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A FUNCTIONAL EQUATION AND DEGENERATE PRINCIPAL SERIES

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ABSTRACT. A functional equation between the ζ distributions can be obtained from the theory of prehomogeneous vector spaces. We show that the functional equation can be extended from the Schwartz space to certain degenerate principal series.

1. Introduction. The fundamental theorem of prehomogeneous vector spaces gives a functional equation for the Fourier transformation of a complex power of associated relative invariants as a distribution. For a class of prehomogeneous vector spaces of particular interest in representation theory, the explicit form of the functional equation is known; see equation (1.1). This class of prehomogeneous vector spaces occurs as $(L, \mathrm{Ad}, \mathfrak{n})$, where $P = LN$ is a certain maximal parabolic subgroup of a reductive Lie group; the cases under consideration are listed in Table 1 in Appendix A. The main purpose of this paper is to extend the domain of the functional equation of some prehomogeneous vector spaces from the Schwartz space to the space of functions of certain degenerate principal series representations $\mathrm{Ind}_P^G(s)$, realized as functions on $\overline{\mathfrak{n}}$, for all $s \in \mathbf{C}$.

The groups G that we consider are those groups for which there is a parabolic subgroup P so that P and its opposite parabolic are G-conjugate, N is abelian and the symmetric space corresponding to G is not of tube type. Such groups G arise from simple non-Euclidean Jordan algebras $\mathfrak{n}$ in the sense that each $\mathfrak{n}$ occurs as the abelian nilradical of a maximal parabolic subalgebra of a reductive Lie algebra $\mathfrak{g}$, where $\mathfrak{g} = \mathrm{Lie}(G)$ for some conformal group G. It follows from Vinberg's theorem that $(L, \mathrm{Ad}, \mathfrak{n})$ is a prehomogeneous vector space.

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These prehomogeneous vector spaces also have a relative invariant ∇. The Levi subgroup L has a finite number of orbits on $\mathfrak{n}$, and in fact, there is an open dense orbit defined by $\nabla \neq 0$. The contragradient $(L, \mathrm{Ad}, \overline{\mathfrak{n}})$ with a relative invariant $\overline{\nabla}$ has the same properties.

It is useful to consider the case $\mathrm{GL}(2n, \mathbf{R})$. Then L and $\mathfrak{n}$ may be identified with $\mathrm{GL}(n, \mathbf{R}) \times \mathrm{GL}(n, \mathbf{R})$ and $M_{n\times n}(\mathbf{R})$, respectively, and the relative invariant is the determinant. The open dense orbit is given by $\mathcal{O}_n = \{X \in M_{n\times n}(\mathbf{R}) : \det(X) \neq 0\}$.

As tempered distributions, the ζ distributions are defined by the integrals:

$$\overline{\mathbf{Z}}(f,t) = \int_{\overline{\mathfrak{n}}} f(X)\overline{\nabla}(X)^t dX, \quad \text{for } f \in \mathcal{S}(\overline{\mathfrak{n}})$$

and

$$\mathbf{Z}(h,t) = \int_{\mathfrak{n}} h(Y)\nabla(Y)^t dY, \quad \text{for } h \in \mathcal{S}(\mathfrak{n}).$$

Here, $\mathcal{S}(\overline{\mathfrak{n}})$ (respectively, $\mathcal{S}(\mathfrak{n})$) denotes the Schwartz space on $\overline{\mathfrak{n}}$ (respectively, $\mathfrak{n}$). The integers m, n, d and e in the following discussion are listed in Table 1 for each group. It is well known that $\overline{\mathbf{Z}}(f,t)$ and $\mathbf{Z}(h,t)$ converge absolutely for $\mathrm{Re}\,(t) > -(e+1)$, and both expressions are complex analytic functions of t. Moreover, these analytic functions in t extend meromorphically to the whole complex plane and satisfy the functional equation:

$$(1.1) \quad \frac{\pi^{nt/2}}{\Gamma_n(t)}\overline{\mathbf{Z}}(f, t - m/n) = \frac{\pi^{n(-t+m/n)/2}}{\Gamma_n(-t+m/n)}\mathbf{Z}(\widehat{f}, -t), \quad \text{for } f \in \mathcal{S}(\overline{\mathfrak{n}}).$$

