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EXTREMAL RADII, DIAMETER AND MINIMUM WIDTH IN GENERALIZED MINKOWSKI SPACES

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ABSTRACT. We discuss the notions of circumradius, inradius, diameter and minimum width in generalized Minkowski spaces (that is, with respect to gauges), i.e., we measure the "size" of a given convex set in a finite-dimensional real vector space with respect to another convex set. This is done via formulating some kind of containment problem incorporating homothetic bodies of the latter set or strips bounded by parallel supporting hyperplanes thereof. This paper can be seen as a theoretical starting point for studying metric problems of convex sets in generalized Minkowski spaces.

1. Introduction. The celebrated *Sylvester problem*, which was originally posed in **[26]**, asks for a point that minimizes the maximum distance to points from a given finite set in the Euclidean plane. There are at least two ways to generalize this problem. From a first point of view, we might keep the participating geometric configuration–given a set, we are searching a point–but change the distance measurement. Classically, distance measurement is provided by the Euclidean norm or, equivalently, by its unit ball, which is a centered, compact, convex set having the origin as an interior point. Then the Sylvester problem asks for the least scaling factor, called *circumradius*, such that there is a correspondingly scaled version of the unit ball that contains the given set. In the literature, this setting has already been relaxed by using norms **[1, 15, 20]** and even by dropping the centeredness and the boundedness of the unit ball as well as the finite cardinality of the given set **[6, 7, 8]**. Vector spaces equipped with such a unit ball shall be called *generalized Minkowski spaces*. The corresponding analogue

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of the norm is the *Minkowski functional of the unit ball*, which is also called *gauge* or *convex distance function* in the literature.

A second possibility for changing the setting of the Sylvester problem is as follows. We keep the Euclidean distance measurement but, instead of asking for a point which approximates the given set in a minimax sense, we ask for an affine flat of a certain dimension doing this. In this paper, we focus on generalizing the distance measurement and, after obtaining an appropriate notion of circumradius, discuss how to define the notions of inradius, diameter, and minimum width within the general setting of generalized Minkowski spaces. The second method of generalizing the Sylvester problem, namely, by involving affine flats of certain dimension, is investigated in **[16]**.

The paper is organized as follows. In Section 2, we introduce the notation and recall some basic facts regarding support functions and width functions. Results on the four classical quantities of circumradius, inradius, diameter and minimum width are presented in Section 3. The paper is finished by a collection of open questions in Section 4.

2. Preliminaries. Four classical quantities for measuring the size of a given set are: the maximum distance between two of its points (its *diameter*), the minimal distance between two parallel supporting hyperplanes (its *minimum width* or *thickness*), the radius of the smallest ball containing the set (its *circumradius*) and the radius of the largest ball that is contained in the set (its *inradius*). In the framework of convex geometry, the definitions of these quantities refer to Euclidean distance measurement, that is, the size of the given set is compared with the size of the Euclidean unit ball. In the following, we will describe how diameter, minimum width, circumradius and inradius can be defined precisely, when comparing sizes with a centered convex body (not necessarily the Euclidean unit ball), and what may be done when we also drop the centeredness of the measurement body. First, we look at support and width functions, which, for convex sets, are related to some kind of signed Euclidean distances between supporting hyperplanes and the origin and between parallel supporting hyperplanes, respectively.

Throughout this paper, we shall be concerned with the vector space $\mathbb{R}^d$, with the topology generated by the usual inner product $\langle \cdot \mid \cdot \rangle$ and the norm $|\cdot| = \sqrt{\langle \cdot \mid \cdot \rangle}$ or, equivalently, its unit ball B. For the *extended real line*, we write $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ with the conventions

$$0(+\infty) := +\infty, \quad 0(-\infty) := 0 \quad \text{and} \quad (+\infty) + (-\infty) := +\infty.$$

We use the notation $\mathcal{C}^d$ for the family of *non-empty closed convex sets* in $\mathbb{R}^d$. We denote the class of bounded sets that belong to $\mathcal{C}^d$ by $\mathcal{K}^d$. We also write $\mathcal{C}^d_0$ and $\mathcal{K}^d_0$ for the classes of sets having non-empty interior and belonging to $\mathcal{C}^d$ and $\mathcal{K}^d$, respectively. The *line segment* between x and y shall be denoted by $[x, y]$. The abbreviations cl, int and co stand for *closure*, *interior* and *convex hull*, respectively. For $K, K' \subseteq \mathbb{R}^d$, $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$, we define $K \pm \alpha K' := \{x \pm \alpha y \mid x \in K, y \in K'\}$ and write $x \pm K$ for $\{x\} \pm K$. A set K is *centrally symmetric* if and only if there is a point $z \in \mathbb{R}^d$ such that $K = 2z - K$, and K is said to be *centered* if and only if $K = -K$.

The *support function* of a set $K \subseteq \mathbb{R}^d$ is defined as $h_K : \mathbb{R}^d \to \overline{\mathbb{R}}$, $h_K(x) := \sup\{\langle x \mid y \rangle \mid y \in K\}$. Its sublevel set

$$K^\circ := \left\{x \in \mathbb{R}^d \mid h_K(x) \leq 1\right\}$$

is called the *polar set* of K.

Lemma 2.1 ([**3,** Proposition 7.11], [**4,** Section 15])**.** *Let $K, K' \subseteq \mathbb{R}^d$, $x, y \in \mathbb{R}^d$ and $\alpha > 0$. We have*

(a) $h_K = h_{\mathrm{cl}(K)} = h_{\mathrm{co}(K)}$,
(b) $h_{K+K'} = h_K + h_{K'}$,
(c) *sublinearity*: $h_K(x + y) \leq h_K(x) + h_K(y)$, $h_K(\alpha x) = \alpha h_K(x)$,
(d) $h_{\alpha K}(x) = \alpha h_K(x)$, $h_{-K}(x) = h_K(-x)$.

Lemma 2.1 states that it suffices to consider closed and convex sets for the study of support functions. The equation $|h_K(u)| = \mathrm{dist}(0, H_K(u))$, which is valid for $K \in \mathcal{C}^d$ and $u \in \mathbb{R}^d$ with $|u| = 1$ and $h_K(u) < +\infty$, links the support function to the Euclidean distance between the origin and *supporting hyperplanes*

$$H_K(u) = \left\{y \in \mathbb{R}^d \mid \langle u, y \rangle = h_K(u)\right\}$$

of K, see **[24,** Theorem 1.1]. Here, the *Euclidean distance function* of a set K evaluated at $x \in \mathbb{R}^d$ is given via minimal distances, namely, $\operatorname{dist}(x, K) := \inf\{|y - x| \mid y \in K\}$.

