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STRONGLY COPURE PROJECTIVE, INJECTIVE AND FLAT COMPLEXES

XIN MA AND ZHONGKUI LIU

ABSTRACT. In this paper, we extend the notions of strongly copure projective, injective and flat modules to that of complexes and characterize these complexes. We show that the strongly copure projective precover of any finitely presented complex exists over n-FC rings, and a strongly copure injective envelope exists over left Noetherian rings. We prove that strongly copure flat covers exist over arbitrary rings and that $(\mathcal{SCF}, \mathcal{SCF}^{\perp})$ is a perfect hereditary cotorsion theory where $\mathcal{SCF}$ is the class of strongly copure flat complexes.

1. Introduction. Enochs and Jenda [**7**] introduced the notions of strongly copure injective and flat modules. A left R-module M is said to be *strongly copure injective* if $\mathrm{Ext}_R^i(E,M)=0$ for all injective left R-modules E and all $i\geq 1$. A left R-module M is said to be *strongly copure flat* if $\mathrm{Tor}_i^R(E,M)=0$ for all injective right R-modules E and all $i\geq 1$. They showed the existence of strongly copure injective preenvelopes over Noetherian rings and strongly copure flat preenvelopes over commutative Artinian rings. Mao and Ding [**15**] introduced the notion of strongly P-projective modules. M is said to be strongly P-projective if $\mathrm{Ext}_R^i(M,P)=0$ for all projective left R-modules P. We find that the notion happens to be the duality of strongly copure injective modules and call them strongly copure projective modules in this paper. Mao [**13**] studied homological properties of strongly copure projective modules and the existence of strongly copure projective precovers.

In this paper, we extend the notions of strongly copure projective modules, strongly copure injective and flat modules to complexes. They

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are called *strongly copure projective*, *injective* and *flat complexes*, respectively, which are generalizations of Gorenstein projective, injective and flat complexes. However, the existence of envelopes and covers with respect to the classes of Gorenstein projective, injective and flat complexes has been studied extensively (see [**9, 10, 11, 12**]). Based on these studies, in this paper, we will concentrate our effort on the existence of covers and envelopes of strongly copure projective, injective and flat complexes. We prove that any finitely presented complex C has a strongly copure projective precover over n-FC rings. We show the existence of the strongly copure injective envelope over Noetherian rings. We also show that the strongly copure flat cover exists over arbitrary rings and that $(\mathcal{SCF}, \mathcal{SCF}^{\perp})$ is a perfect hereditary cotorsion theory, where $\mathcal{SCF}$ is the class of strongly copure flat complexes.

Throughout this paper, R denotes a ring with unitary. $\mathcal{C}$ will be the abelian category of complexes of R-modules. This category has enough projectives and injectives. For objects C and D of $\mathcal{C}$, $\mathrm{Hom}(C, D)$ is the abelian group of morphisms from C to D in $\mathcal{C}$, and $\mathrm{Ext}^i(C, D)$ for $i \geq 0$ will denote the groups we obtain from the right derived functors of Hom.

A complex

$$\cdots \longrightarrow C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \cdots$$

of left R-modules will be denoted (C, δ) or C. We will use subscripts to distinguish complexes. So, if $\{C_i\}_{i\in I}$ is a family of complexes, C_i will be

$$\cdots \longrightarrow C_i^{-1} \xrightarrow{\delta_i^{-1}} C_i^0 \xrightarrow{\delta_i^0} C_i^1 \xrightarrow{\delta_i^1} \cdots .$$

Given a left R-module M, we will denote by $\overline{M}$ the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{id}} M \longrightarrow 0 \longrightarrow \cdots$$

with the M in the -1 and 0th position. Also, by $\underline{M}$ we mean the complex with M in the 0th place and 0 in the other places. Given a complex C and an integer m, $C[m]$ denotes the complex such that $C[m]^n = C^{m+n}$ whose boundary operators are $(-1)^m\delta^{m+n}$. The nth cycle module is defined as $\mathrm{Ker}\delta^n$ and is denoted Z^nC. The nth boundary module is defined as $\mathrm{Im}\,\delta^{n-1}$ and is denoted B^nC.

We recall that a complex P is projective if the functor $\mathrm{Hom}(P,-)$ is exact. Equivalently, P is projective if and only if P is exact and, for each $n \in \mathbb{Z}$, $\mathrm{Ker}(P^n \to P^{n+1})$ is a projective module. For example, if M is a projective module, then the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{id}} M \longrightarrow 0 \longrightarrow \cdots$$

is projective. In fact, any projective complex is uniquely, up to isomorphism, the direct sum of such complexes (one such complex for each $n \in \mathbb{Z}$). The dual notion is that of injective complex. A complex E is injective if and only if it is exact and if, for every $n \in \mathbb{Z}$, $\mathrm{Ker}(E^n \to E^{n+1})$ is an injective module. If N is a injective module, then

$$\cdots \longrightarrow 0 \longrightarrow N \xrightarrow{\mathrm{id}} N \longrightarrow 0 \longrightarrow \cdots$$

is an injective complex. Up to an isomorphism, any injective complex is a direct sum of such complexes. This direct sum is also the direct product of these complexes.

Let C be a complex of left R-modules (respectively, of right R-modules), and let D be a complex of left R-modules. We will denote by $\mathrm{Hom}^\bullet(C,D)$ (respectively, $C \otimes^\bullet D$) the usual homomorphism complex (respectively, tensor product) of the complexes C and D.

Given two complexes C and D, let $\underline{\mathrm{Hom}}(C,D) = Z(\mathrm{Hom}^\bullet(C,D))$. Then, we see that $\underline{\mathrm{Hom}}(C,D)$ can be made into a complex with $\underline{\mathrm{Hom}}(C,D)^m$ the abelian group of morphisms from C to $D[m]$ and with boundary operator given by $f \in \underline{\mathrm{Hom}}(C,D)^m$. Then

$$\delta^m(f) : C \longrightarrow D[m+1]$$

with

$$\delta^m(f)^n = (-1)^m \delta_D^{n+m} f^n, \quad \text{for any } n \in \mathbb{Z}.$$

We note that the new functor $\underline{\mathrm{Hom}}(C,D)$ will have right derived functors whose values will be complexes. These values shall be denoted $\underline{\mathrm{Ext}}^i(C,D)$. It is easy to see that $\underline{\mathrm{Ext}}^i(C,D)$ is the complex

$$\cdots \longrightarrow \mathrm{Ext}^i(C, D[n-1]) \longrightarrow \mathrm{Ext}^i(C, D[n]) \longrightarrow \mathrm{Ext}^i(C, D[n+1]) \longrightarrow \cdots$$

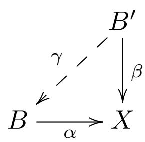
with boundary operator induced by the boundary operator of D.

