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INVERSE SEMIGROUP ACTIONS ON GROUPOIDS

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ABSTRACT. We define inverse semigroup actions on topological groupoids by partial equivalences. From such actions, we construct saturated Fell bundles over inverse semigroups and non-Hausdorff étale groupoids. We interpret these as actions on C^*-algebras by Hilbert bimodules and describe the section algebras of these Fell bundles.

Our constructions give saturated Fell bundles over non-Hausdorff étale groupoids that model actions on locally Hausdorff spaces. We show that these Fell bundles are usually not Morita equivalent to an action by automorphisms, that is, the Packer-Raeburn stabilization trick does not generalize to non-Hausdorff groupoids.

1. Introduction. Two of the most obvious actions of a groupoid G are those by left and right translations on its arrow space G^1. If G is Hausdorff, they induce continuous actions of G on the C*-algebra $C_0(G^1)$. What happens if G is non-Hausdorff?

Let G be a non-Hausdorff, étale groupoid with Hausdorff, locally compact object space G^0. Then G^1 is locally Hausdorff, that is, it has an open covering $\mathfrak{U} = (U_i)_{i\in I}$ by Hausdorff subsets: we may choose U_i so that the range and source maps restrict to homeomorphisms from U_i onto open subsets of the Hausdorff space G^0.

The covering $\mathfrak{U}$ yields an étale, locally compact, Hausdorff groupoid H with object space $H^0 := \bigsqcup_{i\in I} U_i$, arrow space $H^1 := \bigsqcup_{i,j\in I} U_i \cap U_j$, range and source maps $r(i,j,x) := (i,x)$ and $s(i,j,x) := (j,x)$, and multiplication $(i,j,x)\cdot(j,k,x) = (i,k,x)$. The groupoid H is known as the *Čech groupoid* for the covering $\mathfrak{U}$. In noncommutative geometry, we view the groupoid C*-algebra $C^*(H)$ as the algebra of functions on the

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non-Hausdorff space G^1. Is there some kind of action of G on $\mathrm{C}^*(H)$ that corresponds to the translation action of G on G^1?

There is no action of G on $\mathrm{C}^*(H)$ in the usual sense because there is no action of G on H by automorphisms. The problem is that arrows $g \in G^1$ have many liftings $(i, g) \in H^0$. To let $g \in G$ act on H, we must choose $k \in I$ with $gh \in U_k$ for $h \in U_j$ with $r(h) = s(g)$. It may, however, be impossible to choose k continuously when h varies in U_j. This article introduces *actions by partial equivalences* in order to make sense of the actions of G on H and $\mathrm{C}^*(H)$.

First, we replace G by its inverse semigroup of bisections $S = \mathrm{Bis}(G)$. This inverse semigroup cannot act on H by partial groupoid *isomorphisms* for the same reasons as above. It does, however, act on H by partial *equivalences* because the equivalence class of H is independent of the covering, see also [**16,** Lemma 4.1]; thus, partial homeomorphisms on G^1 lift to partial equivalences of H in a canonical way. We will see that an S-action by partial equivalences on a Čech groupoid for a locally Hausdorff space Z is equivalent to an S-action on Z by partial homeomorphisms.

Let S act on a groupoid H by partial equivalences. Then we build a *transformation groupoid* $H \rtimes S$. Special cases of this construction are the groupoid of germs for an action of S on a space by partial homeomorphisms, the semidirect product for a group(oid) action on another group(oid) by automorphisms, and the linking groupoid of a single Morita-Rieffel equivalence. The original action is encoded in the transformation groupoid $L := H \rtimes S$ and open subsets $L_t \subseteq L$ with

$$L_t \cdot L_u = L_{tu}, \qquad L_t^{-1} = L_{t^*},$$
$$L_t \cap L_u = \bigcup_{v \le t,u} L_v, \qquad L^1 = \bigcup_{t\in S} L_t$$

and $H = L_1$. We call such a family of subsets an *S-grading* on L with *unit fiber H*. Any S-graded groupoid is a transformation groupoid for an essentially unique action of S by partial equivalences on its unit fiber. This is a very convenient characterization of actions by partial equivalences.

An action of an inverse semigroup S on H by partial equivalences cannot induce, in general, an action of S on $\mathrm{C}^*(H)$ by partial automorphisms in the usual sense, as defined by Sieben [**33**]. But we do get an action by partial Morita-Rieffel equivalences, that is, by Hilbert bimodules. We show that actions of S by Hilbert bimodules are equivalent to (saturated) Fell bundles over S. Along the way, we also drastically simplify the definition of Fell bundles over inverse semigroups in [**13**]. Our approach clarifies in what sense a Fell bundle over an inverse semigroup is an "action" of the inverse semigroup on a C*-algebra.

In the end, we want an action of the groupoid G itself, not of the inverse semigroup $\mathrm{Bis}(G)$. For actions by automorphisms, Sieben and Quigg [**28**] characterize which actions of $\mathrm{Bis}(G)$ come from actions of G. We extend this characterization to Fell bundles: a Fell bundle over $\mathrm{Bis}(G)$ comes from a Fell bundle over G if and only if the restriction of the action to idempotents in $\mathrm{Bis}(G)$ commutes with suprema of arbitrarily large subsets. This criterion only works for $\mathrm{Bis}(G)$ itself. In practice, we may want to "model" G by a smaller inverse semigroup S such that $G^0 \rtimes S \cong G$. We characterize which Fell bundles over such S come from Fell bundles over G.

In particular, our action of $\mathrm{Bis}(G)$ on $\mathrm{C}^*(H)$ for the Čech groupoid associated to G^1 does come from an action of G, so we get Fell bundles over G that describe the left and the right translation actions on G^1. For these Fell bundles, we show that the section C*-algebras are Morita equivalent to $\mathrm{C}_0(G^0)$. More generally, for any principal G-bundle $X \to Z$, the section algebra of the Fell bundle over G that describes the action of G on a Čech groupoid for X is Morita-Rieffel equivalent to $\mathrm{C}_0(Z)$, just as in the more classical Hausdorff case (see Proposition 6.5).

For any action of an inverse semigroup S on a locally compact groupoid H by partial equivalences, we identify the section C*-algebra of the resulting Fell bundle over S with the groupoid C*-algebra of the transformation groupoid. In brief notation,

$$C^*(H) \rtimes S \cong C^*(H \rtimes S).$$

This generalizes the well-known isomorphism

$$\mathrm{C}_0(X) \rtimes S \cong C^*(X \rtimes S)$$

for inverse semigroup actions on Hausdorff locally compact spaces by partial homeomorphisms.

For a Hausdorff locally compact groupoid, any Fell bundle is equivalent to an ordinary action on a stabilization (Packer-Raeburn stabilization trick, see also [**6,** Proposition 5.2]). In contrast, our Theorem 7.1 shows that a non-Hausdorff groupoid has no action by automorphisms that describes its translation action on G^1. Thus, we really need Fell bundles to treat these actions of a non-Hausdorff groupoid.

Now we give an overview of the individual sections of the paper.

In Section 2, we study partial equivalences between topological groupoids. We show, in particular, that the involution that exchanges the left and right actions on a partial equivalence behaves like the involution in an inverse semigroup.

Section 3 introduces inverse semigroup actions by partial equivalences. We show that the rather simple-minded definition implies further structure, which is needed to construct the transformation groupoid. Once we know that actions by partial equivalences are essentially the same as S-graded groupoids, we treat many examples. This includes actions on spaces and Čech groupoids; in particular, an S-action on a space by partial homeomorphisms induces an action by partial equivalences on any Čech groupoid for a covering of the space. We describe a group action by (partial) equivalences as a kind of extension by the group. We show that any (locally) proper Lie groupoid is a transformation groupoid for an inverse semigroup action on a very simple kind of groupoid: a disjoint union of transformation groupoids of the form $V \rtimes K$, where V is a vector space, K a compact Lie group, and the K-action on V is by an $\mathbb{R}$-linear representation. This is meant as an example for gluing together groupoids along partial equivalences.

In Section 4, we define inverse semigroup actions on C^*-algebras by Hilbert bimodules. The theory is parallel to that for actions on groupoids by partial equivalences because both cases have the same crucial algebraic features. We show that actions by Hilbert bimodules are equivalent to saturated Fell bundles. This simplifies the original definition of Fell bundles over inverse semigroups in [**13**].

In Section 5, we turn inverse semigroup actions on groupoids by partial equivalences into actions on groupoid C^*-algebras by Hilbert bimodules. We do this in two different (but equivalent) ways, by using transformation groupoids and abstract functorial properties of our constructions. The approach using transformation groupoids suggests that the section C*-algebra of the resulting Fell bundle is simply the groupoid C*-algebra of the transformation groupoid:

$$\mathrm{C}^*(H) \rtimes S \cong \mathrm{C}^*(H \rtimes S).$$

We prove this, and a more general result, for Fell bundles over $H \rtimes S$.

In Section 6, we relate inverse semigroup actions to actions of corresponding étale groupoids. In particular, we characterize when an action of $\mathrm{Bis}(G)$ comes from an action of G. Finally, we can then treat our motivating example and turn a groupoid action on a locally Hausdorff space Z into a Fell bundle over the groupoid. We may also describe the section C*-algebra in this case, which plays the role of the crossed product. If the action is free and proper, then the result is Morita-Rieffel equivalent to $\mathrm{C}_0(Z/G)$. We also define "proper actions" of inverse semigroups on groupoids. We show that a free and proper action can only occur on a groupoid that is equivalent to a locally Hausdorff and locally quasi-compact space.

Section 7 shows that the translation action of a non-Hausdorff étale groupoid on its arrow space cannot be described by a groupoid action by automorphisms in the usual sense. Our previous theory shows, however, that we may describe such actions by groupoid Fell bundles. Thus, the no-go theorem in Section 7 shows that the Packer-Raeburn stabilization trick fails for non-Hausdorff groupoids, so Fell bundles are really more general than ordinary actions in that case.

In Section 8, we examine a very simple explicit example to illustrate the no-go theorem and to see how our main results avoid it.

Appendix A deals with topological groupoids, their actions on spaces and equivalences between them. The main point is to define principal bundles and (Morita) equivalence for *non-Hausdorff* groupoids in such a way that the theory works just as well as in the Hausdorff case. Among others, we show that a non-Hausdorff space is equivalent to a Hausdorff, locally compact groupoid if and only if it is locally Hausdorff and locally quasi-compact, answering a question in [**8**].

Appendix B contains a general technical result regarding upper semicontinuous fields of Banach spaces over locally Hausdorff spaces and uses it to prove $\mathrm{C}^*(H) \rtimes S \cong \mathrm{C}^*(H \rtimes S)$ for inverse semigroup actions on groupoids and a more general statement involving Fell bundles over $H \rtimes S$.

2. Partial equivalences. In this section and the next, we work in the category of topological spaces and continuous maps, without assuming spaces to be Hausdorff or locally compact. Appendix A shows how topological groupoids, their actions, principal bundles, and the equivalences between them should be defined so that the theory goes through smoothly without extra assumptions on the underlying topological spaces.

Our main applications deal with groupoids that have a Hausdorff, locally compact object space and a locally Hausdorff, locally quasi-compact arrow space. We care about actions of such groupoids G on locally Hausdorff spaces Z. It is very convenient to encode such an action by the transformation groupoid $G \ltimes Z$. Its object space Z is only locally Hausdorff. When we allow such topological groupoids, the usual definition of equivalence for topological groupoids breaks down because orbit spaces of proper actions are always Hausdorff, so the actions on an equivalence bispace cannot be proper unless the object spaces of the two groupoids are Hausdorff. Tu's definition [**36**] works; it is equivalent to what we do. But the theory becomes more elegant if we also drop the local compactness assumption and thus no longer use proper maps in our basic definitions. The replacement for free and proper actions are "basic" actions, which are characterized by the map

$$G \times_{s,G^0,r} X \longrightarrow X \times X, \qquad (g,x) \longmapsto (gx,x),$$

being a homeomorphism onto its image with the subspace topology from $X \times X$.

Readers already familiar with the usual theory of locally compact groupoids may read on and only turn to Appendix A in cases of doubt; they should note that range and source maps of groupoids are assumed to be open, whereas anchor maps of groupoid actions are not assumed open. Less experienced readers should read Appendix A first.

Definition 2.1. Let G and H be topological groupoids. A *partial equivalence* from H to G is a topological space with *anchor maps* $r\colon X \to G^0$ and $s\colon X \to H^0$ and multiplication maps $G^1 \times_{s,G^0,r} X \to X$ and $X \times_{s,H^0,r} H^1 \to X$, which we write multiplicatively, that satisfy the following conditions:

(P1) $s(g \cdot x) = s(x)$, $r(g \cdot x) = r(g)$ for all $g \in G^1$, $x \in X$ with $s(g) = r(x)$, and $s(x \cdot h) = s(h)$, $r(x \cdot h) = r(x)$ for all $x \in X$,

$h \in H^1$ with $s(x) = r(h)$;

(P2) associativity: $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$, $g_2 \cdot (x \cdot h_1) = (g_2 \cdot x) \cdot h_1$, $x \cdot (h_1 \cdot h_2) = (x \cdot h_1) \cdot h_2$ for all $g_1, g_2 \in G^1$, $x \in X$, $h_1, h_2 \in H^1$ with $s(g_1) = r(g_2)$, $s(g_2) = r(x)$, $s(x) = r(h_1)$, $s(h_1) = r(h_2)$;

(P3) the following two maps are homeomorphisms:

$$G^1 \times_{s,G^0,r} X \longrightarrow X \times_{s,H^0,s} X, \qquad (g, x) \longmapsto (x, g \cdot x),$$
$$X \times_{s,H^0,r} H^1 \longrightarrow X \times_{r,G^0,r} X, \qquad (x, h) \longmapsto (x, x \cdot h);$$

(P4) s and r are open.

The first two conditions say that X is a G, H-bispace. The only difference between a partial and a global equivalence is whether the anchor maps are assumed surjective or not: conditions (P1)–(P4) are the same as conditions (E1)–(E4) in Proposition A.5.

We view a partial equivalence X from H to G as a generalized map from H to G. Indeed, there is a bicategory with partial equivalences as arrows $H \to G$ (Theorem 2.15).

Definition 2.2. Let G be a groupoid. A subset $U \subseteq G^0$ is *G-invariant* if $r^{-1}(U) = s^{-1}(U)$. In this case, U and $r^{-1}(U) = s^{-1}(U)$ are the object and arrow spaces of a subgroupoid of G, which we denote by G_U.

The canonical projection $p\colon G^0 \to G^0/G$ induces a bijection between G-invariant subsets $U \subseteq G^0$ and subsets $p(U) \subseteq G^0/G$. We are mainly interested in open invariant subsets. Since p is open and continuous, open G-invariant subsets of G^0 correspond to open subsets of G^0/G.

Lemma 2.3. *Let G and H be topological groupoids. A partial equivalence X from H to G is the same as an equivalence from H_V to G_U for open, invariant subsets $U \subseteq G^0$, $V \subseteq H^0$. Here, $U = r(X)$, $V = s(X)$.*

Proof. Let $U \subseteq G^0$ be G-invariant. A left G_U-action is the same as a left G-action for which the anchor map takes values in U because $G_U^0 = U$ and $G^1 \times_{s,G^0,r} X \cong G_U^1 \times_{s,G^0,r} X$ if $r(X) \subseteq U$. Thus, the commuting actions of G_U and H_V for an equivalence from H_V to G_U may also be viewed as commuting actions of G and H, respectively.

This gives a partial equivalence, see Definition 2.1. Conversely, let X be a partial equivalence. Let $U := r(X) \subseteq G^0$ and $V := s(X) \subseteq H^0$. These are open subsets because r and s are open, and they are invariant by (P1). The actions of G and H are equivalent to actions of G_U and H_V, respectively. After replacing G and H by G_U and H_V, respectively, all conditions (E1)–(E5) in Proposition A.5 hold; thus, X is an equivalence from H_V to G_U. □

Lemma 2.4. *Let X be a partial equivalence from H to G, and let $U \subseteq G^0$ and $V \subseteq H^0$ be invariant open subsets. Then*

$$_U|X|_V := \{x \in X \mid r(x) \in U,\ s(x) \in V\}$$

is again a partial equivalence from H to G.

We also write $_U|X$ and $X|_V$ for $_U|X|_{H^0}$ and $_{G^0}|X|_V$, respectively.

Proof. The subset $_U|X|_V$ is open because r and s are continuous and U and V are open, and it is invariant under the actions of G and H because U and V are invariant and the two anchor maps are either invariant or equivariant with respect to the two actions. Hence, we may restrict the actions of G and H to $_U|X|_V$. Conditions (P1)–(P2) and (P4) in Definition 2.1 are inherited by an open invariant subspace. The inverse to the first homeomorphism in (P3) maps

$$_U|X|_V \times_{s,H^0,s} {}_U|X|_V$$

into

$$G^1 \times_{s,G^0,r} {}_U|X|_V,$$

and the inverse to the second homeomorphism maps

$$_U|X|_V \times_{r,G^0,r} {}_U|X|_V$$

into

$$_U|X|_V \times_{s,H^0,r} H^1.$$

Thus, $_U|X|_V$ also inherits (P3) and is a partial equivalence from H to G. □

Equivalences are partial equivalences, of course. In particular, the *identity equivalence* G^1 with G acting by left and right multiplication is also a partial equivalence.

Let X and Y be partial equivalences from H to G and from K to H, respectively. Their composite is defined as for global equivalences and still denoted by $\times_H$:

$$X \times_H Y := X \times_{s,H^0,r} Y \,/\, (x \cdot h, y) \sim (x, h \cdot y),$$

equipped with the quotient topology and the induced actions of G and K by left and right multiplication. The canonical map $X \times_{s,H^0,r} Y \to X \times_H Y$ is a principal H-bundle for the H-action defined by $(x, y) \cdot h := (x \cdot h, h^{-1} \cdot y)$; this follows from the general theory in [**21**].

Example 2.5. We associate an equivalence H_f from G to H to a groupoid isomorphism $f\colon G \to H$. The functor f consists of homeomorphisms $f^i\colon G^i \to H^i$ for $i = 0, 1$. We take $X = H^1$ with the usual left H-action and the right G-action by $h \cdot g := h \cdot f^1(g)$ for all $h \in H^1$, $g \in G^1$ with $s(h) = r(f^1(g)) = f^0(r(g))$; so the right anchor map is $(f^0)^{-1} \circ s = s \circ (f^1)^{-1}$.

We claim that an equivalence is of this form if and only if it is isomorphic to H^1 as a left H-space. Since $H\backslash H^1 \cong H^0$, the right anchor map gives a homeomorphism $H^0 \to G^0$ in this case; let $f^0\colon G^0 \to H^0$ be its inverse. The right action of $g \in G^1$ on $h \in H^1$ with $s(h) = f^0(r(g))$ must be of the form $h \cdot g = h \cdot f^1(g)$ for a unique $f^1(g) \in H^1$ with $r(f^1(g)) = f^0(r(g))$ and $s(f^1(g)) = s(h \cdot g) = f^0(s(g))$. It is routine to check that f^0 and f^1 give a topological groupoid isomorphism.

When do two isomorphisms $f, \varphi\colon G \to H$ give isomorphic equivalences? Let $u\colon H^1_f \xrightarrow{\sim} H^1_\varphi$ be an isomorphism. Define a continuous map $\sigma\colon G^0 \to H^1$ by $\sigma(x) := u(1_{f^0(x)})$ for all $x \in G^0$. This satisfies $r(\sigma(x)) = f^0(x)$ and $s(\sigma(x)) = \varphi^0(x)$ for all $x \in G^0$ because u is compatible with anchor maps. Since u is left H-invariant, $u(h) = u(h \cdot 1_{s(h)}) = h \cdot (\sigma \circ (f^0)^{-1} \circ s)(h)$ for all $h \in H^1$, so σ determines u. The right G-invariance of u translates to $\sigma(r(g)) \cdot \varphi^1(g) = f^1(g) \cdot \sigma(s(g))$ for all $g \in G$. Thus,

$$\varphi^0(x) = s(\sigma(x)), \qquad \varphi^1(g) = \sigma(r(g))^{-1} \cdot f^1(g) \cdot \sigma(s(g)). \tag{2.1}$$

Roughly speaking, f and φ differ by an inner automorphism.

Let an equivalence $f\colon G \to H$ and a continuous map $\sigma\colon G^0 \to H^1$ with $r(\sigma(x)) = f^0(x)$ for all $x \in H^0$ be given. Assume that $H^0 \to G^0$,

$x \mapsto s(\sigma(x))$, is a homeomorphism. Then (2.1) defines an isomorphism $\varphi\colon G \to H$ such that $h \mapsto h \cdot \sigma((f^0)^{-1}(s(h)))$ is an isomorphism between the equivalences H_f and H_φ.

Example 2.6. If G and H are *minimal* groupoids in the sense that G^0 and H^0 have no proper open invariant subsets, then any partial equivalence is either empty or a full equivalence $G \xrightarrow{\sim} H$. This holds, in particular, if G and H are groups.

Example 2.7. Any non-empty (partial) equivalence between two groups is isomorphic to one coming from a group isomorphism $G \cong H$. Indeed, since $X/H \cong G^0$ and $G\backslash X \cong H^0$ are a single point, both actions on X are free and transitive. Fix $x_0 \in X$. Since the actions are free and transitive and part of principal bundles, the maps $G \to X$, $g \mapsto g \cdot x_0$ and $H \to X$, $h \mapsto x_0 \cdot h$, are homeomorphisms. The composite map $G \xrightarrow{\sim} X \xrightarrow{\sim} H$ is an isomorphism of topological groups. This isomorphism depends on the choice of x_0. The isomorphisms $G \xrightarrow{\sim} H$ for different choices of x_0 differ by an inner automorphism.

Lemma 2.8. *The composition $\times_H$ is associative and unital with the identity equivalence as a unit, up to the usual canonical bibundle isomorphisms*

$$(X \times_H Y) \times_K Z \cong X \times_H (Y \times_K Z),$$
$$G^1 \times_G X \cong X \cong X \times_H H^1.$$

Proof. For global equivalences with arbitrary topological spaces, this is contained in [**21,** Proposition 7.10]. The proofs in [**21**] can be extended to the partial case as well. Alternatively, we may reduce the partial to the global case by restricting our partial equivalences to global equivalences between open subgroupoids as in Lemma 2.4. This works because

$$_U|(X \times_H Y)|_V \cong (_U|X) \times_H (Y|_V)$$

for $U \subseteq G^0$, $V \subseteq K^0$ open and invariant and partial equivalences X from H to G and Y from K to H. The details are left to the reader. ☐

Proposition 2.9. *Let G and H be topological groupoids. Let X_1 and X_2 be partial equivalences from H to G. There is no bibundle map*

$X_1 \to X_2$ *unless* $r(X_1) \subseteq r(X_2)$ *and* $s(X_1) \subseteq s(X_2)$. *Any* G, H-*bibundle map* $\varphi\colon X_1 \to X_2$ *is an isomorphism onto the open sub-bibundle* ${}_{r(X_1)}|X_2 = X_2|_{s(X_1)}$. *The map* φ *is invertible if* $r(X_2) \subseteq r(X_1)$ *or* $s(X_2) \subseteq s(X_1)$. *In this case,* $r(X_2) = r(X_1)$ *and* $s(X_2) = s(X_1)$.

Proof. Since $r_{X_2} \circ \varphi = r_{X_1}$ and $s_{X_2} \circ \varphi = s_{X_1}$, we must have $r(X_1) \subseteq r(X_2)$ and $s(X_1) \subseteq s(X_2)$ if there is a bibundle map $\varphi\colon X_1 \to X_2$. Assume this from now on. The image of a bibundle map is contained in ${}_{r(X_1)}|X_2$ and in $X_2|_{s(X_1)}$. Since $r(X_1) \subseteq r(X_2)$ and $s(X_1) \subseteq s(X_2)$, we have $r({}_{r(X_1)}|X_2) = r(X_1)$ and $s(X_2|_{s(X_1)}) = s(X_1)$. All remaining assertions now follow once we prove that a bibundle map $\varphi\colon X_1 \to X_2$ is invertible if $r(X_1) = r(X_2)$ or $s(X_1) = s(X_2)$. We treat the case $r(X_2) = r(X_1)$; the other is proved in the same way, exchanging left and right.

Since X_i is a partial equivalence, it is a principal H-bundle over $X_i/H \cong r(X_i)$. The map φ induces a homeomorphism on the base spaces because $r(X_2) = r(X_1)$ both carry the subspace topology from G^0. Hence φ is a homeomorphism by **[21,** Proposition 5.9]. □

In particular, the restricted multiplication maps $G^1_U \times_H X \subseteq G^1 \times_G X \to X$ and $X \times_H H^1_V \subseteq X \times_H H^1 \to X$ are bibundle maps. Proposition 2.9 shows that they induce bibundle isomorphisms

$$G^1_U \times_G X \cong {}_U|X, \qquad X \times_H H^1_V \cong X|_V. \tag{2.2}$$

Partial equivalences carry extra structure similar to an inverse semigroup. The adjoint operation is the following:

Definition 2.10. Given a partial equivalence X from H to G, we define the *dual partial equivalence* X^* by exchanging the left and right actions on X. More precisely, X^* is X as a space, the anchor maps $r^*\colon X^* \to H^0$ and $s^*\colon X^* \to G^0$ are $r^* = s_X$ and $s^* = r_X$, and the left H- and right G-actions are defined by $h \cdot^* x = x \cdot h^{-1}$ and $x \cdot^* g := g^{-1} \cdot x$, respectively.

If X gives an equivalence from H_V to G_U for open invariant subsets $U \subseteq G^0$, $V \subseteq H^0$, then X^* gives the "inverse" equivalence from G_U to H_V.

The following properties of duals are trivial:

- naturality: a bibundle map $X \to Y$ induces a bibundle map $X^* \to Y^*$;
- $(X^*)^* = X$;
- there is a natural isomorphism $\sigma\colon (X \times_H Y)^* \cong Y^* \times_H X^*$, $(x, y) \mapsto (y, x)$, with $\sigma^2 = \mathrm{Id}$.

Let $\mathrm{Map}(Y_1, Y_2)$ be the space of bibundle maps between two partial equivalences Y_1, Y_2 from H to G.

Proposition 2.11. *Let X be a partial equivalence from H to G. Then there are natural isomorphisms*

$$X \times_H X^* \cong G^1_{r(X)}, \qquad X^* \times_G X \cong H^1_{s(X)}$$

that make the following diagrams of isomorphisms commute:

$$\begin{array}{ccc} X \times_H X^* \times_G X & \longrightarrow & X \times_H H^1_{s(X)} \\ \downarrow & & \downarrow \\ G^1_{r(X)} \times_G X & \longrightarrow & X, \end{array}$$

$$\begin{array}{ccc} X^* \times_G X \times_H X^* & \longrightarrow & X^* \times_G G^1_{r(X)} \\ \downarrow & & \downarrow \\ H^1_{s(X)} \times_G X^* & \longrightarrow & X^*. \end{array} \tag{2.3}$$

If K is another groupoid and Y and Z are partial equivalences from K to G and from K to H, respectively, with $r(Y) \subseteq r(X)$ and $r(Z) \subseteq s(X)$, then there are natural isomorphisms

$$\mathrm{Map}(X \times_H Z, Y) \cong \mathrm{Map}(Z, X^* \times_G Y),$$
$$\mathrm{Map}(Y, X \times_H Z) \cong \mathrm{Map}(X^* \times_G Y, Z).$$

Both map the subsets of bibundle isomorphisms onto each other.

Proof. Lemma 2.3 shows that X is an equivalence from $H_{s(X)}$ to $G_{r(X)}$. Hence, the usual theory of groupoid equivalence gives canonical isomorphisms $X \times_H X^* \cong G^1_{r(X)}$ and $X^* \times_G X \cong H^1_{s(X)}$. The first maps the class of (x_1, x_2) with $s(x_1) = s(x_2)$ to the unique $g \in G^1$ with

$x_1 = g \cdot x_2$. In particular, it maps $[x,x] \mapsto 1_{r(x)}$. The second maps the class of (x_1,x_2) with $r(x_1) = r(x_2)$ to the unique $h \in H^1$ with $x_2 = x_1 \cdot h$. In particular, it maps $[x,x] \mapsto 1_{s(x)}$. Then the composite isomorphisms $X \times_H X^* \times_G X \to X$ and $X^* \times_G X \times_H X^* \to X^*$ map $[x,x,x] \mapsto x$, respectively. Since any element in $X \times_H X^* \times_G X$ or $X^* \times_G X \times_H X^*$ has a representative of the form (x,x,x), we get the two commuting diagrams in equation (2.3).

The assumption $r(Y) \subseteq r(X)$ implies $s(X^* \times_G Y) = s(Y)$ because, for any $y \in Y$, there is an $x \in X^*$ with $(x,y) \in X^* \times_G Y$. Similarly, $r(Z) \subseteq s(X)$ implies $s(X \times_H Z) = s(Z)$. By Proposition 2.9, a bibundle map $X \times_H Z \to Y$ exists only if $s(Z) \subseteq s(Y)$, and then it is an isomorphism onto $Y|_{s(Z)}$; and a bibundle map $Z \to X^* \times_G Y$ exists only if $s(Z) \subseteq s(Y)$, and then it is an isomorphism onto $X^* \times_G Y|_{s(Z)}$. Thus, we may as well replace Y by $Y|_{s(Z)}$ to achieve $s(Y) = s(Z)$; then, all bibundle maps $X \times_H Z \to Y$ or $Z \to X^* \times_G Y$ are bibundle isomorphisms. The second isomorphism reduces in a similar way to the case where also $s(Y) = s(Z)$ and where we are dealing only with bibundle isomorphisms.

A bibundle map $\varphi\colon X \times_H Z \to Y$ induces $\mathrm{Id}_{X^*} \times_G \varphi\colon X^* \times_G X \times_H Z \to X^* \times_G Z$; we compose this with the natural isomorphism

$$X^* \times_G X \times_H Z \cong H^1_{s(X)} \times_H Z \cong {}_{s(X)}|Z = Z$$

to get a bibundle map $Z \to X^* \times_G Y$; here, we used $s(X) \supseteq r(Z)$. We claim that this construction gives the desired bijection between $\mathrm{Map}(X \times_H Z, Y)$ and $\mathrm{Map}(Z, X^* \times_G Y)$. Since composing with an isomorphism is certainly a bijection, it remains to show that

$$\begin{aligned} \mathrm{Map}(X \times_H Z, Y) &\longrightarrow \mathrm{Map}(X^* \times_G X \times_H Z, X^* \times_G Y), \\ \varphi &\longmapsto \mathrm{Id}_{X^*} \times_G \varphi, \end{aligned}$$

is bijective. Since $X \times_H X^* \cong G^1_{r(X)}$ and $r(X) \supseteq r(Y)$, we have natural isomorphisms $X \times_H X^* \times_G Y \cong Y$ and $X \times_H X^* \times_G X \times_H Z \cong X \times_H Z$. *Naturality* means that they intertwine $\varphi \mapsto \mathrm{Id}_{X \times_H X^*} \times_G \varphi$ and φ. Since $\mathrm{Id}_{X \times_H X^*} \times_G \varphi = \mathrm{Id}_X \times_H \mathrm{Id}_{X^*} \times_G \varphi$, we see that $\varphi \mapsto \mathrm{Id}_{X^*} \times_G \varphi$ is injective and has $\psi \mapsto \mathrm{Id}_X \times_H \psi$ for $\psi\colon Z \to X^* \times_G Y$ as a one-sided inverse. The same argument also shows that $\psi \mapsto \mathrm{Id}_X \times_H \psi$ is injective, so both constructions are bijective. $\square$

Applying duality, we also get bijections $\mathrm{Map}(Z^* \times_H X^*, Y^*) \cong \mathrm{Map}(Z^*, Y^* \times_G X)$ and $\mathrm{Map}(Y^*, Z^* \times_H X^*) \cong \mathrm{Map}(Y^* \times_G X, Z^*)$ under the same hypotheses.

The canonical isomorphisms

$$X \times_H X^* \times_G X \cong X, \qquad X^* \times_G X \times_H X^* \cong X^* \tag{2.4}$$

from Proposition 2.11 characterize X^* uniquely in the following sense:

Proposition 2.12. *Let X and Y be partial equivalences from H to G and from G to H, respectively. If there are bibundle isomorphisms*

$$X \times_H Y \times_G X \cong X, \qquad Y \times_H X \times_G Y \cong Y,$$

then there is a unique bibundle isomorphism $X^* \cong Y$ such that the composite map

$$X \cong X \times_H X^* \times_G X \cong X \times_H Y \times_G X \cong X \tag{2.5}$$

is the identity map.

Proof. When we multiply the inverse of the isomorphism $X \times_H Y \times_G X \cong X$ on both sides by X^* and use equation (2.2), we get an isomorphism

$$\begin{aligned} X^* \cong X^* \times_G X \times_H X^* &\cong X^* \times_G X \times_H Y \times_G X \times_H X^* \\ &\cong H^1_{s(X)} \times_H Y \times_G G^1_{r(X)} \cong {}_{s(X)}|Y|_{r(X)}. \end{aligned}$$

This implies $s(X) = r(X^*) \subseteq r(Y)$ and $r(X) = s(X^*) \subseteq s(Y)$ by Proposition 2.9. Exchanging X and Y, the isomorphism $Y \times_H X \times_G Y \cong Y$ gives $s(Y) \subseteq r(X)$ and $r(Y) \subseteq s(X)$. Hence, $r(Y) = s(X)$ and $s(Y) = r(X)$, so ${}_{s(X)}|Y|_{r(X)} = Y$. This gives an isomorphism $\alpha\colon X^* \xrightarrow{\sim} Y$.

A diagram chase using the commutative diagrams in (2.3) shows that the composite of the map $X \times_H X^* \times_G X \to X \times_H Y \times_G X$ induced by the isomorphism α and the given isomorphism $X \times_H Y \times_G X \to Y$ (which we used to construct α) is the canonical map $X \times_H X^* \times_G X \to X$ as in equation (2.4). Hence, the composite in equation (2.5) is the identity map for the isomorphism α.