The distances between parallel supporting hyperplanes of a set K are encoded by its width function $w_K : \mathbb{R}^d \to \overline{\mathbb{R}}$, $w_K(x) := h_K(x) + h_K(-x)$. Proving its basic properties is straightforward by using Lemma 2.1.

Lemma 2.2. *Let $K, K' \subseteq \mathbb{R}^d$, $u \in \mathbb{R}^d$, and $\alpha > 0$. We have:*

(a) $w_K = h_{K-K}$,
(b) $w_K = w_{\operatorname{co}(K)} = w_{\operatorname{cl}(K)}$,
(c) w_K *is sublinear and non-negative,*
(d) $w_{K+K'} = w_K + w_{K'}$,
(e) $w_{\alpha K} = \alpha w_K$, $w_{-K} = w_K$.
(f) *If* $K \in \mathcal{C}^d$, $u \in \mathbb{R}^d$, $|u| = 1$ *and* $w_K(u) < +\infty$, *then* $w_K(u) = \operatorname{dist}(y, H_K(-u)) = \operatorname{dist}(0, H_K(u)) + \operatorname{dist}(0, H_K(-u))$ *for all* $y \in H_K(u)$.

3. The four classical quantities.

3.1. Circumradius: Measuring from outside. The definition of the circumradius can be found, e.g., in **[9, 10, 13]** for the case $C = B$ (Euclidean space), and in **[11, 20]** for the case $C = -C \in \mathcal{K}_0^d$ (normed spaces).

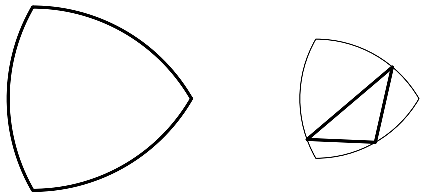


FIGURE 1. Circumradius: The set C is a Reuleaux triangle (bold line, left), the set K is a triangle (bold line, right). The circumradius $R(K, C)$ is determined by the smallest homothet of C that contains K (thin line).

Definition 3.1 (**[6, 7, 8]**)**.** The *circumradius* of $K \subseteq \mathbb{R}^d$ with respect to $C \in \mathcal{C}^d$ is defined as

$$R(K, C) = \inf_{x \in \mathbb{R}^d} \inf\{\lambda > 0 \mid K \subseteq x + \lambda C\}.$$

If $K \subseteq x + R(K, C)C$, then x is a *circumcenter* of K with respect to C.

Proposition 3.2. *Let $K, K' \subseteq \mathbb{R}^d$, $C, C' \in \mathcal{C}^d$ and $\alpha, \beta > 0$. Then,*

(a) $R(K', C') \leq R(K, C)$ *if* $K' \subseteq K$ *and* $C \subseteq C'$,
(b) $R(K, C) = R(\mathrm{cl}(K), C) = R(\mathrm{co}(K), C)$,
(c) $R(K + K', C) \leq R(K, C) + R(K', C)$,
(d) $R(x + K, y + C) = R(K, C)$ *for all* $x, y \in \mathbb{R}^d$,
(e) $R(\alpha K, \beta C) = (\alpha/\beta)R(K, C)$,
(f) $R(K, C') \leq R(K, C)R(C, C')$.

Proof. For $x \in \mathbb{R}^d$ and $\lambda \geq 0$, we have

$$\mathrm{cl}(K) \subseteq x + \lambda C \iff K \subseteq x + \lambda C \iff \mathrm{co}(K) \subseteq x + \lambda C.$$

This proves (b). Statement (c) is also rather simple: if $K \subseteq z + \lambda C$ and $K' \subseteq z' + \lambda' C$ for some $z, z' \in \mathbb{R}^d$, $\lambda, \lambda' > 0$, then $K + K' \subseteq (z + z') + (\lambda + \lambda')C$. In order to show (f), note that there exist numbers $\lambda, \lambda' > 0$ and points $z, z' \in \mathbb{R}^d$ such that $K \subseteq z + \lambda C$ and $C \subseteq z' + \lambda' C'$. Substituting the latter inclusion into the former one, we obtain $K \subseteq z + \lambda z' + \lambda\lambda' C'$. □

Remark 3.3.

(a) The following implication does not hold: If $K, K' \in \mathcal{C}^d$ and $C \in \mathcal{K}_0^d$, then $R(K + K', C + K') = R(K, C)$. For example, take

$$C = [-e_1, e_1] + \cdots + [-e_d, e_d],$$
$$K = (\mathrm{co}\{0, e_1, \ldots, e_d\}), \quad K' = [0, e_d],$$

where $e_i \in \mathbb{R}^d$ denotes the vector whose entries are 0 except for the ith one, which is 1. Then $R(K, C) = 1/2$ and $R(K + K', C + K') = 2/3$. However, the following implication holds: if $K, C' \in \mathcal{C}^d$, $K \in \mathcal{K}^d$ and $R(K, C) = 1$, then $R(K + K', C + K') = R(K, C)$. This is due to the cancellation rule, which reads as:

Let $K_1, K_2 \in \mathcal{C}^d$ and $K \in \mathcal{K}^d$. If $K_1 + K \subseteq K_2 + K$, then $K_1 \subseteq K_2$.

It can easily be proved via support functions. For all $\lambda > 1$, there exists a $z \in \mathbb{R}^d$ such that $K \subseteq z + \lambda C$. It follows that $K + K' \subseteq z + \lambda C + K' \subseteq z + \lambda(C + K')$. In other words, $R(K+K', C+K') \leq 1$. Conversely, assume that $R(K+K', C+K') < 1$. Then there are $\lambda < 1$ and $z \in \mathbb{R}^d$ such that $K + K' \subseteq z + \lambda(C + K') \subseteq z + \lambda C + K'$. By virtue of the cancellation rule, we have $K \subseteq z + \lambda(C + K') \subseteq z + \lambda C$, which is a contradiction to $R(K, C) = 1$.

(b) Proposition 3.2 (c) holds with equality if $K' = \alpha C$ for all $\alpha > 0$.

