu}{\to} Y \overset{\nu}{\to} Z \overset{\omega}{\to} \Sigma X$ is a triangle if and only if $Y \overset{\nu}{\to} Z \overset{\omega}{\to} \Sigma X \overset{-\Sigma\mu}{\to} \Sigma Y$ is so.

(TR3) Given triangles $X \overset{\mu}{\to} Y \overset{\nu}{\to} Z \overset{\omega}{\to} \Sigma X$ and $X' \overset{\mu'}{\to} Y' \overset{\nu'}{\to} Z' \overset{\omega'}{\to} \Sigma X'$, each commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\mu"] \arrow[d, "f"] & Y \arrow[r, "\nu"] \arrow[d, "g"] & Z \arrow[r, "\omega"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "\mu'"] & Y' \arrow[r, "\nu'"] & Z' \arrow[r, "\omega'"] & \Sigma X'
\end{tikzcd}$$

can be completed to a morphism of triangles (but not necessarily uniquely).

(TR4) The octahedral axiom. For this formulation, we refer the reader to Proposition 1.1.

Proposition 1.1 ([**4,** Proposition 2.1])**.** *Let $\mathcal{T}$ be an additive category equipped with an autoequivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ and a class of diagrams $\Delta \subseteq \mathrm{Diag}(\mathcal{T}, \Sigma)$. Suppose that the triple $(\mathcal{T}, \Sigma, \Delta)$ satisfies all the axioms of a triangulated category except possibly of the octahedral axiom. Then the following are equivalent*:

(1) **Base change.** *For any diagram $X \overset{\mu}{\to} Y \overset{\nu}{\to} Z \overset{\omega}{\to} \Sigma X \in \Delta$ and any morphism $\alpha : Z' \to Z$, there exists a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X' \arrow[r, equal] \arrow[d, "\beta'"] & X' \arrow[r] \arrow[d, "\beta"] & 0 \arrow[d] \\
X \arrow[r, "\mu'"] \arrow[d, equal] & Y' \arrow[r, "\nu'"] \arrow[d, "\alpha'"] & Z' \arrow[r, "\omega'"] \arrow[d, "\alpha"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "\mu"] \arrow[d] & Y \arrow[r, "\nu"] \arrow[d, "\gamma'"] & Z \arrow[r, "\omega"] \arrow[d, "\gamma"] & \Sigma X \arrow[d] \\
0 \arrow[r] & \Sigma X' \arrow[r, equal] & \Sigma X' \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and vertical diagrams are triangles in Δ.

(2) **Cobase change.** *For any diagram* $X \xrightarrow{\mu} Y \xrightarrow{\nu} Z \xrightarrow{\omega} \Sigma X \in \Delta$ *and any morphism* $\beta : X \to X'$, *there exists a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Sigma^{-1}Z' \arrow[r, equal] \arrow[d, "-\Sigma^{-1}\gamma"] & \Sigma^{-1}Z' \arrow[r] \arrow[d, "-\Sigma^{-1}\gamma'"] & 0 \arrow[d] \\
\Sigma^{-1}Z \arrow[r, "-\Sigma^{-1}\omega"] \arrow[d, equal] & X \arrow[r, "\mu"] \arrow[d, "\beta"] & Y \arrow[r, "\nu"] \arrow[d, "\beta'"] & Z \arrow[d, equal] \\
\Sigma^{-1}Z \arrow[r, "-\Sigma^{-1}\omega'"] \arrow[d] & X' \arrow[r, "\mu'"] \arrow[d, "\alpha"] & Y' \arrow[r, "\nu'"] \arrow[d, "\alpha'"] & Z \arrow[d] \\
0 \arrow[r] & Z' \arrow[r, equal] & Z' \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and vertical diagrams are triangles in Δ.

(3) **Octahedral axiom.** *For any two morphisms* $\mu : X \to Y$ *and* $\nu : Y \to Z$, *there exists a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "\mu"] \arrow[d, equal] & Y \arrow[r, "\mu'"] \arrow[d, "\nu"] & Z' \arrow[r, "\mu''"] \arrow[d, "\alpha"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "\nu\mu"] \arrow[d, "\mu"] & Z \arrow[r, "\omega"] \arrow[d, equal] & Y' \arrow[r, "\omega'"] \arrow[d, "\beta"] & \Sigma X \arrow[d, "\Sigma\mu"] \\
Y \arrow[r, "\nu"] \arrow[d] & Z \arrow[r, "\nu'"] \arrow[d, "0"] & X' \arrow[r, "\nu''"] \arrow[d, "(\Sigma\mu')\nu''"] & \Sigma Y \arrow[d] \\
0 \arrow[r] & \Sigma Z' \arrow[r, equal] & \Sigma Z' \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and the third vertical diagrams are triangles in Δ.

Proper class of triangles. Let $\mathcal{T} = (\mathcal{T}, \Sigma, \Delta)$ be a triangulated category, where Σ is the suspension functor and Δ is the triangulation.

A triangle $(T) : X \xrightarrow{\mu} Y \xrightarrow{\nu} Z \xrightarrow{\omega} \Sigma X$ is called *split* if it is isomorphic to the triangle $X \xrightarrow{\binom{1}{0}} X \oplus Z \xrightarrow{(0,1)} Z \xrightarrow{0} \Sigma X$. It is easy to see that (T) is split if and only if μ is a section, ν is a retraction or $\omega = 0$. The full subcategory of Δ consisting of the split triangles will be denoted by Δ_0. A class of triangles ξ is *closed under base change* if, for any triangle $X \overset{\mu}{\to} Y \overset{\nu}{\to} Z \overset{\omega}{\to} \Sigma X \in \xi$ and any morphism $\alpha : Z' \to Z$ as in Proposition 1.1 (1), the triangle $X \overset{\mu'}{\to} Y' \overset{\nu'}{\to} Z' \overset{\omega'}{\to} \Sigma X$ is in ξ.

Dually, a class of triangles ξ is *closed under cobase change* if, for any triangle $X \overset{\mu}{\to} Y \overset{\nu}{\to} Z \overset{\omega}{\to} \Sigma X \in \xi$ and any morphism $\beta : X \to X'$ as in Proposition 1.1 (2), the triangle $X' \overset{\mu'}{\to} Y' \overset{\nu'}{\to} Z \overset{\omega'}{\to} \Sigma X'$ is in ξ. A class of triangles ξ is *closed under suspensions* if, for any triangle $X \overset{\mu}{\to} Y \overset{\nu}{\to} Z \overset{\omega}{\to} \Sigma X \in \xi$ and any $i \in \mathbb{Z}$, the triangle

$$\Sigma^i X \xrightarrow{(-1)^i \Sigma^i \mu} \Sigma^i Y \xrightarrow{(-1)^i \Sigma^i \nu} \Sigma^i Z \xrightarrow{(-1)^i \Sigma^i \omega} \Sigma^{i+1} X$$

is in ξ. A class of triangles ξ is called *saturated* if, in the situation of base change in Proposition 1.1, whenever the third vertical and the second horizontal triangles are in ξ, then the triangle $X \overset{\mu}{\to} Y \overset{\nu}{\to} Z \overset{\omega}{\to} \Sigma X$ is in ξ.

The next concept is inspired by the definition of an exact category **[6]**.

A full subcategory $\xi \subseteq \mathrm{Diag}(\mathcal{T}, \Sigma)$ is called a *proper class of triangles* if the following conditions hold:

(i) ξ is closed under isomorphisms, finite coproducts and $\Delta_0 \subseteq \xi \subseteq \Delta$.

(ii) ξ is closed under suspensions and is saturated.

(iii) ξ is closed under base and cobase change.

For example, the class Δ_0 of split triangles and the class Δ of all triangles in $\mathcal{T}$ are proper classes of triangles.

From now on, we fix a triangulated category $\mathcal{T} = (\mathcal{T}, \Sigma, \Delta)$ and a proper class of triangles ξ in $\mathcal{T}$, where Σ is the suspension functor and Δ is the triangulation.

An object $P \in \mathcal{T}$ (respectively, $I \in \mathcal{T}$) is called *ξ-projective* (respectively, *ξ-injective*) if, for any triangle $X \to Y \to Z \to \Sigma X$ in ξ, the induced sequence

$$0 \to \mathcal{T}(P, X) \to \mathcal{T}(P, Y) \to \mathcal{T}(P, Z) \to 0,$$

respectively,

$$0 \to \mathcal{T}(Z, I) \to \mathcal{T}(Y, I) \to \mathcal{T}(X, I) \to 0,$$

is exact in the category $\mathcal{A}$b of abelian groups. We say that $\mathcal{T}$ has *enough ξ-projectives* if, for any object $X \in \mathcal{T}$, there exists a triangle $K \to P \to X \to \Sigma K$ in ξ with P a ξ-projective object.

Dually, we define when $\mathcal{T}$ has enough ξ-injectives. In this case, a triangle $X \to Y \to Z \to \Sigma X$ is in ξ if and only if, for any ξ-projective object P, the induced sequence

$$0 \to \mathcal{T}(P, X) \to \mathcal{T}(P, Y) \to \mathcal{T}(P, Z) \to 0$$

is exact in $\mathcal{A}$b if and only if, for any ξ-injective object I, the induced sequence

$$0 \to \mathcal{T}(Z, I) \to \mathcal{T}(Y, I) \to \mathcal{T}(X, I) \to 0$$

is exact in $\mathcal{A}$b (see [**4,** Lemma 4.2] and its dual).

Let $X \overset{\mu}{\to} Y \overset{\nu}{\to} Z \overset{\omega}{\to} \Sigma X$ be a triangle in ξ. The morphism $\nu : Y \to Z$ is called a *ξ-proper epic*, and $\mu : X \to Y$ is called a *ξ-proper monic*, see [**4**]; μ is called the *hokernel* of ν and ν is called the *hocokernel* of μ, see [**13**].

Proposition 1.2 ([**16**])**.** *The class of ξ-proper monics is closed under compositions. Dually, the class of ξ-proper epics is closed under compositions.*

Proposition 1.3 ([**16**])**.** *Consider morphisms $\mu : X \to Y$ and $\nu : Y \to Z$.*

(1) *If $\nu\mu$ is a ξ-proper monic, then μ is a ξ-proper monic.*

(2) *If $\nu\mu$ is a ξ-proper epic, then ν is a ξ-proper epic.*

Proposition 1.4 ([**16**])**.** *Given a commutative diagram*:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Z' \arrow[r, equal] \arrow[d, "\alpha'"] & Z' \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[d] \\
\Sigma^{-1}Z \arrow[r, "-\Sigma^{-1}\omega'"] \arrow[d, equal] & X' \arrow[r, "\mu'"] \arrow[d, "\beta'"] & Y' \arrow[r, "\nu'"] \arrow[d, "\beta"] & Z \arrow[d, equal] \\
\Sigma^{-1}Z \arrow[r, "-\Sigma^{-1}\omega"] \arrow[d] & X \arrow[r, "\mu"] \arrow[d, "\gamma'"] & Y \arrow[r, "\nu"] \arrow[d, "\gamma"] & Z \arrow[d] \\
0 \arrow[r] & \Sigma Z' \arrow[r, equal] & \Sigma Z' \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and vertical diagrams are in Δ.

(1) *If the third vertical triangle and the triangle* $X \overset{\mu}{\to} Y \overset{\nu}{\to} Z \overset{\omega}{\to} \Sigma X$ *are in* ξ, *then the triangle* $X' \overset{\mu'}{\to} Y' \overset{\nu'}{\to} Z \overset{\omega'}{\to} \Sigma X'$ *is also in* ξ.

(2) *If the second vertical triangle and the triangle* $X' \overset{\mu'}{\to} Y' \overset{\nu'}{\to} Z \overset{\omega'}{\to} \Sigma X'$ *are in* ξ, *then the third vertical triangle is also in* ξ.

Definition 1.5. Let $\mathcal{C}$ be an additive full subcategory of the triangulated category $\mathcal{T}$ closed under isomorphisms and Σ-stable, i.e., $\Sigma(\mathcal{C}) = \mathcal{C}$.

A ξ-*exact complex* X is a diagram

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots$$

in $\mathcal{T}$ such that, for all integers n, there exist triangles $K_{n+1} \overset{g_n}{\to} X_n \overset{f_n}{\to} K_n \overset{h_n}{\to} \Sigma K_{n+1}$ in ξ and the differential d_n is defined as $d_n = g_{n-1} f_n$ for any n.

