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ARITHMETIC AND GEOMETRY OF RATIONAL ELLIPTIC SURFACES

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ABSTRACT. Let $\mathscr{E}$ be a rational elliptic surface over a number field k. We study the interplay between a geometric property, the configuration of its singular fibers, and arithmetic features such as its Mordell-Weil rank over the base field and its possible minimal models over k.

1. Introduction. Let k be a number field and $\overline{k}$ a fixed algebraic closure. Let X be a smooth, projective, geometrically integral surface over k. Let B be a smooth projective curve over k. We say that (X, π, σ) is an *elliptic surface with base* B if $\pi : X \to B$ is a flat morphism defined over k and is a relatively minimal elliptic fibration, i.e., the fibers do not contain (-1)-curves as components, σ is a section to π and there is at least one (geometric) singular fiber. If, moreover, the surface X is geometrically rational, i.e., $\overline{X} := X \times_k \overline{k}$ is birational to $\mathbb{P}^2_{\overline{k}}$, then X is called a *rational elliptic surface*. In this case, we clearly have $B \times_k \overline{k} \simeq \mathbb{P}^1_{\overline{k}}$. This note is dedicated to connecting different arithmetic and geometric features of such surfaces.

Over $\overline{k}$, X is isomorphic to $\mathbb{P}^2_{\overline{k}}$ blown-up at the nine, not necessarily distinct, base points of a linear pencil of cubics. In this setting, supposing moreover, that char $(k) = 0$, Miranda [**8**] classified pencils of cubic curves in $\mathbb{P}^2$ using geometric invariant theory, and Persson gave a list of all possible configurations of singular fibers, see [**11**]. Later, Oguiso and Shioda [**10**] classified all Mordell-Weil lattices that could arise in rational elliptic surfaces.

Over k, the surface X is not necessarily birational to $\mathbb{P}^2_k$, but to del Pezzo surfaces or conic bundles, cf., [**3, 7**]. The singular fibers of the elliptic fibration might not be defined over the ground field.

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Nevertheless, they form a set that is invariant under the action of $\mathrm{Gal}\,(\overline{k}/k)$.

Our goal is to study the interplay between an arithmetic and a geometric property of rational elliptic surfaces, namely, the possible minimal models over k, i.e., *k-minimal models*, and the configuration of the singular fibers.

The paper is organized as follows. We first recall some arithmetic and geometric facts as the construction of rational elliptic surfaces over algebraically closed fields, the possible types of singular fibers, the theory of minimal models over perfect fields and the Shioda-Tate formula. Sections 3 and 4 contain the core of the paper. In Section 3, we establish the connection between k-minimal models of rational elliptic surfaces and the configuration of points in the projective plane blown-up to obtain such surfaces. Section 4 relates k-minimal models to singular fibers. In Section 5, we present some examples to illustrate the theory discussed in the previous sections. Finally, in the appendix, we treat the special case of rational elliptic surfaces that have a del Pezzo surface of degree 2 as a k-model. We describe the possible fiber types of such elliptic fibrations.

2. Preliminaries. Let k be a number field. By an *elliptic surface with base* $\mathbb{P}^1_k$, we mean a triple (X, π, σ) such that X is a smooth projective and geometrically integral surface, $\pi : X \to \mathbb{P}^1_k$ is a relatively minimal elliptic fibration with at least one geometric singular fiber and $\sigma : \mathbb{P}^1_k \to X$ is a section. The generic fiber of X is an elliptic curve over $k(\mathbb{P}^1) \simeq k(t)$. Therefore, it can be written as a Weierstrass equation:

$$Y^2 = X^3 + A(t)X + B(t),$$

with $4A(t)^3 + 27B(t)^2 \notin k$.

The converse also holds, namely, any elliptic curve E over $k(t)$ extends uniquely to an elliptic surface with base $\mathbb{P}^1$, its Kodaira-Néron model.

If $\overline{X} = X \times_k \overline{k}$ is birational to $\mathbb{P}^2_{\overline{k}}$, then the triple above is called a *rational elliptic surface*. We shall deal solely with rational surfaces with base $\mathbb{P}^1$ throughout this text. Therefore, we omit the term *with base* $\mathbb{P}^1$. For these surfaces, $A(t)$ and $B(t)$ are polynomials in $k(t)$ such that $\deg(A(t)) \leq 4$ and $\deg(B(t)) \leq 6$ in the equation above. We focus on them in what follows.