Here, $\Gamma_n(t) = \prod_{j=0}^{n-1} \Gamma((t-jd)/2)$ (Γ is the gamma function on $\mathbf{C}$) and $\widehat{\ }$ denotes the Fourier transform. The functional equation concerning the Fourier transform of a complex power of the relative invariants as a tempered distribution was established by Mikio Sato and is called the fundamental theorem of prehomogeneous vector spaces, see, for example, [**6,** Theorem 4.17]. For our choices of the groups G the explicit functional equation (1.1) is contained in [**3, 11**].

Consider the family of degenerate principal series representations $\mathrm{Ind}_P^G(s)$ for $s \in \mathbf{C}$. Each can be realized as a space of certain functions on $\overline{\mathfrak{n}}$, which is denoted by $I(s)$. The Schwartz space $\mathcal{S}(\overline{\mathfrak{n}})$ is contained in

$I(s)$ for all s. The compact picture $C^\infty(K/M)$ is given by a restriction of functions in $\mathrm{Ind}_P^G(s)$ and we fix an element φ in $C^\infty(K/M)$. The function φ is independent of s, and we let F_s be the corresponding function in $I(s)$. In this paper, we consider the family of integrals $\overline{\mathbf{Z}}(F_s, t)$ and prove the following theorems.

Theorem 1.1. *Let $F_s \in I(s)$. Then the family of integrals $\overline{\mathbf{Z}}(F_s, t)$ is complex analytic on $-(e+1) < \mathrm{Re}\,(t) < \mathrm{Re}\,(s) - d(n-1)$ and has a meromorphic continuation to all of $\mathbf{C}^2$.*

Theorem 1.2. *Let $F_s \in I(s)$. Then the functional equation*

$$\frac{\pi^{nt/2}}{\Gamma_n(t)}\overline{\mathbf{Z}}\left(F_s, t - \frac{m}{n}\right) = \frac{\pi^{n(-t+m/n)/2}}{\Gamma_n(-t+m/n)}\mathbf{Z}(\widehat{F_s}, -t) \tag{1.2}$$

holds as meromorphic functions in $(s,t) \in \mathbf{C}^2$.

The organization of our paper is as follows. We set up some notation and give some properties about the groups that we consider, together with an integral formula for 'polar coordinates' in Section 2 in a form convenient for our purposes. We define principal series representations $I(s)$, and give formulas for the action by elements of certain commuting copies of $\mathfrak{sl}(2, \mathbf{R})$ in Section 3. Section 4 contains the functional equation between the ζ distributions and the convergent range of the integral $\overline{\mathbf{Z}}(F_s, t)$ for $F_s \in I(s)$. Section 5 contains the proofs of the main results.

We apply the polar coordinates to the integral $\overline{\mathbf{Z}}(F_s, t)$ to reduce it to the integrals over a noncompact radial set and a compact set. Then the main part of the proof shows the meromorphic continuation of the integral over the noncompact set. Our main technique is applying a string of differential operators to extend the defining range of the integral $\overline{\mathbf{Z}}(F_s, t)$ in two variables, s and t. Such differential operators are obtained from the Lie algebra action given in Section 3. Section 6 contains some comments on representation theory.

2. Preliminaries. The choices for the groups G with which we work are given in Appendix A, Table 1. These groups G arise from simple non-Euclidean Jordan algebras. Such a G is characterized by the existence of a parabolic subgroup $P = LN$ (a Levi decomposition) such that P and its opposite parabolic $\overline{P} = L\overline{N}$ are G-conjugate, N is abelian and the symmetric space corresponding to G is not of tube type. Much of the background material in Sections 2, 3 and 4 is contained in [**1, 4, 8**]. G has a Cartan involution θ so that θ sends P to the opposite parabolic. We let K be the fixed point group of θ, a maximal compact subgroup of G. The real Lie algebras of various Lie groups are expressed by the corresponding Fraktur letters. The Cartan involution determines a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$.

Following [**8**], there is a maximal abelian subalgebra $\mathfrak{b}$ of $\mathfrak{l} \cap \mathfrak{s}$ with the following properties.