A triangle $X \to Y \to Z \to \Sigma X$ in ξ is called $\mathcal{T}(\mathcal{C}, -)$-*exact* if, for any $C \in \mathcal{C}$, the induced complex

$$0 \longrightarrow \mathcal{T}(C, X) \longrightarrow \mathcal{T}(C, Y) \longrightarrow \mathcal{T}(C, Z) \longrightarrow 0$$

is exact in $\mathcal{A}$b.

A ξ-exact complex $X : \cdots \to X_{n+1} \overset{d_{n+1}}{\to} X_n \overset{d_n}{\to} X_{n-1} \to \cdots$ is called $\mathcal{T}(\mathcal{C}, -)$-*exact* if there are $\mathcal{T}(\mathcal{C}, -)$-exact triangles $K_{n+1} \overset{g_n}{\to} X_n \overset{f_n}{\to} K_n \overset{h_n}{\to} \Sigma K_{n+1}$ in ξ, where the differential d_n is defined as $d_n = g_{n-1} f_n$ for any n.

Let X be an object of $\mathcal{T}$. A $\mathcal{C}(\xi)$-*resolution* of X is a ξ-exact complex

$$\cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} X \longrightarrow 0$$

with all $C_i \in \mathcal{C}$ such that there are triangles $K_{n+1} \overset{g_n}{\to} C_n \overset{f_n}{\to} K_n \overset{h_n}{\to} \Sigma K_{n+1}$ in ξ and the differentials $d_n = g_{n-1} f_n$ for $n \geq 0$, where $K_0 = X$ and $d_0 = f_0$. The above ξ-exact complex is called a *proper* $\mathcal{C}(\xi)$-*resolution* of X if it is $\mathcal{T}(\mathcal{C}, -)$-exact.

Dually, the notions of a $\mathcal{T}(-,\mathcal{C})$-exact triangle in ξ, a $\mathcal{T}(-,\mathcal{C})$-exact ξ-exact complex, a $\mathcal{C}(\xi)$-coresolution and a coproper $\mathcal{C}(\xi)$-coresolution are defined.

We say that $\mathcal{C}$ is *closed under hokernels of ξ-proper epics* if, whenever the triangle

$$X \xrightarrow{\mu} Y \xrightarrow{\nu} Z \xrightarrow{\omega} \Sigma X$$

is in ξ with $Y, Z \in \mathcal{C}$, then X is in $\mathcal{C}$. Dually, we say that $\mathcal{C}$ is *closed under hocokernels of ξ-proper monics* if, whenever the triangle

$$X \xrightarrow{\mu} Y \xrightarrow{\nu} Z \xrightarrow{\omega} \Sigma X$$

is in ξ with $X, Y \in \mathcal{C}$, then Z is in $\mathcal{C}$.

2. Proper resolutions and coproper coresolutions. In this section, we provide a method for constructing a proper $\mathcal{C}(\xi)$-resolution (respectively, coproper $\mathcal{C}(\xi)$-coresolution) of the first (respectively, last) term in a triangle in ξ from those of the other two terms, and a method for constructing a proper $\mathcal{C}(\xi)$-resolution (respectively, coproper $\mathcal{C}(\xi)$-coresolution) of the last (respectively, first) term in a triangle in ξ from those of the other two terms.

We first give the following easy observations.

Lemma 2.1. *Let C be an object in $\mathcal{T}$. Consider the commutative diagram*:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X' \arrow[r, equal] \arrow[d] & X' \arrow[r] \arrow[d] & 0 \arrow[d] \\
X \arrow[r] \arrow[d, equal] & Y \arrow[r, "f"] \arrow[d, "g'"] & Z \arrow[r] \arrow[d, "g"] & \Sigma X \arrow[d, equal] \\
X \arrow[r] \arrow[d] & Y' \arrow[r, "f'"] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
0 \arrow[r] & \Sigma X' \arrow[r, equal] & \Sigma X' \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and vertical diagrams are in Δ. If $\mathcal{T}(C,g)$ is epic, then $\mathcal{T}(C,g')$ is also epic. If $\mathcal{T}(f,C)$ is epic, then $\mathcal{T}(f',C)$ is also epic.

Proof. We prove the first statement since the second follows by duality.

Assume that $\mathcal{T}(C,g)$ is epic. Let $\alpha \in \mathcal{T}(C,Y')$. There exists a $\beta \in \mathcal{T}(C,Z)$ such that $f'\alpha = \mathcal{T}(C,g)(\beta) = g\beta$. Thus, [**11,** Axiom D'] and [**13,** Lemma 27] imply that there is a $\gamma \in \mathcal{T}(C,Y)$ such that $g'\gamma = \alpha$, and hence, $\mathcal{T}(C,g')$ is epic. $\square$

Lemma 2.2. *Let $X \xrightarrow{\mu} Y \xrightarrow{\nu} Z \xrightarrow{\omega} \Sigma X$ be a triangle in Δ.*

(1) *If there exist morphisms $\alpha \in \mathcal{T}(C,X)$, $\gamma \in \mathcal{T}(C'',Z)$ and $f \in \mathcal{T}(C'',Y)$ such that $\gamma = \nu f$, then we have the next morphism of triangles*:

$$\begin{tikzcd}
C \arrow[r, "\binom{1}{0}"] \arrow[d, "\alpha"] & C \oplus C'' \arrow[r, "(0{,}1)"] \arrow[d, "\beta"] & C'' \arrow[r, "0"] \arrow[d, "\gamma"] & \Sigma C \arrow[d, "\Sigma\alpha"] \\
X \arrow[r, "\mu"] & Y \arrow[r, "\nu"] & Z \arrow[r, "\omega"] & \Sigma X.
\end{tikzcd}$$

(2) *If there exist morphisms $\alpha' \in \mathcal{T}(X,D)$, $\gamma' \in \mathcal{T}(Z,D'')$ and $g \in \mathcal{T}(Y,D)$ such that $\alpha' = g\mu$, then we have the next morphism of triangles*:

$$\begin{tikzcd}
X \arrow[r, "\mu"] \arrow[d, "\alpha'"] & Y \arrow[r, "\nu"] \arrow[d, "\beta'"] & Z \arrow[r, "\omega"] \arrow[d, "\gamma'"] & \Sigma X \arrow[d, "\Sigma\alpha'"] \\
D \arrow[r, "\binom{1}{0}"] & D \oplus D'' \arrow[r, "(0{,}1)"] & D'' \arrow[r, "0"] & \Sigma D.
\end{tikzcd}$$

Proof. Straightforward. $\square$

The next result provides a method for constructing a proper $\mathcal{C}(\xi)$-resolution of the first term in a triangle in ξ from those of the last two terms.

Theorem 2.3. *Given a triangle in ξ,*

$$X \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \Sigma X. \tag{2.0}$$

Assume that $\mathcal{C}$ is closed under hokernels of ξ-proper epics and

$$\cdots \longrightarrow C_i^0 \xrightarrow{d_i^0} \cdots \longrightarrow C_1^0 \xrightarrow{d_1^0} C_0^0 \xrightarrow{d_0^0} X^0 \longrightarrow 0, \tag{2.1}$$

$$(2.2) \qquad \cdots \longrightarrow C_i^1 \xrightarrow{d_i^1} \cdots \longrightarrow C_1^1 \xrightarrow{d_1^1} C_0^1 \xrightarrow{d_0^1} X^1 \longrightarrow 0$$

are proper $\mathcal{C}(\xi)$-resolutions of X^0 and X^1, respectively.

(1) *We obtain the following proper $\mathcal{C}(\xi)$-resolution of X*

$$(2.3) \qquad \cdots \longrightarrow C_{i+1}^1 \oplus C_i^0 \longrightarrow \cdots \longrightarrow C_2^1 \oplus C_1^0 \longrightarrow C \longrightarrow X \longrightarrow 0$$

and the following triangle in ξ

$$(2.4) \qquad C \longrightarrow C_1^1 \oplus C_0^0 \longrightarrow C_0^1 \longrightarrow \Sigma C.$$

(2) *If both ξ-exact complexes* (2.1), (2.2) *and the triangle* (2.0) *are $\mathcal{T}(-,\mathcal{C})$-exact, then so is the ξ-exact complex* (2.3).

Proof.

(1) By assumption, there exist $\mathcal{T}(\mathcal{C},-)$-exact triangles

$$K_{i+1}^0 \xrightarrow{g_i^0} C_i^0 \xrightarrow{f_i^0} K_i^0 \xrightarrow{h_i^0} \Sigma K_{i+1}^0,$$
$$K_{i+1}^1 \xrightarrow{g_i^1} C_i^1 \xrightarrow{f_i^1} K_i^1 \xrightarrow{h_i^1} \Sigma K_{i+1}^1$$

in ξ with the differentials $d_i^0 = g_{i-1}^0 f_i^0$ and $d_i^1 = g_{i-1}^1 f_i^1$ for all $i \geq 0$, where $K_0^0 = X^0, d_0^0 = f_0^0$ and $K_0^1 = X^1, d_0^1 = f_0^1$. Applying the base change for the triangle (2.0) along f_0^1, we obtain the following commutative diagram:

$$(2.5) \qquad \begin{tikzcd}
0 \arrow[r] \arrow[d] & K_1^1 \arrow[r, equal] \arrow[d] & K_1^1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
X \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & C_0^1 \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r] \arrow[d] & X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
0 \arrow[r] & \Sigma K_1^1 \arrow[r, equal] & \Sigma K_1^1 \arrow[r] & 0
\end{tikzcd}$$

in which the second horizontal and the second vertical triangles are in ξ. Since the third vertical triangle in diagram (2.5) is $\mathcal{T}(\mathcal{C},-)$-exact, so is the second vertical triangle by Lemma 2.1. Thus, Lemma 2.2 (1) yields the following morphism of triangles:

$$
\begin{tikzcd}
C_1^1 \arrow[r] \arrow[d] & C_1^1 \oplus C_0^0 \arrow[r] \arrow[d] & C_0^0 \arrow[r, "0"] \arrow[d] & \Sigma C_1^1 \arrow[d] \\
K_1^1 \arrow[r] & M \arrow[r] & X^0 \arrow[r] & \Sigma K_1^1.
\end{tikzcd}
$$

Using that Σ is an automorphism and the 3×3 lemma, the commutative square on the top left corner below is embedded in a diagram:

$$
\begin{tikzcd}
\Sigma^{-1}C_0^0 \arrow[r, "0"] \arrow[d, "\Sigma^{-1}f_0^0"] & C_1^1 \arrow[r, "\binom{1}{0}"] \arrow[d, "f_1^1"] & C_1^1 \oplus C_0^0 \arrow[r, "(0{,}1)"] \arrow[d, "f"] & C_0^0 \arrow[d, "f_0^0"] \\
\Sigma^{-1}X^0 \arrow[r, "-\Sigma^{-1}h_0"] \arrow[d, "\Sigma^{-1}h_0^0"] & K_1^1 \arrow[r, "g_0"] \arrow[d, "h_1^1"] & M \arrow[r, "f_0"] \arrow[d, "h"] & X^0 \arrow[d, "h_0^0"] \\
K_1^0 \arrow[r, "h_1"] \arrow[d, "g_0^0"] & \Sigma K_2^1 \arrow[r, "\Sigma g_1"] \arrow[d, "-\Sigma g_1^1"] & \Sigma W_1 \arrow[r, "\Sigma f_1"] \arrow[d, "-\Sigma g"] & \Sigma K_1^0 \arrow[d, "-\Sigma g_0^0"] \\
C_0^0 \arrow[r, "0"] & \Sigma C_1^1 \arrow[r, "\binom{1}{0}"] & \Sigma(C_1^1 \oplus C_0^0) \arrow[r, "(0{,}-1)"] & \Sigma C_0^0
\end{tikzcd}
$$

which is commutative except for the lower right square which anticommutes and where all the rows and columns are in Δ. Then, we have the following commutative diagram:

$$
\begin{tikzcd}
K_2^1 \arrow[r, "g_1"] \arrow[d, "g_1^1"] & W_1 \arrow[r, "f_1"] \arrow[d, "g"] & K_1^0 \arrow[r, "h_1"] \arrow[d, "g_0^0"] & \Sigma K_2^1 \arrow[d, "-\Sigma g_1^1"] \\
C_1^1 \arrow[r, "\binom{1}{0}"] \arrow[d, "f_1^1"] & C_1^1 \oplus C_0^0 \arrow[r, "(0{,}1)"] \arrow[d, "f"] & C_0^0 \arrow[r, "0"] \arrow[d, "f_0^0"] & \Sigma C_1^1 \arrow[d, "\Sigma f_1^1"] \\
K_1^1 \arrow[r, "g_0"] \arrow[d, "h_1^1"] & M \arrow[r, "f_0"] \arrow[d, "h"] & X^0 \arrow[r, "h_0"] \arrow[d, "h_0^0"] & \Sigma K_1^1 \arrow[d, "\Sigma h_1^1"] \\
\Sigma K_2^1 \arrow[r, "\Sigma g_1"] & \Sigma W_1 \arrow[r, "\Sigma f_1"] & \Sigma K_1^0 \arrow[r, "-\Sigma h_1"] & \Sigma^2 K_2^1
\end{tikzcd}
\tag{2.6}
$$

in which both the first and third vertical triangles and the second and third horizontal triangles are in ξ.

