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ON TORSION FREE AND COTORSION DISCRETE MODULES

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ABSTRACT. We prove that, if $\mathcal{F}$ is the class of torsion free discrete modules over a profinite group G, that is, the class of discrete G-modules which are torsion free as abelian groups, then $(\mathcal{F}, \mathcal{F}^{\perp})$ is a complete cotorsion pair. Moreover, we find a structure theorem for torsion free and cotorsion discrete G-modules and for finitely generated cotorsion discrete G-modules.

1. Introduction. Recently, the flat cover conjecture in the category of discrete G-modules for any profinite group G has been positively answered by Enochs and Khan [**6**]. However, little is known about flat discrete G-modules; thus, the information obtained from these types of modules does not provide significant information about discrete G-module categories. Moreover, it is not clear whether flat covers are surjective in these categories; consequently, there has been little progress on the development of a homology theory using flat objects.

Thus, it is of interest to find another class of objects that makes the construction of a nice theory of homology in the category of discrete G-modules possible for any profinite group G. In this setting, we have found that the class of torsion free discrete G-modules (where torsion free means abelian groups) satisfies the necessary premises to get the nice homology theory we are pursuing; besides, it is also a very natural class with which to work. Furthermore, when G is finite, torsion free (discrete) G-modules are precisely the Gorenstein flat G-modules, which makes this theory much more attractive.

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By [**5,** Definition 5.1.1], given any category $\mathcal{C}$ and a class of objects $\mathcal{F}$ closed under isomorphisms, an $\mathcal{F}$-precover of an object X of $\mathcal{C}$ is a morphism $\varphi : F \to X$ with $F \in \mathcal{F}$ such that $\mathrm{Hom}(A, F) \to \mathrm{Hom}(A, X)$ is surjective for all $A \in \mathcal{F}$. If, in addition, any $f : F \to F$ with $\varphi \circ f = \varphi$ is an automorphism of F, then φ is said to be an $\mathcal{F}$-cover of X.

Note that:

(i) if $\mathcal{F}$ is the class of all injective (projective, Gorenstein injective, etc.) objects, then $\mathcal{F}$-(pre)covers are referred to as injective (projective, Gorenstein injective, etc.) (pre)covers.
(ii) $\mathcal{F}$-(pre)covers may not exist, and, if they do exist, they can be 0, for instance, the injective cover of any finite abelian group is 0. Clearly, an $\mathcal{F}$-cover of X (when it exists) is unique up to isomorphism.
(iii) $\mathcal{F}$-(pre)envelopes are defined dually.

Recall that, given a class of objects $\mathcal{F}$ of a category, a right $\mathcal{F}$-resolution of X is a complex

$$0 \longrightarrow X \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots ,$$

such that the sequence

$$\cdots \longrightarrow \mathrm{Hom}(F^1, F) \longrightarrow \mathrm{Hom}(F^0, F) \longrightarrow \mathrm{Hom}(X, F) \longrightarrow 0$$

is exact for any F of $\mathcal{F}$. We label these types of complexes $\mathrm{Hom}(-, \mathcal{F})$-*exact complexes*. When $\mathcal{I}$ is the class of all injective objects, a right $\mathcal{I}$-resolution is simply called an *injective resolution*.

Note that:

(i) a right $\mathcal{F}$-resolution of X is said to be *minimal*, provided that the maps $\mathrm{coker}(F^n \to F^{n+1}) \to F^{n+2}$, $n \geq -1$, are all $\mathcal{F}$-envelopes, where we use $F^{-1} = X$.
(ii) Dually (minimal) left $\mathcal{F}$-resolutions are defined, see [**5,** Definition 8.1.2].
(iii) A complete $\mathcal{F}$-resolution of X is exact and a $\mathrm{Hom}(\mathcal{F}, -)$-exact complex

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots ,$$

with each $F_i,\ F^j \in \mathcal{F}$ and $M = \ker(F^0 \to F^1)$.