Rational surfaces can be elliptic in at most one way, i.e., they admit at most one elliptic fibration. This allows us to write X instead of (X, π, σ) when dealing with such surfaces.

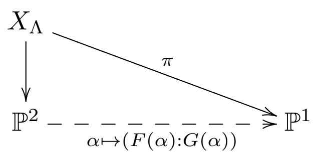
We proceed to the geometric construction of such objects.

2.1. Construction of rational elliptic surfaces. We work, during this subsection, over $\overline{k}$. Let F and G be two distinct homogeneous cubic polynomials in $\overline{k}[X, Y, Z]$. Suppose F is smooth.

The linear pencil of cubics generated by F and G:

$$\Lambda = \{tF + uG = 0;\ (t : u) \in \mathbb{P}^1\}$$

has nine base points (counted with multiplicities). The blow up of $\mathbb{P}^2$ in those points defines a rational surface X_Λ which, when endowed with the obvious morphism π,



becomes an elliptic surface. This motivates the next definition.

Definition 2.1. Let Λ be a smooth pencil of cubics in $\mathbb{P}^2$, and let $X \to \mathbb{P}^1$ be a rational elliptic surface. We say that Λ *induces* X if there exists an isomorphism (of elliptic surfaces) between X and the elliptic surface X_Λ defined by the above construction.

The surface X_Λ is birational to the elliptic surface:

$$X'_\Lambda = \{([x, y, z], [t, u]) \in \mathbb{P}^2 \times \mathbb{P}^1 : tF(x, y, z) + uG(x, y, z) = 0\}.$$

Conversely, we have:

Proposition 2.2 ([**8**])**.** *Every smooth rational elliptic surface, over an algebraically closed field, is induced by some smooth pencil of cubics in* $\mathbb{P}^2$.

Proof. See, for example, [**2**]. □

2.2. Singular fibers. By definition, there are only finitely many singular fibers. Also, in the definition, we assumed that elliptic surfaces have at least one singular fiber. In the case of rational elliptic surfaces, giving a bound for the number of singular fibers and describing which types in Tate's list, cf., [**14**], for example, might occur amounts to the exercise of splitting the Euler number as sums of positive integers and then going through the list of singular fibers. The Euler number, in this case, is 12. In particular, there are at most 12 singular fibers. This information is encoded, for example, in the discriminant of the Weierstrass equation, which for rational elliptic surfaces has degree at most 12 as expected.

After a further examination of the discriminant, we easily see that an additive, respectively multiplicative, singular fiber has at most 10, respectively 9, components.

The next items follow trivially from the above discussion.

(i) There are at least 2 and at most 12 singular fibers.
(ii) The types of singular fibers which might occur are: I_n, with $n \leq 9$, II, III, IV, I_n^*, with $n \leq 4$, IV^*, III^* and II^*.

2.3. Minimal models of rational surfaces over non-closed fields. The theory in this subsection was developed by Enriques followed by Iskovskikh [**3**], Manin and Tsfasman [**7**]. It extends the classical theory of minimal models of surfaces to the case of arbitrary fields. We assume throughout this subsection that k is an arbitrary non-algebraically closed field.

Definition 2.3. A smooth projective surface X is said to be *k-minimal* if every k-birational morphism $f : X \to X'$, where X' is a k-surface, is an isomorphism, or, equivalently, if X does not contain a set of pairwise skew (-1) curves invariant under the action of $\mathrm{Gal}\,(\overline{k}/k)$.

Definition 2.4. We call the *degree* of a surface the self-intersection of its canonical bundle. It is denoted by $d_X := \omega_X^2$.

We only state the main results without proofs; for that purpose, we remind the reader that a *del Pezzo surface* X is a complete smooth surface whose anti-canonical bundle ω_X^{-1} is ample, i.e., it is a *Fano* surface.

Theorem 2.5. *Let X be a k-minimal rational surface. Then X is isomorphic to a surface in one of the following families*:

(**F1**) *A del Pezzo surface with* $\mathrm{Pic}\,(X/k) \simeq \mathbb{Z}$.
(**F2**) *A conic bundle with* $\mathrm{Pic}\,(X/k) \simeq \mathbb{Z}^2$.