We now show that the first horizontal and the second vertical triangles in diagram (2.6) are in ξ. First, Proposition 1.2 implies that $gg_1 = \left(\begin{smallmatrix} g_1^1 \\ 0 \end{smallmatrix}\right)$ is a ξ-proper monic. It follows from Proposition 1.3 that g_1 is a ξ-proper monic and the first horizontal triangle in diagram (2.6) is in ξ.

Next, by the proof of the 3×3 lemma, see **[13,** Corollary 32], we have the following commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & K_2^1 \arrow[r, equal] \arrow[d, "g_1^1"] & K_2^1 \arrow[r] \arrow[d, "c"] & 0 \arrow[d] \\
\Sigma^{-1} C_0^0 \arrow[r, "0"] \arrow[d, equal] & C_1^1 \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, "f_1^1"] & C_1^1 \oplus C_0^0 \arrow[r, "{(0,1)}"] \arrow[d, "a"] & C_0^0 \arrow[d, equal] \\
\Sigma^{-1} C_0^0 \arrow[r, "0"] \arrow[d] & K_1^1 \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, "h_1^1"] & K_1^1 \oplus C_0^0 \arrow[r, "{(0,1)}"] \arrow[d, "b"] & C_0^0 \arrow[d] \\
0 \arrow[r] & \Sigma K_2^1 \arrow[r, equal] & \Sigma K_2^1 \arrow[r] & 0
\end{tikzcd}
$$

in which all horizontal and vertical diagrams are in Δ. Hence, Proposition 1.4 implies that the third vertical triangle is in ξ. We also have the following commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Sigma^{-1} M \arrow[r, equal] \arrow[d, "-\Sigma^{-1} f_0"] & \Sigma^{-1} M \arrow[r] \arrow[d, "-\Sigma^{-1} i"] & 0 \arrow[d] \\
\Sigma^{-1} C_0^0 \arrow[r, "\Sigma^{-1} f_0^0"] \arrow[d, equal] & \Sigma^{-1} X^0 \arrow[r, "\Sigma^{-1} h_0^0"] \arrow[d, "-\Sigma^{-1} h_0"] & K_1^0 \arrow[r, "g_0^0"] \arrow[d, "d"] & C_0^0 \arrow[d, equal] \\
\Sigma^{-1} C_0^0 \arrow[r, "0"] \arrow[d] & K_1^1 \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, "g_0"] & K_1^1 \oplus C_0^0 \arrow[r, "{(0,1)}"] \arrow[d, "e"] & C_0^0 \arrow[d] \\
0 \arrow[r] & M \arrow[r, equal] & M \arrow[r] & 0
\end{tikzcd}
$$

in which all horizontal and vertical diagrams are in Δ. Note that $(0,1)d = g_0^0$ is a ξ-proper monic. Then, Proposition 1.3 shows that d is a ξ-proper monic.

Finally, by the proof of the 3×3 lemma again, we have the following commutative diagram:

$$\begin{tikzcd}
K_1^0 \arrow[r, "d"] \arrow[d, equal] & K_1^1 \oplus C_0^0 \arrow[r, "e"] \arrow[d, "b"] & M \arrow[r, "i"] \arrow[d, "h"] & \Sigma K_1^0 \arrow[d, equal] \\
K_1^0 \arrow[r, "h_1"] & \Sigma K_2^1 \arrow[r, "\Sigma g_1"] \arrow[d, "-\Sigma c"] & \Sigma W_1 \arrow[r, "\Sigma f_1"] \arrow[d, "-\Sigma g"] & \Sigma K_1^0 \arrow[d, "\Sigma d"] \\
 & \Sigma(C_1^1 \oplus C_0^0) \arrow[r, equal] \arrow[d, "-\Sigma a"] & \Sigma(C_1^1 \oplus C_0^0) \arrow[r, "-\Sigma a"] \arrow[d, "-\Sigma f"] & \Sigma(K_1^1 \oplus C_0^0) \\
 & \Sigma(K_1^1 \oplus C_0^0) \arrow[r, "\Sigma e"] & \Sigma M &
\end{tikzcd}$$

in which the first horizontal triangle is in ξ. Also, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & K_2^1 \arrow[r, equal] \arrow[d, "g_1"] & K_2^1 \arrow[r] \arrow[d, "c"] & 0 \arrow[d] \\
\Sigma^{-1} M \arrow[r, "-\Sigma^{-1} h"] \arrow[d, equal] & W_1 \arrow[r, "g"] \arrow[d, "f_1"] & C_1^1 \oplus C_0^0 \arrow[r, "f"] \arrow[d, "a"] & M \arrow[d, equal] \\
\Sigma^{-1} M \arrow[r, "-\Sigma^{-1} i"] \arrow[d] & K_1^0 \arrow[r, "d"] \arrow[d, "h_1"] & K_1^1 \oplus C_0^0 \arrow[r, "e"] \arrow[d, "b"] & M \arrow[d] \\
0 \arrow[r] & \Sigma K_2^1 \arrow[r, equal] & \Sigma K_2^1 \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and vertical diagrams are in Δ. Hence, Proposition 1.4 implies that the second vertical triangle in diagram (2.6) is in ξ. Since both the first and third vertical triangles and the second and third horizontal triangles in the diagram are $\mathcal{T}(\mathcal{C}, -)$-exact, so are the second vertical and the first horizontal triangles in this diagram.

On one hand, applying the base change for the triangle $\Sigma^{-1} C_0^1 \to X \to M \to C_0^1$ along $C_1^1 \oplus C_0^0 \to M$, we have the following commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & W_1 \arrow[r, equal] \arrow[d] & W_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma^{-1}C_0^1 \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d] & C_1^1 \oplus C_0^0 \arrow[r] \arrow[d] & C_0^1 \arrow[d, equal] \\
\Sigma^{-1}C_0^1 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & C_0^1 \arrow[d] \\
0 \arrow[r] & \Sigma W_1 \arrow[r, equal] & \Sigma W_1 \arrow[r] & 0
\end{tikzcd} \tag{2.7}
$$

in which the second vertical triangle is in ξ. Then, Proposition 1.4 shows that the triangle $C \to C_1^1 \oplus C_0^0 \to C_0^1 \to \Sigma C$ is in ξ and $C \in \mathcal{C}$ by assumption. Since the third vertical triangle is $\mathcal{T}(\mathcal{C}, -)$-exact, it follows from Lemma 2.1 that the second vertical triangle is so.

On the other hand, Lemma 2.2 (1) again yields the following morphism of triangles:

$$
\begin{tikzcd}
C_2^1 \arrow[r] \arrow[d] & C_2^1 \oplus C_1^0 \arrow[r] \arrow[d] & C_1^0 \arrow[r, "0"] \arrow[d] & \Sigma C_2^1 \arrow[d] \\
K_2^1 \arrow[r] & W_1 \arrow[r] & K_1^0 \arrow[r] & \Sigma K_2^1.
\end{tikzcd}
$$

Using that Σ is an automorphism and the 3×3 lemma, the commutative square on the top left corner is embedded in a diagram:

$$
\begin{tikzcd}
\Sigma^{-1}C_1^0 \arrow[r, "0"] \arrow[d] & C_2^1 \arrow[r] \arrow[d] & C_2^1 \oplus C_1^0 \arrow[r] \arrow[d] & C_1^0 \arrow[d] \\
\Sigma^{-1}K_1^0 \arrow[r, "-"] \arrow[d] & K_2^1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] & K_1^0 \arrow[d] \\
K_2^0 \arrow[r] \arrow[d] & \Sigma K_3^1 \arrow[r] \arrow[d, "-"] & \Sigma W_2 \arrow[r] \arrow[d, "-"] & \Sigma K_2^0 \arrow[d, "-"] \\
C_1^0 \arrow[r] & \Sigma C_2^1 \arrow[r] & \Sigma(C_2^1 \oplus C_1^0) \arrow[r, "-"] & \Sigma C_1^0
\end{tikzcd}
$$

which is commutative except for the lower right square which anticommutes and where all the rows and columns are in Δ. Then, we have the following commutative diagram:

$$\begin{tikzcd}
K_3^1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & K_2^0 \arrow[r] \arrow[d] & \Sigma K_3^1 \arrow[d, "-"] \\
C_2^1 \arrow[r] \arrow[d] & C_2^1 \oplus C_1^0 \arrow[r] \arrow[d] & C_1^0 \arrow[r, "0"] \arrow[d] & \Sigma C_2^1 \arrow[d] \\
K_2^1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] & K_1^0 \arrow[r] \arrow[d] & \Sigma K_2^1 \arrow[d] \\
\Sigma K_3^1 \arrow[r] & \Sigma W_2 \arrow[r] & \Sigma K_2^0 \arrow[r, "-"] & \Sigma^2 K_3^1
\end{tikzcd} \tag{2.8}$$

in which the second vertical and the first horizontal triangles are in ξ by analogy with the preceding proof. Since both the first and third vertical triangles and the second and third horizontal triangles in the above diagram are $\mathcal{T}(\mathcal{C},-)$-exact, so are the second vertical and the first horizontal triangles in this diagram. Continuing this process, we obtain the desired proper $\mathcal{C}(\xi)$-resolution (2.3) and triangle (2.4).

(2) Note that the third vertical and the third horizontal triangles in diagram (2.5) are $\mathcal{T}(-,\mathcal{C})$-exact, so the second vertical and the second horizontal triangles in this diagram are also $\mathcal{T}(-,\mathcal{C})$-exact. Since both the first and third vertical triangles and the second and third horizontal triangles in diagram (2.6) are $\mathcal{T}(-,\mathcal{C})$-exact, a simple diagram chasing argument shows that the first horizontal and the second vertical triangles in this diagram are also $\mathcal{T}(-,\mathcal{C})$-exact. Thus, the second vertical triangle in diagram (2.7) is $\mathcal{T}(-,\mathcal{C})$-exact. Also, by assumption, both the first and third vertical triangles and the second and third horizontal triangles in diagram (2.8) are $\mathcal{T}(-,\mathcal{C})$-exact. Thus, the second vertical and the first horizontal triangles in this diagram are also $\mathcal{T}(-,\mathcal{C})$-exact. Finally, we deduce that the ξ-exact sequence (2.3) is $\mathcal{T}(-,\mathcal{C})$-exact. $\square$

Based on Theorem 2.3, by using induction on n, it is not difficult to obtain:

Corollary 2.4. *Given a ξ-exact complex*

$$0 \longrightarrow X \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^n \longrightarrow 0. \tag{2.9}$$

Assume that $\mathcal{C}$ is closed under hokernels of ξ-proper epics, and

$$\cdots \longrightarrow C_i^j \longrightarrow \cdots \longrightarrow C_1^j \longrightarrow C_0^j \longrightarrow X^j \longrightarrow 0 \tag{2.10 (j)}$$

is a proper $\mathcal{C}(\xi)$-resolution of X^j for $0 \le j \le n$. Then,

$$\cdots \longrightarrow \bigoplus_{i=0}^{n} C_{i+3}^i \longrightarrow \bigoplus_{i=0}^{n} C_{i+2}^i \longrightarrow \bigoplus_{i=0}^{n} C_{i+1}^i \longrightarrow C \longrightarrow X \longrightarrow 0 \tag{2.11}$$

is a proper $\mathcal{C}(\xi)$-resolution of X, and there exists a ξ-exact complex

$$\begin{aligned} 0 \longrightarrow C \longrightarrow \bigoplus_{i=0}^{n} C_i^i \longrightarrow \bigoplus_{i=1}^{n} C_{i-1}^i \longrightarrow \bigoplus_{i=2}^{n} C_{i-2}^i \longrightarrow \cdots \\ \longrightarrow C_0^{n-1} \oplus C_1^n \longrightarrow C_0^n \longrightarrow 0. \end{aligned}$$

If the ξ-exact complex (2.9) *and all $\mathcal{C}(\xi)$-resolutions* (2.10 (j)) *are $\mathcal{T}(-,\mathcal{C})$-exact, then so is the ξ-exact complex* (2.11).