Let C be a complex of right R-modules, and let D be a complex of left R-modules. We define $C \otimes D$ to be $C \otimes^\bullet D / \mathrm{B}(C \otimes^\bullet D)$. Then, with the maps

$$\frac{(C \otimes^\bullet D)^n}{\mathrm{B}^n(C \otimes^\bullet D)} \longrightarrow \frac{(C \otimes^\bullet D)^{n+1}}{\mathrm{B}^{n+1}(C \otimes^\bullet D)}, \qquad x \otimes y \longmapsto \delta_C(x) \otimes y,$$

where $x \otimes y$ is used to denote the coset in $(C \otimes^\bullet D)^n/\mathrm{B}^n(C \otimes^\bullet D)$, we obtain a complex. Note that the functor will have left derived functors which we denote by $\mathrm{Tor}_i(-, C)$.

Let $\mathcal{B}$ be a class of objects in an abelian category $\mathcal{D}$, and let X be an object of $\mathcal{D}$. A homomorphism $\alpha : B \to X$, where B is in $\mathcal{B}$, is called a $\mathcal{B}$-*precover* of X if the diagram



can be completed for each homomorphism $\beta : B' \to X$ with B' in $\mathcal{B}$. If, furthermore, when $B' = B$ and $\beta = \alpha$, the only such γ is an automorphism of B, then $\alpha : B \to X$ is called a $\mathcal{B}$-cover of X. Dually, we have the concepts of $\mathcal{B}$-preenvelopes and $\mathcal{B}$-envelopes.

2. Strongly copure projective complexes.

Definition 2.1. We say that a complex C is strongly copure projective if $\mathrm{Ext}^i(C, P) = 0$ for all projective complexes P and all $i \geq 1$.

Remark 2.2.

(i) The class of strongly copure projective complexes is closed under direct sums, extensions and kernels of epimorphisms.
(ii) Every Gorenstein projective complex is strongly copure projective.
(iii) $\mathrm{Ext}^i(C, P) = 0 \Leftrightarrow \underline{\mathrm{Ext}}^i(C, P) = 0$ for any projective complex P.

Lemma 2.3. **[11,** Lemma 2.7]. *Let k be a positive integer. If a complex C satisfies $\mathrm{Ext}^k(C, Q) = 0$ for all projective complexes Q, then $\mathrm{Ext}^k(C^m, M) = 0$ for all $m \in \mathbb{Z}$ and all projective left R-modules M.*

Theorem 2.4. *Let C be a complex of left R-modules. Then the following conditions are equivalent*:

(i) *C is a strongly copure projective complex*;
(ii) *C^m is a strongly copure projective left R-module for all $m \in \mathbb{Z}$.*

Proof.

(i) $\Rightarrow$ (ii) follows from Lemma 2.3.

(ii) $\Rightarrow$ (i). Now, assume that C is a complex such that each term C^m is a strongly copure projective module. We only need to show that $\mathrm{Ext}^i(C, Q) = 0$ for any projective complex Q. Let P be a projective module. Let

$$0 \longrightarrow P \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

be an injective resolution of P. Then,

$$0 \longrightarrow \overline{P} \longrightarrow \overline{I_0} \longrightarrow \overline{I_1} \longrightarrow \cdots$$

is an injective resolution of the projective complex $\overline{P}$. If we apply $\mathrm{Hom}(C, -)$ to this injective resolution, we obtain the complex:

$$0 \longrightarrow \mathrm{Hom}(C, \overline{P}) \longrightarrow \mathrm{Hom}(C, \overline{I_0}) \longrightarrow \mathrm{Hom}(C, \overline{I_1}) \longrightarrow \cdots$$

and the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(C, \overline{P}) & \longrightarrow & \mathrm{Hom}(C, \overline{I_0}) & \longrightarrow & \mathrm{Hom}(C, \overline{I_1}) & \longrightarrow & \cdots \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(C^0, P) & \longrightarrow & \mathrm{Hom}_R(C^0, I_0) & \longrightarrow & \mathrm{Hom}_R(C^0, I_1) & \longrightarrow & \cdots \end{array}$$

Since C^0 is strongly copure projective, we obtain $\mathrm{Ext}^i_R(C^0, P) = 0$ for all $i \geq 1$. This results in $\mathrm{Ext}^i(C, \overline{P}) = 0$ for $i \geq 1$. Now, we can repeat the argument with $\overline{P}[n]$ for any $n \in \mathbb{Z}$ and obtain $\mathrm{Ext}^i(C, \overline{P}[n]) = 0$ for any projective module P, any $n \in \mathbb{Z}$ and any $i \geq 1$. Now, if Q is an arbitrary projective complex, we have $Q = \oplus \overline{P_n}[n]$, where P_n is a projective left R-module for each $n \in \mathbb{Z}$. Furthermore,

$$\bigoplus \overline{P_n}[n] = \prod \overline{P_n}[n].$$

Thus, from this observation and from the above, we see that $\mathrm{Ext}^i(C, Q) = 0$. Hence, C is a strongly copure projective complex. $\square$

Now, we provide the definition of strongly copure projective dimension.

Definition 2.5. Let C be a complex of left R-modules. The strongly copure projective dimension, $\mathrm{Scpd}(C)$, of C is defined as:

$$\mathrm{Scpd}(C) = \inf\Big\{n \mid \text{there exists an exact sequence: } 0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow C \longrightarrow 0, \text{ with each } P_i \text{ strongly copure projective}\Big\}.$$

If no such n exists, set $\mathrm{Scpd}(C)=\infty$.

Details and results on strongly copure projective dimension of modules appeared in **[13]**. The strongly copure projective dimension of R-module M is also denoted as $\mathrm{Scpd}(M)$.

Theorem 2.6. *Let C be a complex of left R-modules. Then,* $\mathrm{Scpd}(C) = \sup\{\mathrm{Scpd}(C^m) \mid m \in \mathbb{Z}\}$.

Proof. If $\sup\{\mathrm{Scpd}(C^m) \mid m \in \mathbb{Z}\} = \infty$, then

$$\mathrm{Scpd}(C) \leq \sup\{\mathrm{Scpd}(C^m) \mid m \in \mathbb{Z}\}.$$

Naturally, we may assume that $\sup\{\mathrm{Scpd}(C^m) \mid m \in \mathbb{Z}\} = n$ is finite. Consider a projective resolution

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow C \longrightarrow 0$$

of C, where each P_i is a projective complex. Then, there is an exact sequence of modules

$$0 \longrightarrow K_n^m \longrightarrow P_{n-1}^m \longrightarrow P_{n-2}^m \longrightarrow \cdots \longrightarrow P_0^m \longrightarrow C^m \longrightarrow 0,$$

for all $m \in \mathbb{Z}$. Then K_n^m is strongly copure projective for all $m \in \mathbb{Z}$ by [**13,** Proposition 3.2]. Now, by Theorem 2.4, K_n is strongly copure projective. This shows that $\mathrm{Scpd}(C) \leq n$, and so,

$$\mathrm{Scpd}(C) \leq \sup\{\mathrm{Scpd}(C^m) \mid m \in \mathbb{Z}\}.$$

Now, it is enough to show that $\sup\{\mathrm{Scpd}(C^m) \mid m \in \mathbb{Z}\} \leq \mathrm{Scpd}(C)$. Naturally, we may assume that $\mathrm{Scpd}(C) = n$ is finite. Then there exists an exact sequence:

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow C \longrightarrow 0,$$

with each P_i strongly copure projective. By Theorem 2.4, P_i^m are strongly copure projective for all $m \in \mathbb{Z}$ and all $i = 0, 1, \ldots, n$. Thus, $\mathrm{Scpd}(C^m) \leq n$, and so, $\sup\{\mathrm{Scpd}(C^m) \mid m \in \mathbb{Z}\} \leq \mathrm{Scpd}(C)$. □

Theorem 2.7. *The following are equivalent for a complex C and $n \geq 0$:*

(i) $\mathrm{Scpd}(C) \leq n$;
(ii) $\mathrm{Ext}^i(C, P) = 0$ *for any projective complex P and any $i > n$;*
(iii) $\mathrm{Ext}^i(C, L) = 0$ *for all complexes L with finite projective dimension and any $i > n$;*
(iv) *For any exact sequence*

$$0 \longrightarrow K_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow C \longrightarrow 0$$

with each Q_i strongly copure projective, then K_n is a strongly copure projective complex.

Proof.

(i) ⇒ (iii). By the definition, there exists an exact sequence:

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow C \longrightarrow 0,$$

with each P_j strongly copure projective. It is easy to see that $\mathrm{Ext}^i(P_j, L) = 0$ for all $i \geq 1$ and all complexes L with finite projective dimension. Thus, $\mathrm{Ext}^i(C, L) \cong \mathrm{Ext}^{i-n}(P_n, L) = 0$ for all $i > n$ and all complexes L with finite projective dimension.

(iii) ⇒ (ii). It is clear.

(ii) ⇒ (i). Since $\mathrm{Ext}^i(C, P) = 0$ for all $i > n$ and all projective complexes P, by Lemma 2.3, we have $\mathrm{Ext}_R^i(C^m, Q) = 0$ for $i > n$, all $m \in \mathbb{Z}$ and all projective left R-modules Q. Thus, by [**13,** Proposition 3.2], $\mathrm{Scpd}(C^m) \leq n$. Hence,

$$\mathrm{Scpd}(C) = \sup\{\mathrm{Scpd}(C^m \mid m \in \mathbb{Z}\} \leq n$$

by Theorem 2.4.

(i) ⇔ (iv). It follows from Theorems 2.4 and 2.6. □

Proposition 2.8. *Let C be a complex with* $\mathrm{pd}(C) < \infty$.

(i) *If C is strongly copure projective, then C is projective.*
(ii) $\mathrm{Scpd}(C) = \mathrm{pd}(C)$.

Proof. Analogous to the proof of [**13,** Proposition 3.4]. □

Proposition 2.9. *Let C be an exact complex with* $\mathrm{Hom}_R(-, Q)$ *exact for all projective R-modules Q. Then, C is a strongly copure projective complex if and only if* $\mathrm{Ker}(\delta^m)$ *is a strongly copure projective R-module for all $m \in \mathbb{Z}$.*

Proof.

$\Leftarrow$. Since $\mathrm{Ker}(\delta^m)$ is strongly copure projective for each $m \in \mathbb{Z}$, then C^m is strongly copure projective. By Theorem 2.4, C is strongly copure projective.

$\Rightarrow$. Since C is strongly copure projective, there exists an exact sequence:

$$\cdots \longrightarrow P_2 \xrightarrow{\mu_2} P_1 \xrightarrow{\mu_1} P_0 \xrightarrow{\mu_0} C \longrightarrow 0,$$

with P_i projective such that the sequence is $\mathrm{Hom}(-, P)$ exact for any projective complex P. Note that, for any complex D, $\mathrm{Ext}^n(\underline{R}[-m], D) \cong H^{n+m}(D)$. Particularly, for any exact complex C, $\mathrm{Ext}^n(\underline{R}[-m], C) \cong H^{n+m}(C) = 0$, $n \geq 0$. So, we obtain an exact sequence:

$$\cdots \longrightarrow \mathrm{Hom}(\underline{R}[-m], P_2) \longrightarrow \mathrm{Hom}(\underline{R}[-m], P_1) \longrightarrow \mathrm{Hom}(\underline{R}[-m], P_0) \longrightarrow \mathrm{Hom}(\underline{R}[-m], C) \longrightarrow 0.$$

However, $\mathrm{Hom}(\underline{R}[-m], D) \cong \mathrm{Ker}(\delta_D^m)$ for any complex D. So, there is an exact sequence:

$$\cdots \longrightarrow \mathrm{Ker}(\delta_{P_2}^m) \longrightarrow \mathrm{Ker}(\delta_{P_1}^m) \longrightarrow \mathrm{Ker}(\delta_{P_0}^m) \longrightarrow \mathrm{Ker}(\delta_C^m) \longrightarrow 0,$$

with $\mathrm{Ker}(\delta_{P_i}^m)$ projective, where P_i are projective complexes.

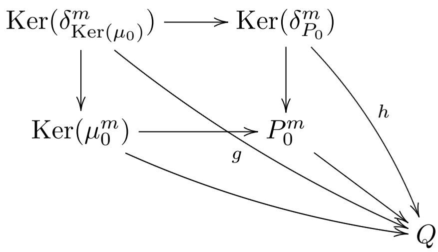
Now, we need to show that the above complex is $\mathrm{Hom}(-, Q)$ exact for any projective complex Q. Consider the short exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\delta^m_{\mathrm{Ker}(\mu_0)}) \longrightarrow \mathrm{Ker}(\delta^m_{P_0}) \longrightarrow \mathrm{Ker}(\delta^m_C) \longrightarrow 0.$$

Suppose that $g : \mathrm{Ker}(\delta^m_{\mathrm{Ker}(\mu_0)}) \to Q$ with Q a projective R-module. Since C is exact and strongly copure projective, then the short exact sequence

$$0 \longrightarrow \mathrm{Ker}(\mu_0)^m \longrightarrow P_0^m \longrightarrow C^m \longrightarrow 0$$

is $\mathrm{Hom}_R(-, Q)$ exact. Since $\mathrm{Ker}(\mu_0)$ is exact, then $\mathrm{Ker}(\mu_0)$ is $\mathrm{Hom}_R(-, Q)$ exact. Thus, we have a morphism h such that the following diagram is commutative.



Analogously with the above discussion, we can show that

$$\mathrm{Ext}^1(\mathrm{Ker}(\delta^m_{\mathrm{Ker}(\mu_i)}, Q)) = 0.$$

Then, the proof is finished. □

The following theorems show the existence of strongly copure precovers under some special conditions.