Given any profinite group G, recall that a given G-module C is discrete, provided that $C = \cup_{N \trianglelefteq_O G} C^N$, where, for each open normal subgroup N, the module C^N denotes the fixed submodule of C under the action of N, that is,

$$C^N = \{c \in C;\ g \cdot c = c \text{ for all } g \in N\}.$$

We denote the category of discrete G-modules by DMod(G); more information about this category may be found in [**9, 11**]. It is well known that DMod(G) is a Grothendieck category so, with enough injectives and that the set $\{\mathbb{Z}[G/N]; N \trianglelefteq_O G\}$ generates the whole category DMod(G), [**8,** Proposition 4], we see that DMod(G) is actually locally Noetherian, and then injective covers always exist, [**4,** Theorems 2.17, 2.18]. Moreover, the existence of Gorenstein injective covers and envelopes has been studied [**4**], where it is shown that Gorenstein injective covers and envelopes always exist whenever G has finite virtual cohomological dimension, see [**12**] for a definition of the virtual cohomological dimension of a group.

The category DMod(G) does not have enough projectives, so the study of homological algebra must be restricted to injectives. We then wish to find a good substitute of projectives that lead to the development of a nice relative homological algebra in this category. As such, a substitute we will consider is the class of torsion free discrete modules, that is, discrete modules that are torsion free as abelian groups. The aim of Section 2 is to prove the existence of torsion free covers, which will be illustrated with several examples.

Section 3 is then devoted to the study of the existence of cotorsion envelopes (being a discrete module C cotorsion provided that $\mathrm{Ext}_{\mathrm{DMod(G)}}(F, C) = 0$ for every torsion free discrete F), and the structure of cotorsion discrete modules under certain circumstances.

2. Torsion free covers. This section is devoted to proving that torsion free covers always exist in DMod(G), where *torsion free* means torsion free as an abelian group. For this, we will prove that Enochs' reduction of the problem in the case of torsion free abelian groups, [**1**], later proved in the more general case of hereditary torsion theories in R-Mod by Teply [**13**], holds in our specific setting of discrete G-modules.

Proposition 2.1. *The injective envelope in* DMod(G) *of any torsion free discrete G-module is torsion free.*

Proof. If $E(F)$ were not torsion free we could find a nonzero element $x \in E(F)$ and a nonzero integer $z \in \mathbb{Z}$ such that $zx = 0$. But, $E(F)$ is an essential extension of F in DMod(G), so we can find $r \in \mathbb{Z}[G]$ such that $0 \neq rx \in F$ and then rx is torsion free. Now, $zrx = rzx = 0$, a contradiction. $\square$

Theorem 2.2. *Every discrete G-module has a torsion free cover.*

Proof. Direct limits of torsion free discrete modules are torsion free. Thus, by [**14,** Theorem 2.2.12], it suffices to find torsion free precovers.

If we find a torsion free precover for every injective discrete module, then every discrete module will have a torsion free precover. If we take any discrete M, its injective envelope $E(M)$, and a torsion free precover F of $E(M)$, we construct the pullback

$$\begin{tikzcd} F' \arrow[r, tail] \arrow[d, "\varphi"'] & F \arrow[d] \\ M \arrow[r, tail] & E(M) \end{tikzcd}$$

and show that F' is torsion free since it is a subgroup of F. Thus, by the pullback properties, φ is a torsion free precover of M.

However, when E is injective, to find a torsion free precover it suffices to find an $\mathcal{F}I$-precover, $\mathcal{F}I$ being the class of all torsion free injective discrete modules, since, if $\varphi : FI \to E$ is such a precover and $f : F \to E$ is any homomorphism with F torsion free, we can take the injective envelope $F \to E(F)$ and the induced homomorphism $g : E(F) \to E$ by the injectivity of E. Then, since $E(F)$ is torsion free, g factors through $h : E(F) \to FI$. It is then immediate to check that the diagram

$$\begin{tikzcd} & F \arrow[dl, "h"'] \arrow[d, "f"] \\ FI \arrow[r, "\varphi"'] & E \end{tikzcd}$$

is commutative and so φ is a torsion free precover.

Thus, if E is an injective discrete module, and we let I be a set of representatives of all indecomposable torsion free injective discrete modules, we see that the natural map

$$\varphi : \bigoplus_{F \in I} F^{(\mathrm{Hom}_{\mathbb{Z}[G]}(F,E))} \longrightarrow E$$

is a torsion free precover. $\square$

Remark 2.3. It is easy to see that, if M is a discrete $\mathbb{Z}[G]$-module with a torsion free cover F in $\mathbb{Z}[G]$-Mod, then its torsion free cover in DMod(G) is given by the discrete torsion submodule $t(F)$ of F.

Now we consider the abelian groups $\mathbb{Z}/(p^n)$ (p a prime) as discrete modules over a profinite group with the trivial action and give their torsion free covers. With this purpose in mind, recall that injective modules in DMod(G) may be characterized as those E such that E^N (the fixed submodule of E under the action of N) is an injective $\mathbb{Z}[G/N]$-module for every open normal subgroup $N \trianglelefteq_O G$ [**4,** Proposition 2.1].