The isomorphisms in Proposition 2.11 give a canonical bijection

$$\begin{aligned}\mathrm{Map}(X^*, Y) &\cong \mathrm{Map}(X^* \times_G X \times_H X^*, Y) \cong \mathrm{Map}(X \times_H X^*, X \times_H Y)\\ &\cong \mathrm{Map}(X, X \times_H Y \times_G X) \cong \mathrm{Map}(X, X).\end{aligned}$$

Inspection shows that it maps an isomorphism $X^* \xrightarrow{\sim} Y$ to the composite map in equation (2.5). Hence, there is only one isomorphism $X^* \xrightarrow{\sim} Y$ for which the composite map in equation (2.5) is the identity map. □

Proposition 2.13. *Let X be a partial equivalence from G to itself, and let $\mu\colon X \times_G X \to X$ be a bibundle isomorphism. Then there is a unique isomorphism $\varphi\colon X \xrightarrow{\sim} G^1_U$ for an open G-invariant subset $U \subseteq G^0$ such that the next diagram commutes:*

$$\begin{array}{ccc} X \times_G X & \xrightarrow{\mu} & X \\ {\scriptstyle \varphi \times_G \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ G^1_U \times_G G^1_U & \xrightarrow{\mu_0} & G^1_U, \end{array} \tag{2.6}$$

$\mu_0(g_1, g_2) = g_1 \cdot g_2$. Hence, $r(X) = s(X)$, and μ is associative.

Proof. The isomorphism μ induces an isomorphism

$$X \times_G X \times_G X \xrightarrow{\mu \times_G \mathrm{Id}_X} X \times_G X \xrightarrow{\mu} X.$$

Hence, $Y = X$ satisfies the two conditions in Proposition 2.11 that ensure $X = Y \cong X^*$. This gives an isomorphism $\varphi\colon X \cong X \times_G X \cong X \times_G X^* \cong G^1_{r(X)}$. Since φ is a bibundle map, the diagram (2.6) commutes if and only if μ is the composite map

$$X \times_G X \xrightarrow{\varphi \times_G \mathrm{Id}_X} G^1_{r(X)} \times_G X \cong X,$$

where the map $G^1_{r(X)} \times_G X \cong X$ is the left multiplication map, $[g, x] \mapsto g \cdot x$. Sending an isomorphism $\varphi\colon X \to G^1_{r(X)} \cong X \times_G X^*$ to this composite map is one of the bijections in Proposition 2.11, namely, the first one for $X = Y = Z$:

$$\mathrm{Map}(X, G^1_{s(X)}) \cong \mathrm{Map}(X, X^* \times_G X) \cong \mathrm{Map}(X \times_G X, X).$$

Hence, there is exactly one isomorphism φ that corresponds under this bijection to μ. $\square$

Proposition 2.11 implies that isomorphism classes of partial equivalences from G to itself form an inverse semigroup $\widetilde{\mathfrak{peq}}(G)$. The idempotents in this inverse semigroup are in bijection with G-invariant open subsets of G^0 by Proposition 2.13. These are, in turn, in bijection with open subsets of the orbit space G^0/G by the definition of the quotient topology on G^0/G. These also correspond to the idempotents of the inverse semigroup $\mathrm{pHomeo}(G^0/G)$ of partial homeomorphisms of the topological space G^0/G.

A partial equivalence X from H to G induces a partial homeomorphism

$$X_*\colon H^0/H \subseteq s(X) \longrightarrow r(X) \subseteq G^0/G,$$

by $X_*([h]) = [g]$ if there is an $x \in X$ with $s(x) \in [h]$ and $r(x) \in [g]$. If Y is another partial equivalence from K to H, then $(X \times_H Y)_* = X_* \circ Y_*$ by definition. This gives a canonical homomorphism of inverse semigroups

$$\widetilde{\mathfrak{peq}}(G) \longrightarrow \mathrm{pHomeo}(G^0/G).$$

Remark 2.14. The homomorphism $\widetilde{\mathfrak{peq}}(G) \longrightarrow \mathrm{pHomeo}(G^0/G)$ is neither injective nor surjective in general, although it is always an isomorphism on the semilattice of idempotents. Consider, for instance, the disjoint union $G = \mathbb{Z}/3 \sqcup \{\mathrm{pt}\}$. This groupoid is a group bundle, and G^0/G has two points. The partial homeomorphism that maps one point to the other does not lift to a partial equivalence because the stabilizers are not the same and equivalences must preserve the stabilizer groups. The group $\mathbb{Z}/3$ has non-inner automorphisms, so there are non-isomorphic partial equivalences of G defined on $\mathbb{Z}/3$ that induce the same partial homeomorphism on G^0/G.

In our definition of an inverse semigroup action (see Sections 3 and 4), certain isomorphisms of partial equivalences are a crucial part of the data. We could not construct transformation groupoids and Fell bundles without them. If we identify isomorphic partial equivalences as above, then we can no longer talk about two isomorphisms of partial equivalences being equal. The correct way to take into account isomorphisms of partial equivalences is through a bicategory (see [**1, 19, 6**]).

The next remarks are intended for readers familiar with bicategories.

Our bicategory has topological groupoids as objects and partial equivalences as arrows. Let G and H be topological groupoids, and let X_1 and X_2 be partial equivalences from H to G. As 2-arrows $X_1 \Rightarrow X_2$, we take all G,H-bibundle isomorphisms $X_1 \to X_2$, so all 2-arrows are invertible. The vertical product of 2-arrows is the composition of bibundle maps. *Unit 2-arrows* are identity maps on partial equivalences. The composition of arrows is $\times_H$. The unit arrow on a topological groupoid G is G^1 with the standard bibundle structure. Lemma 2.8 provides invertible 2-arrows

$$(X \times_H Y) \times_K Z \Longrightarrow X \times_H (Y \times_K Z),$$
$$G^1 \times_G X \Longrightarrow X \Longleftarrow X \times_H H^1,$$

which we take as associator and left and right unit transformations. Let X_1, X_2 be partial equivalences from H to G, and let Y_1, Y_2 be partial equivalences from K to H. The horizontal product of two bibundle maps

$$f\colon X_1 \to X_2 \quad \text{and} \quad g\colon Y_1 \to Y_2$$

is

$$f \times_H g\colon X_1 \times_H Y_1 \longrightarrow X_2 \times_H Y_2.$$

Theorem 2.15. *The data above defines a bicategory* $\mathfrak{peq}$*.*

Proof. It is routine to check that partial equivalences from H to G with bibundle maps between them form a category $\mathcal{C}(G,H)$ for the vertical product of bibundle maps and that the composition of partial equivalences with the horizontal product of bibundle maps is a functor $\mathcal{C}(G,H) \times \mathcal{C}(H,K) \to \mathcal{C}(G,K)$. The associator and both unit transformations are natural isomorphisms of functors; the associator is clearly compatible with unit transformations and makes the usual pentagon commute, see [**19,** page 2]. □

Remark 2.16. We still get a bicategory if we allow all bibundle maps as 2-arrows. We restrict to invertible 2-arrows to get the correct notion of inverse semigroup actions below.

An arrow $f\colon x \to y$ in a bicategory is called an *equivalence* if there are an arrow $g\colon y \to x$ and invertible 2-arrows $g \circ f \Rightarrow \mathrm{Id}_x$ and $f \circ g \Rightarrow \mathrm{Id}_y$. The equivalences in $\mathfrak{peq}$ are precisely the global bibundle equivalences.

The duality $X \mapsto X^*$ with the canonical flip maps $(X \times_H Y)^* \xrightarrow{\sim} Y^* \times_H X^*$ gives a functor $I\colon \mathfrak{peq} \to \mathfrak{peq}^{\mathrm{op}}$ with $I^2 = \mathrm{Id}_{\mathfrak{peq}}$. It seems useful to formalize the properties of this functor and look for examples in more general bicategories. But we shall not go into this question here.

3. Inverse semigroup actions on groupoids. We give two equivalent definitions for actions of inverse semigroups on topological groupoids by partial equivalences. The first is exactly what it promises to be. The second, more elementary definition, does not mention groupoids or partial equivalences.

Let S be an inverse semigroup with unit 1. Let G be a topological groupoid.

Definition 3.1. An *action* of S on G *by partial equivalences* consists of

- partial equivalences X_t from G to G for $t \in S$;
- bibundle isomorphisms $\mu_{t,u}\colon X_t \times_G X_u \xrightarrow{\sim} X_{tu}$ for $t, u \in S$;

satisfying

(A1) X_1 is the identity equivalence G^1 on G;
(A2) $\mu_{t,1}\colon X_t \times_G G^1 \xrightarrow{\sim} X_t$ and $\mu_{1,u}\colon G^1 \times_G X_u \xrightarrow{\sim} X_u$ are the canonical isomorphisms, that is, the left and right G-actions, for all $t, u \in S$;
(A3) associativity: for all $t, u, v \in S$, the next diagram commutes:

$$\begin{array}{ccccc}
(X_t \times_G X_u) \times_G X_v & \xrightarrow{\mu_{t,u} \times_G \mathrm{Id}_{X_v}} & X_{tu} \times_G X_v & \xrightarrow{\mu_{tu,v}} & \\
\mathrm{ass} \updownarrow & & & & X_{tuv} \\
X_t \times_G (X_u \times_G X_v) & \xrightarrow[\mathrm{Id}_{X_t} \times_G \mu_{u,v}]{} & X_t \times_G X_{uv} & \xrightarrow{\mu_{t,uv}} &
\end{array}$$

If S has a zero object 0, then we may also ask $X_0 = \emptyset$.

Remark 3.2. Let S be an inverse semigroup possibly without 1. We may add a unit 1 formally and extend the multiplication by $1 \cdot s = s = s \cdot 1$ for all $s \in S \cup \{1\}$. If partial equivalences $(X_t)_{t \in S}$ and bibundle isomorphisms $(\mu_{t,u})_{t,u \in S}$ are given satisfying associativity for all $t, u, v \in S$, then we may extend this uniquely to an action of $S \cup \{1\}$: we put $X_1 := G^1$ and let $\mu_{t,1}$ and $\mu_{1,u}$ be the right and left G-action, respectively. The associativity condition is trivial if one of t, u, v is 1, so associativity holds for all $t, u, v \in S \cup \{1\}$. As a result, an action of $S \cup \{1\}$ by partial equivalences is the same as $(X_t)_{t \in S}$ and $(\mu_{t,u})_{t,u \in S}$ satisfying only Condition (A3).

Similarly, we may add a zero 0 to S and extend the multiplication by $0 \cdot s = 0 = s \cdot 0$ for all $s \in S \cup \{0\}$. We extend an S-action by $X_0 := \emptyset$, so that $X_0 \times_G X_t = \emptyset = X_t \times_G X_0$, leaving no choice for the maps $\mu_{t,0}, \mu_{0,u} \colon \emptyset \to \emptyset$. This gives an action of $S \cup \{0\}$ with $X_0 = \emptyset$.

If $0, 1 \in S$ and we ask no conditions on X_0 and X_1, then $r(X_t), s(X_t) \subseteq r(X_1) = s(X_1)$ for all $t \in S$, and X_t restricted to $r(X_0) = s(X_0)$ is the trivial action where all X_t act by the identity equivalence. Hence, all the action is on the locally closed, invariant subset $r(X_1) \setminus r(X_0) \subseteq G^0$. The conditions on X_0 and X_1 merely rule out such degeneracies.

Remark 3.3. An inverse semigroup may be viewed as a special kind of category with only one object, which is also a very special kind of bicategory. An inverse semigroup action by partial equivalences is exactly the same as a functor from this category to the bicategory $\mathfrak{peq}$ of partial equivalences, see [**19**].

Lemma 3.4. *For an inverse semigroup action $(X_t, \mu_{t,u})$, we have $r(X_t) = r(X_{tt^*}) = s(X_{tt^*}) = s(X_{t^*})$ and $s(X_t) = s(X_{t^*t}) = r(X_{t^*t}) = r(X_{t^*})$ for each $t \in S$.*

Proof. If $e \in S$ is idempotent, then Proposition 2.13 applied to the isomorphism $\mu_{e,e} \colon X_e \times_G X_e \cong X_e$ gives $r(X_e) = s(X_e)$. The existence of an isomorphism $\mu_{t,t^*} \colon X_t \times_G X_{t^*} \cong X_{tt^*}$ implies $r(X_t) \supseteq r(X_{tt^*})$ and $s(X_{t^*}) \supseteq s(X_{tt^*})$. Similarly, the isomorphism $\mu_{tt^*,t}$ gives $r(X_{tt^*}) \supseteq r(X_t)$, and μ_{t,t^*t} gives $s(X_{t^*t}) \supseteq s(X_t)$. Now everything follows. $\square$

Definition 3.5. Let S be an inverse semigroup with unit. A *simplified action* of S on a topological groupoid consists of

- a topological space G^0;
- topological spaces X_t for $t \in S$;
- continuous maps $s, r\colon X_t \to G^0$;
- continuous maps
$$\mu_{t,u}\colon X_t \times_{s,G^0,r} X_u \longrightarrow X_{tu}, \quad (x,y) \longmapsto x \cdot y,$$
for $t, u \in S$;

satisfying

(S1) $s(x \cdot y) = s(y)$, $r(x \cdot y) = r(x)$ for all $t, u \in S$, $x \in X_t$, $y \in X_u$ with $s(x) = r(y)$;
(S2) $r\colon X_t \to G^0$ and $s\colon X_t \to G^0$ are open for all $t \in S$;
(S3) the maps $r, s\colon X_1 \to G^0$ are surjective;
(S4) $\mu_{t,u}$ is surjective for each $t, u \in S$;
(S5) the map
$$X_t \times_{s,G^0,r} X_u \to X_u \times_{s,G^0,s} X_{tu}, \quad (x,y) \mapsto (y, x \cdot y),$$
is a homeomorphism if $t = 1$ and $u \in S$;
(S6) the map
$$X_t \times_{s,G^0,r} X_u \longrightarrow X_t \times_{r,G^0,r} X_{tu}, \quad (x,y) \longmapsto (x, x \cdot y),$$
is a homeomorphism if $t \in S$ and $u = 1$;
(S7) for all $t, u, v \in S$, the next diagram commutes:

(3.1)
$$\begin{array}{ccc}
(X_t \times_{s,G^0,r} X_u) \times_{s,G^0,r} X_v & \xrightarrow{\mu_{t,u} \times_{s,G^0,r} \mathrm{Id}_{X_v}} & X_{tu} \times_{s,G^0,r} X_v \\
\updownarrow{\scriptstyle \text{ass}} & & \searrow{\scriptstyle \mu_{tu,v}} \\
 & & X_{tuv} \\
 & & \nearrow{\scriptstyle \mu_{t,uv}} \\
X_t \times_{s,G^0,r} (X_u \times_{s,G^0,r} X_v) & \xrightarrow[\mathrm{Id}_{X_t} \times_{s,G^0,r} \mu_{u,v}]{} & X_t \times_{s,G^0,r} X_{uv}
\end{array}$$

If S has a zero element, we may also ask $X_0 = \emptyset$.

This definition is more elementary because it does not mention groupoids or partial equivalences. It seems less elegant than Definition 3.1 but is simpler because much of the complexity of Definition 3.1 is hidden in the conditions (P1)–(P4) defining partial equivalences of topological groupoids.

It is clear that an inverse semigroup action by partial equivalences gives a simplified action: forget the multiplication on G^1 and the left and right actions of G on the spaces X_t. The isomorphisms in (S5) for $t = 1$ and in (S6) for $u = 1$ are those in Definition 3.5, and all other conditions in Definition 3.5 are evident. The converse is more remarkable:

Proposition 3.6. *Any simplified inverse semigroup action on groupoids comes from a unique action by partial equivalences. Thus, actions and simplified actions of inverse semigroups by partial equivalences are equivalent. Furthermore, the maps in* (S5) *and* (S6) *are isomorphisms, and the maps $\mu_{t,u}$ are open for all $t, u \in S$.*

Proof. The spaces G^0 and $G^1 := X_1$ with range and source maps r and s and multiplication $\mu_{1,1}$ satisfy conditions (Gr1)–(Gr4) in Proposition A.1 because these are special cases of our conditions (S1)–(S7). Hence, this data defines a topological groupoid. Similarly, the anchor maps $r\colon X_t \to G^0$ and $s\colon X_t \to G^0$, and the multiplication maps $\mu_{1,t}$ and $\mu_{t,1}$ satisfy conditions (P1)–(P4) in Definition 2.1, and thus, turn X_t into a partial equivalence from G to itself.

Let $t, u \in S$. The associativity of the maps μ for $t, 1, u$, $1, t, u$ and $t, u, 1$ implies that $\mu_{t,u}$ descends to a G,G-bibundle map $\overline{\mu}_{t,u}\colon X_t \times_G X_u \to X_{tu}$. Since $\mu_{t,u}$ is surjective by (S4), so is $\overline{\mu}_{t,u}$. Hence it is a bibundle isomorphism by Proposition 2.9.

The groupoid structure on X_1 and the left and right actions on X_t are defined so that X_1 is the identity equivalence on G and the maps $\overline{\mu}_{1,u}$ and $\overline{\mu}_{t,1}$ are the canonical isomorphisms. The associativity condition for the bibundle isomorphisms $\overline{\mu}_{t,u}$ follows from the corresponding property of the maps $\mu_{t,u}$. Thus, we have obtained an action by partial equivalences. This is the only action that simplifies to the given data because of assumptions regarding X_1, $\mu_{1,u}$ and $\mu_{t,1}$ in Definition 3.1.

By definition, $X_t \times_G X_u$ is the orbit space of the G-action on $X_t \times_{s,G^0,r} X_u$ by $(x_1, x_2)\cdot g := (x_1\cdot g, g^{-1}\cdot x_2)$. The canonical projection

$$X_t \times_{s,G^0,r} X_u \longrightarrow X_t \times_G X_u$$

is open by Proposition A.3. The map $\mu_{t,u}$ is the composite of this projection with the homeomorphism $\overline{\mu}_{t,u}\colon X_t \times_G X_u \to X_{tu}$; hence, it is also open.

Finally, we check that the maps in (S5) are isomorphisms for all $t, u \in S$; exchanging left and right gives the same for the maps in (S6). The map in (S5) is G-equivariant if we let G act on $X_t \times_{s,G^0,r} X_u$ by $g \cdot (x, y) := (xg^{-1}, gy)$ and on $X_u \times_{s,G^0,s} X_{tu}$ by $g \cdot (y, x) := (gy, x)$. Both actions are part of principal bundles: the bundle projection on $X_t \times_{s,G^0,r} X_u$ is the canonical map to $X_t \times_G X_u$, and the bundle projection on $X_u \times_{s,G^0,s} X_{tu}$ is $s \times_{G^0,s} \mathrm{Id}_{X_{tu}}$ to $X_{tu}|_{r(X_u)}$. Our G-equivariant map induces the map $\mu_{t,u}$ on the base spaces, which is a homeomorphism; hence, so is the map on the total spaces by **[21,** Proposition 5.9]. □

3.1. Compatibility with order and involution. Let S be an inverse semigroup with unit. Define a partial order on S by $t \le u$ if $t = tt^*u$ or, equivalently, $t = ut^*t$. The multiplication and involution preserve this order: $t_1t_2 \le u_1u_2$ and $t_1^* \le u_1^*$ if $t_1 \le u_1$ and $t_2 \le u_2$, see **[18]**.

Let $(X_t)_{t\in S}$, $(\mu_{t,u})_{t,u\in S}$ be an action of S on G. We shall prove that the action is compatible with this partial order and the involution on S. To prepare for the proofs of analogous statements for inverse semigroup actions on C*-algebras, we give rather abstract proofs, which literally carry over to the C*-algebraic case.

Proposition 3.7. *There are unique bibundle maps $j_{u,t}\colon X_t \to X_u$ for $t, u \in S$ with $t \le u$ such that the following diagrams commute for all $t_1, t_2, u_1, u_2 \in S$ with $t_1 \le u_1$, $t_2 \le u_2$:*

$$\begin{array}{ccc} X_{t_1} \times_G X_{t_2} & \xrightarrow{\mu_{t_1,t_2}} & X_{t_1t_2} \\ {\scriptstyle j_{u_1,t_1} \times_G j_{u_2,t_2}}\downarrow & & \downarrow{\scriptstyle j_{u_1u_2,t_1t_2}} \\ X_{u_1} \times_G X_{u_2} & \xrightarrow{\mu_{u_1,u_2}} & X_{u_1u_2} \end{array} \tag{3.2}$$

The map $j_{u,t}$ is a bibundle isomorphism onto $X_u|_{s(X_t)} = {}_{r(X_t)}|X_u$. We have $j_{t,t} = \mathrm{Id}_{X_t}$ for all $t \in S$ and $j_{v,u} \circ j_{u,t} = j_{v,t}$ for $t \le u \le v$ in S.

Proof. Let $E(S) \subseteq S$ be the subset of idempotents, and let $e \in E(S)$. Proposition 2.13 gives a unique isomorphism $X_e \cong G^1_{U_e}$ intertwining $\mu_{e,e}\colon X_e \times_G X_e \xrightarrow{\sim} X_e$ and the multiplication in $G^1_{U_e}$; here, $U_e := r(X_e) = s(X_e)$ is an open G-invariant subset of G^0. Diagram (3.2) for $(e,e) \leq (1,1)$ shows that $j_{1,e}$ must be this particular isomorphism $X_e \cong G^1_{U_e} \subseteq G^1$.

To simplify notation, we now identify X_e with $G^1_{U_e}$ for all $e \in E(S)$ using these unique isomorphisms, and we transfer the multiplication maps $\mu_{s,t}$ for idempotent s, t or st accordingly. This gives an isomorphic action of S by partial equivalences. So we may assume that $X_e = G^1_{U_e}$ and that $\mu_{e,e}\colon X_e \times_G X_e \to X_e$ is the usual multiplication map on $G^1_{U_e}$ for all $e \in E(S)$.

Let $e \in E(S)$, and let $t, u \in S$ satisfy $t^*t \leq e$ and $uu^* \leq e$. Thus, $te = t$, $eu = u$ and $teu = tu$. We show that $\mu_{t,e}\colon X_t \times_G G^1_{U_e} \to X_t$ and $\mu_{e,u}\colon G^1_{U_e} \times_G X_u \xrightarrow{\sim} X_u$ are the obvious maps $\mu^0_{t,e}$ or $\mu^0_{e,u}$ from the left and right G-actions in this case. Associativity of the multiplication maps gives us a commuting diagram of isomorphisms:

$$
\begin{array}{ccc}
X_t \times_G X_e \times_G X_u & \xrightarrow{\operatorname{Id}_{X_t} \times_G \mu_{e,u}} & X_t \times_G X_u \\
{\scriptstyle \mu_{t,e} \times_G \operatorname{Id}_{X_u}} \downarrow & \diagup\!\!\diagup & \downarrow {\scriptstyle \mu_{t,u}} \\
X_t \times_G X_u & \xrightarrow[\mu_{t,u}]{} & X_{tu}
\end{array}
$$

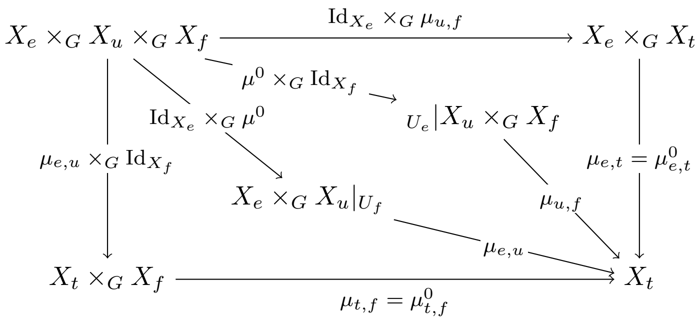
We may cancel the isomorphism $\mu_{t,u}$ to obtain $\operatorname{Id}_{X_t} \times_G \mu_{e,u} = \mu_{t,e} \times_G \operatorname{Id}_{X_u}$.

Now we consider two cases: $t = e$ or $e = u$. If $t = e$, then $\mu_{t,e} = \mu^0_{t,e}$ is the multiplication map on $G^1_{U_e}$, and hence, so is $\mu_{t,e} \times_G \operatorname{Id}_{X_u}$. Thus, $\mu_{e,u}$ and $\mu^0_{e,u}$ induce the same map $G^1_{U_e} \times_G G^1_{U_e} \times_G X_u \to G^1_{U_e} \times_G X_u$. We may use (2.2) to cancel the factor $G^1_{U_e}$ because $s(X_e) = U_e \supseteq r(X_u) \supseteq r(G^1_{U_e} \times_G X_u)$. Thus, $\mu_{e,u} = \mu^0_{e,u}$ if e is idempotent and $e \geq uu^*$. A similar argument in the other case $e = u$ gives $\mu_{t,e} = \mu^0_{t,e}$ if $t^*t \leq e$.

Now let $t \leq u$, that is, $t = tt^*u = ut^*t$. Then we get two candidates for the bibundle map $j_{u,t}\colon X_t \to X_u$:

$$
\begin{aligned}
X_t &\xleftarrow[\cong]{\mu_{tt^*,u}} X_{tt^*} \times_G X_u = G^1_{U_{tt^*}} \times_G X_u \xrightarrow[\cong]{\mu^0_{tt^*,u}} {}_{U_{tt^*}}|X_u \subseteq X_u, \\
X_t &\xleftarrow[\cong]{\mu_{u,t^*t}} X_u \times_G X_{t^*t} = X_u \times_G G^1_{U_{t^*t}} \xrightarrow[\cong]{\mu^0_{u,t^*t}} X_u|_{U_{t^*t}} \subseteq X_u.
\end{aligned}
\tag{3.3}
$$

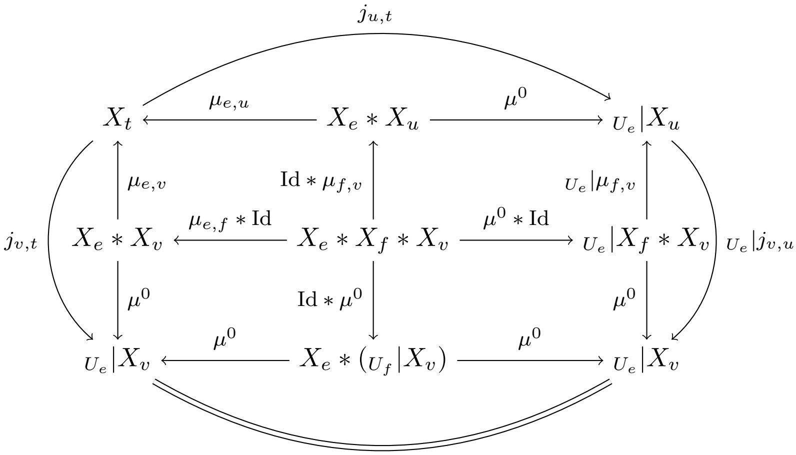
We claim that both maps $X_t \to X_u$ are equal, so we only obtain one map $j_{u,t}\colon X_t \to X_u$. Let $e = tt^*$ and $f = t^*t$. Then there is a commutative diagram of isomorphisms:

(3.4)


The large rectangle commutes by associativity. The above argument gives $\mu_{e,t} = \mu^0_{e,t}$ and $\mu_{t,f} = \mu^0_{t,f}$. The lower left and upper right triangles commute because $\mu_{e,u}$ and $\mu_{u,f}$ are bibundle maps, so they are compatible with μ^0. Hence, the interior quadrilateral commutes. Thus, the two definitions of $j_{u,t}$ in equation (3.3) are equal.

The first construction of $j_{u,t}$ in equation (3.3) gives the unique map for which the diagram in equation (3.2) commutes for $(e,t) \leq (1,u)$ and the inclusion map $j_{1,e}$. Since we already saw that $j_{1,e}$ is unique, the diagrams in equation (3.2) characterize the bibundle maps $j_{u,t}$ uniquely for all $t \leq u$ in S. The map $j_{t,t}$ is the identity on X_t because $\mu_{tt^*,t} = \mu^0_{tt^*,t}$.

Now, let $t \leq u \leq v$, define $e = tt^*$ and $f = uu^*$ and identify X_e and X_f with subsets of G^1. In the next diagram, we abbreviate $\times_G$ to $*$, and μ^0 denotes the left and right actions for subsets of G^1:



The top left square commutes because the multiplication maps are associative, the top right square because they are bibundle maps. The bottom left square commutes because $\mu_{e,f} = \mu^0$, and the bottom right square commutes for trivial reasons. The bent composite arrows are the maps j by construction. Thus, the whole diagram commutes, and this means that $j_{v,u} \circ j_{u,t} = j_{v,t}$.

If $t_1 \le u_1$ and $t_2 \le u_2$ in S, then there is a commuting diagram of isomorphisms

$$
\begin{array}{ccc}
X_{t_1} * X_{t_2} & \xrightarrow{\mu_{t_1,t_2}} & X_{t_1 t_2} \\
\uparrow{\scriptstyle \mu_{t_1t_1^*,u_1} * \mu_{u_2,t_2^*t_2}} & & \uparrow{\scriptstyle \mu_{t_1t_1^*,u_1u_2,t_2^*t_2}} \\
X_{t_1t_1^*} * X_{u_1} * X_{u_2} * X_{t_2^*t_2} & \xrightarrow[\mathrm{Id} * \mu_{u_1,u_2} * \mathrm{Id}]{} & X_{t_1t_1^*} * X_{u_1u_2} * X_{t_2^*t_2} \\
\downarrow{\scriptstyle \mu^0} & & \downarrow{\scriptstyle \mu^0} \\
{}_{U_{t_1t_1^*}}|X_{u_1} * X_{u_2}|_{U_{t_2^*t_2}} & \xrightarrow[\mu_{u_1,u_2}]{} & {}_{U_{t_1t_1^*}}|X_{u_1u_2}|_{U_{t_2^*t_2}}
\end{array}
\tag{3.5}
$$

Here, we abbreviate $\times_G$ to $*$, μ^0 denotes the left and right actions for subsets of G^1, and $\mu_{t_tt_t^*,u_1,u_2,t_2^*t_2}$ denotes the appropriate combination of two multiplication maps, which is well defined by associativity. The upper square commutes by associativity. The lower square commutes because μ_{u_1,u_2} is a bibundle map. The left vertical isomorphism from $X_{t_1} * X_{t_2}$ to ${}_{U_{t_1t_1^*}}|X_{u_1} * X_{u_2}|_{U_{t_2^*t_2}}$ is $j_{u_1,t_1} * j_{u_2,t_2}$ because the two constructions in equation (3.3) coincide. It remains to see that the right vertical isomorphism from $X_{t_1t_2}$ to ${}_{U_{t_1t_1^*}}|X_{u_1u_2}|_{U_{t_2^*t_2}}$ is $j_{u_1u_2,t_1t_2}$.

The proof of this is similar to the proof that the two maps in equation (3.3) coincide. Let $e = (t_1t_2)(t_1t_2)^*$, so $e \le t_1t_1^*$. Since $r(X_{t_1t_2}) = U_e$ and equation (3.5) is a diagram of isomorphisms, we have $X_e * X_{t_1t_1^*} * X_{u_1u_2} * X_{t_2^*t_2} \cong X_{t_1t_1^*} * X_{u_1u_2} * X_{t_2^*t_2}$. Furthermore, the isomorphism

$$\mu_{e,t_1t_1^*} * \mathrm{Id}\colon X_e * X_{t_1t_1^*} * X_{u_1u_2} * X_{t_2^*t_2} \longrightarrow X_e * X_{u_1u_2} * X_{t_2^*t_2}$$

is equal to the standard multiplication map $\mu^0_{e,t_1t_1^*} * \mathrm{Id}$ because $e \le t_tt_1^*$. This fact and associativity show that the right vertical isomorphism in equation (3.5) is equal to the composite map

$$X_{t_1t_2} \xleftarrow[\cong]{\mu_{e,u_1u_2,t_2t_2^*}} X_e * X_{u_1u_2} * X_{t_2^*t_2} \xrightarrow[\cong]{\mu^0} {}_{U_e}|X_{u_1u_2}|_{U_{t_2^*t_2}} = {}_{U_e}|X_{u_1u_2}.$$

Similarly, we obtain the same composite map if we replace $t_2^*t_2$ on the right by the smaller idempotent $f = (t_1t_2)^*(t_1t_2)$. Now diagram (3.4) shows that the map we get is $j_{u_1u_2,t_1t_2}$, as desired. Hence diagram (3.2) commutes. □

Remark 3.8. Let E be a semilattice with unit 1, viewed as an inverse semigroup. An E-action on a topological groupoid G is the same as a unital semilattice map from E to the lattice of open G-invariant subsets of G^0, that is, a map $e \mapsto U_e$ satisfying $U_1 = G^0$ and $U_e \cap U_f = U_{ef}$ for all $e, f \in E$. The corresponding action by partial equivalences is defined by $X_e := G^1_{U_e}$ and $\mu_{e,f} = \mu^0\colon G^1_{U_e} \times_G G^1_{U_f} \to G^1_{U_{ef}}$. Proposition 3.7 implies that every action of E is isomorphic to one of this form.