(1) There are commuting copies of $\mathfrak{sl}(2, \mathbf{R})$, denoted by $\mathfrak{sl}(2, \mathbf{R})_j$, in $\mathfrak{g}$, spanned by $\{F_j, H_j, E_j\}$ a standard basis in the sense that:

$$\theta(E_j) = -F_j \quad \text{and} \quad \theta(H_j) = -H_j$$
$$[E_j, F_j] = H_j, [H_j, E_j] = 2E_j \quad \text{and} \quad [H_j, F_j] = -2F_j,$$

with $E_j \in \mathfrak{n}$ and $F_j \in \overline{\mathfrak{n}}$. Then $\mathfrak{b} = \sum_{j=1}^{n} \mathbf{R}H_j$. Therefore, we can view $\prod_{j=1}^{n} \mathfrak{sl}(2, \mathbf{R})_j$ as a subalgebra of $\mathfrak{g}$ and will denote the element $aH_j + bE_j + cF_j$ by $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)_j$. There is a corresponding Lie group $\prod_{j=1}^{n} \mathrm{SL}\,(2, \mathbf{R})_j$ in G.

(2) For

$$\epsilon_k\left(\sum_{j=1}^{n} a_j H_j\right) \equiv a_k,$$

the $\mathfrak{b}$-roots in $\mathfrak{g}$, $\mathfrak{l}$ and $\mathfrak{n}$ are

$$\Sigma(\mathfrak{g}, \mathfrak{b}) = \{\pm(\epsilon_j - \epsilon_k) : 1 \le j < k \le n\} \cup \{\pm(\epsilon_j + \epsilon_k) : 1 \le j, k \le n\}$$
$$\Sigma(\mathfrak{l}, \mathfrak{b}) = \{\pm(\epsilon_j - \epsilon_k) : 1 \le j < k \le n\}$$

and

$$\Sigma(\mathfrak{n}, \mathfrak{b}) = \{\epsilon_j + \epsilon_k : 1 \le j, k \le n\}.$$

For each G, the roots in $\mathfrak{n}$ have just two possibilities for the multiplicity. This defines integers d and e:

> each short root has multiplicity $2d$, and each long root has multiplicity $e + 1$.

(In the case of $\mathrm{SO}(p, q)$, Case 4 in Tables 1 and 2, d is a half integer. When $n = 1$, d is zero.) We set $m = \dim(\mathfrak{n})$. Then $m = n(d(n-1) +$

$(e+1))$. Define $\Sigma^+(\mathfrak{g},\mathfrak{b}) = \{\epsilon_j - \epsilon_k : 1 \le j < k \le n\} \cup \Sigma(\mathfrak{n},\mathfrak{b})$. Define a character Λ_0 on $\mathfrak{b}$ as

$$\Lambda_0 \equiv \sum_{j=1}^{n} \epsilon_j.$$

Then Λ_0 extends to a character of $\mathfrak{l}$, and we write e^{Λ_0} for the corresponding character of L.

There is a diffeomorphism of $\overline{\mathfrak{n}} \times L \times \mathfrak{n}$ onto a dense open set in G given by $(X, \ell, Y) \mapsto \overline{\mathfrak{n}}_X \ell \mathfrak{n}_Y$, where $\overline{n}_X = \exp(X)$ and $n_Y = \exp(Y)$. Therefore, any $g \in \overline{N}LN$ has a unique decomposition as $g = \overline{N}(g)\ell(g)N(g)$. Furthermore, $L = MA$, where $A = \exp(\mathfrak{a})$,

$$\mathfrak{a} = \bigcap_{j<k} \ker(\epsilon_j - \epsilon_k).$$

Since the L part of the decomposition has a component in A, we define $a(g) \in A$ by $g \in \overline{N}Ma(g)N$. We can see this directly for $SL(2, \mathbf{R})$, as follows.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} \quad \text{if } a \neq 0. \tag{2.1}$$

Here, $a(g) = \mathrm{diag}(a, 1/a)$ with $a > 0$.