Proposition 2.4. *If $p \nmid |G/U|$ for all $U \trianglelefteq_O G$, then the injective envelope of $\mathbb{Z}/(p^n)$ in* DMod(G) *is $\mathbb{Z}_{p^\infty}$, with trivial action.*

Proof. Clearly, we only need to prove that $\mathbb{Z}_{p^\infty}$ is injective in DMod(G) since $E(\mathbb{Z}/(p^n)) = \mathbb{Z}_{p^\infty}$ as abelian groups.

The action of G on $\mathbb{Z}_{p^\infty}$ is trivial so $(\mathbb{Z}_{p^\infty})^U = \mathbb{Z}_{p^\infty}$ for all $U \trianglelefteq_O G$, and then, if any $\mathbb{Z}[G/U]$-homomorphism $f : I \to \mathbb{Z}_{p^\infty}$ can be extended to $\mathbb{Z}[G/U]$ for any ideal $I \leq \mathbb{Z}[G/U]$, $\mathbb{Z}_{p^\infty}$ will be injective in DMod(G) by [**4,** Proposition 2.1]. However, $\mathbb{Z}_{p^\infty}$ is an injective abelian group, so there is an abelian group extension h of f to $\mathbb{Z}[G/U]$.

Now, $p \nmid |G/U|$ means that multiplication by $|G/U|$ is an automorphism of $\mathbb{Z}_{p^\infty}$, so, if we let $1/|G/U|$ be its inverse, then the map $\overline{h} : \mathbb{Z}[G/U] \to \mathbb{Z}_{p^\infty}$ given by

$$\overline{h}(x) = \frac{1}{|G/U|}\Bigg(\sum_{g \in G/U} h(gx)\Bigg),$$

will clearly be a $\mathbb{Z}[G/U]$-extension of f to $\mathbb{Z}[G/U]$. $\square$

More generally, we have the next corollary.

Corollary 2.5. *Let $G = \lim_{\leftarrow i\in I} G/N_i$ be a profinite group. Any divisible abelian group, in which the multiplication by $|G/N_i|$ is an automorphism for all $i \in I$ (for instance, $\mathbb{Q}$) is an injective discrete module with trivial action. Thus, the injective envelope of any abelian group M (thought of as a discrete G-module with the trivial action) coincides with its injective envelope in* $\mathbb{Z}$-Mod *when the multiplication by $|G/N_i|$ is an automorphism of M.*

Proposition 2.6. *If $G = \lim_{\leftarrow i\in I} G/N_i$ is any profinite group such that, for all i, $p \nmid |G/N_i|$, then the torsion free cover $\varphi : \widehat{\mathbb{Z}}_p \to \mathbb{Z}/(p^n)$ in* $\mathbb{Z}$-Mod *is a torsion free cover in* DMod(G) *for every n.*

Proof. Let X be any torsion free discrete G-module, and let $f : X \to \mathbb{Z}/(p^n)$ be any $\mathbb{Z}[G]$-homomorphism. Consider the submodule

$$T = \{\sigma \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Q}, \mathbb{Z}_{p^\infty});\ \sigma(1) \in \mathbb{Z}/(p^n)\}$$

(both $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty}$ with trivial action) and the $\mathbb{Z}[G]$-homomorphism $h : T \to \mathbb{Z}/(p^n)$ given by $h(\sigma) = \sigma(1)$.

Since $\mathbb{Z}_{p^\infty}$ is the injective envelope of $\mathbb{Z}/(p^n)$ in DMod(G), we have the diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "f"'] & X \otimes_{\mathbb{Z}} \mathbb{Q} \\ \mathbb{Z}/(p^n) \arrow[r] & \mathbb{Z}_{p^\infty} \end{tikzcd}$$

(where $X \otimes_{\mathbb{Z}} \mathbb{Q}$ is a discrete G-module with the diagonal action $g(a \otimes b) = ga \otimes gb$ and $\psi(x) = x \otimes 1$) which can be completed commutatively by a G-morphism $\phi : X \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Z}_{p^\infty}$.