Lemma 2.2 states that the circumradius of K with respect to C is invariant under translations of both K and C. Thus, without loss of generality, we may assume that 0 belongs to the relative interior of C. Then the circumradius can be equivalently written as

$$R(K, C) = \inf_{x \in \mathbb{R}^d} \sup_{y \in K} \gamma_C(y - x),$$

where $\gamma_C : \mathbb{R}^d \to \overline{\mathbb{R}}$ is the *Minkowski functional* defined by $\gamma_C(x) := \inf\{\lambda > 0 \mid x \in \lambda C\}$.

In subsection 3.3, we deal with the notion of diameter of a set which is classically defined via the maximum distance between points of this set. For a convex set, this coincides with the maximum length of segments contained in the chosen set. Thus, we now consider how to define the length of a segment in the setting of generalized Minkowski spaces, that is, with respect to the geometry of a given set $C \in \mathcal{K}^d_0$.

Lemma 3.4. *Given $x, y, z \in \mathbb{R}^d$, $\alpha > 0$, and $C \in \mathcal{K}^d_0$ with $0 \in \operatorname{int}(C)$, we define $g(x, y) := 2R(\{x, y\}, C)$. The following statements are true*:

(a) $g(x, y) \geq 0$ *with equality if and only if* $x = y$,
(b) $g(x, y) = g(y, x)$,
(c) $g(x + z, y + z) = g(x, y)$,
(d) $g(\alpha x, \alpha y) = \alpha g(x, y)$,
(e) $g(x, y) \leq g(x, z) + g(z, y)$.

Proof. The non-negativity follows from the definition. The characterization of the equality case is a consequence of the more general result that $R(K, C) = 0$ if and only if K is contained in a translate of the cone $\{y \in \mathbb{R}^d \mid y + C \subseteq C\}$ when the set of extreme points of C is bounded, see [**7,** Lemma 2.2]. The symmetry $g(x, y) = g(y, x)$ is clear. The invariance under translations and the compatibility with scaling follow from Proposition 3.2 (d), (e). Finally, since g is translation-invariant and symmetric, we only need to check $g(0, x + y) \leq g(0, x) + g(0, y)$ for the triangle inequality. However, we have

$$\begin{aligned} g(0, x+y) &= R(\{0, x+y\}, C) \\ &\leq R(\{0, x, y, x+y\}, C) \\ &\leq R(\{0, x\}, C) + R(\{0, y\}, C) \\ &= g(0, x) + g(0, y) \end{aligned}$$

by Proposition 3.2 (c). □

Since the triangle inequality for g is true, the mapping $x \mapsto 2R(\{0, x\}, C)$ defines a norm on $\mathbb{R}^d$. The unit ball of this norm is $(C - C)/2$. This fact can be proved as follows. First, we show that, if $x \in (C - C)/2$, then $R(\{0, x\}, C/2) \leq 1$, namely, there exist $y_1, y_2 \in C/2$ such that $x = y_1 - y_2$. Thus,

$$R(\{0, x\}, \tfrac{1}{2}C) = R(\{y_1, y_2\}, \tfrac{1}{2}C) \leq 1.$$

The reverse implication is as easy as the first. If $R(\{0, x\}, C/2) > 1$, then there is no point $z \in \mathbb{R}^d$ such that $\{0, x\} \subseteq z + C/2$ or, equivalently, such that $\{-z, x - z\} \subseteq C/2$. Thus, there is no representation $x = (x - z) - (-z) \in (C/2) - (C/2)$.

For centered sets K, the maximal circumradius of two-element subsets is attained at antipodal points of K.

Proposition 3.5. *Let $K \subseteq \mathbb{R}^d$ be a bounded set, and let $C \in \mathcal{K}_0^d$. If $K = -K$, then*

$$\sup\{R(\{-x, x\}, C) \mid x \in K\} = \sup\{R(\{x, y\}, C) \mid x, y \in K\}.$$

Proof. Using Proposition 3.2, we have

$$\begin{aligned}\sup\{R(\{x,y\},C) \mid x,y\in K\} &\le \sup\{R(\{0,x,y,x+y\},C) \mid x\in K\}\\ &\le \sup\{R(\{0,x\},C) \mid x\in K\}\\ &\quad + \sup\{R(\{0,y\},C) \mid y\in K\}\\ &= 2\sup\{R(\{0,x\},C) \mid x\in K\}\\ &= \sup\{R(\{0,2x\},C) \mid x\in K\}\\ &= \sup\{R(\{-x,x\},C) \mid x\in K\}\\ &\le \sup\{R(\{x,y\},C) \mid x,y\in K\}. \qquad \square\end{aligned}$$

If both C and K are centered, there is another convenient representation of the circumradius.

Proposition 3.6 ([**11,** (1.1)])**.** *Let $K \subseteq \mathbb{R}^d$, $C \in \mathcal{C}_0^d$, $0 \in \operatorname{int}(C)$, $C = -C$, $K = -K$. Then,*

$$R(K,C) = \sup\{\gamma_C(x) \mid x \in K\}.$$

Proof. If $K \subseteq z + \lambda C$ for suitable $z \in \mathbb{R}^d$ and $\lambda > 0$, then $K \subseteq -z + \lambda C$, due to the centeredness of C and K. It follows that

$$K \subseteq \frac{1}{2}K + \frac{1}{2}K \subseteq \frac{1}{2}(z+\lambda C) + \frac{1}{2}(-z+\lambda C) = \lambda C,$$

in other words, the circumradius is already determined by the sets λC with $\lambda > 0$:

$$R(K,C) = \inf\{\lambda > 0 \mid K \subseteq \lambda C\} = \sup\{\gamma_C(x) \mid x \in K\}. \qquad \square$$

3.2. Inradius: Measuring from inside. The definition of the inradius may be found, e.g., in [**9, 10, 13**] for the case $C = B$ (Euclidean space) and in [**11**] for the case $C = -C \in \mathcal{K}_0^d$ (normed spaces).

Definition 3.7. The *inradius* of $K \subseteq \mathbb{R}^d$ with respect to $C \in \mathcal{C}^d$ is defined as

$$r(K,C) = \sup_{x\in\mathbb{R}^d} \sup\{\lambda \ge 0 \mid x + \lambda C \subseteq K\}.$$

This definition is similar to that of the circumradius, and so are the corresponding basic properties.

FIGURE 2. Inradius: The set C is a triangle (bold line, left), the set K is a Reuleaux triangle (bold line, right). The inradius $r(K, C)$ is determined by the largest homothet of C that is contained in K (thin line).