Proposition 3.9. *There are unique bibundle isomorphisms $J_t\colon X_t^* \to X_{t^*}$ for which the following composite map is the identity*:

(3.6)

$$X_t \cong X_t \times_G X_t^* \times_G X_t \xrightarrow{\mathrm{Id}_{X_t} \times_G J_t \times_G \mathrm{Id}_{X_t}} X_t \times_G X_{t^*} \times_G X_t \xrightarrow{\mu_{t,t^*,t}} X_t.$$

These involutions also make the following diagrams commute:

$$(3.7)\qquad \begin{array}{ccc} X_t \times_G X_t^* & \longrightarrow & G^1_{U_{tt^*}} \\ {\scriptstyle \mathrm{Id}_{X_t} \times_G J_t}\downarrow & & \downarrow \\ X_t \times_G X_{t^*} & \xrightarrow[\mu_{t,t^*}]{} & X_{tt^*} \end{array} \qquad \begin{array}{ccc} X_t^* \times_G X_t & \longrightarrow & G^1_{U_{t^*t}} \\ {\scriptstyle J_t \times_G \mathrm{Id}_{X_t}}\downarrow & & \downarrow \\ X_{t^*} \times_G X_t & \xrightarrow[\mu_{t^*,t}]{} & X_{t^*t} \end{array}$$

Here the unlabeled arrows are the canonical isomorphisms from Propositions 2.11 *and* 2.13. *Furthermore,* $(J_{t^*})^* \circ J_t \colon X_t^* \to X_{t^*} \to X_t^*$ *is the identity map for all* $t \in S$*, and the following diagrams commute for all* $t, u, v \in S$ *with* $t \le u$*:*

$$(3.8)\qquad \begin{array}{ccc} X_u^* \times_G X_v^* & \xrightarrow{\mu^*_{v,u}} & X_{vu}^* \\ {\scriptstyle J_u \times_G J_v}\downarrow & & \downarrow{\scriptstyle J_{vu}} \\ X_{u^*} \times_G X_{v^*} & \xrightarrow[\mu_{u^*,v^*}]{} & X_{u^*v^*} \end{array} \qquad \begin{array}{ccc} X_t^* & \xrightarrow{j^*_{u,t}} & X_u^*|_{U_{tt^*}} \\ {\scriptstyle J_t}\downarrow & & \downarrow{\scriptstyle J_u|_{U_{tt^*}}} \\ X_{t^*} & \xrightarrow[j_{u^*,t^*}]{} & X_{u^*}|_{U_{tt^*}} \end{array}$$

Write $x^* := J_t(x)$ for $x \in X_t$ and $\mu_{t,u}(x, y) = x \cdot y$ for $x \in X_t$, $y \in X_u$ with $s(x) = r(y)$. The above diagrams and equations of maps mean that the involution is characterized by $x \cdot x^* \cdot x = x$ for all $x \in X_t$ and has the properties $x \cdot x^* = 1_{r(x)}$, $x^* \cdot x = 1_{s(x)}$, $(x^*)^* = x$, $(x \cdot y)^* = y^* \cdot x^*$ and $j_{u^*,t^*}(x^*) = j_{u,t}(x)^*$.

Proof. The two isomorphisms $\mu_{t,t^*,t} \colon X_t \times_G X_{t^*} \times_G X_t \to X_t$ and $\mu_{t^*,t,t^*} \colon X_{t^*} \times_G X_t \times_G X_{t^*} \to X_{t^*}$ that we may build from μ are equal by associativity. Proposition 2.12 for these isomorphisms gives a unique isomorphism $J_t \colon X_t^* \cong X_{t^*}$ for which the map in equation (3.6) becomes the identity map.

We claim that the map in equation (3.6) is the identity if and only if either of the diagrams in equation (3.7) commutes. The proofs for both cases differ only by exchanging left and right, so we only write down one of them. Assume that the first diagram in equation (3.7) commutes. Applying the functor $\_ \times_G \mathrm{Id}_{X_t}$ to it, we obtain that the isomorphism (3.6) is the identity map because the multiplication map $\mu_{tt^*,t} \colon X_{tt^*} \times_G X_t \to X_t$ is only the left action if we identify $X_{tt^*} \cong G^1_{U_{tt^*}}$ as usual. Conversely, assume that the isomorphism in equation (3.6) is the identity map. Take a further product with X_{t^*}, and then identify $X_t \times_G X_{t^*} \cong X_{tt^*}$ via μ_{t,t^*}. Using again that the multiplication with X_{tt^*} is only the G-action, this gives the first diagram in equation (3.7).

Next, we show that $J_{t^*} = (J_t^{-1})^*$, which implies $J_{t^*}^* \circ J_t = \mathrm{Id}_{X_t}$. We use the commuting diagram:

$$\begin{array}{ccc}
X_{t^*}^* \times_G X_{t^*} & \longrightarrow & G^1_{U_{tt^*}} \\
{\scriptstyle (J_t^{-1})^* \times_G J_t^{-1}} \big\downarrow & & \big\| \\
X_t \times_G X_t^* & \longrightarrow & G^1_{U_{tt^*}} \\
{\scriptstyle \mathrm{Id}_{X_t} \times_G J_t} \big\downarrow & & \big\downarrow \\
X_t \times_G X_{t^*} & \xrightarrow{\mu_{t,t^*}} & X_{tt^*}
\end{array}$$

The top rectangle commutes because the pairing $X \times_G X^* \to G^1_{r(X)}$ is natural. The bottom diagram is the first one in equation (3.7). The large rectangle is the second diagram in equation (3.7) for t^* with $(J_t^{-1})^*$ instead of J_{t^*}. Since this diagram characterizes J_{t^*}, we obtain $J_{t^*} = (J_t^{-1})^*$ as asserted.

Since the involution J_{vu} is uniquely characterized by a diagram like the first one in equation (3.7), we may prove the first diagram in equation (3.8) by showing that the composite map $\mu_{u^*,v^*} \circ (J_u \times_G J_v) \circ (\mu_{v,u}^*)^{-1} \colon X_{vu}^* \to X_{u^*v^*}$ also makes the diagram in equation (3.7) for $t = vu$ commute. This is a routine computation using the same diagrams for J_u and J_v, and the multiplication maps involving X_e for idempotent $e \in S$ are always given by the left or right action because of the compatibility with $j_{1,e}$. This proof is a variant of the usual proof that $(xy)^{-1} = y^{-1}x^{-1}$ in a group because $y^{-1}x^{-1} \cdot (xy) = 1$.

Similarly, we get the second diagram in equation (3.8) by showing that the composite map $j_{u^*,t^*}^{-1} \circ J_u \circ j_{u,t}^* \colon X_t^* \to X_{t^*}$ satisfies the definition for J_t because $j_{u,t}$ and j_{u^*,t^*} are compatible with the multiplication maps. □

3.2. Transformation groupoids. Let $(X_t, \mu_{t,u})_{t,u\in S}$ be an action of a unital inverse semigroup S on a topological groupoid G by partial equivalences. Define the embeddings $j_{u,t} \colon X_t \to X_u$ for $t \le u$ in S and the involutions $X_t^* \to X_{t^*}$ as in Propositions 3.7 and 3.9.

Let $X := \bigsqcup_{t\in S} X_t$, and define a relation $\sim$ on X by $(t,x) \sim (u,y)$ for $x \in X_t$, $y \in X_u$ if there are $v \in S$ with $v \le t,u$ and $z \in X_v$ with $j_{t,v}(z) = x$ and $j_{u,v}(z) = y$.

Lemma 3.10. *The relation $\sim$ is an equivalence relation. Equip $X_\sim := X/{\sim}$ with the quotient topology. The quotient map $\pi\colon X \to X_\sim$ is a local homeomorphism. It restricts to a homeomorphism from X_t onto an open subset of $X_\sim$ for each $t \in S$. Thus, $X_\sim$ is locally quasi-compact or locally Hausdorff if and only if all X_t are as well.*

Proof. It is clear that $\sim$ is reflexive and symmetric. For transitivity, take $(t_1, x_1) \sim (t_2, x_2) \sim (t_3, x_3)$. Then there are $t_{12} \leq t_1, t_2$, $t_{23} \leq t_2, t_3$, $x_{12} \in X_{t_{12}}$ and $x_{23} \in X_{t_{23}}$ with $j_{t_i,t_{12}}(x_{12}) = x_i$ for $i = 1, 2$ and $j_{t_i,t_{23}}(x_{23}) = x_i$ for $i = 2, 3$. Thus, $s(x_{12}) = s(x_2) = s(x_{23}) \in s(t_{23}) = s(t_{23}^* t_{23})$. Let $t := t_{12} t_{23}^* t_{23}$, so that $t \leq t_{12}$ and $t \leq t_2 t_2^* t_{23} = t_{23}$. We have $x_{12} \in X_{t_{12}}|_{U_{t_{23}^* t_{23}}} \cong X_{t_{12}} \times_G X_{t_{23}^* t_{23}} \cong X_t$. Let x be the image of x_{12} under this isomorphism. Then, $j_{t_{12},t}(x) = x_{12}$. Hence, $j_{t_i,t}(x) = j_{t_i,t_{12}}(j_{t_{12},t}(x)) = x_i$ for $i = 1, 2$. Since $j_{t_2,t_{23}}(x_{23}) = x_2 = j_{t_2,t_{23}}(j_{t_{23},t}(x))$ and $j_{t_2,t_{23}}$ is injective by Proposition 2.9, we obtain $j_{t_{23},t}(x) = x_{23}$, and hence, also $j_{t_3,t}(x) = x_3$. Thus, $x_1 \sim x_3$, as desired.

We prove that π is open. Any open subset of X is a disjoint union of open subsets of the spaces X_t; so π is open if and only if all the maps $X_t \to X_\sim$ are open. Let $U \subseteq X_t$ be open; then we must check that $\pi^{-1}(\pi(U))$ is open. This set is a union over the set of triples $t, v, w \in S$ with $w \leq t, v$, where the set for t, v, w is contained in X_v and consists of all $j_{v,w}(x)$ with $x \in j_{t,w}^{-1}(U)$. The map $j_{v,w}$ is open by Proposition 2.9, and $j_{t,w}$ is continuous, so $j_{v,w}(j_{t,w}^{-1}(U))$ is open. Hence, $\pi^{-1}(\pi(U))$ is open as a union of open subsets of X, showing that π is open.

If $(t, x) \sim (t, y)$, then there are $u \leq t$ and $z \in X_u$ with $x = j_{t,u}(z) = y$; so, the map from X_t to $X_\sim$ is injective. Since π is open and continuous, it restricts to a homeomorphism from X_t onto an open subset of $X_\sim$. Thus π is a local homeomorphism. Since being locally Hausdorff or locally quasi-compact are local properties and π is a local homeomorphism, $X_\sim$ has one of these two properties if and only if X has, if and only if each X_t has. ☐

The space $X_\sim$ need not be Hausdorff, just as for étale groupoids constructed from inverse semigroup actions on spaces, where $X_\sim$ will be the groupoid of germs of the action (by Theorem 3.18).

From now on, we identify X_t with its image in $X_\sim$, using that $\pi|_{X_t}\colon X_t \to X_\sim$ is a homeomorphism onto an open subset by Lemma 3.10.

We will turn $X_\sim$ into a topological groupoid with the same object space G^0 as G. Since $j_{u,t}$ is a bibundle map, it is compatible with range and source maps. Thus, the maps $r, s\colon X_t \rightrightarrows G^0$ induce well-defined maps $r, s\colon X_\sim \rightrightarrows G^0$.

The multiplication maps $\mu_{t,u}$ give a continuous map $X \times_{s,G^0,r} X \to X$ by mapping the t,u-component of $X \times_{s,G^0,r} X$ to the tu-component of X by $\mu_{t,u}$. Equation (3.2) shows that this descends to a well-defined continuous map $\mu\colon X_\sim \times_{s,G^0,r} X_\sim \to X_\sim$.

Lemma 3.11. *The maps $r, s\colon X_\sim \rightrightarrows G^0$ and $\mu\colon X_\sim \times_{s,G^0,r} X_\sim \to X_\sim$ define a topological groupoid $X_\sim$. It contains G as an open subgroupoid. Hence, $X_\sim$ is étale if and only if G is.*

Proof. The multiplication is associative already on X by (A3) and the associativity of S. The maps r and s are open on $X_\sim$ because they are so on each X_t. The maps r, s, μ restricted to $G^1 = X_1$ reproduce the groupoid structure on G by (A1). Even more, (A2) implies that multiplication in $X_\sim$ with elements of X_1 is the same as the G-action. In particular, unit elements in G^1 act identically, so they remain unit elements in $X_\sim$. If $x \in X_t$, then $x^* \in X_{t^*}$ satisfies $\mu_{t,t^*}(x, x^*) = 1_{r(x)}$ by Proposition 3.9. Hence,

$$\pi(x,t) \cdot \pi(x^*, t^*) := \pi(\mu_{t,t^*}(x, x^*), tt^*) = \pi(1_{r(x)}, tt^*).$$

This is equivalent to the unit element $(1_{r(x)}, 1)$ in $X_\sim$ because j_{1,tt^*} is the usual inclusion map (more precisely, the above computation assumes that we identify $X_{tt^*} \cong G^1_{U_{tt^*}} \subseteq G^1$ using j_{1,tt^*}). Similarly, $\pi(x,t) \cdot \pi(x^*, t^*) \sim (1_{s(x)}, 1)$ is a unit element. Thus, $\pi(x^*, t^*)$ is inverse to $\pi(x,t)$. The map $\pi(x,t) \mapsto \pi(x^*, t^*)$ is continuous. Thus we have a topological groupoid. We have seen above that it contains G as an open subgroupoid. Therefore, $X_\sim$ is étale if and only if G is. $\square$

Definition 3.12. The groupoid $X_\sim$ is called the *transformation groupoid* of the S-action $(X_t, \mu_{t,u})$ on G and denoted by $G \rtimes S$, or by $G \rtimes_{X_t,\mu_{t,u}} S$ if the action must be specified.

Our proof shows that $G \rtimes S$ with the family of open subsets $(X_t)_{t\in S}$ encodes all the algebraic structure of our action by partial equivalences. The next definition characterizes when a groupoid H with a family of subsets $(H_t)_{t\in S}$ is the transformation groupoid of an inverse semigroup action.

Definition 3.13. Let S be an inverse semigroup. A (saturated) S-*grading* on a topological groupoid H is a family of open subsets $(H_t)_{t\in S}$ of H^1 such that

(Gr1) $H_t \cdot H_u = H_{tu}$ for all $t,u \in S$;
(Gr2) $H_t^{-1} = H_{t^*}$ for all $t \in S$;
(Gr3)
$$H_t \cap H_u = \bigcup_{v\le t,u} H_v$$
for all $t,u \in S$;
(Gr4)
$$H^1 = \bigcup_{t\in S} H_t.$$

If S has a zero element 0, we may also require $H_0 = \emptyset$.

Conditions (Gr1) and (Gr2) imply that H_1 is a subgroupoid of H, called the *unit fiber* of the grading. Conditions (Gr4) and (Gr1) imply that $s(H_1) = r(H_1) = H^0$. Condition (Gr3) implies $H_v \subseteq H_u$ for $v \le u$.

A non-saturated S-grading would be defined by weakening (Gr1) to $H_t \cdot H_u \subseteq H_{tu}$ for all $t,u \in S$. We only use saturated gradings and drop the adjective.

Theorem 3.14. *Let S be an inverse semigroup with unit. The transformation groupoid $G \rtimes S$ of an S-action on a groupoid G by partial equivalences is an S-graded groupoid. Any S-graded groupoid $(H,(H_t)_{t\in S})$ is isomorphic to one of this form, where $G^0 = H^0$ and $G^1 = H_1 \subseteq H$. Two actions by partial equivalences are isomorphic if and only if their transformation groupoids are isomorphic in a grading-preserving way.*

Here, an isomorphism between actions $(X_t)_{t\in S}$ and $(Y_t)_{t\in S}$ by partial equivalences on two groupoids G and H means the obvious: a family of homeomorphisms $X_t \cong Y_t$ compatible with the range, source and multiplication maps in Definition 3.5.

Proof. It follows directly from our construction that the subspaces $X_t \subseteq G \rtimes S$ for an S-action by partial equivalences satisfy (Gr1)–(Gr4). It is also clear that the transformation groupoid construction is natural for isomorphisms of S-actions.

Let H with the subspaces H_t for $t \in S$ be an S-graded topological groupoid. Then $G^1 := H_1$ with $G^0 = H^0$ is an open subgroupoid of H. Let $X_t = H_t$ with the restriction of the range and source map of H, and with the G-action and maps $\mu_{t,u} \colon X_t \times_{s,G^0,r} X_u \to X_{tu}$ from the multiplication map in H. This satisfies (S3) by definition, (S4) because $H_t \cdot H_u = H_{tu}$, (S1) and (S7) because H is a groupoid, and (S2) because X_t is open in H and the range and source maps of H are open. If $(y, z) \in X_u \times_{s,G^0,s} X_{tu}$, then $zy^{-1} \in X_{tu}X_{u^*} = X_{tuu^*} \subseteq X_t$ because $tuu^* \leq t$. Hence, $(y, z) \mapsto (zy^{-1}, y)$ gives a continuous inverse for the map in (S5), so that the latter is a homeomorphism. A similar argument shows that the map in (S6) is a homeomorphism. Thus, we obtain an S-action by partial equivalences. This construction is natural in the sense that isomorphic S-graded groupoids give isomorphic actions by partial equivalences.

If we begin with an action by partial equivalences, turn it into a graded groupoid, and then back into an action by partial equivalences, then we get an isomorphic action by construction. When we start with a graded groupoid, go to an action by partial equivalences and back to a graded groupoid, then we also get back our original S-graded groupoid. The only non-trivial point is that the map

$$\pi \colon \bigsqcup_{t \in S} H_t \longrightarrow \Big(\bigsqcup_{t \in S} H_t\Big)_{\sim}$$

identifies $x \in H_t$ and $y \in H_u$ for $t, u \in S$ if and only if $x = y$ in H; this is exactly the meaning of (GR3). □

3.3. Examples: group actions and actions on spaces. The equivalence between actions by partial equivalences and graded groupoids makes it easy to describe all actions of groups on groupoids and all actions of inverse semigroups on spaces.

Theorem 3.15. *Let G be a topological groupoid, and let S be a group, viewed as an inverse semigroup. Then, an S-action on G by (partial) equivalences is equivalent to a groupoid H containing G as an open subgroupoid with $H^0 = G^0$, and with a continuous groupoid homomorphism $\pi\colon H \twoheadrightarrow S$ such that $\pi^{-1}(1) = G$ and, for each $x \in H^0$ and $t \in S$, there is $h \in H^1$ with $s(h) = x$ and $\pi(h) = t$. In this situation, H is the transformation groupoid $G \rtimes S$. If G is also a group, this is the same as a group extension $G \rightarrowtail H \twoheadrightarrow S$.*

Proof. Since $tt^* = 1$ for any $t \in S$, any action of S by partial equivalences will be an action by global equivalences. By Theorem 3.14, we may replace an S-action by partial equivalences by an S-graded groupoid $(H, (H_t)_{t\in S})$. We have $H^0 = G^0$ by construction. Since S is a group, (Gr3) says that $H_t \cap H_u = \emptyset$ for $t \neq u$. Thus, we obtain a well-defined map $\pi\colon H^1 \to S$ with $\pi^{-1}(t) = H_t$; in particular, $G = \pi^{-1}(1)$. The map π is continuous because the subsets H_t are open. The condition on the existence of h for given x, t says that the map $s\colon H_t \to H^0$ is onto, that is, H_t is a global equivalence. Thus, an S-action on G gives $\pi\colon H \to S$ with the asserted properties.

For the converse, let $\pi\colon H \to S$ be a groupoid homomorphism as in the statement. Define $H_t := \pi^{-1}(t) \subseteq H^1$. These are open subsets because π is continuous. If $t, u \in S$, then $H_t H_u \subseteq H_{tu}$ is trivial. If $h \in H_{tu}$, then our technical assumption gives $h_2 \in H_u$ with $s(h_2) = s(h)$. Then, $h_1 := hh_2^{-1} \in H_t$, so $h \in H_t H_u$. Thus, (Gr1) holds. The remaining conditions for an S-grading are trivial in this case, and H is the transformation groupoid $G \rtimes S$ by construction.

If G is also a group, then so is H because $G^0 = H^0$, and then the condition on π simply says that it is a surjection $\pi\colon H \twoheadrightarrow S$ with kernel G. This is the same as a group extension. □

The obvious definition of a group action by automorphisms on another group only covers *split* group extensions. We need some kind of twisted action by automorphisms to allow for non-trivial group extensions as well. Our notion of action by equivalences achieves this very naturally.

For groupoid extensions, one usually requires the kernel to be a group bundle; this need not be the case here. There are many examples of groupoid homomorphisms (or 1-cocycles) with the properties required in Theorem 3.15. We mention one typical case:

Example 3.16. Let H be the groupoid associated to a self-covering $\sigma\colon X \to X$ of a compact space X as in **[9]**. The canonical $\mathbb{Z}$-valued cocycle $\pi\colon H \to \mathbb{Z}$ on it clearly has the properties needed to define a $\mathbb{Z}$-grading on H. The subgroupoid $G := \pi^{-1}(0)$ is the groupoid that describes the equivalence relation generated by $x \sim y$ if $\sigma^k(x) = \sigma^k(y)$ for some $k \in \mathbb{N}$. The action of σ on X preserves this equivalence relation and hence gives an endomorphism of G; this endomorphism is an equivalence, and our $\mathbb{Z}$-action on G by equivalences is generated by this self-equivalence of G. But, unless σ is a homeomorphism, σ is not invertible on G, so it gives no action of $\mathbb{Z}$ by automorphisms.

Now we turn to actions of inverse semigroups on topological spaces. Let S be an inverse semigroup with unit, and let Z be a topological space. First, we recall Exel's construction of the *groupoid of germs* for an inverse semigroup action by partial homeomorphisms **[12]**.

Let $\mathrm{pHomeo}(Z)$ be the inverse semigroup of partial homeomorphisms of Z. An action of S on Z by partial homeomorphisms is a monoid homomorphism $\theta\colon S \to \mathrm{pHomeo}(Z)$. This gives partial homeomorphisms $\theta_t\colon D_{t^*t} \to D_{tt^*}$ for $t \in S$ with open subsets $D_e \subseteq Z$ for $e \in E(S)$. The groupoid of germs has object space Z, and its arrows are the "germs" $[t, z]$ for $t \in S$, $z \in D_{t^*t}$; by definition, $[t, z] = [u, z']$ if and only if there is an $e \in E(S)$ with $z = z' \in D_e$ and $te = ue$. The groupoid structure is defined by

$$s[t, z] = z, \qquad r[t, z] = \theta_t(z), \qquad [t, z] \cdot [u, z'] = [tu, z']$$

if $z = \theta_u(z')$, and $[t, z]^{-1} = [t^*, \theta_t(z)]$. The subsets $\{[t, z] \mid z \in U\}$ for $t \in S$ and an open subset $U \subseteq D_{t^*t}$ form a basis for the topology on the arrow space.

Remark 3.17. Many authors use another germ relation that only requires an open subset V of Z with $z \in V$ and $\theta_t|_V = \theta_u|_V$. This may give a different groupoid, of course. Exel's germ groupoids need not be essentially principal, see **[32]**.

Theorem 3.18. *Let Z be a topological space viewed as a topological groupoid, and let S be an inverse semigroup with unit. Isomorphism classes of actions of S on Z by partial equivalences are in natural bijection with actions of S on Z by partial homeomorphisms. The transformation groupoid $Z \rtimes S$ for an action by partial equivalences is the groupoid of germs defined by Exel* [**12**].

Proof. Let $\theta\colon S \to \mathrm{pHomeo}(Z)$ be an action of S by partial homeomorphisms. Exel's groupoid of germs carries an obvious S-grading by the open subsets $X_t := \{[t, z] \mid z \in D_{t^*t}\}$ with $X_1 = Z$. The conditions in Definition 3.13 are trivial to check. Hence, Exel's groupoid is the transformation groupoid $Z \rtimes S$ for an action of S on Z by partial equivalences by Theorem 3.14. Conversely, an S-action on Z is equivalent to the S-graded groupoid $Z \rtimes S$. This groupoid is étale. The assumptions of an S-grading imply that the subsets $X_t \subseteq Z \rtimes S$ form an inverse semigroup of bisections that satisfies the assumptions in [**12,** Proposition 5.4], which ensures that the groupoid of germs is $Z \rtimes S$. □

Corollary 3.19. *Let G be an étale groupoid, let S be an inverse semigroup and let $f\colon S \to \mathrm{Bis}(G)$ be a semigroup homomorphism. This induces an isomorphism $G^0 \rtimes S \cong G$ if and only if*

$$\bigcup_{t \in S} f(t) = G$$

and

$$f(t) \cap f(u) = \bigcup_{\substack{v \in S \\ v \le t,u}} f(v)$$

for all $t, u \in S$.

Here $G^0 \rtimes S$ uses the action of S on G^0 induced by f and the usual action of $\mathrm{Bis}(G)$.

Proof. Add a unit to S and map it to the unit bisection $G^0 \subseteq G$, so that we may apply Theorem 3.14. For $t \in S$, let $G_t := f(t) \subseteq G^1$; these are open subsets because each $f(t)$ is a bisection. Since f is a semigroup homomorphism, (Gr1) and (Gr2) hold. The other two conditions are exactly the technical assumptions of Corollary 3.19. Thus, these two assumptions are equivalent to $(G_t)_{t\in S}$ being an S-grading on G. If they hold, then Theorem 3.14 says that $G \cong G_1 \rtimes S = Z \rtimes S$. Conversely, the transformation groupoid $Z \rtimes S$ is S-graded by Theorem 3.14, so if $G \cong Z \rtimes S$, then it satisfies the two technical assumptions. □

A subsemigroup $S \subseteq \mathrm{Bis}(G)$ with the properties required in Corollary 3.19 is called *wide*. Corollary 3.19 explains why they appear so frequently, see, for instance, [**12, 28, 3**]. It has already been shown [**12,** Proposition 5.4] that $Z \rtimes S = G$ if S is wide, but we have not yet seen the converse statement.

Since the proof of Theorem 3.18 is not explicit, we give another pedestrian proof.

Let $\theta_t\colon D_{t^*t} \to D_{tt^*}$ for $t \in S$ give an action of S on Z by partial homeomorphisms. This is a groupoid isomorphism from D_{t^*t} to D_{tt^*}, which we turn into a partial equivalence from Z to itself as in Example 2.5. Here, this means that we take $X'_t := D_{tt^*}$ with anchor maps $r'(z) := z$, $s'(z) := \theta_t^{-1}(z)$. Since all arrows in Z are units, the range and source maps determine the partial equivalence. The homeomorphism θ_t gives a bibundle isomorphism from X'_t to $X_t = D_{t^*t}$ with $r(z) := \theta_t(z)$ and $s(z) := z$. The comparison with Exel's groupoid is more obvious for the second choice, which we take from now on.

There is an obvious homeomorphism

$$X_t \times_Z X_u \xrightarrow{\sim} \{z \in D_{tt^*} \mid \theta_t(z) \in D_{uu^*}\} = D_{(tu)^*(tu)},$$

such that the range and source maps are θ_{tu} and the inclusion map, respectively. We choose this isomorphism for $\mu_{t,u}$ to define our action by partial equivalences. Actually, this is no choice at all because the range and source maps are injective here, so there is at most one bibundle map between any two partial equivalences. (We will see more groupoids with this property in subsection 3.5.) Hence, the associativity condition in the definition of an inverse semigroup action holds automatically. Thus, we have turned an action by partial homeomorphisms on Z into an action by partial equivalences on Z, viewed as a topological groupoid.

Our construction of the transformation groupoid above is *exactly* the construction of the groupoid of germs in this special case, so the isomorphism between $Z \rtimes S$ and the groupoid of germs from [**12**] is trivial.

Next, we check that every partial equivalence X of Z is isomorphic to one coming from a partial homeomorphism. Since all arrows in Z are units, we have $X/Z = X = Z\backslash X$. Hence, the anchor maps $G^0 \leftarrow X \to H^0$ are continuous, open and injective by condition (P3) in Definition 2.1. The map

$$\theta := r \circ s^{-1} \colon s(X) \longrightarrow r(X)$$

is a partial homeomorphism from G^0 to H^0, and

$$r \colon X \longrightarrow r(X)$$

is an isomorphism of partial equivalences from X to X_θ.

Since there is always only one isomorphism between partial equivalences coming from the same partial homeomorphism, an inverse semigroup action on Z is uniquely determined by the isomorphism classes of the X_t, which are in bijection with partial homeomorphisms of Z. This proves the first statement in Theorem 3.18.

3.4. Morita invariance of actions by partial equivalences.

Proposition 3.20. *Let Y be an equivalence from H to G, and let $(X_t, \mu_{t,u})$ be an action of an inverse semigroup with unit on G. Let $X'_t := Y \times_G X_t \times_G Y^*$, and let $\mu'_{t,u} \colon X'_t \times_H X'_u \to X'_{tu}$ be the composite isomorphism*

$$\begin{aligned} Y \times_G X_t \times_G Y^* \times_H Y \times_G X_u \times_G Y^* &\xrightarrow{\sim} Y \times_G X_t \times_G G^1 \times_G X_u \times_G Y^* \\ &\xrightarrow{\sim} Y \times_G X_t \times_G X_u \times_G Y^* \xrightarrow[\cong]{\mu_{t,u}} Y \times_G X_{tu} \times_G Y^*, \end{aligned}$$

where the first two isomorphisms are canonical from Proposition 2.11 and Lemma 2.8. Then, $\mu'_{t,u}$ is an action of S on H by partial equivalences. Its transformation groupoid $H \rtimes S$ is equivalent to $G \rtimes S$.

When we translate the action on Y back to X using the inverse equivalence Y^*, we get an action on G that is isomorphic to the original one.

Proof. More precisely, X'_1 as defined above is only isomorphic to H^1 in a very obvious way. We should only use the above definition of X'_t for $t \neq 1$, let $X'_1 := H^1$ for $t = 1$, and let $\mu'_{1,t}$ and $\mu'_{t,1}$ be the canonical isomorphisms. We should also put in associators for the composition of partial equivalences, which only cause notational complications, however. Up to these technicalities, it is clear that the maps μ' inherit associativity from the maps μ. The action that we get by translating μ' back to G with Y^* is canonically isomorphic to the original action because $Y^* \times_H Y \cong G^1$.

It remains to prove the equivalence of the transformation groupoids $G \rtimes S$ and $H \rtimes S$. Here, we use the linking groupoid L of the equivalence; its object space is $L^0 = G^0 \sqcup H^0$, its arrow space is $G^1 \sqcup Y \sqcup Y^* \sqcup H^1$, its range and source maps are r and s on each component, and its multiplication consists of the multiplications in G and H, the G,H-bibundle structure on Y, the H,G-bibundle structure on Y^*, and the canonical isomorphisms $Y \times_G Y^* \xrightarrow{\sim} H^1$ and $Y^* \times_H Y \xrightarrow{\sim} G^1$ from Proposition 2.11. This gives a topological groupoid L. There is a canonical right action of L on $G^1 \sqcup Y = r^{-1}(G^1) \subseteq L^1$ that provides an equivalence from L to G when combined with the left actions of G on G^1 and Y; there is a similar canonical equivalence $H^1 \sqcup Y^*$ from L to H.

We may transport the S-action on G to L because it is equivalent to G. When we transport this action on L further to H, we get the action described above because the composite equivalence $(G^1 \sqcup Y) \times_L (H^1 \sqcup Y^*)^*$ from H to G is isomorphic to Y.

The action on L is given by bibundles

$$(G^1 \sqcup Y)^* \times_G X_t \times_G (G^1 \sqcup Y) \\ \cong X_t \sqcup (X_t \times_G Y) \sqcup (Y^* \times_G X_t) \sqcup (Y^* \times_G X_t \times_G Y),$$

where we canceled factors of G^1 using Lemma 2.8. When we restrict the transformation groupoid $L \rtimes S$ to $G^0 \subseteq L^0$ or to $H^0 \subseteq L^0$, then we only pick the components X_t and $Y^* \times_G X_t \times_G Y$ in the above decomposition, so we get the transformation groupoids $G \rtimes S$ and $H \rtimes S$, respectively. Routine computations show that the other two parts $r^{-1}(G^0) \cap s^{-1}(H^0)$ and $r^{-1}(H^0) \cap s^{-1}(G^0)$ of $L \rtimes S$ give an equivalence from $H \rtimes S$ to $G \rtimes S$, such that $L \rtimes S$ is the resulting linking groupoid.

It can be shown with less routine computations that the embedding $G \rtimes S \hookrightarrow L \rtimes S$ is fully faithful and essentially surjective. We checked "fully faithful" above. Being *essentially surjective* means that the map

$$G^0 \times_{\subset, L^0, r} L^1 \longrightarrow L^0, \qquad (x, l) \longmapsto s(l),$$

is open and surjective. It is open because $r\colon L^1 \to L^0$ is open and $G^0 \subset L^0$ is open, and surjective because already $G^0 \times_{L^0} Y \subset G^0 \times_{L^0} L^1$ surjects onto H^0. Since both $G \rtimes S \hookrightarrow L \rtimes S$ and $H \rtimes S \hookrightarrow L \rtimes S$ are fully faithful and essentially surjective, they induce equivalence bibundles by **[21,** Proposition 6.8], which we may compose to an equivalence from $H \rtimes S$ to $G \rtimes S$. Of course, this gives the same equivalence as the argument above. □

Corollary 3.21. *Let S be an inverse semigroup with unit. Let $f\colon X \to Z$ be an open continuous surjection, and let $G(f)$ be its covering groupoid, see Definition* A.8. *Then S-actions by partial equivalences on $G(f)$ are canonically equivalent to S-actions on Z by partial homeomorphisms, such that $G(f) \rtimes S$ is equivalent to $Z \rtimes S$.*

Here *equivalent* means an equivalence of categories, where the arrows are isomorphisms of S-actions that fix the underlying groupoid.

Proof. $G(f)$ is canonically equivalent to Z viewed as a groupoid, so the assertion follows from Theorem 3.18 and Proposition 3.20. □

In particular, Corollary 3.21 applies to the Čech groupoid $G_{\mathfrak{U}}$ of an open covering $\mathfrak{U}$ of a locally Hausdorff space Z by Hausdorff open subsets. Thus, we may replace an S-action by partial homeomorphisms on a locally Hausdorff space Z by an "equivalent" action by partial equivalences on a Hausdorff groupoid $G_{\mathfrak{U}}$, and the resulting transformation groupoids $Z \rtimes S$ and $G_{\mathfrak{U}} \rtimes S$ are equivalent.