Every X in (**F1**) *is minimal. If X belongs to* (**F2**)*, then it is not minimal precisely when one of the two cases hold*: (1) $d_X = 3, 5, 6$ *or,* (2) *X is isomorphic to $\mathbb{P}^2$ blown-up in a single point, and hence, $d_X = 8$. There are no minimal surfaces with $d_X = 7$.*

Certain surfaces endowed with a conic fibration are also del Pezzo surfaces, namely, if $d_X = 3, 5, 6$ or $d_X = 1, 2, 4$ and X has two distinct fibrations, or $d_X = 8$ and $\overline{X} \simeq \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$ or $\mathbb{P}^2_{\bar{k}}$ is blown-up in a point.

Note that, since they are Fano varieties, del Pezzo surfaces have degree $1 \leq d_X \leq 9$. Over an algebraically closed field, a del Pezzo surface X of degree $1 \leq d_X \leq 7$ or $d_X = 9$ is isomorphic to $\mathbb{P}^2$ blown-up in $9 - d_X$ distinct points in general position; if $d_X = 8$, then either $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, or X is $\mathbb{P}^2$ blown-up at a single point, see [**4,** Theorem 24.3].

Definition 2.6. We say that a surface X is *k-birationally trivial*, or *k-rational* if there is a birational map $\mathbb{P}^2 \to X$ defined over k.

Theorem 2.7 ([**7**], Theorem 3.3.1)**.** *Let X be a k-minimal del Pezzo surface or a k-minimal conic bundle. If $d_X \leq 4$, then X is not k-birationally trivial. If $d_X \geq 5$ and $X(k) \neq \emptyset$, then X is k-rational.*

Remark 2.8. A priori, most k-minimal rational surfaces with $d_X \geq 1$ with $X(k) \neq \emptyset$ can occur as a k-minimal model of a rational elliptic surface $\mathscr{E}$. The condition $X(k) \neq \emptyset$ is imposed by the existence of a section of $\mathscr{E} \to B$ defined over k that is contracted to a k-rational point. The condition $d_X \geq 1$ follows from $d_{\mathscr{E}} = 0$.

2.4. The Shioda-Tate formula. We recall the Shioda-Tate formula over an algebraically closed field and its version over a number field k.

Let $\pi : \mathscr{E} \to B$ be an elliptic surface defined over a number field k with zero section σ_0. Let

$$F_v = \pi^{-1}(v) = n_1 C_1 + \cdots + n_s C_s$$

be the fiber over $v \in B(\overline{k})$, where C_i denote its irreducible components. Let

$$F_v = \bigoplus_i \frac{\mathbb{Z}C_i}{\mathbb{Z}F}.$$

Let G denote the absolute Galois group of k. Since the action of G sends fibral divisors to fibral divisors, the finite sum

$$\mathscr{F} = \bigoplus_{\text{reducible fibers}} F_v$$

is stable under the action of G. Taking the embedding of the Mordell-Weil group in the Néron-Severi group that sends a section σ to $\sigma - \sigma_0$ gives the isomorphism below.

Theorem 2.9 (Shioda-Tate formula)**.** *Let $\mathscr{E}$ be an elliptic surface. Identify the Mordell-Weil group $\mathscr{E}(\overline{k}(B))$ with its image in $\mathrm{NS}(\mathscr{E}/\overline{k})$ by means of the map $\sigma \mapsto \sigma - \sigma_0$. Then, we have the following decomposition of G-modules*

$$\mathrm{NS}(\mathscr{E}/\overline{k}) \otimes \mathbb{Q} \simeq (\mathscr{E}(\overline{k}(B)) \otimes \mathbb{Q}) \oplus (\langle O, F\rangle + \mathscr{F}),$$

where $\langle O, F\rangle$ is the subspace of $NS(\mathscr{E}/\overline{k})$ generated by the image of the zero section and an irreducible fiber.

In particular, we have the following.

Corollary 2.10. *Let $\mathscr{E}$ be an elliptic surface. Let ρ be the rank of its Néron-Severi group, r the rank of its generic fiber over $\overline{k}$ and m_v the number of irreducible components of the fiber F_v. Then*

$$\rho = r + 2 + \sum_v (m_v - 1).$$

For the proof of Theorem 2.9, see **[12, 13]**.