Proposition 2.10. *Every complex C with $\mathrm{pd}(C) < \infty$ has a strongly copure projective precover.*

Proof. Let C be a complex with $\mathrm{pd}(C) < \infty$. There is an exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow C \longrightarrow 0$$

with a projective complex P. Note that $\mathrm{pd}(K) < \infty$. So, for any strongly copure projective complex N, there is an exact sequence:

$$0 \longrightarrow \mathrm{Hom}(N, K) \longrightarrow \mathrm{Hom}(N, P) \longrightarrow \mathrm{Hom}(N, C) \longrightarrow 0.$$

Thus, $P \to C$ is a strongly copure projective precover of C. □

Definition 2.11. [**9,** Definition 4.1.1]. We will say that a complex C is finitely generated if, in the case

$$C = \sum_{i \in I} D_i,$$

with $D_i \in \mathcal{C}$ subcomplexes of C, there exists a finite subset $J \subseteq I$ such that

$$C = \sum_{i \in J} D_i.$$

We say that a complex C is finitely presented if C is finitely generated and for any exact sequence of complexes

$$0 \longrightarrow K \longrightarrow L \longrightarrow C \longrightarrow 0$$

with L finitely generated, K is also finitely generated.

Lemma 2.12. [**9,** Theorem 5.2.2]. *Let R be a ring. The following conditions are equivalent*:

(i) *R is right coherent.*
(ii) *Any complex of left R-modules has a flat preenvelope.*

Recall that R is an n-FC ring [**13**] if R is a left and right coherent ring with $FP\text{-id}({}_RR) \leq n$ and $FP\text{-id}(R_R) \leq n$ for an integer $n \geq 0$.

Lemma 2.13. *Let R be an n-FC ring, and let C be a finitely generated strongly copure projective complex of left R-modules. Then, there is an exact sequence*

$$0 \longrightarrow C \longrightarrow P_0 \longrightarrow P_1 \longrightarrow \cdots$$

with all P_i finitely generated projective complexes for all $i \geq 0$.

Proof. By Lemma 2.12, C has a flat preenvelope $C \to F$. But, then, by [**9,** Theorem 4.1.3], $C \to F$ has a factorization $C \to P \to F$ where P is a finitely generated projective complex. So $C \to P$ is a finitely generated projective preenvelope. Then, there is a complex

$$0 \longrightarrow C \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots ,$$

with each P_i a finitely generated projective such that $\mathrm{Hom}(-, P)$ leaves the sequence exact whenever P is a projective complex. We claim that the complex is exact. Consider the mth degree. There is a monomorphism $C^m \xrightarrow{\lambda} E$ with E an injective left R-module, and there is an exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\pi} E \longrightarrow 0$$

with B a projective left R-module. Since $FP\text{-id}({}_RR) < \infty$, $FP\text{-id}(B) < \infty$. So $FP\text{-id}(A) < \infty$. Thus, $fd(A) < \infty$ by [**3,** Theorems 3.5, 3.8]. Since C is a strongly copure projective complex, then C^m is a strongly copure projective R-module by Theorem 2.4. Therefore, $\mathrm{Ext}^1_R(C^m, A) = 0$ by [**13,** Propostion 2.3], and so, there exists a $C^m \xrightarrow{\alpha} B$ such that $\pi\alpha = \lambda$. Hence, α is a monomorphism. So, $C \to P_0$ is a monomorphism. Note that coker$\,(C \to P_0)$ is also a finitely generated strongly copure projective. So, proceeding in the above manner, we obtain the exact sequence

$$0 \longrightarrow C \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots . \quad \square$$

Theorem 2.14. *Let R be an n-FC ring, and let C be a finitely presented complex. Then, there is an exact sequence*

$$0 \longrightarrow K \longrightarrow X \longrightarrow C \longrightarrow 0$$

such that X is a finitely presented strongly copure projective complex and $\mathrm{pd}(K) \leq n - 1$.

Proof. First, suppose that C is a finitely presented complex in $\mathcal{C}$. By [**17,** Theorem 2.4], we can find an epimorphism $P_0 \to C$ with P_0 bounded and ${P_0}^m$ is free for all $m \in \mathbb{Z}$. Then, the induced short exact sequence

$$0 \longrightarrow K \longrightarrow P_0 \longrightarrow C \longrightarrow 0$$

has K finitely generated. Hence, K is finitely presented. Proceeding in this manner, we obtain an exact sequence:

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow C \longrightarrow 0,$$

with P_i and K_n finitely presented. Moreover, by [**13,** Corollary 3.3], K_n is strongly copure projective. By Lemma 2.13, there is an exact sequence:

$$0 \longrightarrow K_n \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow \cdots \longrightarrow Q_{n-2} \longrightarrow Q_{n-1} \longrightarrow L \longrightarrow 0,$$

with all Q_i finitely generated projective complexes such that $\mathrm{Hom}(-, P)$ leaves the sequence exact whenever P is a projective complex. In fact, Q_i is finitely presented. Thus, we obtain the following commutative diagram:

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & K_n & \longrightarrow & Q_0 & \longrightarrow & \cdots & \longrightarrow & Q_{n-2} & \longrightarrow & Q_{n-1} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

By **[6,** Proposition 1.4.14], we obtain the exact sequence:

$$0 \longrightarrow K_n \longrightarrow Q_0 \oplus K_n \longrightarrow Q_1 \oplus P_{n-1} \longrightarrow \\ \cdots \longrightarrow Q_{n-1} \oplus P_1 \longrightarrow L \oplus P_0 \longrightarrow C \longrightarrow 0.$$

which gives the exact sequence:

$$0 \longrightarrow Q_0 \longrightarrow Q_1 \oplus P_{n-1} \longrightarrow \cdots \longrightarrow Q_{n-1} \oplus P_1 \longrightarrow L \oplus P_0 \longrightarrow C \longrightarrow 0.$$

Note that L is a finitely presented strongly copure projective, so $L \oplus P_0$ is a finitely presented strongly copure projective. Consider the short exact sequence

$$0 \longrightarrow K \longrightarrow L \oplus P_0 \longrightarrow C \longrightarrow 0.$$

It is clear that $\mathrm{pd}(K) \leq n-1$. So, $\mathrm{Ext}^1(G, K) = 0$ for any strongly copure projective complex G. Thus, $L \oplus P_0 \to C$ is a strongly copure projective precover of C. $\square$

Corollary 2.15. *Let R be an n-FC ring. Any finitely presented complex has a special $\mathcal{L}$-preenvelope, where $\mathcal{L}$ is denoted as the class of complexes with finite projective dimension.*

Proof. Let C be a finitely presented complex. By Theorem 2.14, there is an exact sequence

$$0 \longrightarrow K \longrightarrow Q \longrightarrow C \longrightarrow 0$$

with Q a finitely presented strongly copure projective and $\mathrm{pd}(K) < \infty$. There is also an exact sequence

$$0 \longrightarrow Q \longrightarrow P \longrightarrow L \longrightarrow 0$$

with P a finitely generated projective complex, by Lemma 2.13. Consider the push-out diagram of $Q \to C$ and $Q \to P$:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & Q & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & D & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & L & = & L & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

So, L is a strongly copure projective and $\mathrm{pd}(D) < \infty$. Hence, $C \to D$ is an $\mathcal{L}$-preenvelope. $\square$

In what follows, $\mathcal{SCP}$ stands for the class of strongly copure projective complexes.