The quickest way to describe the resulting S-action on $G_{\mathfrak{U}}$ explicitly is by describing $G_{\mathfrak{U}} \rtimes S$ and an S-grading on it. Let

$$X := \bigsqcup_{U \in \mathfrak{U}} U,$$

and let $p\colon X \to Z$ be the canonical map, which is an open surjection. The pull-back $p^*(Z \rtimes S)$ of $Z \rtimes S$ along p is a groupoid with object space X, arrow space

$$X \times_{p,Z,r} (Z \rtimes S)^1 \times_{s,Z,p} X,$$

$$r(x_1, g, x_2) = x_1,$$
$$s(x_1, g, x_2) = x_2,$$

and

$$(x_1, g, x_2) \cdot (x_2, h, x_3) = (x_1, g \cdot h, x_3),$$

see [**21,** Example 3.13]. Let

$$X_t := \{(x_1, g, x_2) \in X \times_{p,Z,r} (Z \rtimes S)^1 \times_{s,Z,p} X \mid g \in t\}.$$

Proposition 3.22. *The subspaces $X_t \subseteq p^*(Z \rtimes S)^1$ form an S-grading on $p^*(Z \rtimes S)$. The resulting S-graded groupoid is the transformation groupoid for the S-action on $G_{\mathfrak{U}}$ that we obtain by translating the S-action on Z along the equivalence to $G_{\mathfrak{U}}$.*

Proof. The subspaces X_t form an S-grading because the bisections $t \in S$ give an S-grading on $Z \rtimes S$ and p is surjective. Hence, they describe an S-action on $G_{\mathfrak{U}}$. The equivalence from $G_{\mathfrak{U}}$ to Z is given by the canonical action of $G_{\mathfrak{U}}$ on $G^0_{\mathfrak{U}} = X$ and the projection $p\colon X \to Z$. Hence, X_t is exactly what we get when we translate $t \subseteq Z \rtimes S$ along the equivalence. □

For instance, let H be an étale groupoid with locally Hausdorff arrow space, and let S be some inverse semigroup of bisections with $H \cong H^0 \rtimes S$; we could take $S = \mathrm{Bis}(H)$. Let $Z = H^1$ with the action of H by left multiplication. This induces an action of S on Z. Its transformation groupoid $H^1 \rtimes S$ is $H^1 \rtimes H$ with the obvious S-grading by $H^1 \times_{H^0} t$ for $t \in S$.

The left multiplication action of H on H^1 with the bundle projection $s\colon H^1 \to H^0$ is a trivial principal bundle. In particular, the transformation groupoid $H^1 \rtimes S \cong H^1 \rtimes H$ is isomorphic to the covering groupoid of the cover $s\colon H^1 \twoheadrightarrow H^0$. Hence, it is equivalent to the space H^0, viewed as a groupoid with unit arrows only. The S-grading on $H^1 \rtimes S$ does, however, not carry over to H^0.

If we replace the S-action on H^1 by an equivalent S-action on $G_{\mathfrak{U}}$ for a Hausdorff open cover of H^1, then the transformation groupoid $G_{\mathfrak{U}} \rtimes S$ is equivalent to $H^1 \rtimes S$, and hence, also equivalent to the space H^0. In particular, the groupoid $G_{\mathfrak{U}} \rtimes S$ is *basic*, see Section A.2. If a groupoid is equivalent to a space, then this space must be its orbit space. So if H^0 is Hausdorff, then the groupoid $G_{\mathfrak{U}} \rtimes S$ is a free and proper, Hausdorff groupoid by Proposition A.7.

3.5. Local centralizers. We shall show that, for many groupoids G, the bibundles X_t already determine the multiplication maps $\mu_{t,u}$ and thus the inverse semigroup action. This happens, among others, for essentially principal topological groupoids (meaning that the isotropy group bundle has no interior; see [**32**]) and for groups with trivial center.

Definition 3.23. A *local centralizer* of G is a map $\gamma\colon U \to G^1$ defined on an open G-invariant subset U of G^0 with $s(\gamma(x)) = r(\gamma(x)) = x$ for all $x \in U$ and $\gamma(r(g)) \cdot g = g \cdot \gamma(s(g))$ for all $g \in G$. We say that G has *no local centralizers* if all local centralizers are given by $\gamma(x) = 1_x$ for $x \in U$ and some U as above.

Local centralizers defined on the same subset U form an Abelian group under pointwise multiplication. All local centralizers form an Abelian inverse semigroup. It is the center of $\operatorname{Bis}(G)$ if G is étale.

Lemma 3.24. *Let X be a partial equivalence from H to G. Then* $\operatorname{Map}(X, X)$ *is isomorphic to the group of local centralizers of G defined on $r(X)$, and to the group of local centralizers of H defined on $s(X)$.*

If G has no local centralizers and X and Y are partial equivalences from G or to G, then there is at most one bibundle map $X \to Y$, so bibundle isomorphisms are unique if they exist.

Proof. The two descriptions of $\operatorname{Map}(X, X)$ are equivalent by taking X^*, so we only prove one. Every bibundle map $X \to X$ is invertible by Proposition 2.9. The canonical group homomorphisms

$$\operatorname{Map}(X,X) \xrightarrow{\llcorner\times_H X^*} \operatorname{Map}(X\times_H X^*, X\times_H X^*) \cong \operatorname{Map}(G^1_{r(X)}, G^1_{r(X)}) \xrightarrow{\llcorner\times_G X} \operatorname{Map}(X,X)$$

are inverse to each other by the proof of Proposition 2.11. Thus, it remains to identify the set $\operatorname{Map}(G^1_U, G^1_U)$ for an open G-invariant subset U of G^0 with the group of local centralizers defined on U. We may view G^1_U as the equivalence from G^1_U to itself associated to the identity functor on G^1_U. We described all bibundle isomorphisms between such equivalences in Example 2.5. Specializing Example 2.5 to the automorphisms of the identity functor gives exactly the local centralizers defined on U. A quick computation shows that the composition of bibundle isomorphisms corresponds to the pointwise multiplication of local centralizers.

Let $f_1, f_2\colon X \to Y$ be bibundle maps. Then both are bibundle isomorphisms $X \to Y|_{s(X)}$, and we may form a composite bibundle isomorphism $f_2^{-1} \circ f_1\colon X \to X$. Since there are no local centralizers, the first part of Lemma 3.24 shows that this is the identity map, so $f_1 = f_2$. In particular, if two partial equivalences $G \to H$ or $H \to G$ are isomorphic, then the isomorphism is unique. □

Recall that $\widetilde{\mathfrak{peq}}(G)$ denotes the inverse semigroup of *isomorphism classes* of partial equivalences on G.

Theorem 3.25. *Let G be a topological groupoid without local centralizers. An action of an inverse semigroup S on G is equivalent to a homomorphism $S \to \widetilde{\mathfrak{peq}}(G)$. More precisely, isomorphism classes of S-actions on G by partial equivalences are in canonical bijection with homomorphisms $S \to \widetilde{\mathfrak{peq}}(G)$.*

Proof. A homomorphism $f\colon S \to \widetilde{\mathfrak{peq}}(G)$ gives us bibundles X_t with $X_t \times_G X_u \cong X_{tu}$ and $X_1 \cong G^1$; we may as well assume $X_1 = G^1$. By Lemma 3.24, the isomorphisms $\mu_{t,u}\colon X_t \times_G X_u \xrightarrow{\sim} X_{tu}$ above are unique, so there is no need to specify them. Conditions (A2) and (A3) hold because any two parallel bibundle isomorphisms are equal. Thus, f determines an S-action by partial equivalences. Conversely, an action by partial equivalences determines such a homomorphism by taking the isomorphism classes of the X_t and forgetting the $\mu_{t,u}$. Since isomorphisms of partial equivalences are unique if they exist, this forgetful functor is actually not forgetting anything here, so we get a bijection between isomorphism classes of actions by partial equivalences and homomorphisms $S \to \widetilde{\mathfrak{peq}}(G)$. □

The results in this section are inspired by the notion of a "quasi-graphoid" used by Debord [**10**]. Debord already treated partial equivalences of groupoids as arrows between groupoids and used them to glue together groupoids constructed locally. She restricts, however, to a situation where bibundle isomorphisms are uniquely determined. Even more, she wants the range and source maps to determine a partial equivalence uniquely. For this, she assumes that a smooth map $\gamma\colon U \to G^1$ defined on an open subset U of G^0 must already be the unit section if it only satisfies $s(\gamma(x)) = r(\gamma(x)) = x$ for all $x \in U$. This condition holds for holonomy groupoids of foliations–even for the mildly singular foliations that she considers.

3.6. Decomposing proper Lie groupoids. A manifold may be constructed by taking a disjoint union of local charts and gluing them together along the coordinate change maps, which are partial homeomorphisms, or diffeomorphisms in the smooth case. When constructing groupoids locally, it is more likely that the coordinate change maps are no longer partial isomorphisms but only partial equivalences. Actually, it may well be that the local pieces are, to begin with, only local groupoids and not groupoids, see [**10**]; this is not covered by our theory. Therefore, we know no good examples where groupoids have been constructed by gluing together smaller groupoids along partial equivalences.

Instead, we take a groupoid as given and analyze it using local information. The local information should say that the groupoid locally is *equivalent* to one of a particularly simple form. Then the groupoid is globally equivalent to a transformation groupoid for an inverse semigroup action by partial equivalences on a disjoint union of groupoids having the desired simple form.

We now get more concrete and consider a proper Lie groupoid H. To formulate stronger results, we shall work with (partial) equivalences of Lie groupoids in this section; that is, spaces are replaced by smooth manifolds, continuous maps by smooth maps and open maps by submersions. This does not change the theory significantly, see [**21**].

First we formulate the local linearisability of proper Lie groupoids. This was conjectured by Weinstein [**37**] and proved by Zung [**38**]. Both authors try to describe the local structure of proper Lie groupoids *up to isomorphism.* Following Trentinaglia [**35**], we only aim for a description up to Morita equivalence:

Theorem 3.26. *Let H be a proper Lie groupoid. For every $x \in H^0$, there are an open H-invariant neighborhood U_x of x in H^0, a linear* *representation of the stabilizer group H_x on a finite-dimensional vector space W_x, and a Lie groupoid equivalence from the transformation groupoid $W_x \rtimes H_x$ to H_{U_x}.*

The vector space W_x is the normal bundle to the H-orbit Hx of x, with its canonical representation of H_x.

Weinstein and Zung impose extra assumptions on H to describe H_{U_x} up to isomorphism. The argument in [**35,** Section 4] shows how to deduce Theorem 3.26 quickly from [**38,** Theorem 2.3] without extra assumptions.

Actually, we do not need H to be proper. Since we only need local structure, it is enough for H to be *locally proper*, that is, each $x \in H^0$ has an H-invariant open neighborhood U such that H_U is proper; this allows the orbit space H^0/H to be a locally Hausdorff but non-Hausdorff manifold.

Assume now that H is a locally proper groupoid. By Theorem 3.26, there is a covering $\mathfrak{U}$ of H^0 by open, H-invariant subsets and, for each $U \in \mathfrak{U}$, a Lie groupoid equivalence X_U from a transformation groupoid $W_U \rtimes G_U$ for a compact Lie group G_U and a linear representation W_U of G_U to the restriction H_U. Now let

$$G := \bigsqcup_{U \in \mathfrak{U}} W_U \rtimes G_U.$$

This disjoint union is a groupoid with object space $\bigsqcup W_U$.

Let K be the covering groupoid of H^0 for the covering $\mathfrak{U}$. Since $H_U|_{U\cap V} = H_{U\cap V} = H_V|_{U\cap V}$, the inverse semigroup $S := \mathrm{Bis}(K)$ acts on $\bigsqcup_{U\in\mathfrak{U}} H_U$: each element of $\mathrm{Bis}(K)$ acts by the identity equivalence between the appropriate restrictions of H_U and H_V, and all the multiplication maps are the canonical isomorphisms. The disjoint union $X := \bigsqcup_{U\in\mathfrak{U}} X_U$ gives an equivalence from G to $\bigsqcup_{U\in\mathfrak{U}} H_U$, so we may transfer this S-action to G.

We make the action on G more concrete. Any bisection of K is a disjoint union of bisections of the form

$$(U_1, D, U_2) := \{(U_1, x, U_2) \mid x \in D\}$$

for $U_1, U_2 \in \mathfrak{U}$ and an open subset $D \subseteq U_1 \cap U_2$. The product $(U_1, D_1, U_2) \cdot (U_2', D_2, U_3)$ is empty if $U_2 \neq U_2'$ and is equal to $(U_1, D_1 \cap$

$D_2, U_3)$ if $U_2 = U_2'$.

The partial equivalence X_{U_1,D,U_2} on G associated to (U_1, D, U_2) is the composite partial equivalence

$$G \supseteq W_{U_1} \rtimes G_{U_1} \xrightarrow{{}_D|X^*_{U_1}} H_D \xrightarrow{X_{U_2}|_D} W_{U_2} \rtimes G_{U_2} \subseteq G.$$

The composites of X_{U_1,D_1,U_2} and X_{U_2',D_2,U_3} are clearly empty for $U_2 \neq U_2'$, as they should be. If $U_2 = U_2'$, then there is a canonical isomorphism of partial equivalences

$$\mu_{(U_1,D_1,U_2),(U_2,D_2,U_3)} \colon X_{U_1,D_1,U_2} \times_G X_{U_2,D_2,U_3} \longrightarrow X_{U_1,D_1\cap D_2,U_3},$$

using the restriction of the canonical pairing $X_{U_2} \times_G X^*_{U_2} \to H_{U_2}$ to remove the extra two factors in the middle. This is exactly what happens if we translate the "trivial" action of S on $\bigsqcup H_U$ described above to G along the equivalence $\bigsqcup X_U$.

Theorem 3.27. *The locally proper Lie groupoid H is equivalent to the transformation groupoid $G \rtimes S$ for the action of S on G described above.*

Proof. Since we constructed the action of S on G by translating the action on $\bigsqcup_{U\in\mathfrak{U}} H_U$, Proposition 3.20 shows that $G \rtimes S$ is equivalent to $\bigsqcup_{U\in\mathfrak{U}} H_U \rtimes S$. Since S acts "trivially" on $\bigsqcup H_U$, this transformation groupoid is easy to understand: it is the pull-back $p^*(H)$ of H for the canonical map $p\colon \bigsqcup_{U\in\mathfrak{U}} U \to H^0$. Since p is a surjective submersion, $p^*(H)$ is equivalent to H. □

As a result, any locally proper Lie groupoid is equivalent to a transformation groupoid for an inverse semigroup action on a disjoint union of linear actions of compact groups. Such transformation groupoids need not be locally proper, however, so we do not have a characterization of locally proper Lie groupoids. The groupoid $G \rtimes S$ is étale if and only if G is, if and only if the stabilizers H_x are finite. This means that H is an orbifold, see **[22]**.

4. Inverse semigroup actions on C*-algebras. We now define inverse semigroup actions on C*-algebras by Hilbert bimodules, in parallel to actions on groupoids by partial equivalences.

Definition 4.1. A *Hilbert A,B-bimodule* $\mathcal{H}$ is a left Hilbert A-module and a right Hilbert B-module such that the left and right multiplications commute, and

$$\langle\langle x|y\rangle\rangle_A \cdot z = x \cdot \langle y|z\rangle_B \quad \text{for all } x,y,z \in \mathcal{H}.$$

A *Hilbert A,B-bimodule map* is a bimodule map that also intertwines both inner products.

Let $\mathcal{H}$ be a Hilbert A,B-bimodule. Let $I \lhd A$ and $J \lhd B$ be the closed linear spans of the elements $\langle\langle x|y\rangle\rangle_A$ and $\langle x|y\rangle_B$ with $x,y \in \mathcal{H}$, respectively. These are closed ideals in A and B, and $\mathcal{H}$ is an I,J-imprimitivity bimodule by restricting the left multiplications to I and J. Ideals in a C*-algebra are in bijection with open subsets of its primitive ideal space, so ideals are the right analogues of open invariant subsets of groupoids. Hence, we denote the ideals I and J above as

$$I := r(\mathcal{H}) \quad \text{and} \quad J := s(\mathcal{H}),$$

and we think of Hilbert A,B-bimodules as partial Morita equivalences from B to A.

Given an ideal $K \lhd A$, we define the restriction of a Hilbert bimodule $\mathcal{H}$ to K as ${}_K|\mathcal{H} := K \cdot \mathcal{H} \subseteq \mathcal{H}$, which is canonically isomorphic to $K \otimes_A \mathcal{H}$. We restrict to ideals in B in a similar way.

The left action of A on a Hilbert bimodule is by a nondegenerate *-homomorphism $A \to \mathbb{B}(\mathcal{H})$ into the adjointable operators on $\mathcal{H}$. Thus, a Hilbert A,B-bimodule becomes a correspondence by forgetting the left inner product.

Lemma 4.2. *A correspondence $\mathcal{H}$ carries a Hilbert bimodule structure if and only if there is an ideal $I \lhd A$ such that the left action $\varphi\colon A \to \mathbb{B}(\mathcal{H})$ restricts to an isomorphism from I onto $\mathbb{K}(\mathcal{H})$. This ideal and the left inner product are uniquely determined by the correspondence.*

Proof. First, let $\mathcal{H}$ be a Hilbert bimodule. Then $\mathcal{H}$ is an imprimitivity bimodule from $s(\mathcal{H})$ to $r(\mathcal{H})$, so $\varphi|_{r(\mathcal{H})}$ is an isomorphism from $r(\mathcal{H})$ onto $\mathbb{K}(\mathcal{H})$. If $I \lhd A$ is another ideal with $\varphi(I) = \mathbb{K}(\mathcal{H})$, then $\varphi(r(\mathcal{H}) \cdot I) = \mathbb{K}(\mathcal{H})$ as well. Thus, $r(\mathcal{H})$ is the minimal ideal that φ maps onto $\mathbb{K}(\mathcal{H})$, and the only one on which this happens isomorphically. Thus, $r(\mathcal{H})$ is already determined by the underlying correspondence.

Let $\mathcal{H}'$ be another Hilbert A, B-bimodule with the same underlying correspondence as $\mathcal{H}$ and with left A-valued inner product $\langle\langle x|y\rangle\rangle'_A$. Then

$$\varphi(\langle\langle x|y\rangle\rangle'_A)z = x \cdot \langle y|z\rangle_B = \varphi(\langle\langle x|y\rangle\rangle_A)z$$

for all $x, y, z \in \mathcal{H}$. Since $r(\mathcal{H}) = r(\mathcal{H}')$ depends only on the correspondence and the restriction of φ to $r(\mathcal{H})$ is faithful, we get $\mathcal{H} = \mathcal{H}'$ as Hilbert bimodules.

Now let $\mathcal{H}$ be a correspondence, and let $I \triangleleft A$ be an ideal that is mapped isomorphically onto $\mathbb{K}(\mathcal{H})$. Transfer the usual $\mathbb{K}(\mathcal{H})$-valued left inner product on $\mathcal{H}$ through this isomorphism to one with values in $A \supseteq I$. This turns $\mathcal{H}$ into a Hilbert A, B-bimodule. □

Proposition 4.3. *Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert A, B-bimodules. If there is a Hilbert bimodule map $f\colon \mathcal{H} \to \mathcal{H}'$, then $s(\mathcal{H}) \subseteq s(\mathcal{H}')$ and $r(\mathcal{H}) \subseteq r(\mathcal{H}')$. Such a Hilbert bimodule map is an isomorphism from $\mathcal{H}$ onto the submodule $\mathcal{H}' \cdot s(\mathcal{H}) = r(\mathcal{H}) \cdot \mathcal{H}'$ in $\mathcal{H}'$. So it is an isomorphism onto $\mathcal{H}'$ if and only if $s(\mathcal{H}') \subseteq s(\mathcal{H})$, if and only if $r(\mathcal{H}') \subseteq r(\mathcal{H})$, if and only if the map $\mathbb{K}(\mathcal{H}) \to \mathbb{K}(\mathcal{H}')$ induced by f is an isomorphism.*

Proof. Since the norm on a Hilbert bimodule is generated by the inner products, Hilbert bimodule maps are norm isometries and thus injective. Moreover,

$$f(\mathcal{H}) = f(r(\mathcal{H}) \cdot \mathcal{H}) = r(\mathcal{H}) \cdot f(\mathcal{H}) \subseteq r(\mathcal{H}) \cdot \mathcal{H}'.$$

Thus, $r(\mathcal{H}') \subseteq r(\mathcal{H})$ is necessary for f to be an isomorphism. Conversely, if $r(\mathcal{H}') \subseteq r(\mathcal{H})$, then even $r(\mathcal{H}') = r(\mathcal{H})$ because a bimodule map preserves the left inner product. Then the map from $r(\mathcal{H}) \cong \mathbb{K}(\mathcal{H})$ to $r(\mathcal{H}') \cong \mathbb{K}(\mathcal{H}')$ that sends $|\xi\rangle\langle\eta|$ to $|f(\xi)\rangle\langle f(\eta)|$ for $\xi, \eta \in \mathcal{H}$ is an isomorphism $\mathbb{K}(\mathcal{H}) \cong r(\mathcal{H}') \cong \mathbb{K}(\mathcal{H}')$. Since $\mathbb{K}(\mathcal{H}') \cdot \mathcal{H}' = \mathcal{H}'$, the linear span of elements of the form $|f(\xi)\rangle\langle f(\eta)|\zeta' = f(\xi) \cdot \langle f(\eta)|\zeta'\rangle$ for $\xi, \eta \in \mathcal{H}$, $\zeta' \in \mathcal{H}'$ is dense in $\mathcal{H}'$. Since $f(\mathcal{H})$ is a right B-module, this implies that f is surjective. Hence, it is an isomorphism of Hilbert bimodules. A similar argument for the right inner product instead of the left one shows that all the conditions listed for f are indeed equivalent to f being an isomorphism.

If $r(\mathcal{H}) \neq r(\mathcal{H}')$, then we may restrict f to a Hilbert bimodule map $\mathcal{H} \to r(\mathcal{H})\cdot\mathcal{H}'$. Since $r(\mathcal{H})\cdot r(\mathcal{H})\cdot\mathcal{H}' = r(\mathcal{H})\cdot\mathcal{H}'$, this is an isomorphism by the first statement. A similar argument on the other side shows that f is an isomorphism onto $\mathcal{H}' \cdot s(\mathcal{H})$, so $\mathcal{H}' \cdot s(\mathcal{H}) = r(\mathcal{H}) \cdot \mathcal{H}'$. □

A Hilbert A,B-bimodule $\mathcal{H}$ has a dual Hilbert B,A-bimodule $\mathcal{H}^*$, where we exchange left and right structures using adjoints:

$$b \cdot x^* \cdot a := (a^* \cdot x \cdot b^*)^* \quad \text{for } a \in A,\ b \in B,\ x \in \mathcal{H},$$

and

$$\langle x^*|y^*\rangle_A = \langle\langle y|x\rangle\rangle_A, \qquad \langle\langle x^*|y^*\rangle\rangle_B = \langle y|x\rangle_B.$$

We will see that this construction has the same formal properties as the dual for partial equivalences of groupoids. To begin with, a Hilbert bimodule map $X \to Y$ remains a Hilbert bimodule map $X^* \to Y^*$, and $(X^*)^* = X$. Furthermore,

$$(\xi \otimes \eta)^* \longmapsto \eta^* \otimes \xi^*$$

defines a Hilbert bimodule map

$$\sigma\colon (X \otimes_B Y)^* \longrightarrow Y^* \otimes_B X^*$$

with dense range, hence an isomorphism. Applying σ twice gives the identity map. (More precisely, $\sigma_{Y^*,X^*} \circ \sigma_{X,Y} = \mathrm{Id}_{(X\otimes_B Y)^*}$.)

Proposition 4.4. *Let $\mathcal{H}$ be a Hilbert A,B-bimodule. The inner products on $\mathcal{H}$ give Hilbert bimodule isomorphisms $\mathcal{H} \otimes_B \mathcal{H}^* \cong r(\mathcal{H})$ and $\mathcal{H}^* \otimes_A \mathcal{H} \cong s(\mathcal{H})$, and the restrictions of the left and right actions give Hilbert bimodule isomorphisms*

$$r(\mathcal{H}) \otimes_A \mathcal{H} \cong \mathcal{H} \cong \mathcal{H} \otimes_B s(\mathcal{H}),$$
$$s(\mathcal{H}) \otimes_B \mathcal{H}^* \cong \mathcal{H}^* \cong \mathcal{H}^* \otimes_A r(\mathcal{H}).$$

that make the following diagrams of isomorphisms commute:
(4.1)

$$\begin{array}{ccc} \mathcal{H} \otimes_B \mathcal{H}^* \otimes_A \mathcal{H} & \to & \mathcal{H} \otimes_B s(\mathcal{H}) \\ \downarrow & & \downarrow \\ r(\mathcal{H}) \otimes_A \mathcal{H} & \longrightarrow & \mathcal{H}, \end{array} \qquad \begin{array}{ccc} \mathcal{H}^* \otimes_A \mathcal{H} \otimes_B \mathcal{H}^* & \to & \mathcal{H}^* \otimes_A r(\mathcal{H}) \\ \downarrow & & \downarrow \\ s(\mathcal{H}) \otimes_B \mathcal{H}^* & \longrightarrow & \mathcal{H}^*. \end{array}$$

Let D be another C*-*algebra, let $\mathcal{K}$ be a Hilbert A,D-bimodule, and let $\mathcal{L}$ be a Hilbert B,D-bimodule with $r(\mathcal{K}) \subseteq r(\mathcal{H})$ and $r(\mathcal{L}) \subseteq s(\mathcal{H})$. Then Hilbert A,D-bimodule maps $\mathcal{H} \otimes_B \mathcal{L} \to \mathcal{K}$ are naturally in bijection with Hilbert B,D-bimodule maps $\mathcal{L} \to \mathcal{H}^* \otimes_A \mathcal{K}$, and this bijection maps isomorphisms again to isomorphisms. Similarly, Hilbert A,D-bimodule maps $\mathcal{H} \otimes_B \mathcal{L} \leftarrow \mathcal{K}$ are naturally in bijection with Hilbert B,D-bimodule maps $\mathcal{L} \leftarrow \mathcal{H}^* \otimes_A \mathcal{K}$.*

Proof. The Hilbert bimodule isomorphisms

$$\begin{aligned} \mathcal{H} \otimes_B \mathcal{H}^* &\cong r(\mathcal{H}), \\ \mathcal{H}^* \otimes_A \mathcal{H} &\cong s(\mathcal{H}), \\ r(\mathcal{H}) \otimes_A \mathcal{H} &\cong \mathcal{H} \cong \mathcal{H} \otimes_B s(\mathcal{H}) \end{aligned}$$

and

$$s(\mathcal{H}) \otimes_B \mathcal{H}^* \cong \mathcal{H}^* \cong \mathcal{H}^* \otimes_A r(\mathcal{H})$$

are routine to check using that $\mathcal{H}$ is full as a Hilbert $r(\mathcal{H}), s(\mathcal{H})$-bimodule. The diagrams in equation (4.1) are equivalent to the requirement $\langle\langle x|y\rangle\rangle_A \cdot z = x \cdot \langle y|z\rangle_B$ in the definition of a Hilbert bimodule. The claim about Hilbert bimodule maps is proved like the analogous one about partial equivalences of groupoids in Proposition 2.12; now, we use the canonical isomorphisms just established and Proposition 4.3 instead of Proposition 2.9. □

Proposition 4.5. *Up to isomorphism, $\mathcal{H}^*$ is the unique Hilbert B,A-bimodule $\mathcal{K}$ for which there are isomorphisms:*

$$\mathcal{H} \otimes_B \mathcal{K} \otimes_A \mathcal{H} \cong \mathcal{H}, \qquad \mathcal{K} \otimes_A \mathcal{H} \otimes_B \mathcal{K} \cong \mathcal{K}.$$

More precisely, if there are such isomorphisms, then there is a unique Hilbert bimodule isomorphism $\mathcal{H}^* \xrightarrow{\sim} \mathcal{K}$ such that the following map is the identity map:

$$\mathcal{H} \xrightarrow{\sim} \mathcal{H} \otimes_B \mathcal{K} \otimes_A \mathcal{H} \xrightarrow{\sim} \mathcal{H} \otimes_B \mathcal{H}^* \otimes_A \mathcal{H} \xrightarrow{\sim} r(\mathcal{H}) \otimes_A \mathcal{H} \xrightarrow{\sim} \mathcal{H}.$$

Proof. Repeat the proof of Proposition 2.12, replacing $\times_G$ by $\otimes_A$. □

Proposition 4.6. *Let $\mathcal{H}$ be a Hilbert A,A-bimodule, and let $\mu\colon \mathcal{H}\otimes_A \mathcal{H} \to \mathcal{H}$ be a bimodule isomorphism. Then, there is a unique isomorphism from $\mathcal{H}$ onto an ideal $I \triangleleft A$ that intertwines μ and the multiplication map $I\otimes_A I \xrightarrow{\sim} I$. We have $I = r(\mathcal{H}) = s(\mathcal{H})$, and the multiplication μ is associative.*

Proof. This is proved exactly like Proposition 2.13. □

Definition 4.7. Let S be an inverse semigroup with unit, and let A be a C*-algebra. An *S-action on A by Hilbert bimodules* consists of

- Hilbert A,A-bimodules $\mathcal{H}_t$ for $t \in S$;
- bimodule isomorphisms $\mu_{t,u}\colon \mathcal{H}_t \otimes_A \mathcal{H}_u \xrightarrow{\sim} \mathcal{H}_{tu}$ for $t,u \in S$;

satisfying

(AH1) $\mathcal{H}_1$ is the identity Hilbert A,A-bimodule A;
(AH2) $\mu_{t,1}\colon \mathcal{H}_t \otimes_A A \xrightarrow{\sim} \mathcal{H}_t$ and $\mu_{1,u}\colon A \otimes_A \mathcal{H}_u \xrightarrow{\sim} \mathcal{H}_u$ are the canonical isomorphisms for all $t,u \in S$;
(AH3) associativity: for all $t,u,v \in S$, the next diagram commutes:

$$\begin{array}{ccccc}
(\mathcal{H}_t \otimes_A \mathcal{H}_u) \otimes_A \mathcal{H}_v & \xrightarrow{\mu_{t,u}\otimes_A \mathrm{Id}_{\mathcal{H}_v}} & \mathcal{H}_{tu} \otimes_A \mathcal{H}_v & \searrow^{\mu_{tu,v}} & \\
\mathrm{ass}\updownarrow & & & & \mathcal{H}_{tuv} \\
\mathcal{H}_t \otimes_A (\mathcal{H}_u \otimes_A \mathcal{H}_v) & \xrightarrow[\mathrm{Id}_{\mathcal{H}_t}\otimes_A \mu_{u,v}]{} & \mathcal{H}_t \otimes_A \mathcal{H}_{uv} & \nearrow_{\mu_{t,uv}} &
\end{array}$$

If S has a 0 element, we may also require $\mathcal{H}_0 = \{0\}$.

Theorem 4.8. *Let S be an inverse semigroup with unit, and let A be a C*-algebra. Then actions of S on A by Hilbert bimodules are equivalent to saturated Fell bundles over S (as defined in* **[13]***) with unit fiber A.*

More precisely, let $(\mathcal{H}_t)_{t\in S}$ and $(\mu_{t,u})_{t,u\in S}$ be an S-action by Hilbert bimodules on A. Then there are unique Hilbert bimodule maps $j_{u,t}\colon \mathcal{H}_t \to \mathcal{H}_u$ for $t \le u$ that make the following diagrams commute for all

$t_1, t_2, u_1, u_2 \in S$ with $t_1 \le u_1$, $t_2 \le u_2$:

$$\begin{array}{ccc} \mathcal{H}_{t_1} \otimes_A \mathcal{H}_{t_2} & \xrightarrow{\mu_{t_1,t_2}} & \mathcal{H}_{t_1 t_2} \\ {\scriptstyle j_{u_1,t_1} \otimes_A j_{u_2,t_2}} \downarrow & & \downarrow {\scriptstyle j_{u_1 u_2, t_1 t_2}} \\ \mathcal{H}_{u_1} \otimes_A \mathcal{H}_{u_2} & \xrightarrow{\mu_{u_1,u_2}} & \mathcal{H}_{u_1 u_2} \end{array} \tag{4.2}$$

The map $j_{u,t}$ is a Hilbert bimodule isomorphism onto $\mathcal{H}_u \cdot s(\mathcal{H}_t) = r(\mathcal{H}_t) \cdot \mathcal{H}_u$. We have $j_{t,t} = \mathrm{Id}_{\mathcal{H}_t}$ for all $t \in S$ and $j_{v,u} \circ j_{u,t} = j_{v,t}$ for $t \le u \le v$ in S, and there are unique Hilbert bimodule isomorphisms $J_t \colon \mathcal{H}_t^* \xrightarrow{\sim} \mathcal{H}_{t^*}$, $x \mapsto x^*$, such that

$$\mu_{t,t^*,t}(x, x^*, x) = x \cdot \langle x|x\rangle_A = \langle\langle x|x\rangle\rangle_A \cdot x \quad \textit{for all } x \in \mathcal{H}_t.$$

These also satisfy

$$\begin{aligned} \mu_{t,t^*}(x \otimes x^*) &= \langle\langle x|x\rangle\rangle_A, \\ \mu_{t^*,t}(x^*, x) &= \langle x|x\rangle_A; \\ (x^*)^* &= x \quad \textit{for all } x \in \mathcal{H}_t; \\ \mu_{t,u}(x, y)^* &= \mu_{u^*,t^*}(y^*, x^*) \quad \textit{for all } x \in \mathcal{H}_t, y \in \mathcal{H}_u, t, u \in S; \end{aligned}$$

and

$$j_{u,t}(x)^* = j_{u^*,t^*}(x^*) \quad \textit{for all } t \le u \quad \textit{in } S, x \in \mathcal{H}_t.$$

Conversely, a saturated Fell bundle $(\mathcal{A}_t)_{t \in S}$ over S with $A = \mathcal{A}_1$ becomes an S-action by Hilbert bimodules by taking $\mathcal{H}_t = \mathcal{A}_t$ with the multiplication maps $\mu_{t,u}$ and the A-bimodule structure induced by the Fell bundle multiplication, and the left and right inner products $\langle\langle x|y\rangle\rangle_A := x \cdot y^*$, $\langle x|y\rangle_A := x^* \cdot y$ for $x, y \in \mathcal{H}_t$.