Taking Galois-invariants in Theorem 2.9, we obtain an equality over number fields.

Corollary 2.11.

$$\mathrm{rank}\,(\mathrm{NS}(\mathscr{E}/k)) = 2 + \mathrm{rank}\,(\mathscr{E}(k(B))) + \mathrm{rank}\,(\mathscr{F}^G).$$

Remark 2.12 (Upper bound for the rank of the MW group of RES)**.** Let $\mathscr{E} \to B \simeq \mathbb{P}^1$ be a rational elliptic surface. Then, by Proposition 2.2, it is isomorphic, over the algebraic closure, to the blow up of $\mathbb{P}^2$ in the nine, not necessarily distinct, base points of a pencil of cubics. It follows that $\mathrm{rank}\,(\mathrm{Pic}\,(\mathscr{E}/\overline{k})) = 10$. Since $\mathrm{NS}\,(\mathscr{E}/\overline{k}) \simeq \mathrm{Pic}\,(\mathscr{E}/\overline{k})/\mathrm{Pic}^0(\mathscr{E}/\overline{k})$, Corollary 2.10 implies that $\mathrm{rank}\,(\mathscr{E}(\overline{k}(B)) \leq 8$.

3. k-minimal models and base points. Let $\mathscr{E} \to B$ be a rational elliptic surface defined over a number field k. Let $p_1, \dots, p_r$, with $r \leq 9$, be the distinct base points of a cubic pencil that induces $\mathscr{E}$. We explore the relations between possible k-minimal models of $\mathscr{E}$ and its Mordell-Weil rank over the base field as well as the configuration of the points $p_1, \dots, p_r$. For that, it is worth recalling the following definition.

Definition 3.1. Let $p_1, \dots, p_r$, with $r \leq 9$, be points in the projective plane. We say that $p_1, \dots, p_r$ are in *general position* if, among these points, there are not three collinear, six lying on a conic, nor eight on a cubic singular in one of them.

Theorem 3.2. *Let $\mathscr{E}$ be as above and X a k-minimal model of $\mathscr{E}$. Then the following statements hold.*

- ia) *X is a del Pezzo surface of degree 1 with $\mathrm{Pic}\,(X/k) \simeq \mathbb{Z}$ if and only if the rank of the generic fiber of $\mathscr{E}$ is 8 over $\overline{k}$ and 0 over k.*
- ib) *If X is a del Pezzo surface of degree 1 with $\mathrm{Pic}\,(X/k) \simeq \mathbb{Z}^2$, then the rank of the generic fiber of $\mathscr{E}$ is 8 over $\overline{k}$ and 1 over k.*
- ic) *If the rank of the generic fiber of $\mathscr{E}$ is 8 over $\overline{k}$ and 1 over k, then there is a k-minimal model X' of $\mathscr{E}$ that is a del Pezzo surface of degree 1 with $\mathrm{Pic}\,(X'/k) \simeq \mathbb{Z}^2$ or a del Pezzo surface of degree 2 with $\mathrm{Pic}\,(X'/k) \simeq \mathbb{Z}$.*
- ii) *If X has degree d, then $\mathrm{rank}\,(\mathscr{E}(k(B))) \leq d$.*

iii) *Suppose* $X \simeq \mathbb{P}^2_k$. *The rank of the generic fiber of* $\mathscr{E}$ *over* k *is* 8 *if and only if* $\mathscr{E}$ *is obtained from* X *by blowing up* 9 *distinct* k*-rational points in general position.*

Proof. The proof consists of several applications of the Shioda-Tate formula. Assume that X is a k-minimal del Pezzo surface of degree 1. Then there is a birational map $f : \mathscr{E} \to X$ which is the contraction of a (-1)-curve, say C, in $\mathscr{E}$. We show that $\mathscr{E}$ does not have a reducible fiber. If $F = F_1 + F_2 + \cdots + F_m$, where F_i are the reduced irreducible components of F in $\mathscr{E}$, then, by the adjunction formula, $F_i^2 = -2$. Suppose, without loss of generality, that $C \cdot F_1 = 1$ and $C \cdot F_j = 0$, for $j \geq 2$. Then, for $j \geq 2$, the image of F_j in X are (-2)-curves, but this contradicts the fact that X is a Fano surface. Hence, there are no reducible fibers in $\mathscr{E}$. In particular, the Mordell-Weil rank of the generic fiber over $\overline{k}$ is as large as possible, i.e., eight.