Proposition 2.16. $(\mathcal{SCP}, \mathcal{SCP}^{\perp})$ *is a hereditary cotorsion theory.*

Proof. $\mathcal{SCP} \subseteq {}^{\perp}(\mathcal{SCP}^{\perp})$ is obvious. We only need to show the converse. Let $X \in \mathcal{SCP}$ and $N \in \mathcal{SCP}^{\perp}$. There is an exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow X \longrightarrow 0$$

with P projective and an exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow L \longrightarrow 0$$

with E injective. Then, $\mathrm{Ext}^2(X, N) \cong \mathrm{Ext}^1(K, N) = 0$ and $\mathrm{Ext}^1(X, L) \cong \mathrm{Ext}^1(X, N) = 0$. So, $L \in \mathcal{SCP}^{\perp}$. Now, let $C \in {}^{\perp}(\mathcal{SCP}^{\perp})$. Then, $\mathrm{Ext}^2(C, N) \cong \mathrm{Ext}^1(C, L) = 0$. Therefore, $\mathrm{Ext}^i(C, N) = 0$ for any $i \geq 1$ by induction. Thus, $\mathrm{Ext}^i(C, P) = 0$ for any projective complex P and $i \geq 1$. So, $C \in \mathcal{SCP}$. $\square$

3. Strongly copure injective complexes.

Definition 3.1. We say that C is a strongly copure injective complex if $\mathrm{Ext}^i(E,C)=0$ for all injective complexes of left R-modules E and for all $i \geq 1$.

Remark 3.2.

(i) The class of a strongly copure injective complex is closed under direct products, extensions and cokernels of monomorphisms.
(ii) Every Gorenstein injective complex is strongly copure injective.
(iii) $\mathrm{Ext}^i(E,C)=0 \Leftrightarrow \underline{\mathrm{Ext}}^i(E,C)=0$ for all injective complexes E.

Theorem 3.3. *Let C be a complex of left R-modules. Thus, the following conditions are equivalent*:

(i) *C is a strongly copure injective complex*;
(ii) *C^m is a strongly copure injective left R-module for all $m \in \mathbb{Z}$.*

We also give the definition of strongly copure injective dimension.

Definition 3.4. The strongly copure injective dimension, $\mathrm{Scid}(C)$, of C is defined as

$$\mathrm{Scid}(C)=\inf\Big\{n \mid \text{ there exists an exact sequence: } 0 \longrightarrow C \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_{n-1} \longrightarrow E_n \longrightarrow 0, \text{ with each } E_i \text{ strongly copure injective}\Big\}.$$

Theorem 3.5. *Let R be a left Noetherian ring, and let C be a complex of left R-modules. Then,* $\mathrm{Scid}(C)=\sup\{\mathrm{Scid}(C^m) \mid m \in \mathbb{Z}\}$.

Theorem 3.6. *Let R be a left Noetherian ring. The following conditions are equivalent for a complex C and $n \geq 0$*:

(i) $\mathrm{Scid}(C) \leq n$;
(ii) $\mathrm{Ext}^i(E,C)=0$ *for all injective complexes E and all $i>n$*;
(iii) $\mathrm{Ext}^i(L,C)=0$ *for all complexes L with finite injective dimension and all $i>n$*;

(iv) *For any exact sequence*

$$0 \longrightarrow C \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_{n-1} \longrightarrow L_n \longrightarrow 0$$

with each E_i strongly copure injective, then L_n is a strongly copure injective complex.

Proposition 3.7. *Let C be an exact complex with $\mathrm{Hom}_R(I, -)$ exact for all injective R-modules I. Then C is strongly copure injective if and only if $\mathrm{Ker}(\delta^m)$ is strongly copure injective for all $m \in \mathbb{Z}$.*

Theorem 3.8. *Let R be a left Noetherian ring, and let C be a complex with $\mathrm{Scid}(C) \leq n$. Then there is an exact sequence*

$$0 \longrightarrow C \longrightarrow X \longrightarrow L \longrightarrow 0$$

with $C \to X$ a strongly copure injective preenvelope and $\mathrm{id}\,(L) \leq n-1$.

Proof. By Theorem 3.8, we have an exact sequence:

$$0 \longrightarrow C \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_{n-1} \longrightarrow K \longrightarrow 0,$$

with $E_0, E_1, \ldots, E_{n-1}$ injective and K strongly copure injective. Since R is left Noetherian, by [**8**], there is a set χ of injective left R-modules such that any injective left R-module is the direct sum of modules, each isomorphic to an element of χ. Set

$$\mathcal{S} = \{\overline{I}[m] \mid I \in \chi, m \in \mathbb{Z}\}.$$

Then it is clear that every injective complex is the direct sum of complexes, each isomorphic to an element of $\mathcal{S}$. Thus, by [**2,** Theorem 3.2], any complex has an injective cover. So, we have an exact sequence:

$$0 \longrightarrow C' \longrightarrow D_0 \longrightarrow D_1 \longrightarrow \cdots \longrightarrow D_{n-1} \longrightarrow K \longrightarrow 0,$$

where

$$D_{n-1} \longrightarrow K, D_{n-2} \longrightarrow \mathrm{Ker}(D_{n-1} \longrightarrow K), \ldots$$

are injective covers. Then C' is strongly copure injective. Consider the commutative diagram of complexes:

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & C & \longrightarrow & E_0 & \longrightarrow & E_1 & \longrightarrow & \cdots & \longrightarrow & E_{n-1} & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & C' & \longrightarrow & D_0 & \longrightarrow & D_1 & \longrightarrow & \cdots & \longrightarrow & D_{n-1} & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

Then, we can form the exact sequence of complexes:

$$\begin{aligned} 0 \longrightarrow C \longrightarrow C' \oplus E_0 \longrightarrow D_0 \oplus E_1 \longrightarrow \\ \cdots \longrightarrow D_{n-2} \oplus E_{n-1} \longrightarrow D_{n-1} \oplus K \longrightarrow K \longrightarrow 0. \end{aligned}$$

Note that the complex $\overline{K}$ is a subcomplex of the above, and it is exact. Hence, there is the exact sequence:

$$\begin{aligned} 0 \longrightarrow C \longrightarrow C' \oplus E_0 \longrightarrow D_0 \oplus E_1 \longrightarrow \\ \cdots \longrightarrow D_{n-2} \oplus E_{n-1} \longrightarrow D_{n-1} \longrightarrow 0. \end{aligned}$$

It follows that

$$0 \longrightarrow C \longrightarrow C' \oplus E_0 \longrightarrow V \longrightarrow 0$$

is exact. We see that $\mathrm{id}\,(V) \leq n-1$. This implies that $C \to C' \oplus E_0$ is a strongly copure injective preenvelope of C. $\square$

Corollary 3.9. *Any complex C with* $\mathrm{Scid}(C) \leq n$ *over a left Noetherian ring has a strongly copure injective envelope which is a quasi-isomorphism.*