Proof. We construct the inclusion maps $j_{t,u}$ and the involutions J_t and show their properties exactly as in the proofs of Propositions 3.7 and 3.9. □

With Theorem 4.8, it becomes easier to construct saturated Fell bundles over inverse semigroups because Definition 4.7 needs far less data and has correspondingly fewer conditions to check.

Remark 4.9. The correspondence bicategory introduced in [**6**] is not suitable for our purposes by the following observation: Let $I \hookrightarrow A \twoheadrightarrow A/I$ be a split extension of C*-algebras. Then $p\colon A \to A/I \to A$ is an idempotent endomorphism. It remains an idempotent arrow in the correspondence bicategory. More generally, if A is Morita equivalent to an ideal in a C*-algebra B, then we can translate p to a correspondence $\mathcal{H}$ from B to itself that is idempotent in the sense that $\mathcal{H} \otimes_B \mathcal{H} \cong \mathcal{H}$ with an associative isomorphism. Thus, there are more idempotent endomorphisms in the correspondence bicategory than usual for inverse semigroup actions. Furthermore, the idempotent arrows no longer commute up to isomorphism; thus, a very basic assumption for inverse semigroups fails in this case. This is why we only allowed Hilbert bimodules above.

Proposition 4.10. *There is a bicategory with* C*-*algebras as objects, Hilbert bimodules as arrows, Hilbert bimodule isomorphisms as* 2-*arrows, and* $\otimes_B$ *as composition of arrows.*

Proof. The correspondence bicategory is already constructed in [**6**]. Lemma 4.2 allows identifying Hilbert bimodules with a subset of correspondences. It is well known that composites of Hilbert bimodules are again Hilbert bimodules. Hence, the Hilbert bimodules form a subbicategory in the opposite of the correspondence bicategory. □

5. Fell bundles from actions of inverse semigroups. All groupoids in this section are assumed to be locally quasi-compact, locally Hausdorff and with (locally compact) Hausdorff object space and a Haar system, so that they have groupoid C^*-algebras. Let G be such a groupoid, and let S be a unital inverse semigroup acting on G by partial equivalences. We want to turn this into an action of S on $\mathrm{C}^*(G)$ by Hilbert bimodules; equivalently, we want a Fell bundle over S with unit fiber $\mathrm{C}^*(G)$. There are two closely related ways to construct this. We will explain one approach in detail and briefly sketch the other in subsection 5.3.

We give details for the construction of the Fell bundle using the transformation groupoid $L = G \rtimes S$ because this also suggests how to describe the section C*-algebra of the resulting Fell bundle. The transformation groupoid L comes with an S-grading $(L_t)_{t\in S}$. Roughly speaking, our Fell bundle over S will involve the subspaces of $\mathrm{C}^*(L)$ of elements supported on the open subsets L_t. Since $G^1 = L_1$, the unit fiber of the Fell bundle will be $\mathrm{C}^*(G)$. This also suggests that the section C*-algebra of the Fell bundle over S is $\mathrm{C}^*(L)$. This is indeed the case, but the technical details need some care.

First, we need a Haar system on L. We show in Proposition 5.1 that the Haar system on G extends uniquely to a Haar system on L. Secondly, it is non-trivial that $\mathrm{C}^*(G)$ is contained in $\mathrm{C}^*(L)$: this means that the maximal C*-norm that defines $\mathrm{C}^*(G)$ extends to a C*-norm on $\mathrm{C}^*(L)$. A related issue is to show that an element of $\mathrm{C}^*(L)$ supported in G actually belongs to $\mathrm{C}^*(G)$. These problems become clearer if we construct a pre-Fell bundle using the dense *-algebra that defines $\mathrm{C}^*(L)$ and then complete it.

In the non-Hausdorff case, continuous functions with compact support are replaced by finite linear combinations of certain functions that are not continuous. The identification of $\mathrm{C}^*(L)$ with the section C^*-algebra of the Fell bundle requires a technical result about these functions. We prove it in Appendix B in the more general setting of sections of upper semicontinuous Banach bundles because this is not more difficult and allows us to generalize our main results to Fell bundles over groupoids.

We write $\mathfrak{S}(X)$ for the space of linear combinations of compactly supported functions on Hausdorff open subsets of a locally Hausdorff, locally quasi-compact space X. This is the space of compactly supported continuous functions on X if and only if X is Hausdorff, and it is often denoted by $\mathrm{C_c}(X)$. We find this notation misleading, however, because its elements are not continuous functions.

5.1. A Haar system on the transformation groupoid. Before we enter the construction of Haar systems, we mention an important trivial case: if G is étale, then so is L. Therefore, L certainly has a canonical Haar system if G is étale. This already covers many examples, and the reader only interested in étale groupoids may skip the construction of the Haar system on L.

We define Haar systems as in **[31,** Section 1]. Thus, our Haar system $(\lambda_G^x)_{x\in G^0}$ on G is left invariant, so

$$\operatorname{supp} \lambda_G^x = G^x = \{g \in G^1 \mid r(g) = x\}$$

and

$$g_*\lambda_G^{s(g)} = \lambda_G^{r(g)} \quad \text{for all } g \in G.$$

The continuity requirement for $(\lambda_G^x)_{x\in G^0}$ is that the function $\lambda_G(f)$ on G^0 defined by $\lambda_G(f)(x) := \int_G f(g)\,\mathrm{d}\lambda_G^x(g)$ is continuous on G^0 for all $f \in \mathfrak{S}(G)$. By the definition of $\mathfrak{S}(G)$ (see Definition B.1), it suffices to check continuity if f is a continuous function with compact support on a Hausdorff open subset U of G.

Proposition 5.1. *The Haar system on G extends uniquely to a Haar system on the transformation groupoid L.*

Proof. Fix $x \in G^0 = L^0$. We will describe the measure λ_L^x on L^x in the Haar system. Since $L = \bigcup_{t\in S} L_t$ is an open cover, the measure λ_L^x is determined by its restrictions to L_t for all $t \in S$. If $x \notin r(L_t)$, then there is nothing to do; so, consider $t \in S$ with $x \in r(L_t)$, and fix $g \in L_t$ with $r(g) = x$. If $A \subseteq L_t^x := L_t \cap L^x$ is measurable, then $A = g \cdot (g^{-1} \cdot A)$ with $g^{-1} \cdot A \subseteq L_t^{-1} \cdot L_t = L_1 = G$. Since we want (λ_L^x) to be left invariant and to extend (λ_G^x), we must have

$$\lambda_L^x(A) = \lambda_G^{s(g)}(g^{-1} \cdot A)$$

if $g \in L_t$ satisfies $r(g) = x$ and $A \subseteq L_t^x$ is measurable. Hence, there is at most one Haar measure on L extending the given Haar measure on G.

If $g_1, g_2 \in L_t$ satisfy $r(g_1) = r(g_2) = x$, then $g_1^{-1} \cdot g_2 \in L_t^{-1} L_t = L_1 = G$; the left invariance of (λ_G^x) with respect to G implies that $\lambda_G^{s(g)}(g^{-1} \cdot A)$ does not depend on the choice of g. If

$$\emptyset \neq A \subseteq L_t^x \cap L_u^x,$$

then we may pick the same element $g \in A$ to define the measure of A as a subset of L_t^x and of L_u^x. Thus, the definitions of λ_L^x on the sets L_t^x for $t \in S$ are compatible. Furthermore, there is a unique measure λ_L^x on L^x with $\lambda_L^x(A) = \lambda_G^{s(g)}(g^{-1} \cdot A)$ whenever $A \subseteq L_t^x$ is measurable and $g \in L_t$ satisfies $r(g) = x$. If $l \in L$ has $s(l) = x$, then $l_*(\lambda^x)$ is a measure on $L^{r(l)}$ with the same properties that characterize $\lambda^{r(l)}$ uniquely; so we obtain the left invariance of our family of measures:

$$l_*(\lambda^{s(l)}) = \lambda^{r(l)} \quad \text{for all } l \in L.$$

Checking continuity by hand is unpleasant, so we use a different description of the same Haar system for this purpose. Recall that L_t is an equivalence between restrictions of G to open invariant subsets of G^0. The proof that equivalent groupoids have Morita-Rieffel equivalent groupoid C*-algebras uses a family of measures on the equivalence bibundle in order to define the right inner product; this measure on L_t is exactly the one described above (see the proof of [**31,** Corollary 5.4]), and its continuity is known, even in the non-Hausdorff case. Thus, our family of measures restricts to a continuous family on each L_t. Since the map $\bigoplus \mathfrak{S}(L_t) \to \mathfrak{S}(L)$ in Proposition B.2 is surjective, the family of measures (λ_L^x) is continuous. □

5.2. Construction of the Fell bundle. We now know that L has a Haar system. So we get a *-algebra structure on $\mathfrak{S}(L)$ as in [**31, 25**]. Since the Haar measure on L extends the one on G, the map $\mathfrak{S}(G) \to \mathfrak{S}(L)$ induced by the open embedding $G \to L$ is a *-algebra isomorphism onto its image. The groupoid C*-algebras of L and G are completions of $\mathfrak{S}(L)$ and $\mathfrak{S}(G)$ for suitable C*-norms.

Lemma 5.2. *The involution on $\mathfrak{S}(L)$ maps $\mathfrak{S}(L_t)$ onto $\mathfrak{S}(L_t^{-1}) = \mathfrak{S}(L_{t^*})$. The convolution product maps $\mathfrak{S}(L_t) \times \mathfrak{S}(L_u)$ to $\mathfrak{S}(L_{tu})$.*

Proof. The claim for the involution is trivial. The claim for the convolution product follows, of course, from $L_t \cdot L_u \subseteq L_{tu}$ but requires some care in the non-Hausdorff case because the convolution product is not defined directly, see the proof of [**25,** Proposition 4.4]. If $f_1 \in \mathfrak{S}(U)$ and $f_2 \in \mathfrak{S}(V)$ for Hausdorff open subsets $U \subseteq L_t$ and $V \subseteq L_u$, and if $U \cdot V$ is also Hausdorff, then we directly obtain $f_1 * f_2 \in \mathfrak{S}(U \cdot V)$ with $U \cdot V \subseteq L_{tu}$. If $U \cdot V$ is non-Hausdorff, a partition of unity is used to write f_1 and f_2 as finite sums of functions on *smaller* Hausdorff open subsets $U' \subseteq U$, $V' \subseteq V$ for which $U' \cdot V'$ is Hausdorff. Since $U' \cdot V' \subseteq U \cdot V \subseteq L_{tu}$, we get $\mathfrak{S}(L_t) * \mathfrak{S}(L_u) \subseteq \mathfrak{S}(L_{tu})$ as desired. □

Lemma 5.2 gives $\mathfrak{S}(G) * \mathfrak{S}(L_t) \subseteq \mathfrak{S}(L_t)$ and $\mathfrak{S}(L_t) * \mathfrak{S}(G) \subseteq \mathfrak{S}(L_t)$, so $\mathfrak{S}(L_t)$ is a $\mathfrak{S}(G)$-bimodule; it also implies $f_1^* * f_2 \in \mathfrak{S}(G)$ and $f_1 * f_2^* \in \mathfrak{S}(G)$ for all $f_1, f_2 \in \mathfrak{S}(L_t)$, which gives $\mathfrak{S}(G)$-valued left and right inner products on $\mathfrak{S}(L_t)$. We also have $f_1 * f_2 \in \mathfrak{S}(L_{tu})$ for $f_1 \in \mathfrak{S}(L_t)$ and $f_2 \in \mathfrak{S}(L_u)$, and these multiplication maps are associative and "isometric" with respect to the $\mathfrak{S}(G)$-valued inner products. We put "isometric" in quotation marks because we have not yet talked about norms.

Lemma 5.3. *$f^* * f \in \mathfrak{S}(G)$ is positive in $\mathrm{C}^*(G)$ for each $t \in S$, $f \in \mathfrak{S}(L_t)$, and the closed linear span of $f_1^* * f_2$ for $f_1, f_2 \in \mathfrak{S}(L_t)$ is dense in $\mathrm{C}^*(G_{r(L_t)})$.*

Proof. We have already used in the proof of Proposition 5.1 that the Haar measure on L restricts to the usual family of measures on the partial equivalence space L_t. In that context, the positivity of such inner products is already proved in **[31, 25]** in order to show that $\mathfrak{S}(L_t)$ may be completed to a Hilbert $\mathrm{C}^*(G)$-bimodule. The proof that an equivalence induces a Morita-Rieffel equivalence also shows that the inner product defined above is full, that is, the closed linear span of $f_1^* * f_2$ for $f_1, f_2 \in \mathfrak{S}(L_t)$ is dense in $\mathrm{C}^*(G_{r(L_t)})$. □

Hence, we may complete $\mathfrak{S}(L_t)$ to a Hilbert bimodule $\mathrm{C}^*(L_t)$ over $\mathrm{C}^*(G)$. The densely defined convolution map $\mathfrak{S}(L_t) \times \mathfrak{S}(L_u) \to \mathfrak{S}(L_{tu})$ extends to a Hilbert bimodule map

$$\mu_{t,u}\colon \mathrm{C}^*(L_t) \otimes_{\mathrm{C}^*(G)} \mathrm{C}^*(L_u) \longrightarrow \mathrm{C}^*(L_{tu})$$

because it is isometric for the $\mathfrak{S}(G)$-valued inner products. Since $\mathrm{C}^*(L_t)$ is full as a Hilbert bimodule over $\mathrm{C}^*(G_{r(L_t)})$ and $\mathrm{C}^*(G_{s(L_t)})$, it follows that the maps $\mu_{t,u}$ above are surjective.

The associativity of the multiplication on the dense subspaces $\mathfrak{S}(L_t)$ extends to $\mathrm{C}^*(L_t)$. Thus, we have constructed an action of S by Hilbert bimodules on $\mathrm{C}^*(G)$. By Theorem 4.8, this is equivalent to a saturated Fell bundle $\mathrm{C}^*(L_t)_{t\in S}$ over S.

Theorem 5.4. *The section C^*-algebra $\mathrm{C}^*(S, \mathrm{C}^*(L_t)_{t\in S})$ is naturally isomorphic to the groupoid C^*-algebra $\mathrm{C}^*(L)$.*

This theorem looks almost trivial from our construction; but the proof requires a technical result about $\mathfrak{S}(L)$ to be proved in Appendix B. Before we turn to that, we first add coefficients in a Fell bundle over L.

The above construction still works in almost literally the same way if we replace $\mathfrak{S}(L_t)$ by $\mathfrak{S}(L_t, \mathfrak{B})$ everywhere, where $\mathfrak{B}$ is a Fell bundle over the groupoid L. Unfortunately, we could not find a reference for the generalization of Lemma 5.3 to this context. The references on groupoid crossed products we could find consider *either* Fell bundles over Hausdorff groupoids, such as **[24]**, *or* a more restrictive class of actions for non-Hausdorff groupoids, such as **[31, 25]**, but not both. In particular, the positivity of the inner product on $\mathfrak{S}(L_t, \mathfrak{B})$ for a partial equivalence L_t is only proved in some cases: for arbitrary upper semicontinuous Fell bundles over Hausdorff groupoids in **[24]**; for Green twisted actions of non-Hausdorff groupoids on continuous fields of C^*-algebras over G^0 in **[31]**; and for untwisted actions by automorphisms of non-Hausdorff groupoids on $\mathrm{C}_0(G^0)$-algebras in **[25]**. This is probably only a technical issue that will be eventually resolved but not in this paper. So we add an assumption about it in our next theorem.

Theorem 5.5. *Let $\mathfrak{B}$ be a Fell bundle over L. Assume that $f^* * f \in \mathfrak{S}(G, \mathfrak{B})$ is positive in $\mathrm{C}^*(G, \mathfrak{B})$ for all $f \in \mathfrak{S}(L_t, \mathfrak{B})$, $t \in S$, and that the linear span of these inner products is dense in $\mathrm{C}^*(G_{s(L_t)}, \mathfrak{B})$. Then there is a Fell bundle $\mathrm{C}^*(L_t, \mathfrak{B})_{t\in S}$ over S that has the section C^*-algebra of the restriction $\mathrm{C}^*(G, \mathfrak{B}|_G)$ as unit fiber. The section C^*-algebra $\mathrm{C}^*(S, \mathrm{C}^*(L_t, \mathfrak{B})_{t\in S})$ is naturally isomorphic to the section C^*-algebra of the groupoid Fell bundle $\mathrm{C}^*(L, \mathfrak{B})$.*

Theorem 5.4 is a special case of Theorem 5.5 for the constant Fell bundle $\mathbb{C}$. It remains to prove Theorem 5.5. This will be done in Appendix B.1, after some preliminary results about Banach bundles in Appendix B.

Corollary 5.6. *Let L be an étale topological groupoid with Hausdorff locally compact object space and with a Haar system. Let S be a wide inverse subsemigroup of* $\mathrm{Bis}(L)$*, that is,*

$$\bigcup_{t\in S} t = L$$

and

$$\bigcup_{t\in S, t\subseteq t_1\cap t_2} \quad = t_1 \cap t_2$$

for all $t_1, t_2 \in S$*. Then, the groupoid* C^**-algebra of* L *is isomorphic to the crossed product* $\mathrm{C}_0(L^0) \rtimes S$*.*

More generally, if $\mathfrak{B}$ *is a Fell bundle over* L*, then the section* C^**-algebra* $\mathrm{C}^*(L, \mathfrak{B})$ *is isomorphic to the section* C^**-algebra of the associated Fell bundle over* S*.*

Proof. The assumptions on S ensure that L is an S-graded groupoid by $L_t := t$ with unit fiber $G = L^0$. So, Theorem 5.4 gives the first assertion, and Theorem 5.5 gives the second. In this case, positivity is not an issue because we are dealing with a space G, so positivity in $\mathrm{C}^*(G, \mathfrak{B})$ is equivalent to pointwise positivity in all $x \in L^0 = G^0$. The value $(f^* * f)(x)$ for $f \in \mathfrak{S}(L_t, \mathfrak{B})$ is either zero or $f(l)^* f(l)$ for the unique $l \in L_t$ with $s(l) = x$. This is assumed to be positive in the definition of a Fell bundle over a groupoid. □

The isomorphism $\mathrm{C}^*(L) \cong \mathrm{C}_0(L^0) \rtimes S$ has already been proved [**12,** Theorem 9.8] (if L^0 is second countable and S is countable). The more general result for (separable) Fell bundles over (second countable) étale groupoids is proved in [**4,** Theorem 2.13].

Another special case worth mentioning is group extensions. Let $G \rightarrowtail H \twoheadrightarrow S$ be an extension of locally compact groups with discrete S. This gives an action of S, viewed as an inverse semigroup, on G by Theorem 3.15. We obtain a Fell bundle over S with unit fiber $\mathrm{C}^*(G)$ and section C^*-algebra $\mathrm{C}^*(H)$. More generally, we get a similar result for a Fell bundle over H, compare with [**5,** Example 3.9]. Our Fell bundle also comes from a Green twisted action of (H, G), and, in this formulation, our theorem is well known in this case, see [**14, 7**].

Corollary 5.7. *In the situation of Theorem* 5.5*, the canonical map from* $\mathrm{C}^*(G, \mathfrak{B})$ *to* $\mathrm{C}^*(L, \mathfrak{B})$ *is injective.*

Proof. The unit fiber of the Fell bundle in Theorem 5.5 is $\mathrm{C}^*(G, \mathfrak{B})$, and the section C^*-algebra is $\mathrm{C}^*(L, \mathfrak{B})$. The unit fiber always embeds into the section C^*-algebras of a Fell bundle over an inverse semigroup, see [**13,** Corollary 8.10]. □

Next, we note a useful variant of Theorem 5.4 for group-valued cocycles.

Let L be a locally quasi-compact, locally Hausdorff groupoid, let S be a group and let $c\colon L \to S$ be a 1-cocycle. Let $L_t := c^{-1}(t) \subseteq L$ for $t \in S$, and let $G = L_1 = c^{-1}(1)$. Since we do not assume anything about c, this need *not* be an S-grading, compare Theorem 3.15. Nevertheless, we may complete $\mathfrak{S}(L_t)$ to a Hilbert bimodule over $\mathrm{C}^*(G)$ and thus obtain a Fell bundle over S. The difference to the situation above is that this Fell bundle need not be saturated.

Theorem 5.8. *The section C^*-algebra of the Fell bundle over S with unit fiber $\mathrm{C}^*(G)$ just described is isomorphic to $\mathrm{C}^*(L)$. Hence, the canonical map $\mathrm{C}^*(G) \to \mathrm{C}^*(L)$ is faithful.*

Proof. The proofs of Theorem 5.4 and Corollary 5.7 still work for non-saturated Fell bundles (even over inverse semigroups). Alternatively, we may replace our non-saturated Fell bundle over G by a saturated Fell bundle over an inverse semigroup associated to G, just as for partial actions, see [**11**]. This does not change the section C^*-algebra, and afterwards Theorem 5.4 literally applies. □

5.3. Another construction of the Fell bundle. The construction of the Fell bundle over S in Section 5.2 used the transformation groupoid. Now, we construct this Fell bundle using the abstract functorial properties of actions on groupoids and their corresponding actions on C^*-algebras. Actually, some aspects of this have been used to prove Lemma 5.3 above.

It is well known that two equivalent groupoids have Morita-Rieffel equivalent C^*-algebras, see [**23**], even in the non-Hausdorff case, see [**31**]. The proof is constructive: given an equivalence X from H to G, the space $\mathfrak{S}(X)$ is completed to a $\mathrm{C}^*(G)$-$\mathrm{C}^*(H)$-imprimitivity bimodule, using certain natural formulas for a $\mathfrak{S}(G)$-$\mathfrak{S}(H)$-bimodule structure and $\mathfrak{S}(G)$- and $\mathfrak{S}(H)$-valued inner products. An important ingredient here is that the Haar measures on G and H give canonical families of measures on the fibers of the range and source maps of X, which may be used to integrate functions on X.

Even if X is only a *partial* equivalence, the same formulas still work and give a Hilbert bimodule $\mathrm{C}^*(X)$ from $\mathrm{C}^*(H)$ to $\mathrm{C}^*(G)$ by completing $\mathfrak{S}(X)$. If $f\colon X \to X'$ is an isomorphism between two partial equivalences, then $f_*\colon \mathfrak{S}(X) \to \mathfrak{S}(X')$ defined by $f_*(h) = h \circ f^{-1}$

is an isomorphism that preserves all structure, so it extends to an isomorphism $\mathrm{C}^*(X) \xrightarrow{\sim} \mathrm{C}^*(X')$.

Theorem 5.9. *The maps $G \mapsto \mathrm{C}^*(G)$ from groupoids to C^*-algebras, $X \mapsto \mathrm{C}^*(X)$ from partial equivalences to Hilbert bimodules, and $f \mapsto f_*$ from bibundle isomorphisms to Hilbert bimodule isomorphisms are part of a functor from the bicategory of partial groupoid equivalences to the bicategory of C^*-algebras and Hilbert bimodules.*

Proof. The above map is strictly compatible with unit arrows: the unit arrow G^1 on G is sent to $\mathrm{C}^*(G^1) = \mathrm{C}^*(G)$, and the unit transformations in both bicategories are also preserved. To complete the above data to a functor of bicategories, it remains to give natural isomorphisms $\mathrm{C}^*(X) \otimes_{\mathrm{C}^*(H)} \mathrm{C}^*(Y) \cong \mathrm{C}^*(X \times_H Y)$ and check that they satisfy the expected associativity condition for three composable partial equivalences. They are constructed by writing the "convolution map" $\mathfrak{S}(X) \odot \mathfrak{S}(Y) \to \mathfrak{S}(X \times_H Y)$, given by the formula

$$(\xi \cdot \eta)(x, y) := \int_{H^1} \xi(x \cdot h)\eta(h^{-1} \cdot y) \, \mathrm{d}\lambda^u(h), \tag{5.1}$$

for all $\xi \in \mathfrak{S}(X)$, $\eta \in \mathfrak{S}(Y)$ and $(x, y) \in X \times_H Y$, where $u = s(x) = r(y)$. It is routine to check that the map (5.1) has dense range and is a bimodule map and an isometry for both inner products; thus, it extends to an isomorphism between the completions:

$$\mathrm{C}^*(X) \otimes_{\mathrm{C}^*(H)} \mathrm{C}^*(Y) \cong \mathrm{C}^*(X \times_H Y).$$

One way to construct convolution maps and check their properties is as in our construction above using the transformation groupoid: build an appropriate linking groupoid containing all the data. For two composable equivalences Y and X from K to H and from H to H, this linking groupoid has object space $G^0 \sqcup H^0 \sqcup K^0$; its arrow space is a disjoint union of G^1, H^1, K^1, X, Y, X^*, Y^*, $X \times_H Y$, and $Y^* \times_H X^*$, the source and range maps are the obvious ones, and the multiplication is defined using the left and right actions of G, H and K and canonical maps. This is indeed a topological groupoid, and it inherits a canonical Haar system if G, H and K have Haar systems. The convolution map is the restriction of the convolution in this larger groupoid to $X \times_H Y$. Given three composable partial equivalences, there is a similar linking groupoid combining all the relevant data, and the associativity of its convolution product on $X \times_H Y \times_K Z$ gives the associativity coherence of the isomorphisms $\mathrm{C}^*(X) \otimes_{\mathrm{C}^*(H)} \mathrm{C}^*(Y) \cong \mathrm{C}^*(X \times_H Y)$. □

Remark 5.10. Theorem 5.9 is extended in the thesis of Holkar [**15**], where a similar functor from a bicategory of groupoid correspondences to the bicategory of C*-correspondences is constructed. This construction is more difficult because the family of measures needed to write the right inner product is no longer canonical and becomes part of the data. Hence, the behavior of the measures under composition must be studied as well.

An inverse semigroup action by partial equivalences may be defined as a functor (of bicategories) from the inverse semigroup to the bicategory of groupoids and partial equivalences. Composing it with the functor in Theorem 5.9 gives a functor from the inverse semigroup to the bicategory of Hilbert bimodules, which is the same as an action by Hilbert bimodules. This is also the same as a saturated Fell bundle over the inverse semigroup by Theorem 4.8 and is the second construction of the Fell bundle over S. It gives an isomorphic Fell bundle because the Haar measure on L used above is the same as the combination of the measure families on the partial equivalences L_t that are used to define the convolution maps in Theorem 5.9.

More concretely, an action $(X_t, \mu_{t,u})$ of S on G yields the action on $\mathrm{C}^*(G)$ given by the Hilbert bimodules $\mathrm{C}^*(X_t)$ with the multiplication maps

$$\mathrm{C}^*(X_t) \otimes_A \mathrm{C}^*(X_u) \xrightarrow{\sim} \mathrm{C}^*(X_t \times_G X_u) \xrightarrow[\cong]{\mathrm{C}^*(\mu_{t,u})} \mathrm{C}^*(X_{tu}),$$

which involve the convolution isomorphisms

$$\mathrm{C}^*(X_t) \otimes_A \mathrm{C}^*(X_u) \xrightarrow{\sim} \mathrm{C}^*(X_t \times_G X_u).$$

This is associative by associativity coherence of these convolution isomorphisms.

6. Actions of inverse semigroups and groupoids. Let H be an étale groupoid with locally compact Hausdorff object space. Thus far, we have constructed actions of the inverse semigroup $\mathrm{Bis}(H)$ on certain

C*-algebras. Instead, we would like to construct actions of H itself. In this section, we will see that both kinds of actions are very closely related. Here, an action of $\mathrm{Bis}(H)$ is as above: an action by Hilbert bimodules or, equivalently, a saturated Fell bundle over $\mathrm{Bis}(H)$. The corresponding "actions" of H are saturated Fell bundles over H.

First, we explain how to turn a Fell bundle over H into one over $\mathrm{Bis}(H)$. So, let $\mathfrak{B} = (B_h)_{h\in H}$ be a Fell bundle over H, see **[4, 17]**. Let $A := \mathrm{C}_0(H^0, \mathfrak{B})$ be the C*-algebra of C_0-sections of $\mathfrak{B}$ over H^0; by construction, this is a $\mathrm{C}_0(H^0)$-C*-algebra. If $t \in \mathrm{Bis}(H)$, then the Fell bundle operations turn $\mathcal{H}_t := \mathrm{C}_0(t, \mathfrak{B})$ into a Hilbert $\mathrm{C}_0(r(t), \mathfrak{B})$-$\mathrm{C}_0(s(t), \mathfrak{B})$-bimodule. The multiplication in the Fell bundle induces multiplication maps $\mu_{t,u} \colon \mathcal{H}_t \otimes_A \mathcal{H}_u \to \mathcal{H}_{tu}$. This gives an action of $\mathrm{Bis}(H)$ on A by Hilbert bimodules.

Not every action of $\mathrm{Bis}(H)$ by Hilbert bimodules is of this form. The obstruction lies in how idempotents in $\mathrm{Bis}(H)$ act. Idempotents in $\mathrm{Bis}(H)$ are the same as open subsets of H^0. We identify the idempotent semilattice $E(\mathrm{Bis}(H))$ with the complete lattice $\mathbb{O}(H^0)$ of open subsets of H^0. Thus, the action of idempotents in $\mathrm{Bis}(H)$ becomes a map from $\mathbb{O}(H^0)$ to the complete lattice $\mathbb{I}(A)$ of ideals in A.

Theorem 6.1. *An action $(\mathcal{H}_t, \mu_{t,u})_{t\in\mathrm{Bis}(H)}$ of the inverse semigroup* $\mathrm{Bis}(H)$ *on a* C*-*algebra A by Hilbert A-bimodules comes from a Fell bundle over H if and only if the map from $E(\mathrm{Bis}(H)) \cong \mathbb{O}(H^0)$ to $\mathbb{I}(A)$ commutes with suprema. This Fell bundle over H is unique up to isomorphism, and the Fell bundles over* $\mathrm{Bis}(H)$ *and H have the same section* C*-*algebras.*

Proof. A map $\mathbb{O}(H^0) \to \mathbb{I}(A)$ comes from a continuous map $\mathrm{Prim}(A) \to H^0$ if and only if it commutes with finite infima and arbitrary suprema by **[20,** Lemma 2.25]; here, we need H^0 to be a sober space, a very mild condition that certainly allows all locally Hausdorff spaces. Compatibility with finite infima says that it is a morphism of semilattices, which we assume anyway; compatibility with suprema is an extra condition. A continuous map $\mathrm{Prim}(A) \to H^0$ is equivalent to an isomorphism between A and the C*-algebra of C_0-sections of an upper semicontinuous field $(A_x)_{x\in H^0}$ of C*-algebras over H^0, see **[26]**. Thus, the criterion in Theorem 6.1 is necessary and sufficient for A to come from such an upper semicontinuous field. This gives a Fell bundle over $H^0 \subseteq H$. It remains to extend this to all of H.

Let $t \in \mathrm{Bis}(H)$. Then $\mathcal{H}_t$ is a Hilbert A-bimodule. For $h \in t \subseteq H^1$, we define $\mathcal{H}_{h,t} := \mathcal{H}_t \otimes_A A_{s(h)}$; this is a Hilbert $A_{s(h)}$-module. If $\xi \in \mathcal{H}_t$, then $\|\xi\|^2 = \|\langle \xi, \xi\rangle\|$, and, for $\langle \xi, \xi\rangle \in A$, the norm is the supremum of the norms of its images in A_x for all $x \in H^0$. Therefore, the canonical map from $\mathcal{H}_t$ to $\prod_{h \in t} \mathcal{H}_{h,t}$ is isometric. Thus, we view $\mathcal{H}_t$ as a space of sections of the bundle of Banach spaces $\mathcal{H}_{h,t}$ over t. This is an upper semicontinuous bundle on t because $(A_x)_{x \in H^0}$ is, and the norm on $\mathcal{H}_t$ is given by $\|\xi\|^2 = \|\langle \xi, \xi\rangle\|$ with $\langle \xi, \xi\rangle \in A$.

If $t, u \in \mathrm{Bis}(H)$ and $h \in t \cap u$, then both $\mathcal{H}_{h,t}$ and $\mathcal{H}_{h,u}$ are candidates for the fiber $\mathcal{H}_h$ of our Fell bundle at h. These are isomorphic through the canonical isomorphisms $j_{t,t\cap u}\colon \mathcal{H}_{t\cap u} \to \mathcal{H}_t|_{s(t\cap u)}$ and $j_{u,t\cap u}\colon \mathcal{H}_{t\cap u} \to \mathcal{H}_u|_{s(t\cap u)}$ from Theorem 4.8.

For each $h \in H^1$, choose some $t_h \in \mathrm{Bis}(H)$ with $h \in t_h$, and define $\mathcal{H}_h := \mathcal{H}_{h,t_h}$. If $t \in \mathrm{Bis}(H)$, then there are canonical isomorphisms $\mathcal{H}_h \cong \mathcal{H}_{h,t}$ for all $h \in t$. We use them to transport the topology on the bundle $(\mathcal{H}_{h,t})_{h\in t}$ to the bundle $(\mathcal{H}_h)_{h \in t}$. These topologies are compatible on $t \cap u$ for all $t, u \in \mathrm{Bis}(H)$. Since the subsets $t \in \mathrm{Bis}(H)$ form an open cover of H^1, there is a topology on the whole bundle $(\mathcal{H}_h)_{h \in H^1}$ that coincides with the topology on $(\mathcal{H}_h)_{h\in t}$ described above for each $t \in \mathrm{Bis}(H)$, in particular, the space of C_0-sections of $(\mathcal{H}_h)_{h\in H^1}$ on t coincides naturally with $\mathcal{H}_t$.