The next two results, which are due to Theorem 2.3 and Corollary 2.4, respectively, provide a method for constructing a coproper $\mathcal{C}(\xi)$-coresolution of the last term in a triangle in ξ from those of the first two terms.

Theorem 2.5. *Given a triangle in ξ,*

$$Y_1 \longrightarrow Y_0 \longrightarrow Y \longrightarrow \Sigma Y_1. \tag{2.12}$$

Assume that $\mathcal{C}$ is closed under hocokernels of ξ-proper monics and

$$0 \longrightarrow Y_0 \xrightarrow{d_0^0} C_0^0 \xrightarrow{d_0^1} C_0^1 \longrightarrow \cdots \longrightarrow C_0^i \xrightarrow{d_0^{i+1}} \cdots, \tag{2.13}$$

$$0 \longrightarrow Y_1 \xrightarrow{d_1^0} C_1^0 \xrightarrow{d_1^1} C_1^1 \longrightarrow \cdots \longrightarrow C_1^i \xrightarrow{d_1^{i+1}} \cdots \tag{2.14}$$

are coproper $\mathcal{C}(\xi)$-coresolutions of Y_0 and Y_1, respectively.

(1) *We get the following coproper $\mathcal{C}(\xi)$-coresolution of Y*

$$0 \longrightarrow Y \longrightarrow C \longrightarrow C_0^1 \oplus C_1^2 \longrightarrow \cdots \longrightarrow C_0^i \oplus C_1^{i+1} \longrightarrow \cdots \tag{2.15}$$

and the following triangle in ξ

$$C_1^0 \longrightarrow C_0^0 \oplus C_1^1 \longrightarrow C \longrightarrow \Sigma C_1^0. \tag{2.16}$$

(2) *If both of the ξ-exact complexes* (2.13), (2.14) *and the triangle* (2.12) *are $\mathcal{T}(\mathcal{C},-)$-exact, then so is the ξ-exact complex* (2.15).

Proof. By assumption, there exist $\mathcal{T}(-,\mathcal{C})$-exact triangles

$$K_0^i \xrightarrow{g_0^i} C_0^i \xrightarrow{f_0^i} K_0^{i+1} \xrightarrow{h_0^i} \Sigma K_0^i, \qquad K_1^i \xrightarrow{g_1^i} C_1^i \xrightarrow{f_1^i} K_1^{i+1} \xrightarrow{h_1^i} \Sigma K_1^i$$

in ξ with differentials $d_0^i = g_0^i f_0^{i-1}$ and $d_1^i = g_1^i f_1^{i-1}$ for all $i \geq 0$, where $K_0^0 = Y_0, d_0^0 = g_0^0$ and $K_1^0 = Y_1, d_1^0 = g_1^0$. Applying the cobase change for the triangle $\Sigma^{-1}K_1^1 \to Y_1 \to C_1^0 \to K_1^1$ along $Y_1 \to Y_0$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Sigma^{-1}Y \arrow[r, equal] \arrow[d] & \Sigma^{-1}Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma^{-1}K_1^1 \arrow[r] \arrow[d, equal] & Y_1 \arrow[r] \arrow[d] & C_1^0 \arrow[r] \arrow[d] & K_1^1 \arrow[d, equal] \\
\Sigma^{-1}K_1^1 \arrow[r] \arrow[d] & Y_0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & K_1^1 \arrow[d] \\
0 \arrow[r] & Y \arrow[r, equal] & Y \arrow[r] & 0
\end{tikzcd} \tag{2.17}$$

in which the triangles $C_1^0 \to N \to Y \to \Sigma C_1^0$ and $Y_0 \to N \to K_1^1 \to \Sigma Y_0$ are in ξ. Since the triangle $Y_1 \to C_1^0 \to K_1^1 \to \Sigma Y_1$ is $\mathcal{T}(-,\mathcal{C})$-exact, so is the triangle $Y_0 \to N \to K_1^1 \to \Sigma Y_0$ by Lemma 2.1. Thus, Lemma 2.2 (2) yields the following morphism of triangles:

$$\begin{tikzcd}
Y_0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & K_1^1 \arrow[r] \arrow[d] & \Sigma Y_0 \arrow[d] \\
C_0^0 \arrow[r] & C_0^0 \oplus C_1^1 \arrow[r] & C_1^1 \arrow[r, "0"] & \Sigma C_0^0.
\end{tikzcd}$$

Using that Σ is an automorphism and the 3×3 lemma, the commutative

square on the top left corner is embedded in a diagram:

$$\begin{array}{ccccccc}
\Sigma^{-1}K_1^1 & \xrightarrow{-\Sigma^{-1}h_0} & Y_0 & \xrightarrow{g_0} & N & \xrightarrow{f_0} & K_1^1 \\
\downarrow{\scriptstyle \Sigma^{-1}g_1^1} & & \downarrow{\scriptstyle g_0^0} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_1^1} \\
\Sigma^{-1}C_1^1 & \xrightarrow{0} & C_0^0 & \xrightarrow{\binom{1}{0}} & C_0^0 \oplus C_1^1 & \xrightarrow{(0,1)} & C_1^1 \\
\downarrow{\scriptstyle \Sigma^{-1}f_1^1} & & \downarrow{\scriptstyle f_0^0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1^1} \\
\Sigma^{-1}K_1^2 & \xrightarrow{-\Sigma^{-1}h_1} & K_0^1 & \xrightarrow{g_1} & W^1 & \xrightarrow{f_1} & K_1^2 \\
\downarrow{\scriptstyle -\Sigma^{-1}h_1^1} & & \downarrow{\scriptstyle h_0^0} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_1^1} \\
K_1^1 & \xrightarrow{-h_0} & \Sigma Y_0 & \xrightarrow{\Sigma g_0} & \Sigma N & \xrightarrow{-\Sigma f_0} & \Sigma K_1^1
\end{array}$$

which is commutative except for the lower right square which anticommutes and where all the rows and columns are in Δ. Then, we have the following commutative diagram:

$$\begin{array}{ccccccc}
Y_0 & \xrightarrow{g_0} & N & \xrightarrow{f_0} & K_1^1 & \xrightarrow{h_0} & \Sigma Y_0 \\
\downarrow{\scriptstyle g_0^0} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_1^1} & & \downarrow{\scriptstyle -\Sigma g_0^0} \\
C_0^0 & \xrightarrow{\binom{1}{0}} & C_0^0 \oplus C_1^1 & \xrightarrow{(0,1)} & C_1^1 & \xrightarrow{0} & \Sigma C_0^0 \\
\downarrow{\scriptstyle f_0^0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1^1} & & \downarrow{\scriptstyle \Sigma f_0^0} \\
K_0^1 & \xrightarrow{g_1} & W^1 & \xrightarrow{f_1} & K_1^2 & \xrightarrow{h_1} & \Sigma K_0^1 \\
\downarrow{\scriptstyle h_0^0} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_1^1} & & \downarrow{\scriptstyle \Sigma h_0^0} \\
\Sigma Y_0 & \xrightarrow{\Sigma g_0} & \Sigma N & \xrightarrow{\Sigma f_0} & \Sigma K_1^1 & \xrightarrow{-\Sigma h_0} & \Sigma^2 Y_0
\end{array} \tag{2.18}$$

in which both the first and third vertical triangles and the first and second horizontal triangles are in ξ.

We now show that the third horizontal and the second vertical triangles in diagram (2.18) are in ξ. First, Proposition 1.2 implies that $f_1 f = (0, f_1^1)$ is a ξ-proper epic. It follows from Proposition 1.3 that f_1 is a ξ-proper epic, and the third horizontal triangle in diagram (2.6) is in ξ.

Next, by the proof of the 3×3 lemma, see **[13,** Corollary 32], we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Sigma^{-1}K_1^0 \arrow[r, equal] \arrow[d, "-\Sigma^{-1}h_0^0"] & \Sigma^{-1}K_0^1 \arrow[r] \arrow[d, "-\Sigma^{-1}c"] & 0 \arrow[d] \\
\Sigma^{-1}K_1^1 \arrow[r, "-\Sigma^{-1}h_0"] \arrow[d, equal] & Y_0 \arrow[r, "g_0"] \arrow[d, "g_0^0"] & N \arrow[r, "f_0"] \arrow[d, "a"] & K_1^1 \arrow[d, equal] \\
\Sigma^{-1}K_1^1 \arrow[r, "0"] \arrow[d] & C_0^0 \arrow[r, "\binom{1}{0}"] \arrow[d, "f_0^0"] & C_0^0 \oplus K_1^1 \arrow[r, "{(0,1)}"] \arrow[d, "b"] & K_1^1 \arrow[d] \\
0 \arrow[r] & K_0^1 \arrow[r, equal] & K_0^1 \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and vertical diagrams are in Δ. Thus, the cobase change implies that the triangle $N \xrightarrow{a} C_0^0 \oplus K_1^1 \xrightarrow{b} K_0^1 \xrightarrow{c} \Sigma N$ is in ξ. We also have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Sigma^{-1}(C_0^0 \oplus C_1^1) \arrow[r, equal] \arrow[d, "{(0,-1)}"] & \Sigma^{-1}(C_0^0 \oplus C_1^1) \arrow[r] \arrow[d, "-\Sigma^{-1}i"] & 0 \arrow[d] \\
\Sigma^{-1}K_1^1 \arrow[r, "\Sigma^{-1}g_1^1"] \arrow[d, equal] & \Sigma^{-1}C_1^1 \arrow[r, "\Sigma^{-1}f_1^1"] \arrow[d, "0"] & \Sigma^{-1}K_1^2 \arrow[r, "-\Sigma^{-1}h_1^1"] \arrow[d, "-\Sigma^{-1}d"] & K_1^1 \arrow[d, equal] \\
\Sigma^{-1}K_1^1 \arrow[r, "0"] \arrow[d] & C_0^0 \arrow[r, "\binom{1}{0}"] \arrow[d, "\binom{1}{0}"] & C_0^0 \oplus K_1^1 \arrow[r, "{(0,1)}"] \arrow[d, "e"] & K_1^1 \arrow[d] \\
0 \arrow[r] & C_0^0 \oplus C_1^1 \arrow[r, equal] & C_0^0 \oplus C_1^1 \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and vertical diagrams are in Δ. Then, Proposition 1.2 shows that $\Sigma^{-1}i = (\Sigma^{-1}f_1^1)(0,1)$ is a ξ-proper epic and the triangle $C_0^0 \oplus K_1^1 \xrightarrow{e} C_0^0 \oplus C_1^1 \xrightarrow{i} K_1^2 \xrightarrow{d} \Sigma(C_0^0 \oplus K_1^1)$ is in ξ.

Finally, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & N \arrow[r, equal] \arrow[d, "a"] & N \arrow[r] \arrow[d, "g"] & 0 \arrow[d] \\
\Sigma^{-1}K_1^2 \arrow[r, "-\Sigma^{-1}d"] \arrow[d, equal] & C_0^0 \oplus K_1^1 \arrow[r, "e"] \arrow[d, "b"] & C_0^0 \oplus C_1^1 \arrow[r, "i"] \arrow[d, "f"] & K_1^2 \arrow[d, equal] \\
\Sigma^{-1}K_1^2 \arrow[r, "-\Sigma^{-1}h_1"] \arrow[d] & K_0^1 \arrow[r, "g_1"] \arrow[d, "c"] & W^1 \arrow[r, "f_1"] \arrow[d, "h"] & K_1^2 \arrow[d] \\
0 \arrow[r] & \Sigma N \arrow[r, equal] & \Sigma N \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and vertical diagrams are in Δ. Hence, Proposition 1.4 implies that the second vertical triangle in diagram (2.18) is in ξ. Since both the first and third vertical triangles and the first and second horizontal triangles in diagram (2.18) are $\mathcal{T}(-,\mathcal{C})$-exact, so are the third horizontal and the second vertical triangles in this diagram.