Proposition 3.8. *Let $K, K' \subseteq \mathbb{R}^d$, $C, C' \in \mathcal{C}^d$ and $\alpha, \beta > 0$. Then*

(a) $r(K', C') \geq r(K, C)$ *if* $K' \subseteq K$ *and* $C \subseteq C'$,
(b) $r(K, C) = r(\mathrm{cl}(K), C)$ *if* K *is convex,*
(c) $r(K + K', C) \geq r(K, C) + r(K', C)$,
(d) $r(x + K, y + C) = r(K, C)$ *for all* $x, y \in \mathbb{R}^d$,
(e) $r(\alpha K, \beta C) = (\alpha/\beta) r(K, C)$,
(f) $r(K, C') \geq r(K, C) r(C, C')$.

Proof. The proof holds using arguments similar to those in the proof of Proposition 3.2. □

3.3. Diameter. In Euclidean geometry, the *diameter* of a given set is usually defined as the maximum distance of two points of this set. There are several other representations of this quantity which do not coincide when replacing the Euclidean unit ball by a convex body C in general (but, at least, if $C = -C$). This offers various possibilities to think about an appropriate extension of the notion of diameter. At first, let us consider the interpretation of the diameter as the maximum distance between the points of the set. Here, the distance notion is provided by the Minkowski functional of C. Then, we can rewrite the expression for the diameter as the supremum of the Euclidean width function over the polar set of C. Another representation involves the maximal chord-length function and the radius function of a convex body which are the maximal Euclidean length of chords with given direction and the Euclidean distance from the origin 0 to the boundary, respectively. More precisely, the *maximal chord-length function* of $K \subseteq \mathbb{R}^d$ is defined via $l_K : \mathbb{R}^d \to \overline{\mathbb{R}}$,

$$l_K(x) = \sup\{\alpha > 0 \mid \alpha x \in K - K\}.$$

The *radius function* $r_K : \mathbb{R}^d \to \overline{\mathbb{R}}$, defined as

$$r_K(u) = \sup\{\alpha > 0 \mid \alpha u \in K\},$$

is the pointwise inverse to the Minkowski functional γ_K.

Theorem 3.9. *For $K \subseteq \mathbb{R}^d$ and $C \in \mathcal{K}_0^d$ with $0 \in \operatorname{int}(C)$, the following numbers are equal*:

(a) $\sup\{\gamma_C(x-y) \mid x, y \in K,$
(b) $\sup\{w_K(u) \mid u \in C^\circ\},$
(c) $\sup\{\langle u \mid x\rangle\} \mid u \in C^\circ, x \in K - K\}.$

If $K \in \mathcal{C}^d$, then the following number also belongs to this set of equal quantities:

(d) $\sup\{l_K(u)/r_C(u) \mid u \in \mathbb{R}^d \setminus \{0\}\}.$

Proof. Using **[17,** Lemma 2.1], we have

$$\begin{aligned}
\sup_{x,y\in K} \gamma_C(x-y) &= \sup_{x,y\in K} \sup_{u\in C^\circ} \langle u \mid x-y\rangle \\
&= \sup_{u\in C^\circ} \sup_{x,y\in K} \langle u \mid x-y\rangle \\
&= \sup_{u\in C^\circ} \sup_{x,y\in K} (\langle u \mid x\rangle + \langle -u \mid y\rangle) \\
&= \sup_{u\in C^\circ} (h_K(u) + h_K(-u)) = \sup_{u\in C^\circ} h_{K-K}(u) \\
&= \sup\{\langle u \mid x\rangle \mid u \in C^\circ, x \in K - K\}.
\end{aligned}$$

If $K \in \mathcal{C}^d$, then

$$\begin{aligned}
\sup_{x,y\in K} \gamma_C(x-y) &= \sup_{u\in\mathbb{R}^d\setminus\{0\}} \sup_{\alpha>0:\alpha u\in K-K} \gamma_C(\alpha u) \\
&= \sup_{u\in\mathbb{R}^d\setminus\{0\}} \sup_{\alpha>0:\alpha u\in K-K} \alpha\gamma_C(u) \\
&= \sup_{u\in\mathbb{R}^d\setminus\{0\}} l_K(u)\gamma_C(u) = \sup_{u\in\mathbb{R}^d\setminus\{0\}} \frac{l_K(u)}{r_C(u)}. \qquad \square
\end{aligned}$$

Other representations of the diameter in the Euclidean case are written in terms of circumradii, see [**2,** Theorem 2]. Together with the representation from Theorem 3.9, we obtain a chain of inequalities.

Theorem 3.10. *If $K \subseteq \mathbb{R}^d$ and $C \in \mathcal{K}_0^d$ with $0 \in \operatorname{int}(C)$, then*

(3.1)
$$\begin{aligned}2\sup\left\{\frac{h_{K-K}(u)}{h_{C-C}(u)}\;\middle|\; u\in\mathbb{R}^d\setminus\{0\}\right\} &= 2\sup\{R(\{x,y\},C)\mid x,y\in K\}\\ &= R\left(K-K,\frac{1}{2}(C-C)\right)\\ &\le R(K-K,C)\\ &\le \sup\{\gamma_C(x-y)\mid x,y\in K\},\end{aligned}$$

with equality if $C=-C$. If $K \in \mathcal{C}^d$, then we also have

$$\sup\{R(\{x,y\},C)\mid x,y\in K\} = \sup\left\{\frac{l_K(u)}{l_C(u)}\;\middle|\; u\in\mathbb{R}^d\setminus\{0\}\right\}. \tag{3.2}$$

Proof. If $C=-C$, then we have

$$\begin{aligned}&2\sup\left\{\frac{h_{K-K}(u)}{h_{C-C}(u)}\;\middle|\; u\in\mathbb{R}^d\setminus\{0\}\right\}\\ &\quad= \sup\left\{\frac{h_{K-K}(u)}{h_{C}(u)}\;\middle|\; u\in\mathbb{R}^d\setminus\{0\}\right\}\\ &\quad= \sup\left\{\frac{h_{K-K}(u)}{h_{C}(u)}\;\middle|\; u\in B\setminus\{0\}\right\}\\ &\quad= \sup\left\{h_{K-K}\left(\frac{u}{h_C(u)}\right)\;\middle|\; u\in B\setminus\{0\}\right\}\\ &\quad= \sup\{h_{K-K}(x)\mid x\in C^\circ\setminus\{0\}\} = \sup\{h_{K-K}(x)\mid x\in C^\circ\}\\ &\quad= \sup\{\gamma_C(x-y)\mid x,y\in K\} = R(K-K,C)\\ &\quad= R\left(K-K,\frac{1}{2}(C-C)\right) = \sup\{\gamma_{(C-C)/2}(x)\mid x\in K-K\}\\ &\quad= 2\sup\{R(\{0,x\},C)\mid x\in K-K\} = 2\sup\{R(\{x,y\},C)\mid x,y\in K\}\end{aligned}$$

by using Theorem 3.9, Proposition 3.6, Lemma 3.4 and Proposition 3.2. Note that Lemma 3.4 is independent of centeredness of C and, therefore, can be similarly used in the general case which follows.