Let $A(U)$ for $U \in \mathbb{O}(G)$ be the ideal of C_0-sections of (A_x) vanishing outside U. Then $A(U) = \mathcal{H}_U$ if we view $U \in E(\mathrm{Bis}(G))$. We have

$$\mathcal{H}_t \otimes_A A(U) = \mathcal{H}_{t\cdot U} = \mathcal{H}_{t(U)\cdot t} = A(t(U)) \otimes_A \mathcal{H}_t \tag{6.1}$$

for all $t \in \mathrm{Bis}(H)$, $U \in \mathbb{O}(H^0)$ with $U \subseteq s(t)$. Here, we view each $t \in \mathrm{Bis}(H)$ as a partial homeomorphism $s(t) \to r(t)$ and write $t(U)$ for the image of U under this map. This is exactly how $\mathrm{Bis}(H)$ acts on H^0. Equation (6.1) implies that $\mathcal{H}_{h,t} \cong A_{r(h)} \otimes_A \mathcal{H}_t$. Thus, $\mathcal{H}_h$ is a Hilbert $A_{r(h)}$-$A_{s(h)}$-bimodule. The isomorphism $\mathcal{H}_t \otimes_A \mathcal{H}_u \to \mathcal{H}_{tu}$ is A-linear, and hence, $\mathrm{C}_0(H^0)$-linear. Thus, it restricts to an isomorphism on the fibers,

$$\mathcal{H}_{g,t} \otimes_A \mathcal{H}_{h,u} \longrightarrow \mathcal{H}_{gh,tu}$$

for all $g \in t$, $h \in u$ with $s(g) = r(h)$.

The compatibility of the multiplication with the inclusion maps from Theorem 4.8 shows that these maps on the fibers do not depend on the choice of t and u with $h \in t$ and $h \in u$. Thus, we obtain well-defined isomorphisms

$$\mathcal{H}_g \otimes_{A_{s(g)}} \mathcal{H}_h \to \mathcal{H}_{gh}$$
$$\text{for all } g, h \in H^1 \text{ with } s(g) = r(h).$$

Since they can be put together to maps $\mathcal{H}_t \otimes_A \mathcal{H}_u \to \mathcal{H}_{tu}$ for all $t, u \in \mathrm{Bis}(H)$, and since $\mathrm{Bis}(H)$ covers H^1, they are locally continuous, hence continuous. Similarly, the isomorphisms $\mathcal{H}_t^* \cong \mathcal{H}_{t^*}$ must come from well-defined, continuous maps $\mathcal{H}_h^* \to \mathcal{H}_{h^{-1}}$ for $h \in H^1$ by restricting them to fibers. The remaining algebraic conditions needed for a Fell bundle over the groupoid H^1 all follow easily because $(\mathcal{H}_t, \mu_{t,u})$ gives a Fell bundle over $\mathrm{Bis}(H)$.

If we turn the Fell bundle over H constructed above into a Fell bundle over $\mathrm{Bis}(H)$ again, we clearly get back the original Fell bundle over $\mathrm{Bis}(H)$ because $\mathcal{H}_t$ is the space of C_0-sections of $(\mathcal{H}_h)_{h \in t}$. Conversely, if we start with a Fell bundle over H, turn it into a Fell bundle over $\mathrm{Bis}(H)$, and then use the above construction to go back, we get an isomorphic Fell bundle over H. Hence, we obtain a bijection between isomorphism classes of the two types of Fell bundles. Theorem 5.5 shows that the passage from Fell bundles over H to Fell bundles over $\mathrm{Bis}(H)$ does not change the section C^*-algebras. ☐

We assumed G^0 to be Hausdorff and locally compact so far because Fell bundles over groupoids have not yet been defined in greater generality. We suggest using the necessary and sufficient criterion in Theorem 6.1 as a definition:

Definition 6.2. Let G be an étale topological groupoid for which G^0 (and hence G^1) is sober. An action of G on a C^*-algebra A is an action of $\mathrm{Bis}(G)$ by Hilbert bimodules for which the resulting map $\mathbb{O}(G^0) \to \mathbb{I}(A)$ commutes with arbitrary suprema.

Sobriety of G^0 is needed to turn a map $\mathbb{O}(G^0) \to \mathbb{I}(A)$ that commutes with suprema into a continuous map $\mathrm{Prim}(A) \to G^0$, see **[20,** Lemma 2.25].

Let G be a sober space G^0 viewed as a groupoid. Then, an action of G is the same as a continuous map $\mathrm{Prim}(A) \to G^0$. In the notation of [**20**], this turns A into a C*-algebra over G^0. It is unclear what the "fibers" of such a C*-algebra over G^0 should be if G^0 is badly non-Hausdorff. Therefore, it is not clear how to describe actions of étale sober groupoids in the sense of Definition 6.2 as Fell bundles over G. If G^0 is locally Hausdorff and locally quasi-compact, then Definition 6.2 seems to work quite well; we plan to discuss this in greater detail elsewhere.

The criterion in Theorem 6.1 also suggests how to define actions of étale groupoids on other groupoids:

Definition 6.3. Let G be an étale topological groupoid for which G^0, and hence G^1, is sober, and let H be an arbitrary topological groupoid. An *action* of G on H is an action of $\mathrm{Bis}(G)$ on H by partial equivalences for which the map $\mathbb{O}(G^0) \to \mathbb{O}(H^0/H)$ that describes the action of $E(\mathrm{Bis}(G))$ commutes with arbitrary suprema.

The extra assumption in Definition 6.3 and [**20,** Lemma 2.25] ensure that the map $\mathbb{O}(G^0) \to \mathbb{O}(H^0/H)$ for an action of G on H comes from a continuous map $H^0/H \to G^0$ or, equivalently, an H-invariant continuous map $H^0 \to G^0$.

Proposition 6.4. *Let H be a locally quasi-compact, locally Hausdorff groupoid with Hausdorff object space and with a Haar system. An action of G on H induces an action of G on $\mathrm{C}^*(H)$ as well.*

Proof. In subsection 5.2, we turn an action of $\mathrm{Bis}(G)$ on H into an action of $\mathrm{Bis}(G)$ on $\mathrm{C}^*(H)$. For any open H-invariant subset U of H^0, the closure of $\mathfrak{S}(H_U)$ in $\mathrm{C}^*(H)$ is an ideal $\mathrm{C}^*(H_U)$ in $\mathrm{C}^*(H)$. The map

$$\mathbb{O}(H^0/H) \longrightarrow \mathbb{I}(\mathrm{C}^*(H)), \qquad U \longmapsto \mathrm{C}^*(H_U),$$

commutes with suprema. Hence, Theorem 6.1 applies to the action of $\mathrm{Bis}(G)$ on $\mathrm{C}^*(H)$ if the action of $\mathrm{Bis}(G)$ satisfies the condition in Definition 6.3. □

6.1. The motivating example. Now we consider our motivating example: an action of a locally Hausdorff, locally quasi-compact, étale groupoid H on a locally Hausdorff, locally quasi-compact space Z. Let $\mathfrak{U}$ be a Hausdorff open covering of Z, and let $G_{\mathfrak{U}}$ be the associated covering groupoid, which is étale, locally compact and Hausdorff. Its C*-algebra $\mathrm{C}^*(G_{\mathfrak{U}})$ is our noncommutative model for the non-Hausdorff space Z. We want to construct an "action" of H on it that models the given action of H on Z.

To construct it, we use the inverse semigroup $S := \mathrm{Bis}(H)$ of bisections of H. First, we turn the action of H on Z into an action of S on Z by partial homeomorphisms in the usual way: a bisection $t \in S$ acts by the homeomorphism

$$r^{-1}(s(t)) \longrightarrow r^{-1}(r(t)), \qquad z \longmapsto g_{r(z)} \cdot z,$$

where g_x is the unique arrow in t with $s(g_x) = x$.

We have seen in Corollary 3.21 that the S-action on Z induces an S-action on $G_{\mathfrak{U}}$ by partial equivalences. The transformation groupoid $G_{\mathfrak{U}} \rtimes S$ for this action is Hausdorff, étale and locally compact. It is equivalent to $Z \rtimes S$ by Corollary 3.21.

Let

$$p\colon X := \bigsqcup_{U\in\mathfrak{U}} U \to Z$$

be the canonical map. Then $G_{\mathfrak{U}} = p^*(Z)$. An idempotent $U \in \mathbb{O}(H^0)$ in $\mathrm{Bis}(H)$ acts on $G_{\mathfrak{U}}$ by the identity map on the open invariant subgroupoid $G_{\mathfrak{U}}|_{(r\circ p)^{-1}(U)}$, that is, $\mathbb{O}(H^0)$ acts on $G_{\mathfrak{U}}$ through the map

$$\mathbb{O}(H^0) \longrightarrow \mathbb{O}(G^0_{\mathfrak{U}}/G_{\mathfrak{U}}), \qquad U \longmapsto (r\circ p)^{-1}(U);$$

this commutes with suprema and infima. Thus, our action of $\mathrm{Bis}(H)$ on $G_{\mathfrak{U}}$ is also an action of H in the sense of Definition 6.3.

We may identify $Z \rtimes S \cong Z \rtimes H$ using the obvious S-grading on $Z \rtimes H$ and Theorem 3.14, so $G_{\mathfrak{U}} \rtimes S$ is equivalent to $Z \rtimes H$.

The S-action on $G_{\mathfrak{U}}$ induces a Fell bundle over S with unit fiber $\mathrm{C}^*(G_{\mathfrak{U}})$, which we view as an action of S on $\mathrm{C}^*(G_{\mathfrak{U}})$. Theorem 5.4 gives an isomorphism between its section C*-algebra $\mathrm{C}^*(G_{\mathfrak{U}}) \rtimes S$ and the groupoid C*-algebra $\mathrm{C}^*(G_{\mathfrak{U}} \rtimes S)$. We may turn our Fell bundle over $\mathrm{Bis}(H)$ into a Fell bundle over the groupoid H by Proposition 6.4.

Theorem 6.1 also says that the section C*-algebra of the Fell bundle over H is isomorphic to $\mathrm{C}^*(G_{\mathfrak{U}}) \rtimes S \cong \mathrm{C}^*(G_{\mathfrak{U}} \rtimes S)$. The restriction to the unit fiber is $\mathrm{C}^*(G_{\mathfrak{U}})$, by construction. We will describe this Fell bundle over H.

We have $G_{\mathfrak{U}} \rtimes S \cong p^*(Z \rtimes H)$, that is, the object space of $G_{\mathfrak{U}} \rtimes S$ is X, and the arrow space is homeomorphic to the space of triples (x_1, h, x_2), $x_1, x_2 \in X$, $h \in H^1$ with $r(p(x_1)) = r(h)$ and $r(p(x_2)) = s(h)$ in H^0. Here, (x_1, h, x_2) is an arrow from x_2 to x_1, and the multiplication is $(x_1, h_1, x_2) \cdot (x_2, h_2, x_3) = (x_1, h_1 h_2, x_3)$. For $h \in H^1$, let K_h be the subspace of triples (x_1, h, x_2) for $x_1, x_2 \in X$, $r(p(x_1)) = r(h)$ and $r(p(x_2)) = s(h)$. Since p and H are étale, this is a discrete set. The fiber at h of our Fell bundle over H is the completion of the space $\mathrm{C_c}(K_h)$ of finitely supported functions on K_h to a Hilbert bimodule over $\mathrm{C}^*(K_{1_{r(h)}})$ and $\mathrm{C}^*(K_{1_{s(h)}})$.

Proposition 6.5. *Let Z be a basic action of H with Hausdorff quotient space $H\backslash Z$, for instance, $Z = H^1$ with the action by left or right multiplication and quotient space H^0. Then, the groupoid $G_{\mathfrak{U}} \rtimes S$ is equivalent to $H\backslash Z$ and $\mathrm{C}^*(G_{\mathfrak{U}}) \rtimes S$ is Morita equivalent to $\mathrm{C}_0(H\backslash Z)$.*

Proof. The groupoid $G_{\mathfrak{U}} \rtimes S$ is equivalent to $Z \rtimes S$. This is the same as $Z \rtimes H$ by Theorem 3.14, using the evident S-grading on $Z \rtimes H$. Since the H-action on Z is basic, $Z \rtimes H$ is equivalent to $H\backslash Z$. This space is assumed to be Hausdorff, and $G_{\mathfrak{U}} \rtimes S$ is also a groupoid with Hausdorff object space. Therefore, the equivalence between them is of the usual type, involving free and proper actions, by Proposition A.7. Hence, it induces a Morita-Rieffel equivalence from $\mathrm{C}_0(H\backslash Z)$ to $\mathrm{C}^*(G_{\mathfrak{U}}) \rtimes S$. □

In the situation of Proposition 6.5, $G_{\mathfrak{U}} \rtimes S$ has Hausdorff arrow space because it must be isomorphic to the covering groupoid of the open surjection $G^0_{\mathfrak{U}} \to (G_{\mathfrak{U}} \rtimes S)\backslash G^0_{\mathfrak{U}} \cong H\backslash Z$ between two Hausdorff spaces. In this case, it is also easy to see that any Fell bundle over the groupoid $G_{\mathfrak{U}} \rtimes S$ is a pull-back of a Fell bundle over $H\backslash Z$, which is the same as a $\mathrm{C}_0(H\backslash Z)$-$\mathrm{C}^*$-algebra $\mathfrak{B}$. The section C^*-algebra of the Fell bundle over $G_{\mathfrak{U}} \rtimes S$ is Morita-Rieffel equivalent to this $\mathrm{C}_0(H\backslash Z)$-$\mathrm{C}^*$-algebra $\mathfrak{B}$. By Theorem 5.5, this is also the section C^*-algebra of the Fell bundle over S associated to $\mathfrak{B}$.

Many properties such as properness, amenability and essential principality are shared by an action of a groupoid on a space and its transformation groupoid. This suggests how to extend these notions to inverse semigroup actions on groupoids. We take this as a definition for proper actions of inverse semigroups on locally compact groupoids:

Definition 6.6. An action of an inverse semigroup S on a topological groupoid G is *proper* if the groupoid $G \rtimes S$ is proper, that is, the following map is proper (that is, stably closed):

$$(s,r)\colon (G \rtimes S)^1 \longrightarrow G^0 \times G^0, \qquad g \longmapsto (s(g), r(g)).$$

The action is called *free* if this map is injective.

Let L be a proper groupoid such that L^0 is a locally compact Hausdorff space. Then, the image of L^1 in $L^0 \times L^0$ is locally compact and Hausdorff because it is a closed subspace of a locally compact Hausdorff space. Since this subspace is closed and the orbit space projection $L^0 \to L\backslash L^0$ is open, it also follows that $L\backslash L^0$ is locally compact Hausdorff (see Proposition A.3). The groupoid L itself need not be Hausdorff: the non-Hausdorff group bundle in Section 8 is proper in this sense because it is quasi-compact and the image of (s,r) is closed. If L acts *freely* and properly on a Hausdorff space L^0, however, then L^1 must be Hausdorff. In this case, we also obtain information about any open subgroupoid, which leads to the next proposition.

Proposition 6.7. *Let S act properly and freely on a locally Hausdorff, locally quasi-compact groupoid G. Then G is a basic groupoid, so that G is equivalent to the locally Hausdorff, locally quasi-compact space $G\backslash G^0$.*

Proof. The map in Definition 6.6 is a homeomorphism onto its image because it is continuous, injective and closed. Hence, its restriction to the open subspace $G^1 \subseteq (G \rtimes S)^1$ is still a homeomorphism onto its image. This means that G is a basic groupoid, so G is equivalent to G^0/G. This is locally Hausdorff and locally quasi-compact by Proposition A.14. □

Thus, the free and proper actions of S all come from actions on locally Hausdorff spaces that are desingularized by replacing the space by a Hausdorff groupoid G.

6.2. Inverse semigroup models for étale groupoids. Let G be an étale groupoid. Thus far, we have described actions of G through actions of the inverse semigroup $\mathrm{Bis}(G)$. Since $\mathrm{Bis}(G)$ is usually quite big, even uncountable, we now replace it by smaller inverse semigroups. The next definition describes which inverse semigroups we allow as "models" for G:

Definition 6.8. An *inverse semigroup model* for an étale groupoid G consists of an inverse semigroup S, an S-action on the space G^0 by partial homeomorphisms, and an isomorphism $G^0 \rtimes S \cong G$ of étale groupoids that is the identity on objects.

In particular, if $S \subseteq \mathrm{Bis}(G)$ is a wide inverse subsemigroup, then S with its usual action on G^0 and the canonical isomorphism $G^0 \rtimes S \cong G$ from Corollary 3.19 is a model for G.

Lemma 6.9. *An inverse semigroup model for G is equivalent to an inverse semigroup S with a homomorphism $\varphi\colon S \to \mathrm{Bis}(G)$ that induces an isomorphism $G^0 \rtimes S \to G^0 \rtimes \mathrm{Bis}(G) \cong G$, where we use the canonical action of $\mathrm{Bis}(G)$ on G^0 and φ to let S act on G^0.*

Proof. Let S act on G^0. There is a canonical homomorphism $S \to \mathrm{Bis}(G^0 \rtimes S)$, see [**12**]. Combined with an isomorphism $G^0 \rtimes S \cong G$, we obtain a homomorphism $\varphi\colon S \to \mathrm{Bis}(G)$. Conversely, such a homomorphism induces an action of S on G^0 and then a continuous groupoid homomorphism $G^0 \rtimes S \to G^0 \rtimes \mathrm{Bis}(G) \cong G$. Routine computations show that these two constructions are inverse to each other. □

The following lemma characterizes inverse semigroup models more concretely when we take $\widehat{S} = \mathrm{Bis}(G)$.

Lemma 6.10. *Let S and $\widehat{S}$ be inverse semigroups, let $\varphi\colon S \to \widehat{S}$ be a homomorphism, and let $\widehat{S}$ act on Z by partial homeomorphisms. The induced groupoid homomorphism $\widetilde{\varphi}\colon Z \rtimes S \to Z \rtimes \widehat{S}$ is an isomorphism if and only if*

(1) *for all* $t_1, t_2 \in S$ *and every* $z \in Z$ *with* $z \in D_{t_1^* t_1} \cap D_{t_2^* t_2}$ *and every* $f \in E(\widehat{S})$ *with* $z \in D_f$ *and* $\varphi(t_1)f = \varphi(t_2)f$*, there is* $e \in E(S)$ *with* $z \in D_e$ *and* $t_1 e = t_2 e$*;*
(2) *for every* $u \in \widehat{S}$ *and every* $z \in Z$ *with* $z \in D_{u^* u}$*, there is* $t \in S$ *with* $z \in D_{t^* t}$ *and there is* $f \in E(\widehat{S})$ *with* $z \in D_f$ *and* $uf = \varphi(t)f$*.*

In this case, we call φ a *Z-isomorphism.*

Proof. The groupoid homomorphism $\widetilde{\varphi}$ is the identity on objects and always continuous and open on arrows, so the only issue is whether $\widetilde{\varphi}$ is bijective on arrows. It is routine to check that (1) is equivalent to injectivity and (2) to surjectivity of $\widetilde{\varphi}$. □

Let S and $\varphi\colon S \to \mathrm{Bis}(G)$ be an inverse semigroup model for an étale topological groupoid G. Which actions of S on groupoids by partial equivalences or on C*-algebras by Hilbert bimodules come from actions of G?

First, we consider a trivial special case to see why we need more data. Let G be merely a topological space, viewed as a groupoid. In this case, the trivial inverse semigroup $\{1\}$ is an inverse semigroup model. An action of S contains no information. An action of G on a topological groupoid H or a C*-algebra is simply a continuous map

$$\psi\colon H^0/H \longrightarrow G^0$$

or

$$\psi\colon \mathrm{Prim}(A) \longrightarrow G^0,$$

respectively.

Theorem 6.11. *Let G be a sober étale topological groupoid, and let S and $\varphi\colon S \to \mathrm{Bis}(G)$ be an inverse semigroup model for G. Let H be a topological groupoid. An action of G on H by partial equivalences is equivalent to a pair consisting of an action of S on H by partial equivalences and an S-equivariant map $\psi\colon H^0/H \to G^0$. The transformation groupoid for an action of G (that is,* $\mathrm{Bis}(G)$*) and its restriction to S are the same.*

The S-equivariance of ψ refers to the actions of S on H^0/H and G^0 by partial homeomorphisms induced by the action on H and by φ.

Proof. First, let G act on H; more precisely, $\mathrm{Bis}(G)$ acts on H and the resulting map

$$\mathbb{O}(G^0) = E(\mathrm{Bis}(G)) \longrightarrow \mathbb{O}(H^0/H)$$

commutes with suprema, see Definition 6.3. It comes from a continuous map $\psi\colon H^0/H \to G^0$, as shown in [**20,** Lemma 2.25]. This map is $\mathrm{Bis}(G)$-equivariant, and hence, S-equivariant.

Now, let S act on H, and let $\psi\colon H^0/H \to G^0$ be an S-equivariant map. Let $L := H \rtimes S$ with its canonical S-grading $(L_t)_{t\in S}$. We claim that there is a unique $\mathrm{Bis}(G)$-grading $(\overline{L}_t)_{t\in\mathrm{Bis}(G)}$ on L with $\overline{L}_{\varphi(t)} = L_t$ for all $t \in S$, and $\overline{L}_U = H^1_{\psi^{-1}(U)}$ for $U \in \mathbb{O}(G^0)$. These two conditions on the $\mathrm{Bis}(G)$-grading say exactly that it corresponds to the given S-action and map ψ. Thus, the proof of the claim will finish the proof of the theorem.

For $t \in \mathrm{Bis}(G)$ and $u \in S$, we may form $t \cap \varphi(u) \in \mathrm{Bis}(G)$. We have

$$\begin{aligned} &t \cap \varphi(u) = t \cdot V_{t,u} = \varphi(u) \cdot V_{t,u} \\ &\text{for } V_{t,u} = s(t \cap \varphi(u)) \in \mathbb{O}(G^0); \end{aligned}$$

here, we also view $V_{t,u}$ as an idempotent element of $\mathrm{Bis}(G)$. Since S models G, we have

$$t = \bigcup_{u\in S} t \cap \varphi(u),$$

and hence,

$$s(t) = \bigcup_{u\in S} V_{t,u}.$$

Any $\mathrm{Bis}(G)$-grading with $\overline{L}_V = H^1_{\psi^{-1}(V)}$ for all $V \in \mathbb{O}(G^0)$ satisfies

$$\overline{L}_t|_{\psi^{-1}(V_{t,u})} = \overline{L}_t \cdot \overline{L}_{V_{t,u}} = \overline{L}_{t\cap\varphi(u)} = \overline{L}_{\varphi(u)}|_{\psi^{-1}(V_{t,u})}$$

for all $t \in \mathrm{Bis}(G)$, $u \in S$. Since $s(t) = \bigcup_{u\in S} V_{t,u}$ and ψ is S-equivariant, this shows that there is at most one $\mathrm{Bis}(G)$-grading with the required properties, namely,

$$\overline{L}_t = \bigcup_{u\in S} L_u|_{\psi^{-1}(V_{t,u})}.$$

More explicitly, $l \in \overline{L}_t$ if and only if $l \in L_u$ for some $u \in S$ for which t and $\varphi(u)$ have the same germ at $\psi(s(l))$. We must prove that $(\overline{L}_t)_{t\in \mathrm{Bis}(G)}$ is a grading with all desired properties.

First, we check $\overline{L}_{\varphi(u)} = L_u$ for $u \in S$. The inclusion $\supseteq$ is trivial. If $l \in \overline{L}_{\varphi(u)}$, then $l \in L_{u'}$ for some $u' \in S$ for which $\varphi(u)$ and $\varphi(u')$ have the same germ at $\psi(s(l)) \in G^0$. Hence, there is an idempotent element $e \in S$ with $\psi(s(l)) \in \varphi(e)$ and $ue = u'e$. Since $L_e = H^1_{\psi^{-1}(e)}$, we obtain $l \in L_{u'}L_e = L_{u'e} = L_{ue} = L_uL_e \subseteq L_u$. This finishes the proof that $\overline{L}_{\varphi(u)} = L_u$ for all $u \in S$.

Next, we check $\overline{L}_W = H^1_{\psi^{-1}(W)}$ for $W \in \mathbb{O}(G^0)$. The inclusion $\supseteq$ holds because $V_{W,1} = W$. Conversely, let $l \in \overline{L}_W$. Then, $l \in L_u$ for some $u \in S$ for which $\varphi(u)$ and Id_W have the same germ at $\psi(s(l))$. Since $G^0 \rtimes S \cong G$, there is an idempotent $e \in S$ with $\psi(s(l)) \in \varphi(e)$ and $ue = e$. An argument as in the previous paragraph shows that $l \in L_uL_e = L_e \subseteq H^1$. Thus, $\overline{L}_W = H^1_{\psi^{-1}(W)}$ for all $W \in \mathbb{O}(G^0)$.

If $t \in \mathrm{Bis}(G)$, $u \in S$, then $(\varphi(u) \cap t)^* = \varphi(u^*) \cap t^*$. Hence, $V_{t^*,u^*} = t(V_{t,u}) = \varphi(u)(V_{t,u})$. This implies $L_{t^*} = L_t^{-1}$ for all $t \in \mathrm{Bis}(G)$.

Let $t_1, t_2 \in \mathrm{Bis}(G)$. We claim that $\overline{L}_{t_1} \cdot \overline{L}_{t_1} = \overline{L}_{t_1t_2}$. The inclusion $\subseteq$ follows because

$$(\varphi(u_1) \cap t_1) \cdot (\varphi(u_2) \cap t_2) \subseteq \varphi(u_1u_2) \cap t_1t_2.$$

For the converse inclusion, take $l \in \overline{L}_{t_1t_2}$. Then $t \in L_u$ for some $u \in S$ for which t_1t_2 and $\varphi(u)$ have the same germ at $\psi(s(l))$. Factor this germ as g_1g_2 with $g_j \in t_j$ for $j = 1, 2$. There are $u_j \in S$ with $g_j \in \varphi(u_j)$ for $j = 1, 2$ because $G \cong G^0 \rtimes S$. Thus, $\varphi(u_1)\varphi(u_2) = \varphi(u_1u_2)$ and t_1t_2 have the same germ g_1g_2 at $\psi(s(l))$. Then u_1u_2 and u also have the same germ there, and an argument as above shows that $l \in L_{u_1u_2}$ as well. Using (Gr1) for the S-grading, we obtain $l_j \in L_{u_j}$ for $j = 1, 2$ with $l = l_1l_2$. Then, $s(l_2) = s(l)$ and $r(l_1) = r(l)$. This allows us to prove $l_2 \in \overline{L}_{t_2}$ and $l_1^{-1} \in \overline{L}_{t_1^*}$, so that $l_1 \in \overline{L}_{t_1}$. Hence, the $\mathrm{Bis}(G)$-grading satisfies (Gr1).

It is clear that $\overline{L}_{t_1} \subseteq \overline{L}_{t_2}$ if $t_1 \le t_2$ in $\mathrm{Bis}(G)$, so

$$\overline{L}_{t_1} \cap \overline{L}_{t_2} \supseteq \bigcup_{v \le t_1, t_2} \overline{L}_v = \overline{L}_{t_1 \cap t_2}$$

for all $t_1, t_2 \in \mathrm{Bis}(G)$.

For the converse inclusion, take $l \in \overline{L}_{t_1} \cap \overline{L}_{t_2}$. Then, there are $u_1, u_2 \in S$ with $l \in L_{u_1} \cap L_{u_2}$, such that t_j and $\varphi(u_j)$ have the same germ at $\psi(s(l))$ for $j = 1, 2$. Condition (Gr3) for the S-grading gives $v \in S$ with $v \le u_1, u_2$ and $l \in L_v$. Since ψ is S-equivariant, $\psi(s(l))$ belongs to the domain of $\varphi(v)$, so the germs of $\varphi(v)$ and $\varphi(u_i)$ at $\psi(s(l))$ are equal. Then, the germs of t_1 and t_2 at $\psi(s(l))$ are equal as well, that is, $t_1 \cap t_2$ is defined at $\psi(s(l))$ and has the same germ there as $\varphi(v)$. This means that $l \in \overline{L}_{t_1 \cap t_2}$. This verifies (Gr3) for the $\mathrm{Bis}(G)$-grading.

Since $\overline{L}_{\varphi(u)} = L_u$ for all $u \in S$ and

$$\bigcup_{u \in S} L_u = L^1,$$

we also obtain

$$\bigcup_{t \in \mathrm{Bis}(G)} \overline{L}_t = L^1,$$

which is (Gr4). □

The next lemma is needed to formulate a similar result for actions on C*-algebras:

Lemma 6.12. *An action of S on a* C*-*algebra A by Hilbert bimodules induces an action of S on* $\mathrm{Prim}(A)$ *by partial homeomorphisms.*

Proof. The Rieffel correspondence, see [**29,** Corollary 3.33], says that an imprimitivity bimodule $\mathcal{H}$ from B to A induces a homeomorphism $\mathrm{Prim}(B) \xrightarrow{\sim} \mathrm{Prim}(A)$. The corresponding lattice isomorphism

$$\mathbb{I}(B) = \mathbb{O}(\mathrm{Prim}(B)) \xrightarrow{\sim} \mathbb{O}(\mathrm{Prim}(A)) = \mathbb{I}(A)$$

sends an ideal $J \subseteq B$ to the unique ideal $I \subseteq A$ with $I \cdot \mathcal{H} = \mathcal{H} \cdot J$. A Hilbert A, B-bimodule induces a *partial* homeomorphism $\mathrm{Prim}(B) \to \mathrm{Prim}(A)$ because it is an imprimitivity bimodule between certain ideals in A and B, which correspond to open subsets of the primitive ideal spaces. Isomorphic Hilbert bimodules induce the same partial homeomorphism, of course. The partial homeomorphism associated to a tensor product bimodule $\mathcal{H}_1 \otimes_B \mathcal{H}_2$ is the composite of the partial homeomorphisms associated to $\mathcal{H}_1$ and $\mathcal{H}_2$. Thus, the map from S to $\mathrm{pHomeo}(\mathrm{Prim}(A))$ induced by an action on A by Hilbert bimodules is a homomorphism. □

Theorem 6.13. *Let G be a sober étale topological groupoid, and let S and $\varphi\colon S \to \mathrm{Bis}(G)$ be an inverse semigroup model for G. Let A be a* C^**-algebra. An action of G on A by Hilbert bimodules is equivalent to a pair consisting of an action of S on A by Hilbert bimodules and an S-equivariant map $\psi\colon \mathrm{Prim}(A) \to G^0$. The section* C^**-algebras of the corresponding Fell bundles over* $\mathrm{Bis}(G)$ *and S are the same.*

The S-equivariance of ψ refers to the action of S on $\mathrm{Prim}(A)$ from Lemma 6.12.

Proof. Assume first that G^0 is locally compact Hausdorff. In this case, an action of G is the same as a Fell bundle over G by Theorem 6.1. This determines an action of $\mathrm{Bis}(G)$, which we may compose with φ to obtain an action of S; we also obtain an S-equivariant map ψ. Conversely, let an action of S and a continuous S-equivariant map $\psi\colon \mathrm{Prim}(A) \to G^0$ be given. Since $G \cong G^0 \rtimes S$, we may carry over the proof of Theorem 6.1. The S-equivariance of ψ gives the compatibility condition (6.1). Hence, literally the same argument still works.

If G^0 is only a sober topological space, we need a different proof because we cannot describe G-actions fiberwise. We first construct the section C^*-algebra B of the Fell bundle over S corresponding to the action by Theorem 4.8. This C^*-algebra is S-graded by construction: it is the Hausdorff completion of the *-algebra $\bigoplus_{t\in S} \mathcal{H}_t$ in the maximal C^*-seminorm that vanishes on $j_{u,t}(\xi)\delta_u - \xi\delta_t$ for all $t, u \in S$ with $t \le u$ and all $\xi \in \mathcal{H}_t$, and we let $B_t \subseteq B$ be the image of $\mathcal{H}_t$ in B. In particular, we may identify $A = B_1$. Now we must construct a $\mathrm{Bis}(G)$-grading $(\overline{B}_t)_{t\in\mathrm{Bis}(G)}$ on B with $\overline{B}_{\varphi(t)} = B_t$ for all $t \in S$ and $\overline{B}_U = A(U)$ for all $U \in \mathbb{O}(G^0)$, where $A(U)$ denotes the ideal in A corresponding to $\psi^{-1}(U) \in \mathbb{O}(\mathrm{Prim}(A))$. This is done similarly to the proof of Theorem 6.11. Since this is rather technical and we already have another proof in the locally compact Hausdorff case, we leave it to the determined reader to spell out the details of this argument. □

7. Actions by automorphisms are not enough. The next theorem shows that the multiplication action of a non-Hausdorff groupoid on its own arrow space cannot be described by a continuous groupoid action by automorphisms.

Theorem 7.1. *Let G be a locally quasi-compact, locally Hausdorff, étale groupoid with Hausdorff G^0 such that G^1 is not Hausdorff. Let A be a* C*-*algebra with* $\mathrm{Prim}(A) \cong G^1$. *There is no continuous (twisted) action of G on A by automorphisms that induces the left multiplication action on* $\mathrm{Prim}(A) \cong G^1$.

Proof. Since $\mathrm{Prim}(A) \cong G^1$, the lattice of ideals in A is order-isomorphic to the lattice of open subsets in G^1. Let $A(U) \lhd A$ for an open subset $U \subseteq G^1$ be the corresponding ideal in A. Then,

$$\mathrm{Prim}(A(U)) \cong U.$$

Part of a continuous action of G on A is a continuous map $\mathrm{Prim}(A) \to G^0$, which is equivalent to a $\mathrm{C}_0(G^0)$-algebra structure. Since we want to obtain the left multiplication action of G^1 on $\mathrm{Prim}(A)$, we assume that this map becomes the range map $G^1 \to G^0$ when we identify $\mathrm{Prim}(A) \cong G^1$. The fiber at $x \in G^0$ is the restriction of A to the closed subset $G^x = \{g \in G^1 \mid r(g) = x\}$, which is denoted by $A|_{G^x}$; we have $\mathrm{Prim}(A|_{G^x}) = G^x$. A G-action on A must provide isomorphisms

$$\alpha_g \colon A|_{G^{s(g)}} \longrightarrow A|_{G^{r(g)}} \quad \text{for } g \in G^1.$$

We assume that α_g induces the map

$$G^{s(g)} \longrightarrow G^{r(g)}, \qquad h \longmapsto gh,$$

on the primitive ideal space.