On one hand, applying the cobase change for the triangle $\Sigma^{-1}W^1 \to N \to C_0^0 \oplus C_1^1 \to W^1$ along $N \to Y$, we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & C_1^0 \arrow[r, equal] \arrow[d] & C_1^0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma^{-1}W^1 \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d] & C_0^0 \oplus C_1^1 \arrow[r] \arrow[d] & W^1 \arrow[d, equal] \\
\Sigma^{-1}W^1 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & W^1 \arrow[d] \\
0 \arrow[r] & \Sigma C_1^0 \arrow[r, equal] & \Sigma C_1^0 \arrow[r] & 0
\end{tikzcd} \tag{2.19}$$

in which the triangle $Y \to C \to W^1 \to \Sigma Y$ is in ξ. Then, Proposition 1.4 shows that the third vertical triangle is in ξ and $C \in \mathcal{C}$ by assumption. Since the triangle $N \to C_0^0 \oplus C_1^1 \to W^1 \to \Sigma N$ is $\mathcal{T}(-,\mathcal{C})$-exact, it follows from Lemma 2.1 that the triangle $Y \to C \to W^1 \to \Sigma Y$ is so. On the other hand, again Lemma 2.2 (2) yields the following morphism of triangles:

$$\begin{tikzcd}
K_0^1 \arrow[r] \arrow[d] & W^1 \arrow[r] \arrow[d] & K_1^2 \arrow[r] \arrow[d] & \Sigma K_0^1 \arrow[d] \\
C_0^1 \arrow[r] & C_0^1 \oplus C_1^2 \arrow[r] & C_1^2 \arrow[r, "0"] & \Sigma C_0^1.
\end{tikzcd}$$

Using that Σ is an automorphism and the 3×3 lemma, the commutative square on the top left corner below is embedded in a diagram:

$$\begin{tikzcd}
\Sigma^{-1}K_1^2 \arrow[r, "-"] \arrow[d] & K_0^1 \arrow[r] \arrow[d] & W^1 \arrow[r] \arrow[d] & K_1^2 \arrow[d] \\
\Sigma^{-1}C_1^2 \arrow[r, "0"] \arrow[d] & C_0^1 \arrow[r] \arrow[d] & C_0^1 \oplus C_1^2 \arrow[r] \arrow[d] & C_1^2 \arrow[d] \\
\Sigma^{-1}K_1^3 \arrow[r, "-"] \arrow[d, "-"] & K_0^2 \arrow[r] \arrow[d] & W^2 \arrow[r] \arrow[d] & K_1^3 \arrow[d] \\
K_1^2 \arrow[r, "-"] & \Sigma K_0^1 \arrow[r] & \Sigma W^1 \arrow[r, "-"] & \Sigma K_1^2
\end{tikzcd}$$

which is commutative except the lower right square which anticommutes and where all the rows and columns are in Δ. Then we have the following commutative diagram:

$$\begin{tikzcd}
K_0^1 \arrow[r] \arrow[d] & W^1 \arrow[r] \arrow[d] & K_1^2 \arrow[r] \arrow[d] & \Sigma K_0^1 \arrow[d, "-"] \\
C_0^1 \arrow[r] \arrow[d] & C_0^1 \oplus C_1^2 \arrow[r] \arrow[d] & C_1^2 \arrow[r, "0"] \arrow[d] & \Sigma C_0^1 \arrow[d] \\
K_0^2 \arrow[r] \arrow[d] & W^2 \arrow[r] \arrow[d] & K_1^3 \arrow[r] \arrow[d] & \Sigma K_0^2 \arrow[d] \\
\Sigma K_0^1 \arrow[r] & \Sigma W^1 \arrow[r] & \Sigma K_1^2 \arrow[r, "-"] & \Sigma^2 K_0^1
\end{tikzcd} \tag{2.20}$$

in which the third horizontal and the second vertical triangles are in ξ by analogy with the preceding proof. Since both the first and third vertical triangles and the first and second horizontal triangles in diagram (2.20) are $\mathcal{T}(-,\mathcal{C})$-exact, so are the third horizontal and the second vertical triangles in this diagram. Continuing this process, we obtain the desired coproper $\mathcal{C}(\xi)$-coresolution (2.15) and triangle (2.16).

(2) Note that the triangle (2.12) and the triangle $Y_1 \to C_1^0 \to K_1^1 \to \Sigma Y_1$ in the diagram (2.17) are $\mathcal{T}(\mathcal{C},-)$-exact; thus, the triangles $Y_0 \to N \to K_1^1 \to \Sigma Y_0$ and $C_1^0 \to N \to Y \to \Sigma C_1^0$ in this diagram are also $\mathcal{T}(\mathcal{C},-)$-exact. Since both the first and third vertical triangles and the first and second horizontal triangles in diagram (2.18) are $\mathcal{T}(\mathcal{C},-)$-exact, a simple diagram chasing argument shows that the third horizontal and the second vertical triangles in this diagram are also $\mathcal{T}(\mathcal{C},-)$-exact. Thus, the triangle $Y \to C \to W^1 \to \Sigma Y$ in diagram (2.19) is $\mathcal{T}(\mathcal{C},-)$-exact.

Also by assumption, both the first and third vertical triangles and the first and second horizontal triangles in diagram (2.20) are $\mathcal{T}(\mathcal{C},-)$-exact. Thus, the second vertical and the third horizontal triangles in this diagram are also $\mathcal{T}(\mathcal{C},-)$-exact. Finally, we deduce that the ξ-exact sequence (2.15) is $\mathcal{T}(\mathcal{C},-)$-exact. $\square$

Based on Theorem 2.5, by using induction on n, it is not difficult to obtain:

Corollary 2.6. *Given a ξ-exact complex*

$$0 \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow Y \longrightarrow 0, \tag{2.21}$$

assume that $\mathcal{C}$ is closed under hocokernels of ξ-proper monics and

$$0 \longrightarrow Y_j \longrightarrow C_j^0 \longrightarrow C_j^1 \longrightarrow \cdots \longrightarrow C_j^i \longrightarrow \cdots \tag{2.22 (j)}$$

is a coproper $\mathcal{C}(\xi)$-coresolution of Y_j for $0 \le j \le n$. Then,

(2.23)
$$0 \longrightarrow Y \longrightarrow C \longrightarrow \bigoplus_{i=0}^{n} C_i^{i+1} \longrightarrow \bigoplus_{i=0}^{n} C_i^{i+2} \longrightarrow \bigoplus_{i=0}^{n} C_i^{i+3} \longrightarrow \cdots$$

is a coproper $\mathcal{C}(\xi)$-coresolution of Y, and there exists a ξ-exact complex

$$0 \longrightarrow C_n^0 \longrightarrow C_{n-1}^0 \oplus C_n^1 \longrightarrow \cdots \longrightarrow \bigoplus_{i=2}^{n} C_i^{i-2} \longrightarrow \bigoplus_{i=1}^{n} C_i^{i-1} \longrightarrow \bigoplus_{i=0}^{n} C_i^{i} \longrightarrow C \longrightarrow 0.$$

If the ξ-exact complex (2.21) *and all $\mathcal{C}(\xi)$-coresolutions* (2.22 (j)) *are $\mathcal{T}(\mathcal{C},-)$-exact, then so is the ξ-exact complex* (2.23).

The next result provides a method for constructing a proper $\mathcal{C}(\xi)$-resolution of the last term in a triangle in ξ from those of the first two terms.

Theorem 2.7. *Given a triangle*

$$X_1 \longrightarrow X_0 \longrightarrow X \longrightarrow \Sigma X_1 \tag{2.24}$$

in ξ, assume that

$$C_0^n \xrightarrow{d_0^n} \cdots \longrightarrow C_0^1 \xrightarrow{d_0^1} C_0^0 \xrightarrow{d_0^0} X_0 \longrightarrow 0, \tag{2.25}$$

$$C_1^{n-1} \xrightarrow{d_1^{n-1}} \cdots \longrightarrow C_1^1 \xrightarrow{d_1^1} C_1^0 \xrightarrow{d_1^0} X_1 \longrightarrow 0 \tag{2.26}$$

are proper $\mathcal{C}(\xi)$-resolutions of X_0 and X_1, respectively.

(1) *If the triangle* (2.24) *is $\mathcal{T}(\mathcal{C},-)$-exact, then we get a proper $\mathcal{C}(\xi)$-resolution of X*

$$\begin{aligned} C_0^n \oplus C_1^{n-1} \longrightarrow \cdots \longrightarrow C_0^2 \oplus C_1^1 \\ \longrightarrow C_0^1 \oplus C_1^0 \longrightarrow C_0^0 \longrightarrow X \longrightarrow 0. \end{aligned} \tag{2.27}$$

(2) *If both the ξ-exact complexes* (2.25), (2.26) *and the triangle* (2.24) *are $\mathcal{T}(-,\mathcal{C})$-exact, then so is the ξ-exact complex* (2.27).

Proof.

(1) By assumption, there exist $\mathcal{T}(\mathcal{C},-)$-exact triangles

$$K_0^{i+1} \xrightarrow{g_0^i} C_0^i \xrightarrow{f_0^i} K_0^i \xrightarrow{h_0^i} \Sigma K_0^{i+1},$$

$$K_1^{i+1} \xrightarrow{g_1^i} C_1^i \xrightarrow{f_1^i} K_1^i \xrightarrow{h_1^i} \Sigma K_1^{i+1}$$

in ξ with the differentials $d_0^i = g_0^{i-1} f_0^i$ for $0 \le i \le n-1$, where $K_0^0 = X_0, d_0^0 = f_0^0$, and the differentials $d_1^i = g_1^{i-1} f_1^i$ for $0 \le i \le n-2$, where $K_1^0 = X_1, d_1^0 = f_1^0$. Applying the base change for the triangle

$\Sigma^{-1}X \to X_1 \to X_0 \to X$ along f_0^0, we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & K_0^1 \arrow[r, equal] \arrow[d] & K_0^1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma^{-1}X \arrow[r] \arrow[d, equal] & W^1 \arrow[r] \arrow[d] & C_0^0 \arrow[r] \arrow[d] & X \arrow[d, equal] \\
\Sigma^{-1}X \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & X \arrow[d] \\
0 \arrow[r] & \Sigma K_0^1 \arrow[r, equal] & \Sigma K_0^1 \arrow[r] & 0.
\end{tikzcd} \tag{2.28}$$

Then Proposition 1.4 implies that the triangle $W^1 \to C_0^0 \to X \to \Sigma W^1$ is in ξ. Since the triangle (2.24) and the third vertical triangle in diagram (2.28) are $\mathcal{T}(\mathcal{C},-)$-exact, it follows from Lemma 2.1 that the triangle $W^1 \to C_0^0 \to X \to \Sigma W^1$ and the second vertical triangle in this diagram are also $\mathcal{T}(\mathcal{C},-)$-exact. Thus, Lemma 2.2 (1) yields a morphism of triangles:

$$\begin{tikzcd}
C_0^1 \arrow[r] \arrow[d] & C_0^1 \oplus C_1^0 \arrow[r] \arrow[d] & C_1^0 \arrow[r, "0"] \arrow[d] & \Sigma C_0^1 \arrow[d] \\
K_0^1 \arrow[r] & W^1 \arrow[r] & X_1 \arrow[r] & \Sigma K_0^1.
\end{tikzcd}$$

Using that Σ is an automorphism and the 3×3 lemma, the commutative square on the top left corner is embedded in a diagram:

$$\begin{tikzcd}
\Sigma^{-1}C_1^0 \arrow[r, "0"] \arrow[d] & C_0^1 \arrow[r] \arrow[d] & C_0^1 \oplus C_1^0 \arrow[r] \arrow[d] & C_1^0 \arrow[d] \\
\Sigma^{-1}X_1 \arrow[r, "-"] \arrow[d] & K_0^1 \arrow[r] \arrow[d] & W^1 \arrow[r] \arrow[d] & X_1 \arrow[d] \\
K_1^1 \arrow[r] \arrow[d] & \Sigma K_0^2 \arrow[r] \arrow[d, "-"] & \Sigma W^2 \arrow[r] \arrow[d, "-"] & \Sigma K_1^1 \arrow[d, "-"] \\
C_1^0 \arrow[r] & \Sigma C_0^1 \arrow[r] & \Sigma(C_0^1 \oplus C_1^0) \arrow[r, "-"] & \Sigma C_1^0
\end{tikzcd}$$

which is commutative except for the lower right square which anticommutes and where all the rows and columns are in Δ. Then, we have the following commutative diagram:

$$
\begin{tikzcd}
K_0^2 \arrow[r] \arrow[d] & W^2 \arrow[r] \arrow[d] & K_1^1 \arrow[r] \arrow[d] & \Sigma K_0^2 \arrow[d, "-"] \\
C_0^1 \arrow[r] \arrow[d] & C_0^1 \oplus C_1^0 \arrow[r] \arrow[d] & C_1^0 \arrow[r, "0"] \arrow[d] & \Sigma C_0^1 \arrow[d] \\
K_0^1 \arrow[r] \arrow[d] & W^1 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & \Sigma K_0^1 \arrow[d] \\
\Sigma K_0^2 \arrow[r] & \Sigma W^2 \arrow[r] & \Sigma K_1^1 \arrow[r, "-"] & \Sigma^2 K_0^2
\end{tikzcd} \tag{2.29}
$$

in which both the first and third vertical triangles and the second and third horizontal triangles are in ξ. By analogy with the proof of Theorem 2.3, we see that the first horizontal and the second vertical triangles in diagram (2.29) are in ξ. Since both the first and third vertical triangles and the second and third horizontal triangles in the above diagram are $\mathcal{T}(\mathcal{C}, -)$-exact, so are the first horizontal and the second vertical triangles in this diagram. Finally, repeated applications of Lemma 2.2 (1) yields the proper $\mathcal{C}(\xi)$-resolution (2.27).