Proof. Let C be a complex, and let

$$0 \longrightarrow C \longrightarrow E \longrightarrow L \longrightarrow 0$$

be a sequence with $C \to E$ a special strongly copure injective preenvelope. Then the class of short exact sequences

$$\{0 \longrightarrow C \longrightarrow H \longrightarrow D \longrightarrow 0 \mid D \in \mathcal{L}\}$$

has a generator where $\mathcal{L}$ is the class of complexes with finite injective dimension (we consider this class as a category with the obvious maps). Since the class of $\mathcal{L}$ is closed under direct limits and extensions, it follows by Zorn's lemma for categories that this class has a "minimal" generator, that is, a generator

$$0 \longrightarrow C \longrightarrow N \longrightarrow K \longrightarrow 0$$

such that, for any commutative diagram, f, g are automorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & N & \longrightarrow & K & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & C & \longrightarrow & N & \longrightarrow & K & \longrightarrow & 0. \end{array}$$

This minimal generator produces a strongly copure injective envelope of C. □

Lemma 3.10. *If $\phi : C \to M$ is a strongly copure injective preenvelope, then $\phi^m : C^m \to M^m$ is a strongly copure injective preenvelope in R-*Mod *for all $m \in \mathbb{Z}$.*

Proof. Let $f : C^m \to E$ be a map with E a strongly copure injective R-module. We consider the diagram:

$$\begin{array}{ccc} C & \xrightarrow{\phi} & M \\ {\scriptstyle (f)}\downarrow & \swarrow{\scriptstyle g} & \\ \overline{E}[m] & & \end{array}$$

where $(f)^m = f$, $(f)^{m-1} = f\delta_C^{m-1}$ and $(f)^i = 0$ for $i \neq m-1, m$. Since $\overline{E}[m]$ is strongly copure injective, then there exists $g : M \to \overline{E}[m]$ such that $g\phi = (f)$. Therefore, $g^m f = \phi^m$, and so $\phi^m : C^m \to M^m$ is a strongly copure injective preenvelope in R-Mod. □

Lemma 3.11. *Let*

$$C \equiv 0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots$$

be a bounded from below complex, and let $C \longrightarrow E$ be a strongly copure injective envelope. Then E is also bounded from below.

Proof. The complex

$$E' \equiv 0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

is clearly strongly copure injective, and the obvious induced map $C \to E'$ is a strongly copure injective preenvelope. So, E is a direct summand of E', and hence, E is bounded from below. □

Proposition 3.12. *Let*

$$(C, \delta) \equiv 0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots$$

be a bounded from below complex, and let $\phi : C \to E$ be a strongly copure injective envelope of C. Then $\phi^0 : C^0 \to E^0$ is a strongly copure injective envelope of E^0 in R-Mod.

Proof. By Lemma 3.11, we know that E is bounded from below and, by Lemma 3.10, $\phi^0 : C^0 \to E^0$ is a strongly copure injective preenvelope. So, let $\psi^0 : C^0 \to E(C^0)$ be a strongly copure injective envelope of C^0 in R-Mod. We consider the splitting monomorphism $\alpha : E(C^0) \to E^0$ such that $\alpha\psi^0 = \phi^0$. We take the complex

$$E' \equiv 0 \longrightarrow E(C^0) \xrightarrow{h^0\alpha} E^1 \xrightarrow{h^1} \cdots,$$

where

$$E \equiv 0 \longrightarrow E^0 \xrightarrow{h^0} E^1 \xrightarrow{h^1} \cdots.$$

Also, we consider the morphism of complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0 & \xrightarrow{\delta^0} & C^1 & \xrightarrow{\delta^1} & C^2 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \psi^0} & & \downarrow{\scriptstyle \phi^1} & & \downarrow{\scriptstyle \phi^2} & & \\ 0 & \longrightarrow & E(C^0) & \xrightarrow{h^0\alpha} & E^1 & \xrightarrow{h^1} & E^2 & \longrightarrow & \cdots. \end{array}$$

Now, it is easy to check that the above map, say $\phi' : C \to E'$, is a strongly copure injective preenvelope. Therefore, there exists a splitting monomorphism $t : E \to E'$ such that $t\phi = \phi'$. However, $t^0\phi^0 = \psi^0$; hence, $t^0\alpha\psi^0 = \psi^0$ and so $t^0\alpha$ is an automorphism. We conclude that t^0 is an automorphism, which means that $\phi^0 : C^0 \to E^0$ is a strongly copure injective envelope in R-Mod. □

4. Strongly copure flat complexes.

Definition 4.1. We will say that C is strongly copure flat if $\mathrm{Tor}_i(E, C) = 0$ for all injective complexes of right R-modules E and for $i \geq 1$.

Remark 4.2. Every Gorenstein flat complex is strongly copure flat. Strongly copure flat complexes are DG-flat by [**9,** Lemma 5.4.1].

Theorem 4.3. *The following conditions are equivalent*:

(i) C *is strongly copure flat*;
(ii) C^+ *is strongly copure injective*;
(iii) C^m *is strongly copure flat in* R-Mod *for all* $m \in \mathbb{Z}$.

Proof.

(i) $\Rightarrow$ (iii). There is an exact sequence

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

with all F_i flat, and for any injective complex E of right R-modules, the sequence remains exact when we apply $E \otimes -$. Consider the mth degree term,

$$\cdots \longrightarrow F_1^m \longrightarrow F_0^m \longrightarrow C^m \longrightarrow 0.$$

The sequence remains exact when we apply $I \otimes_R - \equiv \overline{I} \otimes -$.

(iii) $\Rightarrow$ (ii). For any injective right R-module I, $\mathrm{Tor}_i(I, C^m)^+ \cong \underline{\mathrm{Ext}}^i(I, C^{m+}) = 0$. By Theorem 3.3, C^+ is strongly copure injective.

(ii) $\Rightarrow$ (i). By $\underline{\mathrm{Ext}}^i(E, C^+) \cong \mathrm{Tor}_i(E, C)^+ = 0$. □

Definition 4.4. The strongly copure flat dimension $\mathrm{Scfd}(C)$ of a complex C is defined by declaring that $\mathrm{Scfd}(C) \leq n$ if and only if C has a strongly copure flat resolution of length n.

Theorem 4.5. *Let C be a complex of left R-modules. Then*

$$\mathrm{Scfd}(C) = \sup\{\mathrm{Scfd}(C^m) \mid m \in \mathbb{Z}\}.$$

Proof. It follows from [**7,** Lemma 3.3]. □

Corollary 4.6. *Let C be a complex. Then,*

$$\mathrm{Scfd}(C) = \mathrm{Scid}(C^+).$$

Proposition 4.7. *The following conditions are equivalent for a complex C over any ring R*:

(i) $\mathrm{Scfd}(C) \leq n$;
(ii) $\mathrm{Tor}_i(E, C) = 0$ *for all injective complexes of right R-modules E and for $i > n$*;
(iii) $\mathrm{Tor}_i(E, C) = 0$ *for all complexes of right R-modules E with injective dimension finite and for $i > n$*;
(iv) *Every* nth *syzygy of C is strongly copure flat.*

Recall that a complex M is cotorsion if $\mathrm{Ext}^1(F, M) = 0$ for any flat complex F [**6,** Definition 5.3.22].