We describe the orbits of L in $\overline{n}$. If we set $X_q \equiv F_1 + \cdots + F_q$, $q = 1, 2, \dots, n$ and $X_0 \equiv 0$, then, by **[5, 12]** the L-orbits in $\overline{\mathfrak{n}}$ are, precisely, $\mathcal{O}_q = L(X_q)$, $q = 0, 1, 2, \dots, n$. We write $\mathcal{O}_q = L/S_q$, S_q the stabilizer of X_q. As $\mathrm{ad}(X_n) : \mathfrak{l} \to \mathfrak{n}$ is onto, $\mathcal{O}_n$ is open in $\mathfrak{n}$. Moreover, it is also dense and a semisimple symmetric space of rank n.

Example 2.1. If $G = GL(2n, \mathbf{R})$, then the subgroups L and N are given by:

$$L = \left\{ \begin{pmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{pmatrix} : \ell_j \in GL(n, \mathbf{R}) \right\},$$
$$N = \left\{ n_Y = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} : Y \in M_{n\times n}(\mathbf{R}) \right\}.$$

$\mathfrak{s} = \mathrm{Sym}\,(2n, \mathbf{R})$ in the Cartan decomposition. The n-commuting copies of $\mathfrak{sl}(2, \mathbf{R})$ in $\mathfrak{g}$ are given as follows. Let E_{ij} be the matrix with 1 in the ij-place and $0'$s elsewhere. We set, for $j = 1, \ldots, n$,

$$E_j = E_{j,n+j} \in \mathfrak{n}, \qquad H_j = E_{j,j} - E_{n+j,n+j}, \qquad F_j = E_{n+j,j} \in \overline{\mathfrak{n}}.$$

Then we define $\mathfrak{sl}(2,\mathbf{R})_j$ as the Lie algebra generated by $\{F_j, H_j, E_j\}$. The dimension of $\mathfrak{n}$ is $m = n^2$. For $j \neq k$, the $\epsilon_j + \epsilon_k$'s are short roots in $\mathfrak{n}$ with multiplicity 2 and the $2\epsilon_j$'s are long roots with multiplicity 1. The Bruhat decomposition $g = \overline{N}(g)\ell(g)N(g)$ is given by the identity,

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}, \quad \text{for } \det(A) \neq 0,$$

where $\ell_1 = A$, $Y = A^{-1}B$, $X = CA^{-1}$ and $\ell_2 = D - CA^{-1}B$.

The Iwasawa decomposition of G with respect to P is $G = K\exp(\mathfrak{m} \cap \mathfrak{s})AN$. We will write it as $g = \kappa(g)\mu(g)e^{H(g)}n(g)$. This decomposition for $SL(2,\mathbf{R})$ is given by the identity:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a/\sqrt{a^2+c^2} & -c/\sqrt{a^2+c^2} \\ c/\sqrt{a^2+c^2} & a/\sqrt{a^2+c^2} \end{pmatrix} \begin{pmatrix} \sqrt{a^2+c^2} & 0 \\ 0 & 1/\sqrt{a^2+c^2} \end{pmatrix} \begin{pmatrix} 1 & (ab+cd)/(a^2+c^2) \\ 0 & 1 \end{pmatrix}.$$

In particular, for $X = \left(\begin{smallmatrix} 0 & 0 \\ x & 0 \end{smallmatrix}\right)$,

$$e^{H(\overline{n}_X)} = \begin{pmatrix} \sqrt{1+x^2} & 0 \\ 0 & 1/\sqrt{1+x^2} \end{pmatrix}. \tag{2.2}$$

The following lemma is easily proven by uniqueness of the decomposition.

Lemma 2.2. *For $k \in K \cap L$ and $X \in \overline{\mathfrak{n}}$, $\kappa(\overline{n}_{k\cdot X}) = k\,\kappa(\overline{n}_X)\,k^{-1}$.*

We define

$$w \in \prod_{j=1}^{n} \mathrm{SL}(2,\mathbf{R})_j \quad \text{as} \quad w = \prod_{j=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_j,$$

which is in K and satisfies $\mathrm{Ad}(w)\mathfrak{n} = \overline{\mathfrak{n}}$. We also define functions on dense open subsets of $\overline{\mathfrak{n}}$ and $\mathfrak{n}$ by

$$\overline{\nabla}(X) \equiv e^{\Lambda_0(\log(a(w\overline{n}_X)))}, \quad X \in \overline{\mathfrak{n}},$$

and

$$\nabla(Y) \equiv \overline{\nabla}(\theta(Y)), \quad Y \in \mathfrak{n}.$$

Then both $\overline{\nabla}$ and ∇ are invariant under $K \cap L$. For $G = GL(2n, \mathbf{R})$, $\nabla = \overline{\nabla} = |\det|$. The next lemma follows from equation (2.1).