(2) Since the triangle (2.24) and the third vertical triangle in the diagram (2.28) are $\mathcal{T}(-, \mathcal{C})$-exact, the second vertical triangle and the triangle $W^1 \to C_0^0 \to X \to \Sigma W^1$ in this diagram are also $\mathcal{T}(-, \mathcal{C})$-exact. Also, by assumption, both the first and third vertical triangles and the second and third horizontal triangles in diagram (2.29) are $\mathcal{T}(-, \mathcal{C})$-exact. Thus, the second vertical and the first horizontal triangles in this diagram are also $\mathcal{T}(-, \mathcal{C})$-exact. Finally, we deduce that the ξ-exact sequence (2.27) is $\mathcal{T}(-, \mathcal{C})$-exact. □

Corollary 2.8. *Given a ξ-exact complex*

$$X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X \longrightarrow 0, \tag{2.30}$$

assume that

$$C_j^{n-j} \longrightarrow \cdots \longrightarrow C_j^1 \longrightarrow C_j^0 \longrightarrow X_j \longrightarrow 0 \tag{2.31 (j)}$$

is a proper $\mathcal{C}(\xi)$-resolution of X_j for $0 \leq j \leq n$. If the ξ-exact complex

(2.30) *is* $\mathcal{T}(\mathcal{C},-)$*-exact, then*
(2.32)
$$\bigoplus_{i=0}^{n} C_i^{n-i} \longrightarrow \bigoplus_{i=0}^{n-1} C_i^{(n-1)-i} \longrightarrow \cdots \longrightarrow C_0^1 \oplus C_1^0 \longrightarrow C_0^0 \longrightarrow X \longrightarrow 0$$

is a proper $\mathcal{C}(\xi)$*-resolution of* X*. Furthermore, if the* ξ*-exact complex* (2.30) *and all* $\mathcal{C}(\xi)$*-resolutions* (2.31 (j)) *are* $\mathcal{T}(-,\mathcal{C})$*-exact, then so is the* ξ*-exact complex* (2.32)*.*

Proof. By assumption, there exist a ξ-proper epic $X_n \to K_n$ and triangles $K_{i+1} \to X_i \to K_i \to \Sigma K_{i+1}$ in ξ for $0 \le i \le n-1$, where $K_0 = X$. Also, there is a ξ-proper epic $C_n^0 \to X_n$. Thus, Proposition 1.2 implies that $C_n^0 \to K_n$ is a ξ-proper epic. Now, using Theorem 2.7 and induction on n, we obtain the desired ξ-exact complex (2.32). $\square$

The next two results which are dual to Theorem 2.7 and Corollary 2.8, respectively, provide a method for constructing a coproper $\mathcal{C}(\xi)$-coresolution of the first term in a triangle in ξ from those of the last two terms.

Theorem 2.9. *Given a triangle*

$$Y \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow \Sigma Y \tag{2.33}$$

in ξ*, assume that*

$$0 \longrightarrow Y^0 \xrightarrow{d_0^0} C_0^0 \xrightarrow{d_1^0} C_1^0 \longrightarrow \cdots \xrightarrow{d_n^0} C_n^0, \tag{2.34}$$

$$0 \longrightarrow Y^1 \xrightarrow{d_0^1} C_0^1 \xrightarrow{d_1^1} C_1^1 \longrightarrow \cdots \xrightarrow{d_{n-1}^1} C_{n-1}^1 \tag{2.35}$$

are coproper $\mathcal{C}(\xi)$*-coresolutions of* Y^0 *and* Y^1*, respectively.*

(1) *If the triangle* (2.33) *is* $\mathcal{T}(-,\mathcal{C})$*-exact, then we get a coproper* $\mathcal{C}(\xi)$*-coresolution of* Y
(2.36)
$$0 \longrightarrow Y \longrightarrow C_0^0 \longrightarrow C_0^1 \oplus C_1^0 \longrightarrow C_1^1 \oplus C_2^0 \longrightarrow \cdots \longrightarrow C_{n-1}^1 \oplus C_n^0.$$

(2) *If both of the* ξ*-exact complexes* (2.34), (2.35) *and the triangle* (2.33) *are* $\mathcal{T}(\mathcal{C},-)$*-exact, then so is the* ξ*-exact complex* (2.36)*.*

Proof. By assumption, there exist $\mathcal{T}(-,\mathcal{C})$-exact triangles

$$K_i^0 \xrightarrow{g_i^0} C_i^0 \xrightarrow{f_i^0} K_{i+1}^0 \xrightarrow{h_i^0} \Sigma K_i^0,$$
$$K_i^1 \xrightarrow{g_i^1} C_i^1 \xrightarrow{f_i^1} K_{i+1}^1 \xrightarrow{h_i^1} \Sigma K_i^1$$

in ξ with the differentials $d_i^0 = g_i^0 f_{i-1}^0$ for $0 \leq i \leq n-1$, where $K_0^0 = Y^0, d_0^0 = g_0^0$, and the differentials $d_i^1 = g_i^1 f_{i-1}^1$ for $0 \leq i \leq n-2$, where $K_0^1 = Y^1, d_0^1 = g_0^1$. Applying the cobase change for the triangle $\Sigma^{-1}K_1^0 \to Y^0 \to C_0^0 \to K_1^0$ along $Y^0 \to Y^1$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Y \arrow[r, equal] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma^{-1}K_1^0 \arrow[r] \arrow[d, equal] & Y^0 \arrow[r] \arrow[d] & C_0^0 \arrow[r] \arrow[d] & K_1^0 \arrow[d, equal] \\
\Sigma^{-1}K_1^0 \arrow[r] \arrow[d] & Y^1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] & K_1^0 \arrow[d] \\
0 \arrow[r] & \Sigma Y \arrow[r, equal] & \Sigma Y \arrow[r] & 0.
\end{tikzcd} \tag{2.37}$$

Then Proposition 1.4 implies that the third vertical triangle is in ξ. Since the triangle (2.33) and the triangle $Y^0 \to C_0^0 \to K_1^0 \to \Sigma Y^0$ in diagram (2.37) are $\mathcal{T}(-,\mathcal{C})$-exact, it follows from Lemma 2.1 that the third vertical triangle and the triangle $Y^1 \to W_1 \to K_1^0 \to \Sigma Y^1$ in this diagram are also $\mathcal{T}(-,\mathcal{C})$-exact. Thus, Lemma 2.2 (2) yields a morphism of triangles:

$$\begin{tikzcd}
Y^1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] & K_1^0 \arrow[r] \arrow[d] & \Sigma Y^1 \arrow[d] \\
C_0^1 \arrow[r] & C_0^1 \oplus C_1^0 \arrow[r] & C_1^0 \arrow[r, "0"] & \Sigma C_0^1.
\end{tikzcd}$$

Using that Σ is an automorphism and the 3×3 lemma, the commutative

square on the top left corner below is embedded in a diagram:

$$\begin{tikzcd}
\Sigma^{-1}K_1^0 \arrow[r, "-"] \arrow[d] & Y^1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] & K_1^0 \arrow[d] \\
\Sigma^{-1}C_1^0 \arrow[r, "0"] \arrow[d] & C_0^1 \arrow[r] \arrow[d] & C_0^1 \oplus C_1^0 \arrow[r] \arrow[d] & C_1^0 \arrow[d] \\
\Sigma^{-1}K_2^0 \arrow[r, "-"] \arrow[d, "-"] & K_1^1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & K_2^0 \arrow[d] \\
K_1^0 \arrow[r, "-"] & \Sigma Y^1 \arrow[r] & \Sigma W_1 \arrow[r, "-"] & \Sigma K_1^0
\end{tikzcd}$$

which is commutative except for the lower right square which anticommutes and where all the rows and columns are in Δ. Then, we have the following commutative diagram:

$$\begin{tikzcd}
Y^1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] & K_1^0 \arrow[r] \arrow[d] & \Sigma Y^1 \arrow[d, "-"] \\
C_0^1 \arrow[r] \arrow[d] & C_0^1 \oplus C_1^0 \arrow[r] \arrow[d] & C_1^0 \arrow[r, "0"] \arrow[d] & \Sigma C_0^1 \arrow[d] \\
K_1^1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & K_2^0 \arrow[r] \arrow[d] & \Sigma K_1^1 \arrow[d] \\
\Sigma Y^1 \arrow[r] & \Sigma W_1 \arrow[r] & \Sigma K_1^0 \arrow[r, "-"] & \Sigma^2 Y^1
\end{tikzcd} \tag{2.38}$$

in which both the first and third vertical triangles and the first and second horizontal triangles are in ξ. By analogy with the proof of Theorem 2.5, we have the third horizontal and the second vertical triangles in diagram (2.38) are in ξ. Since both the first and third vertical triangles and the first and second horizontal triangles in the diagram are $\mathcal{T}(-,\mathcal{C})$-exact, so are the third horizontal and the second vertical triangles in this diagram. Finally repeated applications of Lemma 2.2 (2) yields the coproper $\mathcal{C}(\xi)$-coresolution (2.36).

(2) Since the triangle (2.33) and triangle $Y^0 \to C_0^0 \to K_1^0 \to \Sigma Y^0$ in diagram (2.37) are $\mathcal{T}(\mathcal{C},-)$-exact, the third vertical triangle and the triangle $Y^1 \to W_1 \to K_1^0 \to \Sigma Y^1$ in this diagram are also $\mathcal{T}(\mathcal{C},-)$-

exact. Also, by assumption, both the first and third vertical triangles and the first and second horizontal triangles in the diagram (2.38) are $\mathcal{T}(\mathcal{C},-)$-exact. Thus, the second vertical and the third horizontal triangles in this diagram are also $\mathcal{T}(\mathcal{C},-)$-exact. Finally, we deduce that the ξ-exact sequence (2.36) is $\mathcal{T}(\mathcal{C},-)$-exact. $\square$

Corollary 2.10. *Given a ξ-exact complex*

$$0 \longrightarrow Y \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow \cdots \longrightarrow Y^n. \tag{2.39}$$

Assume that

$$0 \longrightarrow Y^j \longrightarrow C_0^j \longrightarrow C_1^j \longrightarrow \cdots \longrightarrow C_{n-j}^j \tag{2.40 (j)}$$

is a coproper $\mathcal{C}(\xi)$-coresolution of Y^j for $0 \le j \le n$. If the ξ-exact complex (2.39) *is $\mathcal{T}(-,\mathcal{C})$-exact, then*

$$\begin{aligned} 0 \longrightarrow Y \longrightarrow C_0^0 \longrightarrow C_1^0 \oplus C_0^1 \longrightarrow \cdots \\ \longrightarrow \bigoplus_{i=0}^{n-1} C^i_{(n-1)-i} \longrightarrow \bigoplus_{i=0}^{n} C^i_{n-i} \end{aligned} \tag{2.41}$$

is a coproper $\mathcal{C}(\xi)$-coresolution of Y. Furthermore, if the ξ-exact complex (2.39) *and all $\mathcal{C}(\xi)$-coresolutions* (2.40 (j)) *are $\mathcal{T}(\mathcal{C},-)$-exact, then so is the ξ-exact complex* (2.41).