Now, we know that

$$\mathrm{Hom}_{\mathbb{Z}[G]}(X \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Z}_{p^\infty}) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(X, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_{p^\infty}))$$

are abelian groups in such a way that

$$\phi \longmapsto \phi' \quad \text{and} \quad \phi'(x)(r) = \phi(x \otimes r).$$

Thus,

$$\phi'(1) = \phi(x \otimes 1) = f(x) \in \mathbb{Z}/(p^n),$$

that is, $\phi'(x) \in T$ for all $x \in X$.

Finally, $(h\phi')(x) = \phi(x \otimes 1) = f(x)$, so the diagram

$$\begin{tikzcd} & X \arrow[dl, "\phi'"'] \arrow[d, "f"] \\ T \arrow[r, "h"'] & \mathbb{Z}/(p^n) \end{tikzcd}$$

is commutative, and this means that h is a torsion free precover in DMod(G).

However, by [**5,** Example 4.3.3], we know that $T \cong \widehat{\mathbb{Z}}_p$ (both as abelian groups and as discrete G-modules since the action of G on both is trivial), so T is indecomposable since $\widehat{\mathbb{Z}}_p$ is a torsion free cover as an abelian group. Thus, $h : T \to \mathbb{Z}/(p^n)$ is a torsion free cover in DMod(G), and then in $\mathbb{Z}$-Mod, since the action of G on $\mathbb{Z}/(p^n)$ and on T are both trivial. $\square$

Example 2.7. We will show that the condition $p \nmid |G/N_i|$ for all i is not trivial and that, indeed, it cannot be dropped in general.

Let $G = \{1, g\}$, $g^2 = 1$, and consider $\mathbb{Z}$ as a discrete G-module with the action $gz = -z$. Thus, the canonical projection $p : \mathbb{Z} \to \mathbb{Z}/(2)$ is a G-morphism, so if $\varphi : \widehat{\mathbb{Z}}_2 \to \mathbb{Z}/(2)$ were a torsion free precover in DMod(G), then we could complete the diagram

$$\begin{tikzcd} & \mathbb{Z} \arrow[dl, "h"'] \arrow[d, "p"] \\ \widehat{\mathbb{Z}}_2 \arrow[r, "\varphi"'] & \mathbb{Z}/(2) \end{tikzcd}$$

commutatively by a G-morphism h. But, then

$$-h(1) = h(-1) = h(g \cdot 1) = g \cdot h(1) = h(1),$$

and this means that $2h(1) = 0$, which is impossible since $\widehat{\mathbb{Z}}_2$ is a torsion free abelian group.

The next result is an easy generalization of Proposition 2.6. The proof uses Corollary 2.5.

Proposition 2.8. *Let* $G = \lim_{\leftarrow i\in I} G/N_i$ *be a profinite group and* $\psi : F \to M$ *a torsion free cover in* $\mathbb{Z}$-Mod *such that the multiplication by* $|G/N_i|$ *is an automorphism of* M *for all* $i \in I$. *Then,* $\psi : F \to M$ *is a torsion free cover in* DMod(G) (*both* F *and* M *with the trivial action*).

3. Cotorsion discrete modules. Given any finite group T, it is well known that a Gorenstein flat $\mathbb{Z}[T]$-module is nothing more than a torsion free T-module. However, nothing is known about the connection between Gorenstein flat and torsion free discrete modules when the group is profinite.

The class of cotorsion $\mathbb{Z}[T]$-modules for a finite group T has been studied and shown to have nice properties [**2**]. Our purpose in this section is the translation of this problem to the category of discrete modules over a profinite group, where the class of cotorsion discrete modules will be understood as the right orthogonal class of that consisting of all torsion free discrete modules. Thus, from now on, we will let $\mathcal{F}$ denote the class of all torsion free discrete modules, so

$$\mathcal{F}^{\perp} = \{C \in \mathrm{DMod(G)};\ \mathrm{Ext}^1_{\mathrm{DMod(G)}}(F, C) = 0 \text{ for all } F \in \mathcal{F}\}$$

will be the class of cotorsion discrete modules.

Next, we prove the existence of cotorsion envelopes.

Proposition 3.1. $(\mathcal{F}, \mathcal{F}^{\perp})$ *is a complete cotorsion pair.*

Proof. Given $M \in {}^{\perp}(\mathcal{F}^{\perp})$, consider the exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where $F \to M$ is a torsion free cover, so $K \in \mathcal{F}^{\perp}$ by Wakamatsu's lemma, see for instance, [**14,** Lemma 2.1.1]. Thus, the sequence splits, and hence, $M \in \mathcal{F}$. Therefore, $(\mathcal{F}, \mathcal{F}^{\perp})$ is actually a cotorsion pair which clearly has enough projectives.