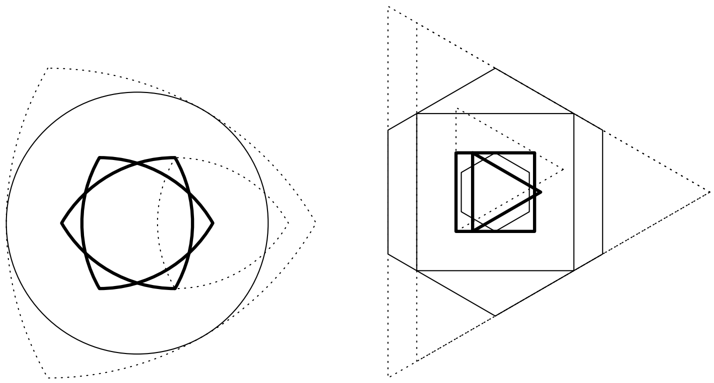
From now on, we do not assume that $C = -C$. We apply calculations for the symmetric case and obtain:

$$
\begin{aligned}
&2\sup\left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\}\right\}\\
&= 2\sup\left\{ \frac{h_{(K-K)-(K-K)}(u)}{h_{(C-C)-(C-C)}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\}\right\}\\
&= R\Big((K-K)-(K-K), \frac{1}{2}((C-C)-(C-C))\Big)\\
&= R\Big(K-K, \frac{1}{2}(C-C)\Big)\\
&= 2\sup\{R(\{x,y\},C) \mid x,y \in K\}\\
&= 2\sup\{R(\{0,x\},C) \mid x \in K-K\}\\
&= \sup\{R(\{-x,x\},C) \mid x \in K-K\} \le R(K-K,C)\\
&\le \inf\{\lambda > 0 \mid K-K \subseteq \lambda C\} = \sup_{x\in K-K} \gamma_C(x)\\
&= \sup_{x,y\in K} \gamma_C(x-y).
\end{aligned}
$$

In order to prove the addendum (3.2), let $K \in \mathcal{C}^d$. Then,

$$
\begin{aligned}
&2\sup\{R(\{x,y\},C) \mid x,y \in K\}\\
&= 2\sup\left\{ \frac{|x-y|}{l_C((x-y)/|x-y|)} \;\middle|\; x,y \in K,\ x \neq y\right\}\\
&= 2\sup_{u\in\mathbb{R}^d\setminus\{0\}} \sup\left\{ \frac{\alpha}{l_C(u)} \;\middle|\; \alpha > 0,\ \alpha u \in K-K\right\}\\
&= 2\sup_{u\in\mathbb{R}^d\setminus\{0\}} \frac{\sup\{\alpha \mid \alpha > 0, \alpha u \in K-K\}}{l_C(u)}\\
&= 2\sup\left\{ \frac{l_K(u)}{l_C(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\}\right\}. \qquad \square
\end{aligned}
$$

The next examples show that the inequalities in (3.1) need not be strict if C and K are not centrally symmetric, but, on the other hand, may be strict even if K is centrally symmetric. An illustration of these examples is provided by Figure 3.



(a) C and K are Reuleaux triangles.

(b) C is an equilateral triangle, K is a square.

FIGURE 3. Illustration of Example 3.11: the sets C and K are depicted in bold lines.

Example 3.11.

(a) Let $d = 2$ and

$$\begin{aligned} C = -K = &((2,0) + 2\sqrt{3}B) \\ &\cap ((-1,\sqrt{3}) + 2\sqrt{3}B) \\ &\cap ((-1,-\sqrt{3}) + 2\sqrt{3}B). \end{aligned}$$

Then $K - K = C - C = 2\sqrt{3}B$, i.e.,

$$2\sup\left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\} \right\} = 2,$$

but $R(K-K,C) = \sup\{\gamma_C(x-y) \mid x,y \in K\} = (3+\sqrt{3})/2 \approx 2.366025$.

(b) Let $d = 2$, $C = \operatorname{co}(\{(2,0),(-1,\sqrt{3}),(-1,-\sqrt{3})\})$, and

$$K = \operatorname{co}\{\{(-\sqrt{3},-\sqrt{3}),(-\sqrt{3},\sqrt{3}),(\sqrt{3},-\sqrt{3}),(\sqrt{3},\sqrt{3})\}\}.$$

Then

$$\sup\{\gamma_C(x-y) \mid x,y \in K\} = 3+\sqrt{3} \approx 4.732,$$
$$R(K-K,C) = 2+\frac{4}{\sqrt{3}} \approx 4.3094,$$
$$2\sup\{R(\{x,y\},C) \mid x,y \in K\} = \frac{2}{3}(3+\sqrt{3}) \approx 3.1547,$$
$$2\sup\left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\}\right\} = \frac{2}{3}(3+\sqrt{3}) \approx 3.1547.$$

Note that usually the diameter is defined on the lines of Theorem 3.9 (a), see **[9, 10, 13]** for the Euclidean case, i.e., $C = B$, and **[11]** for the normed case, i.e., $C = -C \in \mathcal{K}_0^d$. In the general setting, each of the representations may have its own benefits. However, following [**7,** Definition 5.2], we can define the notion of diameter via the circumradii of two-element subsets which is, by Lemma 3.4, the usual diameter with respect to the norm generated by $(C-C)/2$.

Definition 3.12. The *diameter* of K with respect to C is

$$D(K,C) = 2\sup\{R(\{x,y\},C) \mid x,y \in K\}.$$

The diameter also behaves conveniently under hull operations and Minkowski sums in the first arguments, as well as under independent translations and scalings of both arguments.