Proof. By assumption, there exist a ξ-proper monic $L^n \to Y^n$ and triangles $L^i \to Y^i \to L^{i+1} \to \Sigma L^i$ in ξ for $0 \le i \le n-1$, where $L^0 = Y$. Also, there is a ξ-proper monic $Y^n \to C_0^n$. Thus, Proposition 1.2 implies that $L^n \to C_0^n$ is a ξ-proper monic. Now, using Theorem 2.9 and induction on n, we obtain the desired ξ-exact complex (2.41). $\square$

3. Gorensteinness in triangulated categories. In this section, some applications of the results in Section 2 are given. We introduce the Gorenstein category $\mathcal{GC}(\xi)$ in triangulated categories and show the stability of $\mathcal{GC}(\xi)$.

We begin with the next definition.

Definition 3.1. Let X be an object of $\mathcal{T}$. A *complete $\mathcal{C}(\xi)$-resolution* of X is both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact ξ-exact complex

$$\cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots$$

in $\mathcal{C}$ such that $X_1 \to C_0 \to X \to \Sigma X_1$ and $X \to C^0 \to X^1 \to \Sigma X$ are corresponding triangles in ξ.

The *Gorenstein subcategory* $\mathcal{GC}(\xi)$ of $\mathcal{T}$ is defined as

$$\mathcal{GC}(\xi) = \{X \in \mathcal{T} \mid X \text{ admits a complete } \mathcal{C}(\xi)\text{-resolution}\}.$$

Set $\mathcal{GC}^1(\xi) = \mathcal{GC}(\xi)$, and inductively set $\mathcal{GC}^{n+1}(\xi) = \mathcal{G}(\mathcal{GC}^n(\xi))$ for any $n \geq 1$.

Remark 3.2. Let $\mathcal{T}$ be a triangulated category and $\mathcal{P}(\xi)$ (respectively, $\mathcal{I}(\xi)$) the full subcategory of ξ-projective (respectively, ξ-injective) objects of $\mathcal{T}$. Then, $\mathcal{GP}(\xi)$ (respectively, $\mathcal{GI}(\xi)$) coincides with the subcategory of $\mathcal{T}$ consisting of ξ-Gorenstein projective (respectively, injective) objects [**1**].

As a main application of the results in Section 2, we obtain the following result.

Theorem 3.3. *Let $\mathcal{T}$ be a triangulated category with countable coproducts. If $\mathcal{C}$ is closed under countable coproducts, then*

(1) $\mathcal{GC}^n(\xi) = \mathcal{GC}(\xi)$ *for any* $n \geq 1$.

(2) $\mathcal{GC}(\xi)$ *is closed under direct summands.*

Proof.

(1) Let $G \in \mathcal{GC}^n(\xi)$. Note that the triangles $G \xrightarrow{1} G \to 0 \to \Sigma G$ and $0 \to G \xrightarrow{1} G \to 0$ are in ξ. It is easy to check that

$$\cdots \longrightarrow 0 \longrightarrow G \longrightarrow G \longrightarrow 0 \longrightarrow \cdots$$

is a complete $\mathcal{GC}^n(\xi)$-resolution of G, and thus, $G \in \mathcal{GC}^{n+1}(\xi)$. It follows that

$$\mathcal{C} \subseteq \mathcal{GC}(\xi) \subseteq \mathcal{GC}^2(\xi) \subseteq \mathcal{GC}^3(\xi) \subseteq \cdots$$

is an ascending chain of additive subcategories of $\mathcal{T}$.

Let X be an object in $\mathcal{GC}^2(\xi)$ and

$$\cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

a complete $\mathcal{GC}(\xi)$-resolution of X such that

$$\cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow X \longrightarrow$$

and

$$0 \longrightarrow X \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

are both $\mathcal{T}(\mathcal{GC}(\xi), -)$-exact and $\mathcal{T}(-, \mathcal{GC}(\xi))$-exact ξ-exact complexes. Then, for any $j \geq 0$, there exist both $\mathcal{T}(\mathcal{C}, -)$-exact and $\mathcal{T}(-, \mathcal{C})$-exact ξ-exact complexes:

$$\cdots \longrightarrow C_j^i \longrightarrow \cdots \longrightarrow C_j^1 \longrightarrow C_j^0 \longrightarrow G_j \longrightarrow 0,$$
$$0 \longrightarrow G^j \longrightarrow B_0^j \longrightarrow B_1^j \longrightarrow \cdots \longrightarrow B_i^j \longrightarrow \cdots$$

with all C_j^i and B_i^j in $\mathcal{C}$. Thus, Corollaries 2.8 and 2.10 yield the following ξ-exact complexes:

$$\cdots \longrightarrow \bigoplus_{i=0}^{n} C_i^{n-i} \longrightarrow \cdots \longrightarrow C_0^1 \oplus C_1^0 \longrightarrow C_0^0 \longrightarrow X \longrightarrow 0,$$

$$0 \longrightarrow X \longrightarrow B_0^0 \longrightarrow B_1^0 \oplus B_0^1 \longrightarrow \cdots \longrightarrow \bigoplus_{i=0}^{n} B_{n-i}^i \longrightarrow \cdots$$

which are both $\mathcal{T}(\mathcal{C}, -)$-exact and $\mathcal{T}(-, \mathcal{C})$-exact. It follows that

$$\begin{aligned}\cdots \longrightarrow \bigoplus_{i=0}^{n} C_i^{n-i} \longrightarrow \cdots \\ \longrightarrow C_0^1 \oplus C_1^0 \longrightarrow C_0^0 \longrightarrow B_0^0 \longrightarrow B_1^0 \oplus B_0^1 \longrightarrow \cdots \\ \longrightarrow \bigoplus_{i=0}^{n} B_{n-i}^i \longrightarrow \cdots\end{aligned}$$

is a complete $\mathcal{C}(\xi)$-resolution of X, and thus, $X \in \mathcal{GC}(\xi)$. By using induction on n we easily obtain the assertion.

(2) Let

$$X_1 \oplus X_2 = X \in \mathcal{GC}(\xi)$$

and

$$\cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots$$

be a complete $\mathcal{C}(\xi)$-resolution of X with $K_1 \to C_0 \to X \to \Sigma K_1$ and $X \to C^0 \to K^1 \to \Sigma X$ the corresponding triangles in ξ. Applying the base change for the triangle $\Sigma^{-1}X_1 \to X_2 \to X \to X_1$ along $C_0 \to X$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & K_1 \arrow[r, equal] \arrow[d] & K_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma^{-1}X_1 \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & X_1 \arrow[d, equal] \\
\Sigma^{-1}X_1 \arrow[r, "0"] \arrow[d] & X_2 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X_1 \arrow[d] \\
0 \arrow[r] & \Sigma K_1 \arrow[r, equal] & \Sigma K_1 \arrow[r] & 0.
\end{tikzcd}$$

Then Proposition 1.4 implies that the triangle $D \to C_0 \to X_1 \to \Sigma D$ is in ξ. Applying to the above diagram the homological functors $\mathcal{T}(C,-), \mathcal{T}(-,C)$ for any $C \in \mathcal{C}$, it is straightforward to show that the triangle $D \to C_0 \to X_1 \to \Sigma D$ is both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact.

Similarly, we have a triangle $D' \to C_0 \to X_2 \to \Sigma D'$ in ξ which is both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact. Consider the triangle

$$X_i \longrightarrow X \longrightarrow X_j \xrightarrow{0} \Sigma X_i \quad \text{for } i,j = 1,2.$$

Theorem 2.7 yields both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact ξ-exact complexes $C_0 \oplus C_1 \to C_0 \to X_1 \to 0$ and $C_0 \oplus C_1 \to C_0 \to X_2 \to 0$. Again, by Theorem 2.7, we obtain both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact ξ-exact complexes $C_0 \oplus C_1 \oplus C_2 \to C_0 \oplus C_1 \to C_0 \to X_1 \to 0$ and $C_0 \oplus C_1 \oplus C_2 \to C_0 \oplus C_1 \to C_0 \to X_2 \to 0$. Continuing this process, we obtain both of the following $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact ξ-exact complexes

$$\cdots \to \bigoplus_{i=0}^{n-1} C_i \to \cdots \to C_0 \oplus C_1 \oplus C_2 \to C_0 \oplus C_1 \to C_0 \to X_1 \to 0,$$
$$\cdots \to \bigoplus_{i=0}^{n-1} C_i \to \cdots \to C_0 \oplus C_1 \oplus C_2 \to C_0 \oplus C_1 \to C_0 \to X_2 \to 0.$$

Dually, repeated applications of Theorem 2.9 yields both of the following $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact ξ-exact complexes

$$\begin{aligned} 0 \longrightarrow X_1 \longrightarrow C^0 \longrightarrow C^0\oplus C^1 \longrightarrow C^0\oplus C^1\oplus C^2 \longrightarrow \cdots \\ \longrightarrow \bigoplus_{i=0}^{n-1} C^i \longrightarrow \cdots, \\ 0 \longrightarrow X_2 \longrightarrow C^0 \longrightarrow C^0\oplus C^1 \longrightarrow C^0\oplus C^1\oplus C^2 \longrightarrow \cdots \\ \longrightarrow \bigoplus_{i=0}^{n-1} C^i \longrightarrow \cdots. \end{aligned}$$

Consequently, X_1 and X_2 are in $\mathcal{GC}(\xi)$. □

Proposition 3.4. *Given both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact triangle $X \to Y \to Z \to \Sigma X$ in ξ, if any two of X, Y and Z are objects in $\mathcal{GC}(\xi)$, then so is the third.*

Proof. First, assume that $X, Z \in \mathcal{GC}(\xi)$. There exist complete $\mathcal{C}(\xi)$-resolutions

$$\cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots,$$
$$\cdots \longrightarrow B_1 \longrightarrow B_0 \longrightarrow B^0 \longrightarrow B^1 \longrightarrow \cdots$$

of X and Z, respectively. Consider both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact triangles $X_1 \to C_0 \to X \to \Sigma X_1$ and $Z_1 \to B_0 \to Z \to \Sigma Z_1$. By assumption and Lemma 2.2 (1), we obtain the following morphism of triangles:

$$\begin{tikzcd} C_0 \arrow[r] \arrow[d] & C_0\oplus B_0 \arrow[r] \arrow[d] & B_0 \arrow[r, "0"] \arrow[d] & \Sigma C_0 \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & \Sigma X. \end{tikzcd}$$

Using that Σ is an automorphism and the 3×3 lemma, the commutative

square on the top left corner below is embedded in a diagram:

$$\begin{tikzcd}
\Sigma^{-1}B_0 \arrow[r, "0"] \arrow[d] & C_0 \arrow[r] \arrow[d] & C_0 \oplus B_0 \arrow[r] \arrow[d] & B_0 \arrow[d] \\
\Sigma^{-1}Z \arrow[r, "-"] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
Z_1 \arrow[r] \arrow[d] & \Sigma X_1 \arrow[r] \arrow[d, "-"] & \Sigma Y_1 \arrow[r] \arrow[d, "-"] & \Sigma Z_1 \arrow[d, "-"] \\
B_0 \arrow[r] & \Sigma C_0 \arrow[r] & \Sigma(C_0 \oplus B_0) \arrow[r, "-"] & \Sigma B_0
\end{tikzcd}$$

which is commutative except for the lower right square which anticommutes and where all the rows and columns are in Δ. Then, we have the following commutative diagram:

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & \Sigma X_1 \arrow[d, "-"] \\
C_0 \arrow[r] \arrow[d] & C_0 \oplus B_0 \arrow[r] \arrow[d] & B_0 \arrow[r, "0"] \arrow[d] & \Sigma C_0 \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
\Sigma X_1 \arrow[r] & \Sigma Y_1 \arrow[r] & \Sigma Z_1 \arrow[r, "-"] & \Sigma^2 X_1
\end{tikzcd}$$

in which both the first and third vertical triangles and the second and third horizontal triangles are in ξ. By analogy with the proof of Theorem 2.3, we see that the first horizontal and the second vertical triangles in the above diagram are in ξ. Since both the first and third vertical triangles and the second and third horizontal triangles in the above diagram are both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact, so are the second vertical and the first horizontal triangles in this diagram.