In order to prove the existence of enough injectives we use Salce's lemma [**10**]. Let M be any discrete module, let $F \to E(M)/M$ be a torsion free cover and consider the pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d] & F \arrow[d] \\ E(M) \arrow[r] & E(M)/M \end{tikzcd}$$

of $E(M) \to E(M)/M$ and $F \to E(M)/M$. Then, we obtain an exact sequence

$$0 \longrightarrow M \longrightarrow X \longrightarrow F \longrightarrow 0,$$

with $X \in \mathcal{F}^{\perp}$ and $F \in \mathcal{F}$. $\square$

Now, $\mathcal{F}$ is closed under direct limits, so we actually have $\mathcal{F}^{\perp}$-envelopes by [**7,** Theorem 3.1].

Corollary 3.2. *Every discrete module has a cotorsion envelope.*

We shall now provide two interesting relations between cotorsion discrete G-modules and H-modules when $H \leq G$ is a closed subgroup.

Proposition 3.3. *If U is any open subgroup of G, then every cotorsion discrete G-module C is cotorsion in* $\mathrm{DMod}(U)$.

Proof. By [**12,** Proposition 1.3.2], we know that the coinduction functor Coind_U^G is left and right adjoint to the forgetful functor, and it is clear that, if F is torsion free in $\mathrm{DMod}(U)$, then $\mathrm{Coind}_U^G(F)$ is torsion free in DMod(G). Hence, we have

$$0 = \mathrm{Ext}^1_{\mathrm{DMod(G)}}(\mathrm{Coind}_U^G(F), C) \cong \mathrm{Ext}^1_{\mathrm{DMod}(U)}(F, C),$$

and we see that C is cotorsion in $\mathrm{DMod}(U)$. $\square$

Proposition 3.4. *If H is any closed subgroup of G and C is any cotorsion discrete H-module, then* $\mathrm{Coind}_H^G(C)$ *is cotorsion in* DMod(G). *In particular, if C is any cotorsion* (*and torsion free*) *abelian group, then* $\mathrm{Coind}_{\{e\}}^G(C)$ *is cotorsion* (*and torsion free*) *in* DMod(G).

Proof. Let F be a torsion free discrete G-module. By [**12,** Proposition 1.3.1], we know that

$$\mathrm{Hom}_{\mathrm{DMod}(H)}(F,C) \cong \mathrm{Hom}_{\mathrm{DMod(G)}}(F, \mathrm{Coind}_H^G(C)).$$

But, Coind_H^G is exact and preserves injectives so we have

$$0 = \mathrm{Ext}^1_{\mathrm{DMod}(H)}(F,C) \cong \mathrm{Ext}^1_{\mathrm{DMod(G)}}(F, \mathrm{Coind}_H^G(C)).$$

Therefore, $\mathrm{Coind}_H^G(C)$ is cotorsion in DMod(G). □

It turns out that there is a nice connection between cotorsion discrete G-modules and Gorenstein cotorsion $\mathbb{Z}[G/U]$-modules for every $U \trianglelefteq_O G$. Recall that a $\mathbb{Z}[G/U]$-module N is said to be Gorenstein cotorsion if $\mathrm{Ext}^1(L,N) = 0$ for all L Gorenstein flat.

Lemma 3.5. *Let C be in* DMod(G)*, and suppose that C^U is Gorenstein cotorsion in $\mathbb{Z}[G/U]$-*Mod *for all $U \trianglelefteq_O G$. Then,* $\mathrm{H}^1(H,C) = 0$ *for any closed subgroup H of G.*

Proof. By [**2,** Proposition 2.5], C^U is Gorenstein cotorsion in $\mathbb{Z}[H/U \cap H]$-Mod. Then, by [**9,** Proposition 6.5.5], we have

$$\begin{aligned} \mathrm{H}^1(H,C) &= \lim_{\to U} \mathrm{H}^1(H/H \cap U, C^U) \\ &= \lim_{\to U} \mathrm{Ext}^1_{\mathbb{Z}[H/H\cap U]}(\mathbb{Z}, C^U) \\ &= 0. \end{aligned}$$ □