Proposition 3.13. *Let $K, K' \subseteq \mathbb{R}^d$, $C, C' \in \mathcal{C}^d$, and $\alpha, \beta > 0$. Then, we have*

(a) $D(K',C') \leq D(K,C)$ *if* $K' \subseteq K$ *and* $C \subseteq C'$,
(b) $D(K,C) = D(\mathrm{cl}(K),C) = D(\mathrm{co}(K),C)$,
(c) $D(K+K',C) \leq D(K,C) + D(K',C)$,
(d) $D(x+K, y+C) = D(K,C)$ *for all* $x,y \in \mathbb{R}^d$,
(e) $D(\alpha K, \beta C) = (\alpha/\beta) D(K,C)$,
(f) $D(K,C') \leq D(K,C) D(C,C')$.

Proof. Statement (a) is a consequence of Proposition 3.2 (a). Clearly, we have $D(K,C) \leq D(\mathrm{cl}(K),C)$ and $D(K,C) \leq D(\mathrm{co}(K),C)$.

Furthermore, we obtain

$$
\begin{aligned}
&D(\mathrm{co}(K), C)\\
&\quad = \sup\{R(\{x,y\}, C) \mid x, y \in \mathrm{co}(K)\}\\
&\quad = \sup\{R(\{0,z\}, C) \mid z \in \mathrm{co}(K-K)\}\\
&\quad = \sup\left\{ R\left(\left\{0, \sum_{i=1}^{n} \lambda_i x_i\right\}, C\right) \;\middle|\; \begin{array}{l} n \in \mathbb{N},\ x_i \in K-K,\ \lambda_i \geq 0,\\ i \in \{1,\dots,n\},\ \sum_{i=1}^{n} \lambda_i = 1 \end{array} \right\}\\
&\quad \leq \sup\left\{ R\left(\sum_{i=1}^{n} \lambda_i \{0, x_i\}, C\right) \;\middle|\; \begin{array}{l} n \in \mathbb{N},\ x_i \in K-K,\ \lambda_i \geq 0,\\ i \in \{1,\dots,n\},\ \sum_{i=1}^{n} \lambda_i = 1 \end{array} \right\}\\
&\quad \leq \sup\left\{ \sum_{i=1}^{n} \lambda_i R(\{0, x_i\}, C) \;\middle|\; \begin{array}{l} n \in \mathbb{N},\ x_i \in K-K,\ \lambda_i \geq 0,\\ i \in \{1,\dots,n\},\ \sum_{i=1}^{n} \lambda_i = 1 \end{array} \right\}\\
&\quad \leq \sup\left\{ \sum_{i=1}^{n} \lambda_i D(K, C) \;\middle|\; \begin{array}{l} n \in \mathbb{N},\ \lambda_i \geq 0,\\ i \in \{1,\dots,n\},\ \sum_{i=1}^{n} \lambda_i = 1 \end{array} \right\}\\
&\quad = D(K,C) \sup\left\{ \sum_{i=1}^{n} \lambda_i \;\middle|\; \begin{array}{l} n \in \mathbb{N},\ \lambda_i \geq 0,\\ i \in \{1,\dots,n\},\ \sum_{i=1}^{n} \lambda_i = 1 \end{array} \right\}\\
&\quad = D(K,C)
\end{aligned}
$$

and

$$
\begin{aligned}
D(\mathrm{cl}(K), C) &= \sup\{R(\{x,y\}, C) \mid x, y \in \mathrm{cl}(K)\}\\
&= \sup\{R(\{x_i, y_i\}, C) \mid x_i, y_i \in K,\ i \in \mathbb{N},\ x_i \to x,\ y_i \to y\}\\
&\leq \sup\left\{ \sup\{R(\{w,z\}, C) \mid w, z \in K\} \;\middle|\; \begin{array}{c} x_i, y_i \in K,\ i \in \mathbb{N},\\ x_i \to x,\ y_i \to y \end{array} \right\}\\
&= \sup\{R(\{w,z\}, C) \mid w, z \in K\}.
\end{aligned}
$$

This yields claim (b).

In order to prove part (c), we observe that

$$D(K+K', C) = \sup\{R(\{x,y\}, C) \mid x, y \in K+K'\}$$

$$
\begin{aligned}
&= \sup\{R(\{w+w', z+z'\}, C) \mid w, z \in K,\ w', z' \in K'\} \\
&\leq \sup\{R(\{w+w', z+z', w+z, w'+z'\}, C) \mid w, z \in K,\ w', z' \in K'\} \\
&\leq \sup\{R(\{w, z\}, C) + R(\{w', z'\}, C) \mid w, z \in K, w', z' \in K'\} \\
&= \sup\{R(\{w, z\}, C) \mid w, z \in K\} + \sup\{R(\{w', z'\}, C) \mid w', z' \in K'\} \\
&= D(K, C) + D(K', C).
\end{aligned}
$$

In order to prove (f), we use the representation

$$D(K, C) = \sup\{\gamma_{C-C}(x) \mid x \in K - K\}.$$

Without loss of generality, we may assume that K is bounded since, otherwise, $D(K, C) = D(K, C') = +\infty$ due to the positive homogeneity of Minkowski functionals. Furthermore, we assume that $0 \in \operatorname{int}(C) \cap \operatorname{int}(C')$ due to (d). Denoting the boundary of B by $\operatorname{bd}(B)$, we have

$$
\begin{aligned}
D(K, C') &= \sup\{\gamma_{C'-C'}(x) \mid x \in K - K\} \\
&= \sup\{\gamma_{C'-C'}(\alpha u) \mid u \in \operatorname{bd}(B),\ \alpha \in [0, r_{K-K}(u)]\} \\
&= \sup\{\gamma_{C'-C'}(r_{K-K}(u)u) \mid u \in \operatorname{bd}(B)\} \\
&= \sup\left\{ r_{K-K}(u)\gamma_{C-C}(u)\ \frac{\gamma_{C'-C'}(u)}{\gamma_{C-C}(u)}\ \middle|\ u \in \operatorname{bd}(B) \right\} \\
&= \sup\{r_{K-K}(u)\gamma_{C-C}(u) r_{C-C}(u)\gamma_{C'-C'}(u) \mid u \in \operatorname{bd}(B)\} \\
&\leq \sup\{r_{K-K}(u)\gamma_{C-C}(u) \mid u \in \operatorname{bd}(B)\} \\
&\quad \cdot \sup\{r_{C-C}(u)\gamma_{C'-C'}(u) \mid u \in \operatorname{bd}(B)\} \\
&= \sup\{\gamma_{C-C}(x) \mid x \in K - K\} \\
&\quad \cdot \sup\{\gamma_{C'-C'}(x) \mid x \in C - C\} \\
&= D(K, C)D(C, C'). \qquad \square
\end{aligned}
$$

Finally, we remark that a classical upper bound of the diameter in terms of the circumradius is still valid in generalized Minkowski spaces, namely, $D(K, C) \leq 2R(K, C)$ for all $K \subseteq \mathbb{R}^d$ and $C \in \mathcal{K}_0^d$ with $0 \in \operatorname{int}(C)$ with equality if, e.g., $C = -C$ and $K = -K$. This follows immediately from Proposition 3.2 (a) and Theorem 3.10. An estimate of the diameter in terms of the circumradius from below is given by Jung's inequality, see **[5]**.