On the other hand, consider both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact triangles $X \to C^0 \to X^1 \to \Sigma X$ and $Z \to B^0 \to Z^1 \to \Sigma Z$. By assumption and Lemma 2.2 (2), we obtain a morphism of triangles:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
C^0 \arrow[r] & C^0 \oplus B^0 \arrow[r] & B^0 \arrow[r, "0"] & \Sigma C^0.
\end{tikzcd}$$

Using that Σ is an automorphism and the 3×3 lemma, the commutative square on the top left corner is embedded in a diagram:

$$\begin{tikzcd}
\Sigma^{-1}Z \arrow[r, "-"] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
\Sigma^{-1}B^0 \arrow[r, "0"] \arrow[d] & C^0 \arrow[r] \arrow[d] & C^0 \oplus B^0 \arrow[r] \arrow[d] & B^0 \arrow[d] \\
\Sigma^{-1}Z^1 \arrow[r, "-"] \arrow[d, "-"] & X^1 \arrow[r] \arrow[d] & Y^1 \arrow[r] \arrow[d] & Z^1 \arrow[d] \\
Z \arrow[r, "-"] & \Sigma X \arrow[r] & \Sigma Y \arrow[r, "-"] & \Sigma Z
\end{tikzcd}$$

which is commutative except for the lower right square which anticommutes and where all the rows and columns are in Δ. Then, we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d, "-"] \\
C^0 \arrow[r] \arrow[d] & C^0 \oplus B^0 \arrow[r] \arrow[d] & B^0 \arrow[r, "0"] \arrow[d] & \Sigma C^0 \arrow[d] \\
X^1 \arrow[r] \arrow[d] & Y^1 \arrow[r] \arrow[d] & Z^1 \arrow[r] \arrow[d] & \Sigma X^1 \arrow[d] \\
\Sigma X \arrow[r] & \Sigma Y \arrow[r] & \Sigma Z \arrow[r, "-"] & \Sigma^2 X
\end{tikzcd}$$

in which both the first and third vertical triangles and the first and second horizontal triangles are in ξ. By analogy with the proof of Theorem 2.5, we see that the third horizontal and the second vertical triangles in the above diagram are in ξ. Since both the first and third vertical triangles and the first and second horizontal triangles in the above diagram are both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact, so are the second vertical and the third horizontal triangles in this diagram. Continuing this process, we obtain that

$$\cdots \longrightarrow C_1 \oplus B_1 \longrightarrow C_0 \oplus B_0 \longrightarrow C^0 \oplus B^0 \longrightarrow C^1 \oplus B^1 \longrightarrow \cdots$$

is a complete $\mathcal{C}(\xi)$-resolution of Y, as desired.

Next, assume that $Y, Z \in \mathcal{GC}(\xi)$. Then Theorem 2.9 implies that X has a coproper $\mathcal{C}(\xi)$-coresolution which is $\mathcal{T}(\mathcal{C}, -)$-exact. Consider both $\mathcal{T}(\mathcal{C}, -)$-exact and $\mathcal{T}(-, \mathcal{C})$-exact triangles $Y_1 \to C_0 \to Y \to \Sigma Y_1$ in ξ with $C_0 \in \mathcal{C}$. Applying the base change for the triangle $\Sigma^{-1} Z \to X \to Y \to Z$ along $C_0 \to Y$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Y_1 \arrow[r, equal] \arrow[d] & Y_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma^{-1} Z \arrow[r] \arrow[d, equal] & Z' \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & Z \arrow[d, equal] \\
\Sigma^{-1} Z \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
0 \arrow[r] & \Sigma Y_1 \arrow[r, equal] & \Sigma Y_1 \arrow[r] & 0.
\end{tikzcd}$$

Then Proposition 1.4 implies that the triangle $Z' \to C_0 \to Z \to \Sigma Z'$ is in ξ. Applying to the above diagram the homological functors $\mathcal{T}(C, -)$ and $\mathcal{T}(-, C)$ for any $C \in \mathcal{C}$, a simple diagram chasing argument shows that the second vertical triangle and the triangle $Z' \to C_0 \to Z \to \Sigma Z'$ are both $\mathcal{T}(\mathcal{C}, -)$-exact and $\mathcal{T}(-, \mathcal{C})$-exact.

Consider both $\mathcal{T}(\mathcal{C}, -)$-exact and $\mathcal{T}(-, \mathcal{C})$-exact triangles $Z_1 \to B_0 \to Z \to \Sigma Z_1$ in ξ with $B_0 \in \mathcal{C}$; **[11,** Axioms B' and E] yields the following morphism of triangles:

$$\begin{tikzcd}
Z_1 \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & \Sigma Z_1 \arrow[d] \\
Z' \arrow[r] & C_0 \arrow[r] & Z \arrow[r] & \Sigma Z'
\end{tikzcd}$$

such that the triangle $Z_1 \to Z' \oplus B_0 \to C_0 \to \Sigma Z_1$ is in ξ and both $\mathcal{T}(\mathcal{C}, -)$-exact and $\mathcal{T}(-, \mathcal{C})$-exact. Then, $Z' \oplus B_0$ has a proper $\mathcal{C}(\xi)$-resolution which is $\mathcal{T}(-, \mathcal{C})$-exact, and so Z' has a proper $\mathcal{C}(\xi)$-resolution that is $\mathcal{T}(-, \mathcal{C})$-exact by the preceding proof. Now, applying

Theorem 2.7 for the triangle $Y_1 \to Z' \to X \to \Sigma Y_1$, we obtain that X has a proper $\mathcal{C}(\xi)$-resolution which is $\mathcal{T}(-,\mathcal{C})$-exact. It follows that $X \in \mathcal{GC}(\xi)$.

Finally, assume that $X, Y \in \mathcal{GC}(\xi)$. Then Theorem 2.7 implies that Z has a proper $\mathcal{C}(\xi)$-resolution which is $\mathcal{T}(-,\mathcal{C})$-exact. Consider both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact triangles $Y \to B^0 \to Y^1 \to \Sigma Y$ in ξ with $B^0 \in \mathcal{C}$. Applying the cobase change for the triangle $\Sigma^{-1}Y^1 \to Y \to B^0 \to Y^1$ along $Y \to Z$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X \arrow[r, equal] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma^{-1}Y^1 \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & B^0 \arrow[r] \arrow[d] & Y^1 \arrow[d, equal] \\
\Sigma^{-1}Y^1 \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & Y^1 \arrow[d] \\
0 \arrow[r] & \Sigma X \arrow[r, equal] & \Sigma X \arrow[r] & 0.
\end{tikzcd}$$

Then Proposition 1.4 implies that the third vertical triangle is in ξ. Applying to the above diagram the homological functors $\mathcal{T}(C,-)$ and $\mathcal{T}(-,C)$ for any $C \in \mathcal{C}$, a simple diagram chasing argument shows that the third vertical triangle and the triangle $Z \to X' \to Y^1 \to \Sigma Z$ are both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact. Consider both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact triangle $X \to C^0 \to X^1 \to \Sigma X$ in ξ with $C^0 \in \mathcal{C}$; [**11,** Axioms B and E] yields the following morphism of triangles:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & B^0 \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r] & C^0 \arrow[r] & X^1 \arrow[r] & \Sigma X
\end{tikzcd}$$

such that $B^0 \to X' \oplus C^0 \to X^1 \to \Sigma B^0$ is in ξ and both $\mathcal{T}(\mathcal{C},-)$-exact and $\mathcal{T}(-,\mathcal{C})$-exact. Then, $X' \oplus C^0$ has a coproper $\mathcal{C}(\xi)$-coresolution which is $\mathcal{T}(\mathcal{C},-)$-exact, and thus, X' has a coproper $\mathcal{C}(\xi)$-coresolution that is $\mathcal{T}(\mathcal{C},-)$-exact by the preceding proof.

Now, applying Theorem 2.9 for the triangle $Z \to X' \to Y^1 \to \Sigma Z$, we obtain that Z has a coproper $\mathcal{C}(\xi)$-coresolution which is $\mathcal{T}(\mathcal{C},-)$-exact. It follows that $Z \in \mathcal{GC}(\xi)$. $\square$

Let X, Z be two objects of $\mathcal{T}$, and consider the class $\xi^*(Z,X)$ of all triangles $X \overset{\mu}{\to} Y \overset{\nu}{\to} Z \overset{\omega}{\to} \Sigma X$ in ξ. We define a relation in $\xi^*(Z,X)$ as follows. If $(T)_i : X \overset{\mu_i}{\to} Y_i \overset{\nu_i}{\to} Z \overset{\omega_i}{\to} \Sigma X$, $i = 1, 2$, are elements of $\xi^*(Z,X)$, then we define $(T)_1 \sim (T)_2$ if there exists a morphism of triangles:

$$\begin{tikzcd}
(T)_1 : & X \arrow[r, "\mu_1"] \arrow[d, equal] & Y_1 \arrow[r, "\nu_1"] \arrow[d, "g"] & Z \arrow[r, "\omega_1"] \arrow[d, equal] & \Sigma X \arrow[d, equal] \\
(T)_2 : & X \arrow[r, "\mu_2"] & Y_2 \arrow[r, "\nu_2"] & Z \arrow[r, "\omega_2"] & \Sigma X.
\end{tikzcd}$$

Obviously, g is an isomorphism and $\sim$ is an equivalence relation on the class $\xi^*(Z,X)$. Using base and cobase changes, it is easy to see that we can define (as in the case of the classical Baer's theory in an abelian category) a sum in the class $\xi(Z,X) := \xi^*(Z,X)/\sim$ in such a way that $\xi(Z,X)$ becomes an abelian group and $\xi(-,-) : \mathcal{T}^{\mathrm{op}} \times \mathcal{T} \to \mathcal{A}\mathrm{b}$ an additive bifunctor.

Lemma 3.5. *Given a triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \tag{3.1}$$

in ξ, assume $\xi(C, C') = 0$ for any $C, C' \in \mathcal{C}$.

(1) *If $Z \in \mathcal{GC}(\xi)$, then the triangle* (3.1) *is $\mathcal{T}(-,\mathcal{C})$-exact.*

(2) *If $X \in \mathcal{GC}(\xi)$, then the triangle* (3.1) *is $\mathcal{T}(\mathcal{C},-)$-exact.*

Proof. We only need to prove (1) since (2) follows by duality.

Since $Z \in \mathcal{GC}(\xi)$, there exists a $\mathcal{T}(-,\mathcal{C})$-exact triangle $Z_1 \to C \to Z \to \Sigma Z_1$ in ξ with $C \in \mathcal{C}$. Let $C' \in \mathcal{C}$. Then, we have a long exact sequence

$$0 \longrightarrow \mathcal{T}(Z, C') \longrightarrow \mathcal{T}(C, C') \longrightarrow \mathcal{T}(Z_1, C') \overset{f}{\longrightarrow} \xi(Z, C') \longrightarrow 0.$$

Since the triangle $Z_1 \to C \to Z \to \Sigma Z_1$ is $\mathcal{T}(-,\mathcal{C})$-exact, $f = 0$, and thus, $\xi(Z, C') = 0$. It follows that the triangle (3.1) is $\mathcal{T}(-,\mathcal{C})$-exact. $\square$

Corollary 3.6. *Given a triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \tag{3.2}$$

in ξ, assume $\xi(C,C')=0$ for any $C,C'\in\mathcal{C}$.

(1) *If $X,Z\in\mathcal{GC}(\xi)$, then $Y\in\mathcal{GC}(\xi)$.*

(2) *If the triangle* (3.2) *is $\mathcal{T}(\mathcal{C},-)$-exact and $Z\in\mathcal{GC}(\xi)$, then $X\in\mathcal{GC}(\xi)$ if and only if $Y\in\mathcal{GC}(\xi)$.*

(3) *If the triangle* (3.2) *is $\mathcal{T}(-,\mathcal{C})$-exact and $X\in\mathcal{GC}(\xi)$, then $Y\in\mathcal{GC}(\xi)$ if and only if $Z\in\mathcal{GC}(\xi)$.*

As an immediate consequence of Corollary 3.6, we obtain the next result which was obtained under the assumption that the triangulated category has enough ξ-projectives (respectively, ξ-injectives), see [**1**, Theorem 3.11] and its dual.

Corollary 3.7. *Let $X\to Y\to Z\to\Sigma X$ be a triangle in ξ.*

(1) *If $Z\in\mathcal{GP}(\xi)$, then $X\in\mathcal{GP}(\xi)$ if and only if $Y\in\mathcal{GP}(\xi)$.*

(2) *If $X\in\mathcal{GI}(\xi)$, then $Y\in\mathcal{GI}(\xi)$ if and only if $Z\in\mathcal{GI}(\xi)$.*

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