Lemma 2.3. *For positive real numbers of x_j's,*

$$\overline{\nabla}\left(\sum_{j=1}^{n} x_j F_j\right) = \prod_{j=1}^{n} x_j.$$

Proof. In each $\mathrm{SL}(2, \mathbf{R})_j$, by equation (2.1), we have the following $\mathrm{SL}(2, \mathbf{R})$ computations.

$$a(w\overline{n}_{x_i F}) = a\left(w\begin{pmatrix} 1 & 0 \\ x_j & 1 \end{pmatrix}\right) = a\left(\begin{pmatrix} x_j & 1 \\ -1 & 0 \end{pmatrix}\right) = \begin{pmatrix} x_j & 0 \\ 0 & 1/x_j \end{pmatrix}.$$

This proves the lemma. □

We describe a "polar coordinate" expression for the Lebesgue measure on $\overline{\mathfrak{n}}$. In [**10**], it is shown that the elements

$$\left\{\sum_{j=1}^{n} x_j F_j : x_1 > \cdots > x_n > 0\right\}$$

give a complete set of orbit representatives for the action of $K \cap L$ on $\mathcal{O}_n$. We write Ω for the cone $\Omega = \{(x_1, \cdots, x_n) : x_1 > \cdots > x_n > 0\}$. Each dx_j denotes the Lebesgue measure on $\mathbf{R}$.

Proposition 2.4 ([**4,** Proposition 1.2], [**14,** Proposition 7.1.3])**.** *Let dX be the Lebesgue measures on $\overline{\mathfrak{n}}$. Then we have*

$$\int_{\overline{\mathfrak{n}}} f(X)\, dX$$
$$= c\int_{K\cap L}\left[\int_{\Omega} f\left(k \cdot \sum_{j=1}^{n} x_j F_j\right) \prod_{j=1}^{n} x_j^e \prod_{1\le i<j\le n} (x_i^2 - x_j^2)^d \prod_{j=1}^{n} dx_j\right] dk.$$

The scalar c in the above formula is independent of f and depends upon the normalization of the measure dX.

3. Generalized principal series representations and the Lie algebra action. For $s \in \mathbf{C}$, we may define a family of normalized principal series representations as induced representations:

$$\begin{aligned}\mathrm{Ind}_P^G(s) = \{f : G \longrightarrow \mathbf{C} : f \text{ is smooth and} \\ f(gman) = e^{-(s+m/n)\Lambda_0(\log(a))} f(g), man \in P = MAN\}.\end{aligned}$$

The group G acts by left translation $(g \cdot f)(g_1) = f(g^{-1}g_1)$. Then the compact picture is obtained from the induced picture by restricting to K. We set $\varphi = f|_K$, $f \in \mathrm{Ind}_P^G(s)$. The compact picture may be written as

$$\begin{aligned}C^\infty(K/M) = \{\varphi : K \longrightarrow \mathbf{C} : \varphi \text{ is smooth and} \\ \varphi(km) = \varphi(k), k \in K \text{ and } m \in M\}.\end{aligned}$$

Note that the action of G corresponding to the left translation does depend on s, but the function φ in the compact picture is independent of s.