We deal now with the Mordell-Weil rank of $\mathscr{E}$ over k. By Theorem 2.5, the Picard group of X satisfies $\mathrm{Pic}\,(X/k) \simeq \mathbb{Z}$ or $\mathbb{Z}^2$ and therefore, $\mathrm{Pic}\,(\mathscr{E}/k) \simeq \mathbb{Z}^2$ or $\mathbb{Z}^3$. Since $\mathscr{E}$ has no reducible fibers, we conclude, after applying Corollary 2.11, that $\mathrm{rank}\,(\mathscr{E}(k(B))) = 0$ or 1, depending on $\mathrm{rank}\,\mathrm{Pic}\,(\mathscr{E}/k)$. This proves ib) and the first implication of ia).

Conversely, if $\mathrm{rank}\,\mathscr{E}(\overline{k}(B)) = 8$, then $\mathscr{E}$ does not have reducible fibers by Theorem 2.9. If $\mathrm{rank}\,\mathscr{E}(k(B)) = 1$, then the contraction of the zero section of $\mathscr{E}$ yields a del Pezzo surface X of degree 1 with $\mathrm{Pic}\,(X/k) \simeq \mathbb{Z}^m$, $m \leq 2$. This last surface is minimal if and only if there is a generator of the Mordell-Weil group of $\mathscr{E}(k(B))$ that does not intersect the zero section. If it is minimal, then it is a del Pezzo surface of degree 1 that is also a conic bundle. If it is not minimal, then the contraction of a generator of the Mordell-Weil group gives us a del Pezzo surface of degree 2 with $\mathrm{Pic}\,(X/k) \simeq \mathbb{Z}$.

Since X is k-minimal, it satisfies $\mathrm{Pic}\,(X/k) \simeq \mathbb{Z}$ or $\mathbb{Z}^2$. As $\mathscr{E}$ is the blow-up of X in d points, its Néron-Severi group has rank at most $2+d$. By Corollary 2.11, $\mathrm{rank}\,(\mathscr{E}(k(B))) \leq d$, proving ii).

The last statement follows after observing that the blow-up points in non-general position correspond to the existence of reducible fibers. ☐

Remark 3.3. The hypotheses on the geometric generic rank on ia) are crucial for assuring that a k-minimal model of $\mathscr{E}$ cannot be a del Pezzo surface of higher degree. In ic), we cannot assure that every k-minimal model is a del Pezzo of degree at most 2 because the Mordell-Weil group of $\mathscr{E}(k(B))$ may contain a large Galois-orbit. This is not the case in ia) as there are no torsion sections.

4. k-minimal models and singular fibers. Throughout this section, k is a number field. Let $\mathscr{E}$ be a rational elliptic surface defined over k and X a k-minimal model of $\mathscr{E}$. Our goal in this section is to describe the types of reducible fibers that a rational elliptic surface can have according to its minimal models over k and vice-versa.

We first give a restriction of the degree of the minimal model imposed by the presence of a unique reducible fiber of any type.

Proposition 4.1. *Let $\mathscr{E}$ be a rational elliptic surface and F a reducible fiber. If $\mathscr{E}$ does not admit another fiber with same Kodaira type as F, then any k-minimal model of $\mathscr{E}$ has degree at least two.*

Proof. Let C be the 0 section, F the reducible fiber mentioned in the statement and F_0 its component cut by C. Then F_0 is a (-2)-curve in $\mathscr{E}$, which is defined over the ground field k. It is enough to observe that the contraction of C takes F_0 to a (-1)-curve defined over k, which we may contract. □

The following is a trivial consequence of the Shioda-Tate formula together with the fact that rational elliptic surfaces have *degree* 0, i.e., $K_{\mathscr{E}}^2 = 0$.

Lemma 4.2. *Let $\mathscr{E}$ be a rational elliptic surface and X a k-model, not necessarily minimal of $\mathscr{E}$. Suppose that X has degree $d \geq 1$ and* $\mathrm{Pic}\,(X/k) \simeq \mathbb{Z}^n$. *Then* $\mathrm{Pic}\,(\mathscr{E}/k) \simeq \mathbb{Z}^m$ *with $n + 1 \leq m \leq n + d$.*

In what follows, we apply Lemma 4.2 together with a more detailed analysis of how the absolute Galois group acts on the singular fibers.