Proposition 3.6. *Let C be a discrete G-module. Then, C is cotorsion discrete if and only if C^U is Gorenstein cotorsion in $\mathbb{Z}[G/U]$-*Mod *for all $U \trianglelefteq_O G$ and*

$$\lim_{\leftarrow U}{}^1 \mathrm{Hom}_{\mathbb{Z}[G/U]}(F^U, C) = 0 \quad \textit{for all } F \in \mathcal{F}.$$

Proof. Let

$$0 \longrightarrow C \longrightarrow E(C) \longrightarrow L \longrightarrow 0$$

be exact in DMod(G) with $E(C)$ the injective envelope of C. By Proposition 3.3, we know that C is cotorsion in DMod(U) for any $U \leq_O G$, so by Lemma 3.5, $\mathrm{H}^1(U,C) = 0$ for all $U \trianglelefteq_O G$. Therefore, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C^U \arrow[r] \arrow[d] & E(C)^U \arrow[r] \arrow[d] & L^U \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r] & E(C) \arrow[r] & L \arrow[r] & 0.
\end{tikzcd}$$

Now, for any torsion free $\mathbb{Z}[G/U]$-module (Gorenstein flat) F, we have

$$\mathrm{Hom}_{\mathrm{DMod(G)}}(F, M) \cong \mathrm{Hom}_{\mathbb{Z}[G/U]}(F, M^U)$$

for any discrete M. Applying the functor $\mathrm{Hom}_{\mathrm{DMod(G)}}(F, -)$ to the above diagram, by [**4,** Proposition 2.1], we obtain that the induced map

$$\mathrm{Ext}^1_{\mathbb{Z}[G/U]}(F, C^U) \longrightarrow \mathrm{Ext}^1_{\mathrm{DMod(G)}}(F, C) = 0$$

is an isomorphism. Thus, $C^U \in \mathbb{Z}[G/U]$-Mod is Gorenstein cotorsion for any $U \trianglelefteq_O G$.

The sequences

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G/U]}(F^U, C) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G/U]}(F^U, E(C)) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G/U]}(F^U, L) \longrightarrow 0$$

are all exact, so computing the inverse limit, we obtain an exact sequence

$$\cdots \longrightarrow \varprojlim_{U} \mathrm{Hom}_{\mathbb{Z}[G/U]}(F^U, L) \longrightarrow \varprojlim_{U}{}^1 \mathrm{Hom}_{\mathbb{Z}[G/U]}(F^U, C) \longrightarrow 0.$$

On the other hand, the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{DMod(G)}}(F, C) \longrightarrow \mathrm{Hom}_{\mathrm{DMod(G)}}(F, E(C)) \longrightarrow \mathrm{Hom}_{\mathrm{DMod(G)}}(F, L) \longrightarrow 0$$

is also exact and

$$\mathrm{Hom}_{\mathrm{DMod(G)}}(F, M) \cong \varprojlim_{U} \mathrm{Hom}_{\mathbb{Z}[G/U]}(F^U, M),$$

for any discrete module M, so we obtain

$$\varprojlim_{U}{}^1 \mathrm{Hom}_{\mathbb{Z}[G/U]}(F^U, C) = 0.$$

Conversely, let

$$0 \longrightarrow C \longrightarrow V \longrightarrow L \longrightarrow 0$$

be exact in DMod(G) with $V \in \mathcal{F}^{\perp}$ and $L \in \mathcal{F}$, and choose any $F \in \mathcal{F}$. Then, the sequences

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G/U]}(F^U, C) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}[G/U]}(F^U, V) \\ \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G/U]}(F^U, L) &\longrightarrow \mathrm{Ext}^1_{\mathbb{Z}[G/U]}(F^U, C) \longrightarrow 0 \end{aligned}$$

are all exact. However,

$$\mathrm{Hom}_{\mathbb{Z}[G/U]}(F^U, M) \cong \mathrm{Hom}_{\mathbb{Z}[G/U]}(F^U, M^U),$$

for all M in DMod(G) and

$$\mathrm{Ext}^1_{\mathbb{Z}[G/U]}(F^U, V^U) = 0,$$

by the necessary condition that V is cotorsion. It follows that

$$\mathrm{Ext}^1_{\mathbb{Z}[G/U]}(F^U, C) \cong \mathrm{Ext}^1_{\mathbb{Z}[G/U]}(F^U, C^U) = 0.$$

Finally, using a similar argument to that of the necessary condition, we obtain

$$\mathrm{Ext}^1(F, C) \cong \varprojlim_U{}^1 \mathrm{Hom}(F^U, C) = 0. \quad \square$$

Given a prime number p, completions of free $\widehat{\mathbb{Z}}_p$-modules are usually denoted as T_p. The next result generalizes [**2,** Proposition 2.6].