What does continuity of $g \mapsto \alpha_g$ mean? Let $U, V \subseteq G^1$ be bisections. Then $U \cdot V$ is also a bisection. If $g \in U$, $h \in V$ satisfy $s(g) = r(h)$, then α_g restricts to an isomorphism $\alpha_{g,h} \colon A|_h \to A|_{gh}$. Any element of $U \cdot V$ is of the form $g \cdot h$ for unique $g \in U$, $h \in V$. Continuity of (α_g)

means that, for all bisections U, V and all $a = (a_h)_{h\in V}$ in $A(V)$, the section

$$(g \cdot h) \longmapsto \alpha_{g,h}(a_h) \quad \text{for } g \in U,\ h \in V$$

is continuous on $U \cdot V$, that is, it belongs to $A(U \cdot V)$, see also [**27,** Definition 2.3]. Thus, we obtain isomorphisms

$$\alpha_U \colon A(V) \longrightarrow A(U \cdot V).$$

In brief, $\mathrm{Bis}(G)$ acts on A by partial isomorphisms.

Since G^1 is non-Hausdorff, there are $g_1, g_2 \in G^1$ that cannot be separated by open subsets. Then, $r(g_1) = r(g_2)$ and $s(g_1) = s(g_2)$. Let U_1 and U_2 be bisections of G containing g_1 and g_2, respectively. Through shrinking, we may achieve that $s(U_1) = s(U_2)$. Let

$$V := U_1^* U_1 = \{1_x \mid x \in s(U_1)\} = U_2^* U_2;$$

then, $U_1 V = U_1$ and $U_2 V = U_2$. Since g_1 and g_2 cannot be separated, there is a net (h_n) in $U_1 \cap U_2$ that converges both to g_1 and to g_2.

Let $f \in A(V)$ with $f(1_{s(g_1)}) \neq 0$. Then

$$\alpha_{U_1}(f) \in A(U_1 V)$$

and

$$\alpha_{U_2}(f) \in A(U_2 V)$$

by our continuity assumption. Thus,

$$\psi := \alpha_{U_1}(f) \cdot \alpha_{U_2}(f)^* \in A(U_1 V) \cap A(U_2 V) = A(U_1 \cap U_2),$$

so ψ vanishes at g_1 and g_2. At $h_n \in U_1 \cap U_2$, we have

$$\alpha_{U_1}(f)(h_n) = \alpha_{U_1 \cap U_2}(f)(h_n) = \alpha_{U_2}(f)(h_n) = \alpha_{h_n}(f(1_{s(h_n)})).$$

Since each α_{h_n} is an isomorphism, we obtain

$$\|\psi(h_n)\| = \|\alpha_{h_n}(f(1_{s(h_h)}) f(1_{s(h_n)})^*)\| = \|f(1_{s(h_h)})\|^2.$$

If $U \subseteq G^1$ is Hausdorff and $a \in A(U)$, then

$$U \ni x \longmapsto \|a\|_x$$

is continuous by [**26,** Corollary 2.2] because the map

$$\mathrm{Prim}\, A(U) \longrightarrow U$$

is open and U is Hausdorff and locally compact. Therefore, $\|\psi(h_n)\|$ converges towards $\|\psi(g_1)\| = 0$. At the same time, $\|\psi(h_n)\|$ converges towards $\|f(1_{s(g_1)})\|^2 \neq 0$ because $s(h_n) \to s(g_1)$ inside the Hausdorff open subset V. This contradiction shows that there is no continuous action of G on A that lifts the multiplication action on

$$\mathrm{Prim}(A) \cong G^1. \qquad \square$$

Remark 7.2. More generally, if we only assume an open continuous surjection

$$p\colon \mathrm{Prim}(A) \longrightarrow G^1,$$

then there is no continuous action of G on A such that p is G-equivariant for the induced action of G on $\mathrm{Prim}(A)$ and the left multiplication action on G^1; the proof is exactly the same.

The proof of Theorem 7.1 does not use the multiplicativity of the action, so allowing "twisted" actions of G does not help. There are only two ways around this. First, we may allow Fell bundles over G. Second, we may allow actions of the inverse semigroup $\mathrm{Bis}(G)$. After stabilization, every Fell bundle becomes a twisted action by partial automorphisms, see **[3]**. We cannot remove the twist, however, because an untwisted action of $\mathrm{Bis}(G)$ by automorphisms would give an action of G by automorphisms as well, which cannot exist by Theorem 7.1.

8. A simple explicit example. Let G be the group bundle over $G^0 = [0,1]$ with trivial isotropy groups $G(x)$ for $x \neq 0$ and with $G(0) \cong \mathbb{Z}/2 = \{1, -1\}$. So, as a set, G is

$$(0,1] \cup \{0^+, 0^-\}$$

with 0^+ corresponding to $+1 \in \mathbb{Z}/2$ and 0^- to $-1 \in \mathbb{Z}/2$. The topology on G is the quotient topology from $[0,1] \times \mathbb{Z}/2$, where we divide by the equivalence relation generated by

$$(x,1) \sim (x,-1) \quad \text{for } x \neq 0.$$

With this topology, G is an étale, quasi-compact, second countable, locally Hausdorff, non-Hausdorff groupoid (even a group bundle). The points 0^+ and 0^- cannot be topologically separated: any net in $(0,1]$ converging to 0^+ also converges to 0^- and vice versa.

Let H be the groupoid of the equivalence relation $\sim$ on $[0,1]\times\mathbb{Z}/2$ defined above. Its C*-algebra $C^*(H)\cong C^*_{\mathrm{r}}(H)$ is

$$A := \{f \in \mathrm{C}([0,1],\mathbb{M}_2) : f(0) \text{ is diagonal}\}.$$

This can be proved using the same idea as in [**8,** Example 7.1]. This is a C*-algebra over $[0,1]$ with fibers $A_x \cong \mathbb{M}_2$ at $x \neq 0$ and $A_0 \cong \mathbb{C}^2$, and it has

$$\widehat{A} \cong \mathrm{Prim}(A) \cong G^1,$$

which is a special case of [**8,** Corollary 5.4]. Theorem 7.1 shows that there is no action of G on A by automorphisms that would model the left multiplication action of G on G^1.

Since A is the groupoid C*-algebra of the Čech groupoid for the covering

$$[0^+,1]\cup[0^-,1] = H^1,$$

our main results give an action of G on A by Hilbert bimodules. We first describe it as an inverse semigroup action for a very small inverse semigroup S that models G. We consider three special bisections of G:

$$\begin{aligned} 1 &= [0^+,1] = G^1 \setminus \{0^-\},\\ g &= [0^-,1] = G^1 \setminus \{0^+\},\\ e &= (0,1] = g \cap 1. \end{aligned}$$

The bisection 1 is the unit bisection of G, so

$$1x = x = x1 \quad \text{for all } x \in \{1,g,e\}.$$

Moreover, $g^2 = 1$, $e^2 = e$ and $eg = ge = e$. Thus, $S := \{1,e,g\}$ is an inverse semigroup with $x^* = x$ for all $x \in \{1,e,g\}$. A bisection t of G cannot contain both 0^+ and 0^-. Hence, either

$$0^+ \in t \subseteq 1, \qquad 0^- \in t \subseteq g,$$

or

$$t \subseteq e = 1 \cap g.$$

The groupoid G is the étale groupoid associated to the trivial action of S on G^0; here, the trivial action has 1 and g acting by the identity on G^0 and e acting by the identity on $(0,1] \subseteq G^0$. An action of G on a groupoid or a C*-algebra is equivalent to an action of S together with a compatible action of $G^0 = [0,1]$, see Theorem 6.13.

The transformation groupoid L of the S-action on H may be identified with the groupoid of the equivalence relation on $[0,1] \sqcup [0,1]$ that identifies the two copies of $(0,1]$, so that

$$\begin{aligned} L^1 &= [0,1] \times \{(+,+),(+,-),(-,+),(-,-)\} \\ &\subseteq ([0,1] \times \{(+,+),(+,-),(-,+),(-,-)\})^2. \end{aligned}$$

The S-grading on L has

$$\begin{aligned} L_1 &= (0,1] \times \{(+,+),(+,-),(-,+),(-,-)\} \sqcup \{0\} \times \{(+,+),(-,-)\}, \\ L_g &= (0,1] \times \{(+,+),(+,-),(-,+),(-,-)\} \sqcup \{0\} \times \{(+,-),(-,+)\}, \\ L_e &= (0,1] \times \{(+,+),(+,-),(-,+),(-,-)\} = L_1 \cap L_g. \end{aligned}$$

Therefore, $L_1 \cong H$ is open but not closed. The C*-algebra of L is

$$B := \mathrm{C}([0,1], \mathbb{M}_2).$$

To let S act on the C*-algebra A of H, we use the transformation groupoid C*-algebra B and the involution

$$u := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in B.$$

We have $u = u^*$ and $u^2 = 1$, $u \cdot A(0,1] = A(0,1] = A(0,1] \cdot u$ and $uA = Au$ as subsets of B. Let

$$A_1 := A, \qquad A_e := A(0,1] \subseteq A_1,$$

and

$$A_g := uA = Au.$$

These subspaces A_x for $x \in S$ satisfy $A_x^* = A_x = A_{x^*}$ for all $x \in S$ and $A_x \cdot A_y = A_{xy}$ for all $x, y \in S$; in particular, A_g is a full Hilbert bimodule over A_1 with inner products given by the usual formulas $a_1^* \cdot a_2$ and $a_1 \cdot a_2^*$. Furthermore, $A_1 \cap A_g = A_e$ and $A_1 + A_g = B$ because elements of A_g are precisely those of $f \in B$ with off-diagonal $f(0)$. Hence, the map

$$g \longmapsto A_g$$

defines an action of S on A by Hilbert bimodules. Since $A_1 + A_g$ is already complete in the C*-norm of B, there is only one C*-norm on

$A_1 + A_g$ that extends the given C*-norm on A_1. Thus, the sectional C*-algebra for the resulting Fell bundle over S is B, which is Morita-Rieffel equivalent to $C[0,1]$.

The S-action on A extends to all bisections of G because they are all contained in 1 or g. If $t \subseteq G^1$ is a bisection, then let $A_t = A_1|_{s(t)}$ if $t \subseteq 1$ and $A_t = A_g|_{s(t)}$ if $t \subseteq g$; this is consistent for $t \subseteq 1 \cap g = e$ because $A_e = A_1 \cap A_g$.

Next, we describe a twisted S-action by partial automorphisms of A that induces the S-action by Hilbert bimodules described above. (This is possible by [**3,** Corollary 4.16] because our saturated Fell bundle is regular in the notation of [**3**].)

A twisted S-action by partial automorphisms is given by ideals $A_1 = A$ and A_e with isomorphisms $\alpha_x\colon A_{xx^*} \to A_{x^*x}$ and unitary multipliers (the twists) $\omega(x,y)$ in $\mathcal{M}(A_{xyy^*x^*})$ for $x, y \in S$. For the idempotent elements $x = e, 1$, the isomorphism α_x is the identity; for $x = g$, it is the order-2 automorphism

$$\alpha_g\colon A \longrightarrow A, \qquad a \longmapsto uau,$$

because $a_1 \cdot ua_2 = u \cdot (ua_1u \cdot a_2)$ for all $a_1 \in A_1$ and $ua_2 \in A_g$. The automorphism α_g is *not* inner on A_1 because $u \in B$ does not belong to $\mathcal{M}(A)$. The restriction of α_g to the ideal A_e becomes inner, however, because $u \in \mathcal{M}(A(0,1])$. This unitary u enters in the twisting unitaries $\omega(x,y)$ for $x, y \in S$; they are 1 if $x = 1$ or $y = 1$, or if (x,y) is (e,e) or (g,g) ($\alpha_g^2 = \mathrm{Id}_A = \alpha_1$).

The remaining cases are

$$\omega(e,g) = \omega(g,e) = u|_{A_e},$$

that is, u viewed as a multiplier of the ideal $A_e = A(0,1]$. It is routine to check that this data gives a twisted action of S on A in the sense of [**3,** Definition 4.1] and that the resulting saturated Fell bundle over S is isomorphic to that described above. Incidentally, this is not a twisted action in the sense of Sieben [**34**] because $\omega(e,g)$ and $\omega(g,e)$ are non-trivial although e is idempotent.

This twisted S-action cannot be turned into a groupoid action of G by partial automorphisms because, for $x \in 1 \cap g$, the restrictions of α_g and α_1 to $A|_{s(x)}$ differ by a non-trivial inner automorphism. This impossibility is in accord with Theorem 7.1.

Remark 8.1. The Packer-Raeburn stabilization trick replaces a twisted group action by an untwisted action on a suitable C*-stabilization. We claim that this cannot be done for the above inverse semigroup twisted action. Let D be a C*-algebra with an untwisted action of S by automorphisms. Then, 1 and e act by the identity on D and by some ideal $D_e \triangleleft D$, respectively, and g acts by some automorphism α_g on D. If there is no twist, then

$$\alpha_g|_{D_e} = \alpha_1|_{D_e}$$

is the identity on D_e because $eg = e = ge$. Suppose that D is also a C*-algebra over $[0,1]$ with $D((0,1]) = D_e$. This then allows us to define an action of the groupoid G on D by letting elements of g or 1 act by the fiber restrictions of α_g and Id_D, respectively. This gives a well-defined, untwisted action of G on D. Theorem 7.1 implies that $\mathrm{Prim}(D) \ncong G^1$, so that A and D cannot be Morita-Rieffel equivalent. This example therefore shows that the Packer-Raeburn stabilization trick cannot be extended from groups to inverse semigroups or non-Hausdorff groupoids.

APPENDIX

A. Preliminaries on topological groupoids. This appendix defines topological groupoids and equivalences between them, following [**21**]. All of this works smoothly without assuming topological spaces to be Hausdorff or locally (quasi)-compact if appropriate definitions are chosen. The theory of possibly non-Hausdorff topological groupoids becomes very natural if one treats topological groupoids, Lie groupoids, infinite-dimensional Lie groupoids (modeled on Banach or Fréchet manifolds) and other types of groupoids simultaneously as in [**21**]. Here, we recall the results and definitions [**21**] that are relevant for us.

The theory of topological groupoids and their principal bundles and equivalences depends on a choice of "covers" in the category of topological spaces, see [**21**]. We choose the open surjections as covers. This means that we require the *range and source maps in a topological groupoid, the bundle projection in a principal bundle and the anchor maps in a* (*bibundle*) *equivalence to be open surjections.*

Following Bourbaki, we require compact and locally compact spaces to be Hausdorff. Since many authors allow non-Hausdorff locally compact spaces, we usually speak of "Hausdorff locally compact" spaces to avoid confusion. A topological space is *locally quasi-compact* if every point has a neighborhood basis consisting of quasi-compact neighborhoods. Strictly, this is more than having a single quasi-compact neighborhood, but both notions coincide in the locally Hausdorff case, which is the case in which we are interested. Recall that a topological space is *locally Hausdorff* if every point has a Hausdorff neighborhood (and thus, a neighborhood basis consisting of Hausdorff neighborhoods). A space is locally Hausdorff, locally quasi-compact if and only if every point has a compact (hence Hausdorff) neighborhood. It would make sense to call such spaces "locally compact," if it were not for the conflict with other established notation.

A.1. Topological groupoids, principal bundles and equivalences. We now specialize the general definitions of groupoids, groupoid actions, principal bundles, basic groupoid actions and bibundle equivalences in **[21]** to the category of (all) topological spaces with open surjections as covers.

Proposition A.1. *A topological groupoid consists of topological spaces G^0 and G^1 and continuous maps*

$$r, s\colon G^1 \rightrightarrows G^0$$

and

$$m\colon G^1 \times_{s,G^0,r} G^1 \longrightarrow G^1, \quad (g_1, g_2) \longmapsto g_1 \cdot g_2,$$

such that

(Gr1) $s(g_1 \cdot g_2) = s(g_2)$ *and* $r(g_1 \cdot g_2) = r(g_1)$ *for all* $g_1, g_2 \in G^1$;
(Gr2) m *is associative*:

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3) \quad \text{for all } g_1, g_2, g_3 \in G^1,$$

with $s(g_1) = r(g_2)$, $s(g_2) = r(g_3)$;
(Gr3) *the next two maps are homeomorphisms*:

$$G^1 \times_{s,G^0,r} G^1 \longrightarrow G^1 \times_{s,G^0,s} G^1, \qquad (g_1, g_2) \longmapsto (g_1 \cdot g_2, g_2),$$
$$G^1 \times_{s,G^0,r} G^1 \longrightarrow G^1 \times_{r,G^0,r} G^1, \qquad (g_1, g_2) \longmapsto (g_1, g_1 \cdot g_2);$$

(Gr4) *r and s are open surjections.*

Then, m is open and surjective and there are continuous maps

$$G^0 \longrightarrow G^1 \quad \text{and} \quad G^1 \longrightarrow G^1$$

with the usual properties of unit and inversion. Conversely, the maps in (Gr3) *are homeomorphisms if G has continuous unit and inversion maps.*

Proof. Our definition of a groupoid is exactly as in [**21,** Definition 3.4]. It implies that m is open, surjective and equivalent to the usual groupoid with unit and inverse by [**21,** Proposition 3.6]. □

Let G be a topological groupoid as above.

Proposition A.2. *A (right) G-action is a space X with continuous maps*

$$s\colon X \longrightarrow G^0$$

and

$$m\colon X \times_{s,G^0,r} G^1 \longrightarrow X, \qquad (x, g) \longmapsto x \cdot g,$$

such that

(A1) *$s(x \cdot g) = s(g)$ for all $x \in X$, $g \in G^1$ with $s(x) = r(g)$;*
(A2) *m is associative: $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 \cdot g_2)$ for all $x \in X$, $g_1, g_2 \in G^1$ with $s(x) = r(g_1)$ and $s(g_1) = r(g_2)$;*
(A3) *m is surjective.*

Condition (A3) *holds if and only if $x \cdot 1_{s(x)} = x$ for all $x \in X$, if and only if m is an open surjection, if and only if the next map is a homeomorphism:*

$$X \times_{s,G^0,r} G^1 \longrightarrow X \times_{s,G^0,s} G^1, \qquad (x, g) \longmapsto (x \cdot g, g).$$

Proof. This is contained in [**21,** Definition, Lemma 4.1]. □

Left actions are defined similarly and are equivalent to right actions by

$$g \cdot x = x \cdot g^{-1}.$$

The transformation groupoid $X \rtimes G$ of a groupoid action is a topological groupoid [**21,** Definition, Lemma 4.11]. Any groupoid acts on G^0 by

$$r(g) \cdot g := s(g) \quad \text{for all } g \in G^1,$$

and on G^1 both on the left and right by left and right multiplication.

Proposition A.3. *For any G-action on a topological space X, the orbit space projection*

$$X \longrightarrow X/G$$

is an open surjection, and X/G is Hausdorff if and only if

$$X \times_{X/G} X = \{(x_1, x_2) \in X \mid \text{there is } g \in G^1 \text{ with } s(x_1) = r(g) \text{ and } x_1 \cdot g = x_2\}$$

is a closed subset of $X \times X$.

Proof. The orbit space projection is open [**21,** Proposition 9.31] because the range and source maps of G are open. By [**21,** Proposition 9.18], X/G is Hausdorff if and only if $X \times_{X/G} X$ is closed in $X \times X$ (open surjections are clearly biquotient maps, see the discussion in [**21,** subsection 9.6]). □

We now specialize the general concepts of basic actions and principal bundles from [**21**] to our context.

Proposition A.4. *A right G-action is* basic *if the map*

$$X \times_{s,G^0,r} G^1 \longrightarrow X \times X, \qquad (x, g) \longmapsto (x, x \cdot g), \tag{A.1}$$

is a homeomorphism onto its image with the subspace topology.

A principal right G-bundle is a space X with continuous maps

$$s\colon X \longrightarrow G^0, \qquad p\colon X \longrightarrow Z,$$

and

$$m\colon X \times_{s,G^0,r} G^1 \longrightarrow X, \qquad (x, g) \longmapsto x \cdot g,$$

such that

(Pr1) *$s(x \cdot g) = s(g)$ and $p(x \cdot g) = p(x)$ for all $x \in X$, $g \in G^1$ with $s(x) = r(g)$;*

(Pr2) *m is associative:*

$$(x \cdot g_1) \cdot g_2 = x \cdot (g_1 \cdot g_2) \quad \text{for all } x \in X,$$

$g_1, g_2 \in G^1$ with $s(x) = r(g_1)$ and $s(g_1) = r(g_2)$;

(Pr3) *the map*

$$X \times_{s,G^0,r} G^1 \longrightarrow X \times_{p,Z,p} X, \qquad (x, g) \longmapsto (x, x \cdot g),$$

is a homeomorphism;

(Pr4) *the map p is open and surjective.*

Then, $x \cdot 1_{s(x)} = x$ for all $x \in X$, and there is a unique homeomorphism $Z \cong X/G$ intertwining p and the canonical projection

$$X \longrightarrow X/G.$$

Thus, a principal G-bundle is equivalent to a basic G-action with a homeomorphism $X/G \cong Z$.

Proof. A principal bundle in the above sense also satisfies $x \cdot 1_{s(x)} = x$ for all $x \in X$ because of (Pr3), see [**21,** Lemma 5.3]. Hence, m and s give a right G-action, and all conditions for a principal bundle in [**21**] are met. The unique homeomorphism $X/G \cong Z$ intertwining p and the canonical map $X \to X/G$ are given in [**21,** Lemma 5.3]. □

A groupoid action is called *basic* [**21**] if it becomes a principal bundle with

$$X \longrightarrow X/G$$

as a bundle projection. The canonical map $X \to X/G$ is automatically G-invariant, and it is an open surjection [**21,** Proposition 9.31]. Thus, the second half of (Pr1) and (Pr4) hold for any G-action with this choice of p. The first half of (Pr1) and (Pr2) are part of the definition of a groupoid action. The image of the map in (A.1) is $X \times_{X/G} X$ by the definition of X/G, so that (Pr3) is equivalent to (A.1) being a homeomorphism onto its image. □

Next, we consider the notion of equivalence between groupoids as defined in [**21**]. We will relate it to notions of equivalence by other authors in Appendix A.2.

Proposition A.5. *Let G and H be topological groupoids. A* bibundle equivalence *from H to G consists of a topological space X, continuous maps*

$$r\colon X \longrightarrow G^0, \qquad s\colon X \longrightarrow H^0$$

(anchor maps)*,*

$$G^1 \times_{s,G^0,r} X \longrightarrow X \quad \text{and} \quad X \times_{s,H^0,r} H^1 \longrightarrow X$$

(multiplications)*, satisfying the following conditions*:

(E1) $s(g \cdot x) = s(x)$, $r(g \cdot x) = r(g)$ *for all* $g \in G^1$, $x \in X$ *with* $s(g) = r(x)$ *and* $s(x \cdot h) = s(h)$, $r(x \cdot h) = r(x)$ *for all* $x \in X$, $h \in H^1$ *with* $s(x) = r(h)$;

(E2) *associativity*:

$$g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x, \quad g_2 \cdot (x \cdot h_1) = (g_2 \cdot x) \cdot h_1,$$
$$x \cdot (h_1 \cdot h_2) = (x \cdot h_1) \cdot h_2$$

for all $g_1, g_2 \in G^1$, $x \in X$, $h_1, h_2 \in H^1$ *with*

$$s(g_1) = r(g_2), \quad s(g_2) = r(x), \quad s(x) = r(h_1), \quad s(h_1) = r(h_2);$$

(E3) *the following two maps are homeomorphisms*:

$$G^1 \times_{s,G^0,r} X \longrightarrow X \times_{s,H^0,s} X, \qquad (g, x) \longmapsto (x, g \cdot x),$$
$$X \times_{s,H^0,r} H^1 \longrightarrow X \times_{r,G^0,r} X, \qquad (x, h) \longmapsto (x, x \cdot h);$$

(E4) *s and r are open*;

(E5) *s and r are surjective.*

Then,

$$1_{r(x)} \cdot x = x = x \cdot 1_{s(x)} \quad \text{for all } x \in X,$$

and the anchor maps descend to homeomorphisms $G\backslash X \cong H^0$ and $X/H \cong G^0$.

Proof. Conditions (E1) and (E3) are equivalent to (Pr1) and (Pr3) for both the left G-action with $p = s$ and the right H-action with $p = r$, respectively. Condition (E2) means that the left G- and right H-actions satisfy (Pr2) and commute. Conditions (E4) and (E5) together are equivalent to (Pr4) for both actions. Thus, conditions (E1)–(E5) characterize bibundle equivalences in the notation of **[21]**. The last sentence follows from the general properties of principal bundles, see Proposition A.4. □

We abbreviate "bibundle equivalence" to "equivalence" because we do not use any other equivalences between groupoids.

We have switched the direction of a bibundle equivalence compared to [**21**] because this is convenient here. Going from right to left is also consistent with our notation s and r for the right and left anchor maps.

A.2. Basic actions versus free and proper actions. Now, we compare our basic actions with free and proper actions. A continuous map

$$f\colon X \longrightarrow Y$$

is *closed* if it maps closed subsets of X to closed subsets of Y and *proper* if

$$\mathrm{Id}_Z \times f\colon Z \times X \longrightarrow Z \times Y$$

is closed for all topological spaces Z or, equivalently, f is closed and $f^{-1}(y)$ is quasi-compact for all $y \in Y$, see [**2,** I.10.2, Theorem 1]. A map from a Hausdorff space X to a Hausdorff locally compact space Y is proper if and only if preimages of compact subsets are compact. In this case, X is necessarily locally compact, see [**2,** I.10.3, Proposition 7].

Definition A.6. A right action of a topological groupoid G on a topological space X is *proper* if the map in (A.1) is proper. The action is *free* if the map (A.1) is injective.

Groupoids for which the action on their unit space is free, that is, for which the map

$$s \times r\colon G^1 \longrightarrow G^0 \times G^0$$

is injective, are often called *principal*, see [**30**]. This terminology conflicts, however, with the usual notion of a principal bundle, which requires extra topological conditions besides freeness of the action.

We call a groupoid *basic* if its canonical action on the object space is basic, that is, the map $s \times r\colon G^1 \to G^0 \times G^0$ is a homeomorphism onto its image.

Proposition A.7. *A groupoid action is free and proper if and only if it is basic and has Hausdorff orbit space.*

If G and H are topological groupoids with Hausdorff object spaces, then an equivalence from H to G in our sense is the same as a topological space X with commuting free and proper actions of G and H such that the anchor maps induce homeomorphisms $G\backslash X \cong H^0$ and $X/H \cong G^0$.

Proof. The characterization of free and proper actions is [**21,** Corollary 9.32]; the main point of the proof is that the orbit space is Hausdorff if and only if the orbit equivalence relation is closed in $X \times X$, see Proposition A.3. The left and right actions on an equivalence are basic with

$$X/H \cong G^0 \quad \text{and} \quad G\backslash X \cong H^0;$$

hence, they are free and proper if and only if G^0 and H^0 are Hausdorff, respectively. Conversely, if the actions of G and H on X are free and proper, then both actions are basic, and both anchor maps are open because they are equivalent to orbit space projections. Thus, we have an equivalence in our sense. □

For a general action of a groupoid G on a space X, the image of the map (A.1) is the orbit equivalence relation

$$X \times_{X/G} X \subseteq X \times X.$$

Thus, the map (A.1) is a homeomorphism (the action is basic) if and only if the action is free and the map that sends $(x_1, x_2) \in X \times_{X/G} X$ to the unique $g \in G^1$ with $s(x_1) = r(g)$ and $x_1 \cdot g = x_2$ is continuous.

If G, H and X are locally compact Hausdorff, then an equivalence in our sense is the same as a (G, H)-equivalence in the notation of [**23**]; the main result therein is that such an equivalence induces a Morita equivalence between the groupoid C*-algebras of G and H (for any Haar systems).

For non-Hausdorff groupoids, a notion of equivalence was defined by Tu [**36**] using a technical variant of proper actions. He calls a groupoid G *ρ-proper* with respect to a G-invariant continuous map

$$\rho\colon G^0 \longrightarrow T$$

if the map

$$(r, s)\colon G^1 \longrightarrow G^0 \times_{\rho,T,\rho} G^0, \qquad g \longmapsto (r(g), s(g)),$$

is proper. If T is non-Hausdorff, then $G^0 \times_{\rho,T,\rho} G^0$ need not be closed in $G^0 \times G^0$, so this is weaker than properness. In the definition of equivalence, he takes ρ to be the anchor map on the other side and requires the maps in (E3) to be proper. These maps are continuous bijections because the actions are free. A continuous, proper bijection, being closed, must be a homeomorphism. Thus, Tu's notion of equivalence is identical to ours.

A.3. Covering groupoids and equivalence.

Definition A.8. Let $f\colon X \to Z$ be a continuous, open surjection. The *covering groupoid* $G(f)$ has object space X, arrow space

$$X \times_{f,Z,f} X,$$

range and source maps

$$r(x_1, x_2) := x_1, \qquad s(x_1, x_2) := x_2,$$

and multiplication

$$(x_1, x_2) \cdot (x_2, x_3) := (x_1, x_3)$$

for all $x_1, x_2, x_3 \in X$ with $f(x_1) = f(x_2) = f(x_3)$.

The assumption on f implies that it is a quotient map, that is, we may identify Z with the quotient space $X/{\sim}$ by the following equivalence relation: $x \sim y$ if and only if $f(x) = f(y)$; and f becomes the quotient map $X \to X/{\sim}$. The covering groupoid $G(f)$ is the groupoid associated to this equivalence relation. In particular, Z can be identified with the orbit space $X/G(f)$ for the canonical action of $G(f)$ on its unit space X.

Every covering groupoid is basic, that is, its action on the unit space is basic. Conversely, if G is a basic groupoid, then it is isomorphic to a covering groupoid. The map

$$r \times s\colon G^1 \longrightarrow G^0 \times G^0$$

gives a homeomorphism from G^1 onto $G^0 \times_{f,G^0/G,f} G^0$, where

$$f\colon G^0 \longrightarrow G^0/G$$

denotes the quotient map. This yields an isomorphism of topological groupoids $G \cong G(f)$.

Example A.9 (Čech groupoids)**.** Let Z be a topological space, and let $\mathfrak{U}$ be an open covering of Z. Let

$$X := \bigsqcup_{U\in\mathfrak{U}} U,$$

and let

$$f\colon X \longrightarrow Z$$

be the canonical map; f is the inclusion map on each $U \in \mathfrak{U}$. This map is an open surjection. It is even étale, that is, a local homeomorphism. We denote the covering groupoid of f by $G_{\mathfrak{U}}$ and call it the *Čech groupoid* of the covering.

Assume that Z is locally Hausdorff, and choose the open covering $\mathfrak{U}$ to consist of Hausdorff open subsets $U \subset Z$. Then the Čech groupoid $G_{\mathfrak{U}}$ is a Hausdorff, étale topological groupoid, see also [**8,** Lemma 4.2]. If, in addition, Z is locally quasi-compact, then $G_{\mathfrak{U}}$ is a (Hausdorff) locally compact, étale groupoid. This is the situation in which we are primarily interested.

Proposition A.10. *Let*

$$f_i\colon X_i \longrightarrow Z \quad \text{for } i = 1, 2,$$

be two continuous, open surjections. Then, $X_1 \times_{f_1,Z,f_2} X_2$ *with the obvious left and right actions of* $G(f_1)$ *and* $G(f_2)$ *gives an equivalence from* $G(f_2)$ *to* $G(f_1)$*.*

Proof. This is [**21,** Example 6.4]. □

If $G(f_1)$ and $G(f_2)$ are Hausdorff locally compact, then so is the equivalence $X_1 \times_{f_1,Z,f_2} X_2$ between them. If the maps f_1 and f_2 are both étale, for instance, if they come from open coverings of Z, then the groupoids $G(f_1)$ and $G(f_2)$ are étale, and the anchor maps

$$X_1 \longleftarrow X_1 \times_{f_1,Z,f_2} X_2 \longrightarrow X_2, \qquad x_1 \longleftarrow (x_1, x_2) \longrightarrow x_2,$$

are étale as well.

Proposition A.11. *The covering groupoid $G(f)$ of a continuous open surjection $f\colon X \to Z$ is always equivalent (as a topological groupoid) to the space Z viewed as a groupoid with identity arrows only. In particular, the Čech groupoid of a covering of Z is equivalent to Z.*

Conversely, if X is an equivalence from a space Z to a topological groupoid G, then G is isomorphic to the covering groupoid of the anchor map $s\colon X \to Z$.

Hence, covering groupoids are exactly the groupoids that are equivalent to spaces.

Proof. The first part is a consequence of Proposition A.10 applied to $f_1 = f$ and $f_2 = \mathrm{Id}_Z$, see also [**21,** Example 6.3]. For the second part, observe that the action of Z on X is simply the anchor map

$$s\colon X \longrightarrow Z,$$

which must be an open surjection. The anchor map

$$r\colon X \longrightarrow G^0$$

must be a homeomorphism (because it must be the projection map $X \to Z\backslash X = X$), so we may assume $X = G^0$ as well. Then,

$$G^1 \times_{s,G^0,r} X \cong G^1,$$

and the first isomorphism in (E3) identifies G^1 with $X \times_{s,Z,s} X$. This yields an isomorphism from G to the covering groupoid $G(s)$ of $s\colon X \longrightarrow Z$. □

Let Z be a space and view Z as a groupoid with identity arrows only. When is Z equivalent to a locally compact, Hausdorff groupoid? If Z is equivalent to a topological groupoid G, then G is necessarily the covering groupoid $G(f)$ of a cover

$$f\colon X \longrightarrow Z$$

by Proposition A.11.