Proposition 4.3. *Let $\pi : \mathscr{E} \to \mathbb{P}^1$ be a rational elliptic surface and X a k-minimal model of $\mathscr{E}$. Assume that X has degree d.*

The following hold:

A) *If* X *is a del Pezzo surface, then the reducible fibers of* $\mathscr{E}$ *belong to the list*:
   (a) *semistable, i.e., of type* I_n*, with* $n \leq 2d-1$*,*
   (b) I_n^**, if* $n \leq d-3$ *and* $d \geq 3$*,*
   (c) *additive of types* III *or* IV*, if* $2 \leq d \leq 9$*,*
   (d) *additive of type* IV^**, respectively* III^**, respectively* II^**, if* $d \geq 6$*, respectively* 8*, respectively* 9*.*

B) *If* X *is a conic bundle that is not a del Pezzo surface, then the reducible fibers of* $\mathscr{E}$ *belong to the list above including the extra possibilities below*:
   (a) I_4, I_5, I_6*, if* $d = 2$*,*
   (b) I_2, I_3, III *and* IV*, if* $d = 1$*.*

Proof. We deal first with A), that is, we suppose that the elliptic surface $\mathscr{E}$ has a k-minimal model that is isomorphic to a del Pezzo surface X of degree d. Since X does not have (-2)-curves, components of reducible fibers in $\mathscr{E}$ are given by the blow-up of points that lie on (-1)-curves in X or of infinitely near points in X. Note that the most efficient way of producing (-2)-curves is by blowing up points that lie on the intersection of (-1)-curves. Now we analyze this situation according to the type of singular fiber:

(i) *semi-stable*: Since each component of a semi-stable fiber intersects precisely two others, its components arise after the blow-up of points in the intersection of at most two (-1)-curves. As the map $\mathscr{E} \to X$ is the blow-up of d points, a semi-stable fiber has at most $2d$ components, and therefore, is of type I_n with $n \leq 2d-1$.

(ii) *additive of type* I_n^* (*potentially multiplicative*): Since such fibers have at least one component that intersects precisely three other components, its components arise as the blow-up of points in the intersection of at most four (-1)-curves. If $d = 1$, then these fiber types do not occur since it takes at least two blow-ups from the minimal model to obtain such a configuration of (-2)-curves on a rational elliptic surface. If $d = 2$, then, in order to get such a fiber we would have to blow-up infinitely near points in the intersection of four (-1)-curves, but this would yield a section of the elliptic fibration intersecting a non-reduced component of a fiber, which cannot happen. Hence, if $d = 2$, then $\mathscr{E}$ does not have singular fibers of type I_n^*. If $d \geq 3$, then there are at most three (-1)-curves through the same point and an additive fiber of type I_n^* has at most $d+1$ components and therefore $n \leq d-3$.

(iii) *reduced additive* (II, III *and* IV): It follows from the proof of Theorem 3.2 that, if $d = 1$, then X admits only irreducible fibers; therefore, the only additive fibers it admits are of type II. The other configurations occur for any $d \neq 1$. For $d = 2$, see the appendix. From the existence of such fibers for $d = 2$, the same holds for higher values of d.

(iv) *non-reduced additive of type* IV^*, III^* *or* II^*: These types of fibers have several non-reduced components with distinct multiplicities, which guarantee that many of them, if not all, are defined over the ground field. Fibers of type II^*, respectively III^*, have all components defined over the ground field assuring that the Picard rank of the elliptic surface is 10, respectively at least 9 over the ground field. Therefore, any minimal model must have degree 9, respectively 8. A fiber of type IV^* might have components defined over a degree 2 extension of the ground field. Taking that into account, the Picard rank of the elliptic surface over k is al least 7, implying that any k-minimal model has degree at least 6.