Proposition 3.7. *Let p be a prime number and M a discrete $\mathbb{Z}[G]$-module. If $M^U \cong T_p$ (as abelian groups) for all $U \trianglelefteq_O G$, and there exists an open subgroup $P \leq G$ with $p \nmid |G/P|$ (for instance, if G has an open Sylow p-subgroup), such that M is cotorsion in* DMod(P)*, then M is cotorsion in* DMod(G).

Proof. Let

$$0 \longrightarrow M \longrightarrow X \longrightarrow L \longrightarrow 0$$

be exact in DMod(G) with $X \in \mathcal{F}^{\perp}$ and L torsion free. Then the sequence splits in DMod(P). Let $f : X \to M$ be a $\mathbb{Z}[P]$-linear retraction of $M \to X$. Since $p \nmid |G/P|$, $M \xrightarrow{\cdot|G/P|} M$ is an isomorphism of G-modules, so if we call $1/|G/P|$ its inverse, then the standard average morphism

$$\overline{f}(x) = \frac{1}{|G/P|} \sum_{g \in G/P} g f(g^{-1}x)$$

is a $\mathbb{Z}[G]$-linear retraction of $M \to X$. $\square$

Theorem 3.8. *Every cotorsion and torsion free discrete G-module C can be written, uniquely up to isomorphism, as a direct sum $C = V \oplus L$ of an injective discrete G-module V and a cotorsion reduced discrete G-module L, which is also reduced as an abelian group. Moreover, for any $U \trianglelefteq_O G$, the $\mathbb{Z}[G/U]$-module L^U is a product*

$$L^U = \prod T_p \quad \textit{in } \mathbb{Z}[G/U]\text{-}\mathrm{Mod},$$

in such a way that each T_p is the p-adic completion of a free $\mathbb{Z}[P/U]$-module for some (any) Sylow p-subgroup $P/U \leq G/U$. Each T_p is uniquely determined up to isomorphism by the $(\mathbb{Z}/(p))[G/U]$-module T_p/pT_p.

Proof. Let V be the largest divisible abelian subgroup of C. Then, V is a discrete G-submodule of C since gV is a divisible abelian group for all $g \in G$.

Now, for any $U \trianglelefteq_O G$, given $z \in \mathbb{Z}$ and $x \in V^U$, we know that there exists $y \in V$ such that $x = zy$. However, if $u \in U$, then we have $nuy = ux = x \Rightarrow n(uy - y) = 0$, so $uy - y = 0$ since V^U is torsion free and then $y \in V^U$. We see that V^U is divisible as an abelian group. Then, by [**2,** Lemma 2.8], we have that V^U is an injective $\mathbb{Z}[G/U]$-module. Therefore, V is injective in DMod(G) so $C = V \oplus L$ in DMod(G) with L reduced.

By Proposition 3.6, we know that L^U is Gorenstein cotorsion in $\mathbb{Z}[G/U]$-Mod. If each L^U were reduced, then the result would follow by [**2,** Theorem 2.9]. But, if $T \subseteq L^U$ is injective in $\mathbb{Z}[G/U]$-Mod, then T is divisible, and, since $T \subset L$, we have $T = 0$. $\square$

Now, we are able to describe the structure of finitely generated cotorsion modules in DMod(G).

Theorem 3.9. *Let $C \in \mathrm{DMod(G)}$ be finitely generated and cotorsion. Then,*

$$C = C_1 \oplus \cdots \oplus C_n$$

in $\mathrm{DMod(G)}$ *where $C_i \cong (\mathbb{Z}/(p_i^{n_i}))^{m_i}$ as abelian groups for some primes p_i.*

Proof. C is finitely generated, $C = C^U$ for some $U \trianglelefteq_O G$, and this means that C is Gorenstein cotorsion in $\mathbb{Z}[G/U]$-Mod by Proposition 3.6. Hence, by [**2,** Proposition 2.5], C is cotorsion in $\mathbb{Z}$-Mod.