Theorem 4.8. *Let R be a right coherent ring. Then, the following conditions are equivalent for a complex C:*

(i) *C is a strongly copure flat complex*;
(ii) $\underline{\mathrm{Ext}}^i(C, M) = 0$ *for any flat cotorsion complex M and $i \geq 1$*;
(iii) $\underline{\mathrm{Ext}}^i(C, M) = 0$ *for any cotorsion complex M with* $\mathrm{fd}(M) < \infty$ *and $i \geq 1$*;
(iv) $\underline{\mathrm{Ext}}^i(C, N) = 0$ *for any pure injective complex N with* $\mathrm{fd}(N) < \infty$ *and $i \geq 1$.*

Proof.

(i) $\Rightarrow$ (ii). Let M be a flat cotorsion complex. Then M^+ is an injective complex. Since R is right coherent, M^{++} is a flat complex. Note that M is a pure subcomplex of M^{++} by [**9,** Proposition 5.1.4], and hence, M^{++}/M is flat. Since M is cotorsion, the pure exact sequence

$$0 \longrightarrow M \longrightarrow M^{++} \longrightarrow M^{++}/M \longrightarrow 0$$

is split. Therefore, $\underline{\mathrm{Ext}}^i(C, M)$ is a direct summand of $\underline{\mathrm{Ext}}^i(C, M^{++})$. By (i), $\underline{\mathrm{Ext}}^i(C, M^{++}) \cong \mathrm{Tor}_i(M^+, C)^+ = 0$, so $\underline{\mathrm{Ext}}^i(C, M) = 0$ for all $i \geq 1$.

(ii) $\Rightarrow$ (iii). Let M be a cotorsion complex with $\mathrm{fd}(M) = n < \infty$. Then there is a flat resolution,

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where $F_j \to \mathrm{Ker}(F_{j-1} \to F_{j-2})$, $F_1 \to \mathrm{Ker}(F_0 \to M)$ and $F_0 \to M$ are flat covers for $2 \leq j \leq n-1$. Since K_j is cotorsion by Wakamatsu's lemma [**9,** Proposition 1.2.3], F_{j-1} is flat cotorsion. Note that $F_n = K_n$

is flat cotorsion. Therefore, $\mathrm{Ext}^i(C,M) \cong \mathrm{Ext}^{i+1}(C,K_1) \cong \cdots \cong \mathrm{Ext}^{i+n}(C,K_n) = 0$ for any $i \geq 1$ by (ii).

(iii) $\Rightarrow$ (iv). By [**9,** Lemma 5.3.2], any pure injective and flat complex C is cotorsion.

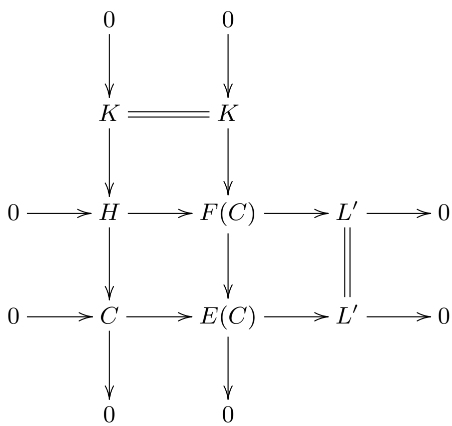
(iv) $\Rightarrow$ (i). Let E be an injective complex of right R-modules. Then E^+ is flat. However, E^+ is pure injective, and, as well, $\mathrm{Tor}_i(E,C)^+ \cong \underline{\mathrm{Ext}}^i(C,E^+) = 0$. □

Lemma 4.9. *Let R be a right coherent ring with FP-id$(R_R) \leq n$, $n \geq 0$, and let C be a strongly copure flat complex. Then there is an exact sequence*

$$0 \longrightarrow C \longrightarrow F \longrightarrow L \longrightarrow 0$$

with F flat and L strongly copure flat.

Proof. Let C be a strongly copure flat complex, and consider the following commutative diagram:



with $C \to E(C)$ an injective envelope of C and $F(C) \to E(C)$ a flat cover of $E(C)$. Clearly, K is cotorsion by Wakamatsu's lemma and exact by the exactness of $E(C)$, $F(C)$. Since R is right coherent and $FP - \mathrm{id}(R_R) \leq n$, $\mathrm{fd}(\mathrm{Ker}\delta^m_{E(C)}) \leq n$ for all $m \in \mathbb{Z}$; hence, $\mathrm{fd}(E(C)) \leq n$. So, $\mathrm{fd}(K) \leq n-1$. However, C is strongly copure flat, and so $\mathrm{Ext}^i(C,K) = 0$, by Theorem 4.8. Then the sequence

$$0 \longrightarrow K \longrightarrow H \longrightarrow C \longrightarrow 0$$

is split. Therefore, C is a subcomplex $F(C)$. Note that C has a flat preenvelope $\alpha : C \to F$ since R is right coherent. So, α is a monomorphism, and we get an exact sequence

$$0 \longrightarrow C \longrightarrow F \longrightarrow L \longrightarrow 0.$$

For degree m, $m \in \mathbb{Z}$, by [**14,** Proposition 2.7] and [**16,** Remark 3.2], L^m is a copure flat R-module. Then $\mathrm{Tor}_1(I, L^m) = 0$ for any injective right R-module I. Note that $\mathrm{Tor}_i(I, L^m) \cong \mathrm{Tor}_{i-1}(I, C^m) = 0$ for $i \geq 2$. Since C^m is strongly copure flat, L^m is strongly copure flat. So, L is strongly copure flat by Theorem 4.3. □

Theorem 4.10. *Let R be a right coherent ring with FP-id$(R_R) \leq n, n \geq 0$. Then the following conditions are equivalent*:

(i) *C is strongly copure flat*;
(ii) *C is Gorenstein flat*;
(iii) *There is an exact sequence*:

$$\cdots \longrightarrow F_{-1} \longrightarrow F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots$$

of flat complexes with $C \cong \mathrm{Ker}(F_0 \to F_1)$ such that $\mathrm{Hom}(-, M)$ leaves the sequence exact for any flat cotorsion complex M;
(iv) *There is an exact sequence*:

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

with F_i flat such that $\mathrm{Hom}(-, M)$ leaves the sequence exact for any flat cotorsion complex M.

Proof.

(ii) ⇒ (i). Obvious.

(i) ⇒ (ii). Let C be a strongly copure flat complex, and let E be an injective complex. By Lemma 4.9, there is an exact sequence

$$0 \longrightarrow C \longrightarrow F_0 \longrightarrow L_1 \longrightarrow 0$$

with F_0 flat and L_1 strongly copure flat. It is clear that $E \otimes -$ preserves the exactness of this sequence. For L_1, by Lemma 4.9 again, we can also get an exact sequence

$$0 \longrightarrow L_1 \longrightarrow F_1 \longrightarrow L_2 \longrightarrow 0$$

with L_2 strongly copure flat such that $E \otimes -$ is exact. Repeating the same procedure, we get an exact sequence:

$$0 \longrightarrow F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots .$$

On the other hand, consider the exact sequence:

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0,$$

where $F_0 \to M$, $F_1 \to \mathrm{Ker}(F_0 \to M)$ and $F_{i+1} \to \mathrm{Ker}(F_i \to F_{i-1})$ are flat covers for any $i \geq 1$. Since $\mathrm{Tor}_1(E, C) = 0$, $E \otimes -$ leaves the sequence

$$0 \longrightarrow \mathrm{Ker}(F_0 \longrightarrow C) \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

exact. It is easy to see that $\mathrm{Ker}(F_0 \to C)$ is strongly copure flat. Then, we can show that $E \otimes -$ leaves the sequence

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

exact by repeating the same procedure. So, C is Gorenstein flat.