3.4. Minimum width. In Euclidean space, the notion of *minimum width* is intimately related to the notion of diameter. The latter is the supremum of the width function (see Theorem 3.9), the former is classically defined as the corresponding infimum. Here, the reference to the (possibly non-centered) "unit ball" is done by considering the ratio of the width functions. First, we collect relations between several representations of minimum width in normed spaces [**2,** Theorem 3] and within the general setting.

Lemma 3.14. *Let $K \subseteq \mathbb{R}^d$ and $C \in \mathcal{K}_0^d$. If $C = -C$, we have*

$$2 \inf \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\} \right\} = \inf \left\{ \frac{h_{K-K}(u)}{\gamma_{C^\circ}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\} \right\},$$

in other words, the minimal ratio of the (*Euclidean*) *width function is equal to the minimal distance of parallel supporting hyperplanes of K, measured by the norm γ_C. If $h_{K-K}(u) > 0$ for all $u \in \mathbb{R}^d \setminus \{0\}$, then the reverse implication is also true.*

Proof. The first statement is clear by the relations $\gamma_{C^\circ} = h_C$, $C - C = 2C$ and Lemma 2.1 (d).

For the reverse statement, assume that $h_{K-K}(u) > 0$ for all $u \in \mathbb{R}^d \setminus \{0\}$ and

$$2 \inf \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\} \right\} < \inf \left\{ \frac{h_{K-K}(u)}{h_C(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\} \right\}. \tag{3.3}$$

Then,

$$2 \frac{h_{K-K}(u)}{h_{C-C}(u)} < \frac{h_{K-K}(u)}{h_C(u)} \quad \text{for all } u \in \mathbb{R}^d \setminus \{0\},$$

which is equivalent to

$$\frac{h_{C-C}}{2} < h_C$$

or $h_C > h_{-C}$, which is impossible. We arrive at the same conclusion if we assume the reverse inequality in (3.3). $\square$

Lemma 3.15. *If $K, C \in \mathcal{K}_0^d$, then*

$$\begin{aligned} r(K-K,C) &= R(C,K-K)^{-1} \\ &= \left(\sup\{\langle u \mid x\rangle \mid u \in (K-K)^\circ,\ x \in C\}\right)^{-1}. \end{aligned}$$

Proof. Note that $K-K$ is centered, and apply Theorem 3.10. □

Remark 3.16. The claim of Lemma 3.15 fails if K is not convex. For example, if K is a finite set, then $r(K-K,C)=0$. However,

$$\begin{aligned} (\mathrm{co}(K)-\mathrm{co}(K))^\circ &= (\mathrm{co}(K-K))^\circ \\ &= \{y \in \mathbb{R}^d \mid h_{\mathrm{co}(K-K)}(y) \leq 1\} \\ &= \{y \in \mathbb{R}^d \mid h_{K-K}(y) \leq 1\} \\ &= (K-K)^\circ, \end{aligned}$$

where we merely used Lemma 2.1 (a). It follows that

$$\begin{aligned} &\sup\{\langle u \mid x\rangle \mid u \in (K-K)^\circ,\ x \in C\} \\ &\qquad = \sup\{\langle u \mid x\rangle \mid u \in (\mathrm{co}(K)-\mathrm{co}(K))^\circ,\ x \in C\}, \end{aligned}$$

which does not equal zero in general.

Lemma 3.17. *Let $K \in \mathcal{K}^d$ and $C \in \mathcal{K}_0^d$. Then,*

$$r(K-K,C) \leq 2\inf\left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\}\right\} \tag{3.4}$$

with equality if $C=-C$.

Proof. For all $\alpha < r(K-K,C)$, there exists a $z \in \mathbb{R}^d$ such that $z+\alpha C \subseteq K-K$. Using Lemma 2.2, we obtain

$$\begin{aligned} & w_{z+\alpha C}(u) \leq w_{K-K}(u) && \text{for all } u \in \mathbb{R}^d \setminus \{0\} \\ \iff & w_{\alpha C}(u) \leq w_{K-K}(u) && \text{for all } u \in \mathbb{R}^d \setminus \{0\} \\ \iff & \alpha w_C(u) \leq 2h_{K-K}(u) && \text{for all } u \in \mathbb{R}^d \setminus \{0\} \\ \iff & \alpha \leq 2\frac{h_{K-K}(u)}{h_{C-C}(u)} && \text{for all } u \in \mathbb{R}^d \setminus \{0\}. \end{aligned}$$

Passing α to $r(K-K,C)$, we obtain (3.4). Now, let $C=-C$, and assume that (3.4) is a strict inequality, i.e., an α exists such that

$$r(K-K,C)<\alpha<2\inf\left\{\frac{h_{K-K}(u)}{h_{C-C}(u)}\;\middle|\;u\in\mathbb{R}^d\setminus\{0\}\right\}.$$

As above, we obtain $w_{\alpha C}<w_{K-K}$. Dividing by 2, we have $h_{\alpha C}<h_{K-K}$. It follows that

$$r(K-K,C)C\subsetneq\alpha C\subsetneq K-K.$$

This is a contradiction to the definition of $r(K-K,C)$. □

Remark 3.18. If $C\neq-C$, we may have a strict inequality in (3.4). In the situation of Example 3.11 (a), we obtain

$$2\inf\left\{\frac{h_{K-K}(u)}{h_{C-C}(u)}\;\middle|\;u\in\mathbb{R}^d\setminus\{0\}\right\}=2,$$

but obviously, $r(K-K,C)=\sqrt{3}\neq 2$, see Figure 4.

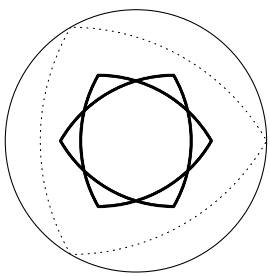


FIGURE 4. Illustration of Remark 3.18: K and C are Reuleaux triangles (bold lines).