Now, C is finitely generated as an abelian group by [**12,** Lemma 1.2.6]. Applying [**3,** Proposition 2.7], we obtain that the torsion abelian subgroups $t(C)$ and $C/t(C)$ both are cotorsion abelian groups, and then, $C/t(C) \cong \mathbb{Z}^n$ must be 0. Therefore, C is a torsion abelian group so $C \cong C_1 \oplus \cdots \oplus C_n$ where $C_i \cong (\mathbb{Z}/(p_i^{n_i}))^{m_i}$ (as abelian groups) for some primes p_i. Then, each C_i is the p_i-primary part of C, which is a discrete $\mathbb{Z}[G]$-submodule; thus, the direct sum decomposition lies in $\mathrm{DMod(G)}$. □

We now see that the cotorsion discrete $\mathbb{Z}[G]$-modules are not necessarily the G-modules whose underlying abelian group is cotorsion.

Proposition 3.10. *If p is a prime number and $(\mathbb{Z}/(p^n))^m$ is the underlying abelian group of a cotorsion discrete $\mathbb{Z}[G]$-module C, then there exists $U \trianglelefteq_O G$ such that $|P/U| \mid m$ for some (any) Sylow p-subgroup $P/U \leq G/U$.*

Proof. Let $C = (\mathbb{Z}/(p^n))^m$. Since C is finitely generated in $\mathrm{DMod(G)}$, then $C = C^U$ for some $U \trianglelefteq_O G$, and then, $C \in \mathbb{Z}[G/U]$-Mod is Gorenstein cotorsion, see Proposition 3.6.

Let $\psi : F \to C$ be the torsion free cover of C in $\mathbb{Z}[G/U]$-Mod. By [**2,** Lemma 3.1], $\mathrm{Ker}(\psi)$ and F are the completion of free $\mathbb{Z}[P/U]$-modules, T_p's, for some Sylow p-subgroup $P/U \leq G/U$. Therefore, we easily deduce that

$$C \cong \frac{T_p}{p^n T_p} \cong \frac{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[P/U], U_p)}{p^n \,\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[P/U], U_p)}$$

$$\cong \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}[P/U], \frac{U_p}{p^n U_p}\right) \cong (\mathbb{Z}/(p^n))^{|I|\cdot|P/U|},$$

where $U_p \cong \widehat{\mathbb{Z}}_p^{(I)}$. Hence, $m = |I| \cdot |P/U|$. □

Then, the next corollary follows immediately.

Corollary 3.11. *If $(\mathbb{Z}/(p^n))^m$ is the underlying abelian group of a cotorsion discrete G-module whose G-action is not trivial and $p \nmid m$, then $p \nmid |G|$.*

REFERENCES

1. E. Enochs, *Torsion free covering modules*, Proc. Amer. Math. Soc. **14** (1963), 884–890.

2. E. Enochs, S. Estrada and B. Torrecillas, *Gorenstein flat covers and Gorenstein cotorsion modules over integral group rings*, Alg. Rep. Theory **8** (2005), 525–539.

3. E. Enochs, J.R. García Rozas and L. Oyonarte, *Finitely generated cotorsion modules*, Proc. Edinburgh Math. Soc. **44** (2001), 43–152.

4. E. Enochs, J.R. García Rozas, L. Oyonarte and B. Torrecillas, *On Gorenstein injective discrete modules over profinite groups*, Acta Math. Hung. **142** (2014), 296–316.

5. E. Enochs and O. Jenda, *Relative homological algebra*, de Gruyter Expos. Math. **30**, Walter de Gruyter, Berlin, 2000.

6. E. Enochs and E. Khan, *Flat discrete modules over profinite groups*, Comm. Algebra **42** (2014), 2331–2337.

7. E. Enochs and L. Oyonarte, *Flat covers and cotorsion envelopes of sheaves*, Proc. Amer. Math. Soc. **130** (2002), 1285–1292.

8. J.J. Martínez, *Cohomological dimension of discrete modules over profinite groups*, Pacific J. Math. **49** (1973), 185–189.

9. L. Ribes and P. Zalesskii, *Profinite groups*, Springer-Verlag, Berlin, 2000.

10. L. Salce, *Cotorsion theories for abelian groups*, Sympos. Math. **23** (1979), 11–32.

11. J.P. Serre, *Cohomologie Galoisienne*, Lect. Notes Math. **5**, Springer-Verlag, Berlin, 1997.

12. C. Studer-De Boer, *Tate cohomology for profinite groups*, Ph.D. Dissertation, Math. Wissen., Zürich, 2001.

13. M.L. Teply, *Torsionfree injective modules*, Pacific J. Math. **28** (1969), 441–453.

14. J. Xu, *Flat covers of modules*, Lect. Notes Math. **1634**, Springer-Verlag, Berlin, 1996.

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