Given a space Z, we thus seek a locally compact, Hausdorff space X and an open, continuous surjection $f\colon X \to Z$ such that $X \times_{f,Z,f} X$ is locally compact. The question of when $X \times_{f,Z,f} X$ is locally compact is asked in [**8,** Section 4]. We answer this question in Proposition A.14: $X \times_{f,Z,f} X$ is locally compact if and only if Z is locally Hausdorff. Proposition A.16 says that the only topological spaces Z that are equivalent to locally compact Hausdorff groupoids are the locally Hausdorff, locally quasi-compact ones; Example A.9 gives such an equivalence, where the groupoid is even étale.

We need some preparation in order to prove Proposition A.14.

Definition A.12. [**2,** I.3.3, Definition 2, Proposition 5]. A subset S of a topological space X is *locally closed* if it satisfies the following equivalent conditions:

(1) any $x \in S$ has a neighborhood U such that $S \cap U$ is relatively closed in U;
(2) S is open in its closure;
(3) S is an intersection of an open and a closed subset of X.

The next proposition generalizes [**2,** I.9.7, Propositions 12, 13] to the locally Hausdorff case.

Proposition A.13. *A subset S of a locally Hausdorff, locally quasi-compact space X is locally quasi-compact in the subspace topology if and only if it is locally closed.*

Proof. First, let S be locally closed. Write $S = A \cap U$ with A closed and U open in X. Let $x \in S$. Since X is locally quasi-compact, the quasi-compact neighborhoods of x in X form a neighborhood basis of X. Since $x \in U$, those quasi-compact neighborhoods of x that are contained in U form a neighborhood basis in U. Their intersections with A remain quasi-compact because A is closed in X. They form a neighborhood basis of x in S, proving that S is locally quasi-compact.

Conversely, assume that S is locally quasi-compact in the subspace topology. Let $x \in S$. Let U be a Hausdorff open neighborhood of x in X. Then, $S \cap U$ is a neighborhood of x in S and, hence, contains a quasi-compact neighborhood K of x in S because S is locally quasi-compact. We have that $K = S \cap V$ for some neighborhood V of x in X, and we may assume that $V \subseteq U$ because $K \subseteq U$. The subset $S \cap V$ is relatively closed in V because $U \supseteq V$ is Hausdorff and $S \cap V$ is quasi-compact. Thus, S is locally closed. □

Proposition A.14. *Let $f\colon X \to Z$ be a continuous, open surjection. The equivalence relation $X_{f,Z,f}X \subseteq X \times X$ defined by f is locally closed if and only if Z is locally Hausdorff. In particular, if X is locally quasi-compact and locally Hausdorff, then $X_{f,Z,f}X$ is locally quasi-compact if and only if Z is locally Hausdorff.*

Proof. First, assume Z to be locally Hausdorff. Let $(x_1, x_2) \in X \times_{f,Z,f} X$, and let $U \subseteq Z$ be a Hausdorff open neighborhood of $f(x_1) = f(x_2)$. Then, $f^{-1}(U) \subseteq X$ is an open subset such that

$$f\colon f^{-1}(U) \longrightarrow U$$

is an open map onto a Hausdorff space. Hence,

$$f^{-1}(U) \times_{f,U,f} f^{-1}(U) = (X \times_{f,Z,f} X) \cap (f^{-1}(U) \times f^{-1}(U))$$

is relatively closed in $f^{-1}(U) \times f^{-1}(U)$ by [**21,** Proposition 9.15]. Thus, $X \times_{f,Z,f} X$ is locally closed in $X \times X$.

Conversely, assume $X \times_{f,Z,f} X$ to be locally closed in $X \times X$. Let $x \in X$. Then, (x, x) has a neighborhood in $X \times X$ so that $X \times_{f,Z,f} X$ restricted to it is relatively closed. Shrinking this neighborhood, we may assume that it is of the form $U \times U$ for an open neighborhood of x, by the definition of the product topology on $X \times X$. The map

$$f|_U\colon U \longrightarrow f(U)$$

is open, and

$$(X \times_{f,Z,f} X) \cap (U \times U) = U \times_{f|_U, f(U), f|_U} U.$$

Since this is relatively closed by assumption, [**21,** Proposition 9.15] shows that $f(U)$ is Hausdorff. Since x was arbitrary, this means that Z is locally Hausdorff.

The last sentence follows from the first and Proposition A.13. □

Corollary A.15. *A topological space X is locally Hausdorff if and only if the diagonal $\{(x, x) \mid x \in X\}$ is a locally closed subset in $X \times X$.*

Proof. Apply Proposition A.14 to the identity map. □

Proposition A.16. *Let G be a locally quasi-compact, locally Hausdorff groupoid, and let X be a basic right G-action. Then, X/G is locally quasi-compact and locally Hausdorff.*

If X is an equivalence from a space Z to G, then $Z \cong X/G$ is locally quasi-compact and locally Hausdorff.

Proof. Since G and X are locally quasi-compact and locally Hausdorff, so is their product $X \times G^1$. Since G^0 is locally Hausdorff, the diagonal in G^0 is locally closed by Corollary A.15. The fiber product $X \times_{s,G^0,r} G^1$ is the preimage of the diagonal in $G^0 \times G^0$ under the continuous map

$$r \times s \colon X \times G^1 \longrightarrow G^0 \times G^0;$$

hence, $X \times_{s,G^0,r} G^1$ is locally closed in $X \times G^1$. Thus, $X \times_{s,G^0,r} G^1$ is locally quasi-compact and locally Hausdorff by Proposition A.13.

Since the G-action on X is basic, $X \times_{s,G^0,r} G^1$ is homeomorphic to the subset $X \times_{X/G} X \subseteq X \times X$. Now, Proposition A.13 shows that $X \times_{X/G} X$ is locally closed in $X \times X$. Then, X/G is locally Hausdorff by Proposition A.14. Since continuous images of quasi-compact subsets are again quasi-compact, X/G is also locally quasi-compact.

An equivalence from a space Z to G is the same as a basic G-action with a homeomorphism $X/G \cong Z$. If this exists, then Z must be locally Hausdorff and locally quasi-compact by the above argument. □

B. Fields of Banach spaces over locally Hausdorff spaces.

Let X be a locally quasi-compact, locally Hausdorff space. Thus, any Hausdorff open subset of X is locally compact.

Definition B.1 (see **[25]** and the references therein)**.** An *upper semicontinuous field* of Banach spaces on X is a family of Banach spaces $(\mathfrak{B}_x)_{x\in X}$ with a topology on

$$\mathfrak{B} = \bigsqcup_{x\in X} \mathfrak{B}_x$$

such that, for each Hausdorff open subset U of X, $\mathfrak{B}|_U$ is an upper semicontinuous field of Banach spaces on U. In particular, the norm of any continuous section of $\mathfrak{B}|_U$ is an upper semicontinuous, scalar-valued function on U.

Let $\mathfrak{S}(U, \mathfrak{B})$ denote the vector space of continuous, compactly supported sections of $\mathfrak{B}|_U$. This is the union (hence, inductive limit) of the subspaces $\mathfrak{S}_0(K, \mathfrak{B})$ of continuous sections on K vanishing on ∂K, where K runs through the directed set of compact subsets of U and

$$\partial K = K \cap \overline{U \setminus K}$$

is the boundary of K in U. Each $\mathfrak{S}_0(K, \mathfrak{B})$ is a Banach space for the supremum norm

$$\|f\|_\infty := \sup\{\|f(x)\| \mid x \in K\}.$$

We call a subset of $\mathfrak{S}(U, \mathfrak{B})$ *bounded* if it is the image of a norm-bounded subset of $\mathfrak{S}_0(K, \mathfrak{B})$ for some K.

If $f \in \mathfrak{S}(U, \mathfrak{B})$ for a Hausdorff open subset U of X, then we always extend f to a section of $\mathfrak{B}$ on all of X by taking $f(x) := 0$ for $x \notin U$. Let $\mathfrak{S}(X, \mathfrak{B})$ be the vector space of all sections of $\mathfrak{B}$ that may be written as finite linear combinations

$$\sum_{i=1}^m f_i \quad \text{for } f_i \in \mathfrak{S}(U_i, \mathfrak{B})$$

and Hausdorff open subset U_i of X. We call such sections of $\mathfrak{B}$ *quasi-continuous.*

A subset A of $\mathfrak{S}(X, \mathfrak{B})$ is *bounded* if there are Hausdorff open subsets $U_1, \dots, U_m$, of X and bounded subsets $A_i \subseteq \mathfrak{S}(U_i, \mathfrak{B})$ for $i = 1, \dots, m$, such that every element of A may be written as a sum

$$\sum_{i=1}^m f_i$$

with $f_i \in A_i$ for $i = 1, \dots, m$.

To simplify our proofs, we use bornological language, that is, we speak of bounded instead of open subsets. For a Hausdorff locally compact space X, $\mathfrak{S}(X, \mathfrak{B})$ with its usual topology is an inductive limit of Banach spaces. The inductive limit topology is determined by its continuous seminorms. A seminorm is continuous if and only if it is bounded in the sense that its supremum over each bounded subset is finite; this is so because a seminorm on a Banach space is continuous if and only if it is bounded. For locally Hausdorff X, the bounded seminorms are those that restrict to bounded seminorms on all the subspaces $\mathfrak{S}(U,\mathfrak{B})$ for $U \subseteq X$ open and Hausdorff; this is the same as the quotient topology from the map

$$\bigoplus_{U} \mathfrak{S}(U,\mathfrak{B}) \longrightarrow \mathfrak{S}(X,\mathfrak{B}),$$

where U runs through the Hausdorff open subsets of X. Thus, the usual topology on $\mathfrak{S}(X,\mathfrak{B})$, which is the quotient topology induced by the inductive limit topologies on the direct sums of the spaces $\mathfrak{S}(U,\mathfrak{B})$, is that generated by all bounded seminorms.

Let $\mathfrak{U}$ be a family of open subsets of X with the next two properties:

(1) $X = \bigcup_{U\in\mathfrak{U}} U$, that is, for each $x \in X$, there is $U \in \mathfrak{U}$ with $x \in U$;
(2) $U_1 \cap U_2 = \bigcup\{U \in \mathfrak{U} \mid U \subseteq U_1 \cap U_2\}$ for all $U_1, U_2 \in \mathfrak{U}$; that is, if $x \in U_1 \cap U_2$, then there is $U \in \mathfrak{U}$ with $U \subseteq U_1 \cap U_2$ and $x \in U$.

In our main application, the open subsets in $\mathfrak{U}$ will not be Hausdorff. Thus, $\mathfrak{S}(U,\mathfrak{B})$ for $U \in \mathfrak{U}$ is defined in the same way as $\mathfrak{S}(X,\mathfrak{B})$, by taking finite linear combinations of continuous compactly supported sections on Hausdorff open subsets of U. We view $\mathfrak{S}(U,\mathfrak{B})$ as a subspace in $\mathfrak{S}(X,\mathfrak{B})$ by extending functions on U by 0 outside U. This gives an injective, bounded linear map

$$\mathfrak{S}(U,\mathfrak{B}) \longrightarrow \mathfrak{S}(X,\mathfrak{B}).$$

Being bounded means that it maps bounded subsets to bounded subsets.

Let

$$\iota_U : \mathfrak{S}(U,\mathfrak{B}) \longrightarrow \bigoplus_{U\in\mathfrak{U}} \mathfrak{S}(U,\mathfrak{B})$$

for $U \in \mathfrak{U}$ denote the inclusion map of the U-summand. We call a subset A of $\bigoplus_{U\in\mathfrak{U}} \mathfrak{S}(U,\mathfrak{B})$ bounded if there are finitely many $U_1,\dots,U_m \in \mathfrak{U}$ and bounded subsets A_i of $\mathfrak{S}(U_i,\mathfrak{B})$ such that any element of A may be written as

$$\sum_{i=1}^{m} \iota_{U_i}(f_i)$$

with $f_i \in A_i$.

Proposition B.2. *The map*

$$E\colon \bigoplus_{U\in\mathfrak{U}} \mathfrak{S}(U,\mathfrak{B}) \longrightarrow \mathfrak{S}(X,\mathfrak{B})$$

is bounded linear and a bornological quotient map in the sense that any bounded subset of $\mathfrak{S}(X,\mathfrak{B})$ *is the image of a bounded subset of*

$$\bigoplus_{U\in\mathfrak{U}} \mathfrak{S}(U,\mathfrak{B});$$

in particular, it is surjective.

The kernel of E *is the closed linear span of the set of elements of the form* $\iota_U(f) - \iota_V(f)$ *for* $f \in \mathfrak{S}(U,\mathfrak{B})$*,* $U, V \in \mathfrak{U}$ *with* $U \subseteq V$*.*

"Closure" in the description of the kernel is bornological, defined using Mackey's notion of convergence in a bornological vector space. For any element $g \in \ker E$, we will find a bounded subset

$$A \subseteq \bigoplus_{U\in\mathfrak{U}} \mathfrak{S}(U,\mathfrak{B})$$

and linear combinations g_n of $\iota_U(f) - \iota_V(f)$ for $f \in \mathfrak{S}(U,\mathfrak{B})$ and $U, V \in \mathfrak{U}$ with $U \subseteq V$ such that $g - g_n \in 2^{-n} \cdot A$. This implies convergence in any bounded seminorm.

Remark B.3. Proposition B.2 implies that E is a quotient map with respect to the canonical topologies on the spaces involved, that is, a seminorm p on $\mathfrak{S}(X,\mathfrak{B})$ is continuous if and only if $p \circ E$ is a continuous seminorm on $\bigoplus_{U\in\mathfrak{U}} \mathfrak{S}(U,\mathfrak{B})$. The proof assumes that continuity and boundedness are equivalent for seminorms on both spaces and that E is a bornological quotient map. It seems inconvenient, however, to prove this directly without bornological language.

Proof. In the proof, we abbreviate $\mathfrak{S}(U) := \mathfrak{S}(U, \mathfrak{B})$ because the Banach space bundle is fixed throughout. First, we show that E is a bornological quotient map.

Let $A \subseteq \mathfrak{S}(X)$ be bounded. By definition, there are finitely many Hausdorff open subsets $V_1, \dots, V_m \subseteq X$, compact subsets $K_i \subseteq V_i$ and scalars $C_i > 0$ such that any $f \in A$ may be written as

$$\sum_{i=1}^{m} f_i$$

with $f_i \in \mathfrak{S}_0(K_i)$ having $\|f_i\|_\infty \leq C_i$.

Since the subsets $U \in \mathfrak{U}$ cover X, they cover the compact subset K_i. Since compact spaces are paracompact, there is a finite subordinate partition of unity $(\psi_{i,U})_{U\in\mathfrak{U}}$, that is,

$$\psi_{i,U} \colon K_i \longrightarrow [0,1]$$

is continuous and has compact support $L_{i,U}$ contained in $U \cap K_i$, only finitely many $\psi_{i,U}$ are non-zero and

$$\sum_{U\in\mathfrak{U}} \psi_{i,U}(x) = 1.$$

If $f_i \in \mathfrak{S}_0(K_i)$, then

$$f_i \cdot \psi_{i,U} \in \mathfrak{S}_0(K_i \cap L_{i,U}) \subseteq \mathfrak{S}(V_i \cap U)$$

and

$$\|f_i \cdot \psi_{i,U}\|_\infty \leq \|f_i\|_\infty.$$

Now, write $f \in A$ first as $\sum_{i=1}^m f_i$ with $f_i \in \mathfrak{S}_0(K_i)$ having $\|f_i\|_\infty \leq C_i$, and then as

$$\sum_{i=1}^{n} \sum_{U\in\mathfrak{U}} f_i \cdot \psi_{i,U}.$$

This sum is still finite because only finitely many $\psi_{i,U}$ are non-zero for each i, and each summand $f_i \cdot \psi_{i,U}$ runs through a bounded subset of $\mathfrak{S}(V_i \cap U)$, and hence, of $\mathfrak{S}(U)$, because we have uniform control on the supports

$$\operatorname{supp} f_i\psi_{i,U} \subseteq K_i \cap L_{i,U}$$

and norms

$$\|f_i \cdot \psi_{i,U}\|_\infty \leq C_i$$

of the summands. Hence, A is contained in the E-image of a bounded subset in $\bigoplus \mathfrak{S}(U)$.

Now, we describe the kernel of E. Let N be the linear span of elements of the form $\iota_U(f) - \iota_V(f)$ for all $f \in \mathfrak{S}(U)$, $U, V \in \mathfrak{U}$ with $U \subseteq V$. Since $E(\iota_U(f) - \iota_V(f)) = 0$, we obtain $N \subseteq \ker E$. If $U_1, U_2, V \in \mathfrak{U}$ satisfy $V \subseteq U_1 \cap U_2$ and $f \in \mathfrak{S}(V)$, then

$$\iota_{U_1}(f) - \iota_{U_2}(f) = -(\iota_V(f) - \iota_{U_1}(f)) + (\iota_V(f) - \iota_{U_2}(f)) \in N.$$

We shall modify a given element of $\ker E$ by adding elements of N so that the norms of its constituents become arbitrarily small, without enlarging their supports.

A generic element

$$f \in \bigoplus_{U \in \mathfrak{U}} \mathfrak{S}(U)$$

is of the form

$$f = \sum \iota_U(f_U)$$

with $f_U \in \mathfrak{S}(U)$ and $f_U = 0$ for all but finitely many U. Each non-zero f_U is a sum

$$f_U = \sum_{j=1}^{k_U} f_{U,j},$$

with $f_{U,j} \in \mathfrak{S}(V_{U,j})$ for finitely many Hausdorff open subsets $V_{U,1}, \dots, V_{U,k_U} \subseteq U$. We renumber the finitely many Hausdorff open subsets $V_{U,j}$ consecutively as $V_1, \dots, V_m$ and relabel our sections $f_i \in \mathfrak{S}(V_i)$ accordingly. Let $U_i \in \mathfrak{U}$ for $i = 1, \dots, m$, be such that

$$f = \sum_{i=1}^{m} \iota_{U_i}(f_i);$$

so, $V_i \subseteq U_i$. Let

$$K_i := \operatorname{supp} f_i \subseteq V_i,$$

and let K_i° be the interior of K_i inside V_i. Thus, $x \in K_i^\circ$ for all $x \in X$ with $f_i(x) \neq 0$.

Now, assume $f \in \ker(E)$, and let $\epsilon > 0$. We will construct a finite sequence

$$f^{(j)} = \sum_{i=1}^{m} \iota_{U_i}(f_i^{(j)}),$$

with $f^{(0)} = f$, $f^{(j+1)} - f^{(j)} \in N$ and

$$\|f_i^{(m)}\| < \epsilon \quad \text{for all } i = 1, \dots, m.$$

Furthermore, our construction ensures that the support of $f_i^{(j)}$ is contained in K_i for all i, j. Letting ϵ run through a sequence going to 0, the differences $f - f^{(m)}$ in N will converge to f in the sense explained above because each constituent $f_i - f_i^{(m)}$ converges to f_i in the normed space $\mathfrak{S}_0(K_i)$. Our construction will be such that $f_i^{(j)} = f_i^{(i)}$ for $j \geq i$, that is, in the jth step we keep $f_1, \dots, f_{j-1}$ fixed. To make the following steps possible we aim for stronger norm estimates $\|f_i^{(j)}\| < 2^{j-m}\epsilon$. Assume that we have already constructed

$$f^{(j)} = \sum_{i=1}^{m} \iota_{U_i}(f_i^{(j)})$$

with $f - f^{(j)} \in N$ and

$$\|f_i^{(j)}\| < 2^{j-m}\epsilon \quad \text{for } i = 1, \dots, j;$$

for $j = 0$, this is satisfied for $f^{(0)} = f$. We shall construct

$$f^{(j+1)} = \sum_{i=1}^{m} \iota_{U_i}(f_i^{(j+1)})$$

with $f^{(j)} - f^{(j+1)} \in N$, and hence, $f - f^{(j+1)} \in N$, with $f_i^{(j+1)} = f_i^{(j)}$ for $i = 1, 2, \dots, j$, and $\|f_{j+1}^{(j+1)}\| < 2^{j+1-m}\epsilon$.

Let

$$A_{j+1} = \{x \in V_{j+1} \mid \|f_{j+1}^{(j)}(x)\| \geq 2^{j+1-m}\epsilon\}.$$

This is a closed subset of K_{j+1}° because the norm function is upper semicontinuous. Since K_{j+1} is compact, A_{j+1} is compact. Since

$E(f) = 0$ and $E(f - f^{(j)}) = 0$, we have

$$\sum_{i=1}^{m} f_i^{(j)}(x) = 0 \quad \text{for all } x \in X.$$

If $x \in A_{j+1}$, then this gives

$$\left\|\sum_{i=j+2}^{m} f_i^{(j)}(x)\right\| = \left\|\sum_{i=1}^{j+1} f_i^{(j)}(x)\right\| \geq \|f_{j+1}^{(j)}(x)\| - \sum_{i=1}^{j} \|f_i^{(j)}\|_\infty > 0.$$

Hence, there must be $i > j+1$ with $f_i^{(j)}(x) \neq 0$, so that $x \in K_i^\circ$. Thus, the open subsets K_i° for $i > j+1$ cover A_{j+1}. If $x \in A_{j+1} \cap K_i^\circ$, then $x \in U_i \cap U_{j+1}$. By our assumption on $\mathfrak{U}$, there is $U \in \mathfrak{U}$ with $x \in U$ and $U \subseteq U_i \cap U_{j+1}$. Thus, the open subsets $K_i^\circ \cap U$ for $i > j+1$ and $U \in \mathfrak{U}$ with $U \subseteq U_i \cap U_{j+1}$ cover A_{j+1}.

Since A_{j+1} is compact and contained in the Hausdorff locally compact space V_{j+1}, there is a subordinate finite partition of unity $(\psi_{i,U})$, that is, all but finitely many $\psi_{i,U}$ are non-zero,

$$\psi_{i,U} \colon A_{j+1} \longrightarrow [0,1]$$

is a continuous function with compact support contained in $K_i^\circ \cap U$, and

$$\sum \psi_{i,U}(x) = 1 \quad \text{for } x \in A_{j+1}.$$

We may extend each non-zero $\psi_{i,U}$ from A_{j+1} to a continuous function

$$\overline{\psi}_{i,U} \colon K_{j+1} \longrightarrow [0,1]$$

vanishing in a neighborhood of ∂K_{j+1} and on $K_{j+1} \setminus (K_i^\circ \cap U)$ because these two compact subsets of A_{j+1} are disjoint from the compact support of $\psi_{i,U}$ in $K_i^\circ \cap U$. If necessary, we multiply all $\overline{\psi}_{i,U}$ with a suitable cut-off function so that

$$\sum \overline{\psi}_{i,U}(x) \leq 1 \quad \text{for all } x \in K_{j+1}.$$

Now, we let

$$f^{(j+1)} = f^{(j)} + \sum_{i,U} \iota_{U_i}(f_{j+1}^{(j)} \overline{\psi}_{i,U}) - \iota_{U_{j+1}}(f_{j+1}^{(j)} \overline{\psi}_{i,U}).$$

By construction, $f^{(j)}_{j+1}\overline{\psi}_{i,U}$ is continuous and supported in a compact subset of $K_i^\circ \cap U$ with $U \subseteq U_i \cap U_{j+1}$, $U \in \mathfrak{U}$. Hence,

$$\iota_{U_i}(f^{(j)}_{j+1}\overline{\psi}_{i,U}) - \iota_{U_{j+1}}(f^{(j)}_{j+1}\overline{\psi}_{i,U}) \in N,$$

so $f^{(j+1)} - f^{(j)} \in N$ as desired. Since only $i > j+1$ appear in the sum, $f_i^{(j+1)} = f_i^{(j)}$ for $i < j+1$. We obtain

$$f^{(j+1)}_{j+1}(x) = f^{(j)}_{j+1}(x) \cdot \left(1 - \sum_{i,U} \overline{\psi}_{i,U}(x)\right).$$

This has a supremum norm less than $2^{j+1-m}\epsilon$ because

$$1 - \sum_{i,U} \overline{\psi}_{i,U}(x)$$

vanishes where $\|f^{(j)}_{j+1}(x)\| \geq 2^{j+1-m}\epsilon$ and is at most 1 everywhere else. The support of $f^{(j+1)}_{j+1}$ is still contained in K_{j+1} by construction.

For $i > j+1$, we obtain

$$f^{(j+1)}_i = f^{(j)}_i + \sum_U f^{(j)}_{j+1} \cdot \overline{\psi}_{i,U}.$$

This still has support K_i because $\overline{\psi}_{i,U}$ is supported there. This completes the induction step, and thus, the proof. □

Remark B.4. If X is Hausdorff, then a partition of unity argument as in the proof of [**4,** Theorem 2.13] shows that $\ker(E)$ is the linear span *without closure* of $\iota_U(f) - \iota_V(f)$ with $U, V \in \mathfrak{U}$. Hence, this linear span is already closed for the natural topology on

$$\bigoplus_{U \in \mathfrak{U}} \mathfrak{S}(U, \mathfrak{B}).$$

A convergent infinite series is needed to generate $\ker E$ from $\iota_U(f) - \iota_V(f)$ with $U, V \in \mathfrak{U}$. This happens in simple examples, such as the space

$$X = [0,1] \bigsqcup_{(0,1]} [0,1]$$

discussed in Section 8 with the trivial bundle $\mathbb{C}$ and the standard open cover by two Hausdorff open subsets with their intersection $(0,1]$.

B.1. Proof of Theorem 5.5. We apply Proposition B.2 to $X = L$, the cover $(L_t)_{t\in S}$, and the given Fell bundle $\mathfrak{B}$ as in the statement of Theorem 5.5. The subsets B^* and $B_1 * B_2$ for bounded subsets $B, B_1, B_2 \subseteq \mathfrak{S}(L, \mathfrak{B})$ are again bounded; this is routine to check. Thus, $\mathfrak{S}(L, \mathfrak{B})$ is a bornological *-algebra. (The continuity of the operations for the inductive limit topology is also known but somewhat more difficult.)

We shall cite below some results of **[31]** which follow from the disintegration and Morita equivalence theorems. We assume that they hold for the Fell bundle $\mathfrak{B}$ in question and its restriction to G; this is not yet proved in the literature, see the discussion before Theorem 5.5. Remark B.6 sketches a slightly more complicated proof that uses only the Morita equivalence theorem, that is, the assumptions in Theorem 5.5.

Lemma B.5. *The* C^**-algebra* $\mathrm{C}^*(L, \mathfrak{B})$ *is the completion of* $\mathfrak{S}(L, \mathfrak{B})$ *in the maximal bounded* C^**-seminorm.*

Proof. Usually, $\mathrm{C}^*(L, \mathfrak{B})$ is defined as the completion of $\mathfrak{S}(L, \mathfrak{B})$ in the maximal C^*-seminorm that is bounded with respect to the I-norm, a certain norm on $\mathfrak{S}(L, \mathfrak{B})$. It has been shown **[31,** Corollary 4.8] that a representation of $\mathfrak{S}(L, \mathfrak{B})$ that is continuous with respect to the inductive limit topology is bounded for the I-norm. Hence, a C^*-seminorm on $\mathfrak{S}(L, \mathfrak{B})$ is continuous with respect to the inductive limit topology if and only if it is bounded with respect to the I-norm. The topology on $\mathfrak{S}(L, \mathfrak{B})$ called *inductive limit topology* in **[31]** is really the quotient topology induced by the inductive limit topology on

$$\bigoplus_{U\in\mathfrak{U}} \mathfrak{S}(U, \mathfrak{B}),$$

where $\mathfrak{U}$ is the set of all Hausdorff open subsets of L and $\bigoplus_{U\in\mathfrak{U}} \mathfrak{S}(U, \mathfrak{B})$ is viewed as the inductive limit of the Banach subspaces

$$\bigoplus_{U\in F} \mathfrak{S}_0(K_U, \mathfrak{B})$$

where F is a finite subset of $\mathfrak{U}$ and $K_U \subseteq U$ for $U \in F$ are compact subsets. As discussed above, a seminorm is continuous in this sense if and only if it is bounded in the canonical bornology on $\mathfrak{S}(L, \mathfrak{B})$ introduced in Appendix B. □

Proof of Theorem 5.5. Let

$$D := \bigoplus_{t\in S} \mathfrak{S}(L_t, \mathfrak{B}).$$

This carries a canonical direct sum bornology as in Appendix B. Fell bundle operations turn it into a *-algebra. The multiplication and involution are bounded, so we also have a bornological *-algebra. The map

$$E\colon D \longrightarrow \mathfrak{S}(L, \mathfrak{B})$$

from Proposition B.2 is a bounded *-homomorphism.

Since E is a bornological quotient map by Proposition B.2, a C*-seminorm p on $\mathfrak{S}(L, \mathfrak{B})$ is bounded if and only if $p \circ E$ is a bounded C*-seminorm on D. A bounded C*-seminorm on D is of the form $p\circ E$ for a C*-seminorm p on $\mathfrak{S}(L, \mathfrak{B})$ if and only if it vanishes on the kernel of E. By Proposition B.2, a bounded seminorm on D vanishes on $\ker E$ if and only if it vanishes on $\iota_t(f) - \iota_u(f)$ for all $f \in \mathfrak{S}(L_t, \mathfrak{B})$, $t, u \in S$, $t \leq u$. Thus, $\mathrm{C}^*(L, \mathfrak{B})$ is isomorphic to the completion of D in the maximal C*-seminorm q on D that is bounded and vanishes on $\iota_t(f) - \iota_u(f)$ for all f, t, u as above.

The restriction of this C*-seminorm q to $\mathfrak{S}(G, \mathfrak{B}) \subseteq D$ is bounded. Since $\mathrm{C}^*(G, \mathfrak{B})$ is defined as the completion of $\mathfrak{S}(G, \mathfrak{B})$ with respect to the maximal bounded C*-seminorm on $\mathfrak{S}(G, \mathfrak{B})$, q extends to a C*-seminorm on $\mathrm{C}^*(G, \mathfrak{B})$. Since

$$q(f)^2 = q(f^* * f) \quad \text{for } f \in \mathfrak{S}(L_t, \mathfrak{B}),$$

the restriction of q to $\mathfrak{S}(L_t, \mathfrak{B})$ is dominated by the Hilbert module norm from $\mathrm{C}^*(L_t, \mathfrak{B})$. Thus, q automatically extends to the sum

$$\bigoplus_{t\in S} \mathrm{C}^*(L_t, \mathfrak{B})_{t\in S}.$$

Furthermore, q still annihilates $\iota_t(f) - \iota_u(f)$ for all $f \in \mathrm{C}^*(L_t, \mathfrak{B})$, $t, u \in S$, $t \leq u$, because $\mathfrak{S}(L_t, \mathfrak{B})$ is dense in $\mathrm{C}^*(L_t, \mathfrak{B})$.

Conversely, a C*-seminorm on

$$\bigoplus_{t\in S} \mathrm{C}^*(L_t, \mathfrak{B})$$

that annihilates $\iota_t(f) - \iota_u(f)$ for all $f \in \mathrm{C}^*(L_t, \mathfrak{B})$, $t, u \in S$, $t \leq u$, restricts to a C*-seminorm q on D that annihilates $\iota_t(f) - \iota_u(f)$ for all $f \in \mathfrak{S}(L_t, \mathfrak{B})$, $t, u \in S$, $t \leq u$. Since D is dense in $\bigoplus_{t\in S} \mathrm{C}^*(L_t, \mathfrak{B})$, this implies that $\mathrm{C}^*(L, \mathfrak{B})$ is isomorphic to the completion of $\bigoplus_{t\in S} \mathrm{C}^*(L_t, \mathfrak{B})$ in the maximal C*-seminorm that annihilates $\iota_t(f) - \iota_u(f)$ for all $f \in \mathrm{C}^*(L_t, \mathfrak{B})$, $t, u \in S$, $t \leq u$. This is exactly the definition of the section C*-algebra of the Fell bundle $\mathrm{C}^*(L_t, \mathfrak{B})_{t\in S}$ over S. This concludes the proof of Theorem 5.5. □

Remark B.6. We may also prove Theorem 5.5 without Lemma B.5, using the standard definition of $\mathrm{C}^*(L, \mathfrak{B})$ involving the I-norm on D. This variant of the proof has the advantage that it does not require the disintegration theorem. We still need the Morita equivalence theorem for our Fell bundles, however, so that our inner products are positive and generate the expected ideals.

We merely explain the new points in this alternative proof. The I-norm on $\mathfrak{S}(L, \mathfrak{B})$ restricts to the I-norm on $\mathfrak{S}(G, \mathfrak{B})$. Consider a C*-seminorm q on D that annihilates $\iota_t(f) - \iota_u(f)$ for all $f \in \mathfrak{S}(L_t, \mathfrak{B})$, $t, u \in S$, $t \leq u$, and satisfies

$$q(f) \leq \|f\|_I \quad \text{for all } f \in \mathfrak{S}(G, \mathfrak{B}).$$

Then,

$$q(f) = q(f^* * f)^{1/2} \leq \|f^* * f\|_I^{1/2} \leq \|f\|_I$$

for all $f \in \mathfrak{S}(L_t, \mathfrak{B})$, $t \in S$. Thus, q is bounded with respect to our bornology as well, so it factors as $\dot{q} \circ E$ for a bounded seminorm $\dot{q}$ on $\mathfrak{S}(L, \mathfrak{B})$ by Proposition B.2. This seminorm satisfies $\dot{q}(f) \leq \|f\|_I$ for all $f \in \mathfrak{S}(L_t, \mathfrak{B})$, $t \in S$. But then $\dot{q}(f) \leq \|f\|_I$ follows for all $f \in \mathfrak{S}(L, \mathfrak{B})$, $t \in S$. □

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