We treat case B), i.e., conic bundles. The conic bundles that occur as minimal models of rational elliptic surfaces have degree $d = 1, 2$ or 4 and do not admit $(-n)$-curves for $n \geq 3$. Their anticanonical divisor is therefore always big and nef, i.e., they are weak del Pezzo surfaces. The (-2)-classes lie in the orthogonal complement of the canonical divisor and form a D_{8-d} root lattice in a conic bundle of degree d, see [**4**]. These (-2)-classes are in general not effective, but in the special case in which they are effective, we get at most $8-d$ (-2)-curves in the conic bundle surface in a D_{8-d} configuration. If $d = 4$, then X is k-birational to a del Pezzo surface of degree 4, and therefore, any rational elliptic surface above it cannot have other singular fibers than the ones stated in the first part of the proof. If X has degree 2, then it admits at most seven (-2)-curves which fit in a D_6.

For conic bundles of degree 2, let F be a fiber of the elliptic surface. Since X has at most six (-2)-curves in a D_6, any fiber of the elliptic surface has at most seven components. If F is non-reduced, then its contribution to the rank of the Picard group over the ground field is at least 3, which means that the Picard group has rank at least 5, and therefore, any k-minimal model has degree at least 3. Also, note that, contracting a (-1)-curve defined over the ground field, e.g., the 0 section, and the subsequent (-1)-curve corresponding to the component of F that intersects this section, then the degree 2 surface one obtains has at least one (-1)-curve defined over k and is therefore non-minimal. If F is reduced, then it cannot contain seven components as it would yield an A_6 which cannot be embedded into D_6.

For degree 1, we use Proposition 4.1 which tells us that any reducible singular fiber of the rational elliptic surface comes in pairs or larger sets. To find which configurations are allowed, we must search for lattices L of ADE-type such that mL, $m \geq 2$, can be embedded in D_7. By **[9]**, we see that this can occur only for $L = A_n$ with $n \leq 2$, which is enough to conclude the proof. □

Corollary 4.4. *Let $\mathscr{E}$ be a rational elliptic surface defined over a number field k. If $\mathscr{E}$ has a non-reduced fiber, then $\mathscr{E}$ is k-rational.*

Proof. This is an immediate consequence of A), part (d), in Proposition 4.3, together with Theorem 2.7. □

We conclude this section with the next simple, yet interesting observation.

Proposition 4.5. *The universal elliptic modular surfaces over $X_1(N)$ of levels $N = 4$, 5 and 6 are $\mathbb{Q}$-rational.*

Proof. The universal elliptic modular surface of level 4 has singular fibers of types $(I_0^*, I_4, 2I_1)$. Each four torsion section intersects a different reduced irreducible component of the I_0^* fiber. As torsion sections do not intersect, we may contract all of them simultaneously, again obtaining four disjoint (-1)-curves. The contraction of the latter yields a rational surface of degree 8, which is always rational over the ground field. For levels 5 and 6, it suffices to simultaneously contract the non-intersecting torsion sections to obtain a rational surface of degree at least 5. These are, as well, always rational over its ground field. □

For more details on the universal modular elliptic surface see **[13]**.

5. Examples.

Example 5.1. We give an example of a rational elliptic surface defined over $\mathbb{Q}$ for which $\mathbb{P}^2$ is not a $\mathbb{Q}$-minimal model.

Consider the cubic surface X in $\mathbb{P}^3$ given by the following equation:

$$T_0^3 + T_1^3 + T_2^3 + 2T_3^3 = 0.$$

It is a $\mathbb{Q}$-minimal surface, see for example, [**4,** Example 21.9], such that $\mathrm{Pic}\,(X) \simeq \mathbb{Z}$ and $\mathrm{Pic}\,(X/\overline{\mathbb{Q}}) \simeq \mathbb{Z}^7$. Let $i : X \hookrightarrow \mathbb{P}^3$ be the closed immersion given by the anti-canonical bundle Ω_X^{-1}. The anti-canonical divisor satisfies:

$$\omega_X^{-1} \simeq i^*(\mathscr{O}_{\mathbb{P}^3}(1)).$$

A linear pencil in $|\omega_X^{-1}|$ has three base points.

We consider a linear pencil such that one base point is defined over $\mathbb{Q}$ and the others are conjugate under $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$.

The pencil is the following:

$$\{tF + uG = 0; (t:u) \in \mathbb{P}^1\}, \quad \text{where } F = T_0 - T_1 \text{ and } G = T_2.$$

The base points are

$$P_1 = (1:1:0:-1),$$
$$P_2 = (1:1:0:-e^{2i\pi/3})$$

and

$$P_3 = (1:1:0:-e^{4i\pi/3}).$$

The blow up of X in these three points gives rise to a (isotrivial) rational elliptic surface whose Weierstrass equation given by

$$\mathscr{E} : Y^2 = X^3 + \left(\frac{28}{27}t^6 + \frac{1}{216}t^3\right).$$

Its singular fibers are of types I_0^* and $3II$.