(i) $\Rightarrow$ (iii). The proof is similar to that of (1) $\Rightarrow$ (2).

(iii) $\Rightarrow$ (iv). Trivial.

(iv) $\Rightarrow$ (i). Let M be a flat cotorsion complex. Consider the exact sequence:

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

of flat complexes such that $\mathrm{Hom}(-, M)$ leaves the sequence exact. Let $K_1 = \mathrm{Ker}(F_0 \to C)$. Then, there is an exact sequence

$$0 \longrightarrow K_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0.$$

Clearly, $\mathrm{Hom}(-, M)$ preserves the exactness of this sequence. Hence, $\mathrm{Ext}^1(C, M) = 0$. Now, by induction, $\mathrm{Ext}^i(C, M) = 0$ for any $i \geq 1$. So $\underline{\mathrm{Ext}}^i(C, M) = 0$. Hence, C is strongly copure flat by Theorem 4.8. $\square$

Theorem 4.11. *Let $\mathcal{SCF}$ be the class of strongly copure flat complexes. Then $(\mathcal{SCF}, \mathcal{SCF}^{\perp})$ is a complete hereditary cotorsion theory.*

Proof. Let $F \in \mathcal{SCF}$ and $S \subseteq F$ be a pure subcomplex. Then

$$0 \longrightarrow (F/S)^+ \longrightarrow F^+ \longrightarrow (S)^+ \longrightarrow 0$$

is split, and F^+ is strongly copure injective; therefore, F and F/S are strongly copure flat, that is, $\mathcal{SCF}$ is closed under pure subcomplexes and pure epimorphisms. Thus, there exists a cardinal $\mathcal{N}$ such that F can be written as the direct union of a continuous chain of subcomplexes $(F_\alpha)_{\alpha<\lambda}$ with λ an ordinal number such that F_0, $F_{\alpha+1}/F_\alpha \in \mathcal{SCF}$ when $\alpha+1<\lambda$ with $\mathrm{Card}\,(F_0)$, $\mathrm{Card}\,(F_{\alpha+1}/F_\alpha) \leq \mathcal{N}$. Therefore, if B is the direct sum of all representatives of $\mathcal{SCF}$ such that their cardinal is less than or equal to $\mathcal{N}$, then $M \in \mathcal{SCF}^\perp$ if and only if $\mathrm{Ext}^1(B, M) = 0$. $\square$

Now, let N be any complex. We will use the procedure in [**4,** Theorem 10] to get an exact sequence

$$0 \longrightarrow N \longrightarrow A \longrightarrow F \longrightarrow 0$$

such that $A \in \mathcal{SCF}^\perp$ and $F \in \mathcal{SCF}$.

Let B be a complex, and let

$$0 \longrightarrow S \longrightarrow P \longrightarrow B \longrightarrow 0$$

be exact with P a projective complex. For any ordinal λ, by transfinite induction, we can construct a continuous chain of subcomplexes $(N_\alpha)_{\alpha<\lambda}$ such that $N_0 = N$ and, if $\alpha+1<\lambda$, any morphism $S \to N_\alpha$ has an extension $P \to N_{\alpha+1}$. Then, each $N_{\alpha+1}/N_\alpha$ is a direct sum of copies of $P/S \cong B$ for $\alpha+1<\lambda$.

We now use [**6,** Corollary 7.3.2] with S the set of that corollary to find a corresponding ordinal λ and define $A = \cup_{\alpha<\lambda} N_\alpha$ and $F = A/N$. Then, for any morphism $S \to A$, there is a factorization $S \to N_\alpha \to A$ for some $\alpha < \lambda$. Since λ is a limit ordinal, we have $\alpha+1<\lambda$, and so, by construction, there is a morphism $P \to N_{\alpha+1}$ which agrees with $S \to N_\alpha$. However, this is just to say that $A \in \mathcal{SCF}^\perp$. If we put $F_\alpha = N_\alpha/N$, we get that F_0 and $F_{\alpha+1}/F_\alpha$ are in $\mathcal{SCF}$. Now, using an easy induction over the exact sequence of Tor's associated with the exact sequence

$$0 \longrightarrow F_\alpha \longrightarrow F_{\alpha+1} \longrightarrow F_{\alpha+1}/F_\alpha \longrightarrow 0,$$

we obtain $F_\alpha \in \mathcal{SCF}$ for each $\alpha < \lambda$. Since $\mathcal{SCF}$ is closed under direct limits, it follows that $F \in \mathcal{SCF}$.

If $X \in {}^\perp(\mathcal{SCF}^\perp)$, there is an exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow X \longrightarrow 0$$

with P a projective complex. If we apply the preceding to K, we obtain an exact sequence

$$0 \longrightarrow K \longrightarrow A \longrightarrow F \longrightarrow 0$$

with $A \in \mathcal{SCF}^{\perp}$ and $F \in \mathcal{SCF}$. Consider the next pushout diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & F & = & F & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

We get that $C \in \mathcal{SCF}$ for $P, F \in \mathcal{SCF}$. Since $A \in \mathcal{SCF}^{\perp}$, the central row splits, and so, $X \in \mathcal{SCF}$. This gives us that $\mathcal{SCF} = {}^{\perp}(\mathcal{SCF}^{\perp})$. Therefore, $(\mathcal{SCF}, \mathcal{SCF}^{\perp})$ is a cotorsion theory with enough injectives and projectives. The fact that $(\mathcal{SCF}, \mathcal{SCF}^{\perp})$ is hereditary is immediate from the definition of $\mathcal{SCF}$.

Corollary 4.12. $(\mathcal{SCF}, \mathcal{SCF}^{\perp})$ *is a perfect hereditary cotorsion theory.*

Proof. By Theorem 4.11, every complex has an $\mathcal{SCF}$-precover and an $\mathcal{SCF}^{\perp}$-preenvelope, and therefore, the result follows from [**6,** Theorem 7.2.6] since it is clear that $\mathcal{SCF}$ is closed under direct limits. □

Remark 4.13. If R is a right coherent ring with FP-id$(R_R) < \infty$, by Theorem 4.10, then $\mathcal{SCF}$ becomes a Gorenstein flat complex.

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Department of Mathematics, Northwest Normal University, Lanzhou 730070, China and Science College, Gansu Agricultural University, Lanzhou 730070, China

Email address: maxin263@126.com

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

Email address: liuzk@nwnu.edu.cn