Summarizing, we obtain the next theorem on the notion of minimum width in normed spaces.

Theorem 3.19 ([**2,** Theorem 3])**.** *For $K\in\mathcal{C}^d$ and $C\in\mathcal{K}_0^d$ with $C=-C$, the following numbers are equal*:

(a) $r(K-K,C)$,

(b) $2\inf\{h_{K-K}(u)/h_{C-C}(u) \mid u \in \mathbb{R}^d \setminus \{0\}\}$,
(c) $\inf\{h_{K-K}(u)/\gamma_{C^\circ}(u) \mid u \in \mathbb{R}^d \setminus \{0\}\}$,
(d) $(\sup\{\langle u \mid x\rangle \mid u \in (K-K)^\circ,\ x \in C\})^{-1}$.

Proof. This is a combination of the previous lemmas. If $h_{K-K} \equiv +\infty$, then $K = \mathbb{R}^d$ and all the numbers equal $+\infty$. Similarly, if there is a $u \in \mathbb{R}^d \setminus \{0\}$ such that $h_{K-K}(u) = 0$, then all of the numbers equal 0. (For the last item, use the conventions $1/0 = +\infty$, $1/(+\infty) = 0$.) □

Since we focus on containment problems, that is, finding in some sense extremal scaling factors, it is useful to take the minimal ratio of support functions as the definition of minimum width.

Definition 3.20 ([**7,** Definition 2.6])**.** The *minimum width* of K with respect to C is:

$$\begin{aligned}\omega(K,C) :=& 2\inf\left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\}\right\}\\ =& 2\inf\{R(K, C+L) \mid L \in \mathcal{L}^d_{d-1}\},\end{aligned}$$

where $\mathcal{L}^d_{d-1}$ denotes the family of $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$.

Proposition 3.21. *Let $K, K' \subseteq \mathbb{R}^d$, $C, C' \in \mathcal{C}^d$, and $\alpha, \beta > 0$. Then, we have*:

(a) $\omega(K', C') \geq \omega(K, C)$ *if* $K' \subseteq K$ *and* $C \subseteq C'$,
(b) $\omega(K, C) = \omega(\mathrm{cl}(K), C)$ *if* K *is convex,*
(c) $\omega(K + K', C) \geq \omega(K, C) + \omega(K', C)$,
(d) $\omega(x + K, y + C) = \omega(K, C)$ *for all* $x, y \in \mathbb{R}^d$,
(e) $\omega(\alpha K, \beta C) = (\alpha/\beta)\omega(K, C)$,
(f) $\omega(K, C') \geq \omega(K, C)\omega(C, C')$.

Proof. For $x \in \mathbb{R}^d$, $\lambda \geq 0$ and $L \in \mathcal{L}^d_{d-1}$, we have

$$\mathrm{cl}(K) \subseteq x + L + \lambda C \iff K \subseteq x + L + \lambda C.$$

This proves (b).

In order to prove part (c), we observe

$$\begin{aligned}
\omega(K+K',C) &= \inf\left\{ \frac{h_{K+K'-K-K'}(u)}{h_{C-C}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\}\right\} \\
&= \inf\left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} + \frac{h_{K'-K'}(u)}{h_{C-C}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\}\right\} \\
&\geq \inf\left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\}\right\} \\
&\quad + \inf\left\{ \frac{h_{K'-K'}(u)}{h_{C-C}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\}\right\} \\
&= \omega(K,C) + \omega(K',C).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\omega(K,C') &= \inf\left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \frac{h_{C-C}(u)}{h_{C'-C'}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\}\right\} \\
&\geq \inf\left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\}\right\} \\
&\quad \cdot \inf\left\{ \frac{h_{C-C}(u)}{h_{C'-C'}(u)} \;\middle|\; u \in \mathbb{R}^d \setminus \{0\}\right\} \\
&= \omega(K,C')\omega(C,C'). \qquad \square
\end{aligned}$$

4. Open questions. An increasing interest in real vector spaces equipped with Minkowski functionals may be observed in various directions. For example, *gauges* or *convex distance functions* occur in computational geometry, operations research and location science, see, e.g., [**12, 14, 17, 19, 24, 25**]. In the present paper, a gentle start is provided for applying this setting to basic metrical notions of convex geometry. Various further natural questions occur immediately. Propositions 3.2 (c), 3.13 (c), 3.8 (c), and 3.21 (c) refer to lower bounds for the circumradius, the diameter, and to upper bounds for the inradius and the minimum width of the Minkowski sum of two sets in terms of the sum of the respective quantities for the single sets. The question of the existence of reverse inequalities can be answered easily as follows. If $K, C \in \mathcal{K}_0^d$ and, as before, B denotes the Euclidean unit ball, we can derive inequality:

$$\begin{aligned}\frac{R(K,C)+R(K',C)}{R(B,C)} &\leq R(K,B)+R(K',B)\\ &\leq \sqrt{2}R(K+K',B)\\ &\leq \sqrt{2}R(C,B)R(K+K',C)\end{aligned}$$

by using Proposition 3.2 (f) and [**9,** Theorem 1.1]. This implies

$$R(K,C)+R(K',C) \leq \sqrt{2}R(C,B)R(B,C)R(K+K',C). \tag{4.1}$$

Similarly, Proposition 3.13 (f) and [**9,** Theorem 1.2] yield

$$D(K,C)+D(K',C) \leq \sqrt{2}D(C,B)D(B,C)D(K+K',C). \tag{4.2}$$

According to [**9,** Theorems 1.1, 1.2], there exist no constants α, β such that

$$\begin{aligned}r(K,B)+r(K',B) &\geq \alpha r(K+K',B),\\ \omega(K,B)+\omega(K',B) &\geq \alpha\omega(K+K',B).\end{aligned}$$

Of course, this implies the non-existence of analogous inequalities if the Euclidean unit ball B is replaced by an arbitrary set $C \in \mathcal{K}_0^d$. However, it remains open whether the constants in inequalities (4.1) and (4.2) can be improved. Furthermore, with the notions of diameter and minimum width, the investigation of diametrical maximal bodies [**22, 23**], constant width bodies [**21,** Section 2] and reduced bodies [**18**] in generalized Minkowski spaces is enabled. (First results in this direction are derived in [**5**].) As pointed out before, the quantities defined in Definitions 3.12 and 3.20 shift the problem to normed spaces. However, Theorems 3.10 and 3.19 provide other expressions which are suitable as definitions for generalizations of diameter and width, respectively, and might have their own benefits for specific purposes.

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