Example 5.2. We present a rational elliptic surface defined over $\mathbb{Q}$ with four Galois-conjugate fibers of type I_2 and arithmetic Mordell-Weil rank 0.

Let S be the double cover of $\mathbb{P}^2$ ramified above the quartic

$$Q : x^4 + f_2(y,z)x^2 + f_4(y,z), \quad \text{where } f_i \text{ has degree } i.$$

Then, by varying the coordinates y, z in $\mathbb{P}^1$, one obtains an elliptic fibration. The rational elliptic surface is given by the blow up of S at the points $(1:0:0:1)$ and $(1:0:0:-1)$ in the weighted projective space $\mathbb{P}(1,1,1,2)$.

The surface S above is a del Pezzo surface of degree 2. We dedicate further attention to rational elliptic surfaces that have a del Pezzo surface of degree 2 as k-minimal models in the appendix that follows.

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APPENDIX

A. del Pezzo surfaces of degree two and rational elliptic surfaces above them. Throughout the appendix, we work over an arbitrary field k with characteristic different from 2.

We focus on rational elliptic surfaces which have a del Pezzo of degree 2 as k-minimal models. The latter can be regarded as a double cover of $\mathbb{P}^2$ ramified over a quartic curve. We describe the possible fiber types of the rational elliptic surfaces according to the behavior of certain lines in relation to the quartic curve mentioned above.

Let X be a del Pezzo surface of degree 2, let p be a point on X, and let $X' \to X$ be the blow-up of X at p. The anticanonical linear system on the surface X' is a pencil of genus 1 curves. Blowing up X' at the unique base point of $|-K_{X'}|$, we obtain a rational elliptic surface $X'' \to \mathbb{P}^1$. In this appendix, we describe the possible singular fibers of the elliptic surface X''. Recall that $\kappa\colon X \to \mathbb{P}^2$ is the double cover branched over a smooth plane quartic B. The fibers of the elliptic fibration correspond to lines in $\mathbb{P}^2$ through the point $q = \kappa(p)$. Thus, to determine the fiber types of the elliptic fibration, it suffices to describe configurations of a line L in $\mathbb{P}^2$ through q with respect to the quartic B.

Singular fibers correspond to degenerate configurations, where the line is not transverse to the quartic. In Table 1, we denote by $(L \cdot B)_q$ the intersection multiplicity of the line L with the quartic B at the point q. We also define

- a *simple tangent line* to be a line with exactly one point of intersection multiplicity 2 with B,
- a *simple inflection line* to be a line with exactly one point of intersection multiplicity 3 with B,
- a *simple bitangent line* to be a line with exactly two points of intersection multiplicity 2 with B,
- a *flex bitangent line* to be a line with exactly one point of intersection multiplicity 4 with B.

TABLE 1. Fiber types.

Point	**Line with respect to** B	**Kodaira type**
Anywhere	Transverse	I_0
$(L \cdot B)_q \leq 1$	Simple tangent line	I_1
$(L \cdot B)_q = 2$	Simple tangent line	I_2
$(L \cdot B)_q \leq 1$	Simple inflection line	II
$(L \cdot B)_q = 3$	Simple inflection line	III
$(L \cdot B)_q = 0$	Simple bitangent line	I_2
$(L \cdot B)_q = 2$	Simple bitangent line	I_3
$(L \cdot B)_q = 0$	Flex bitangent line	III
$(L \cdot B)_q = 4$	Flex bitangent line	IV

The entries of Table 1 corresponding to the cases in which the point q lies on B, namely, $(L \cdot B)_q \geq 1$, can be found in [**11,** Table 3].

Remark A.1. Note the contrast between rational elliptic surfaces that admit a del Pezzo surface of degree 1 as k-minimal models and those that admit as a k-minimal model a del Pezzo surface of degree 2. The former have only irreducible fibers and therefore always have geometric Mordell-Weil rank equal to 8, while the latter admit all the fiber types described in Table 1 (and Theorem 3.2) and can have geometric Mordell-Weil rank ranging between 4 and 8.

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