The noncompact picture is given by restricting the induced picture to $\overline{N}$. For $f \in \mathrm{Ind}_P^G(s)$, we set $F_s(X) = f(\overline{n}_X)$. Then $\mathrm{Ind}_P^G(s)$ may be identified with

$$I(s) = \{F_s \in C^\infty(\overline{\mathfrak{n}}) : F_s(X) = f(\overline{n}_X), \text{ for some } f \in \mathrm{Ind}_P^G(s)\}.$$

By the Bruhat decomposition, $g \in \overline{N}P$ acts by

$$g \cdot F_s(X) = e^{-(s+m/n)\Lambda_0}(a(g^{-1}\overline{n}_X))F_s(\log(\overline{N}(g^{-1}\overline{n}_X))).$$

In particular,

$$(\ell \cdot F_s)(X_1) = e^{-(s+m/n)\Lambda_0}(a(\ell^{-1}))F_s(\ell^{-1} \cdot X_1)$$

and

$$(\overline{n}_X \cdot F_s)(X_1) = F_s(X_1 - X).$$

We may express functions in the noncompact picture in terms of functions in the compact picture. By the Iwasawa decomposition, we have

$$F_s(X) = e^{-(s+m/n)\Lambda_0(H(\overline{n}_X))}\varphi(\kappa(\overline{n}_X)).$$

If $\varphi \equiv 1$, then F_s is called the *spherical function*. We denote it by h_s, that is,

$$h_s(X) = e^{-(s+m/n)\Lambda_0(H(\overline{n}_X))}.$$

Note that h_s is invariant under K. We give an explicit formula for functions in $I(s)$.

Lemma 3.1. *Let $X = k \cdot Y$, for $k \in K \cap L$ and $Y = \sum_{j=1}^n x_j F_j$. Then the spherical function is*

$$h_s(X) = \frac{1}{\prod_{j=1}^n (1 + x_j^2)^{(s+m/n)/2}}. \tag{3.1}$$

Therefore, $F_s \in I(s)$ can be expressed as

$$F_s(X) = \frac{(k^{-1} \cdot \varphi)\,(\kappa(\overline{n}_Y))}{\prod_{j=1}^n (1 + x_j^2)^{(s+m/n)/2}}. \tag{3.2}$$

Proof. Equation (3.1) follows directly from equation (2.2). On the other hand,

$$\varphi(\kappa(\overline{n}_X)) = \varphi(k\kappa(\overline{n}_Y)k^{-1}) = (k^{-1} \cdot \varphi)(\kappa(\overline{n}_Y))$$

by Lemma 2.2. This proves equation (3.2). □

We describe the action by elements of the Lie subalgebras $\mathfrak{sl}(2, \mathbf{R})_j$ of $\mathfrak{g}$. The Lie algebra action will play an important role in proving the main results. For the group representation on $I(s)$, the corresponding Lie algebra representation is given by the formula, for $Y \in \mathfrak{g}$, as

$$\begin{aligned} Y \cdot F_s(X) &= \frac{d}{dt}\Big|_{t=0} \exp(tY) \cdot f(\overline{n}_X) \\ &= \frac{d}{dt}\Big|_{t=0} e^{-(s+m/n)\Lambda_0}(\ell(\exp(-tY)\overline{n}_X)) F_s(\log(\overline{N}(\exp(-tY)\overline{n}_X))). \end{aligned}$$

Lemma 3.2. *Let*

$$X = \sum_{j=1}^n x_j F_j.$$

The action of $\mathfrak{sl}(2, \mathbf{R})_j$ on $I(s)$ is given by:

(i) $E_k \cdot F_s(X) = (s + m/n)x_k F_s(X) + x_k^2 (\partial F_s)/(\partial x_k)(X)$,

(ii) $H_k \cdot F_s(X) = (s + m/n)F_s(X) + 2x_k(\partial F_s)/(\partial x_k)(X)$, *and*

(iii) $F_k \cdot F_s(X) = -(\partial F_s)/(\partial x_k)(X)$.

Proof. For sufficiently small values of t, we may assume that $1 - tx_k > 0$. In $\prod_{j=1}^{n} \mathrm{SL}(2, \mathbf{R})_j$, by the Bruhat decomposition, we have:

$$\exp(-tE_k)\exp(X) = \exp\left(\sum_{j\neq k} x_j F_j + \frac{x_k}{1-tx_k} F_k\right) \begin{pmatrix} 1-tx_k & 0 \\ 0 & 1/(1-tx_k) \end{pmatrix}_k \begin{pmatrix} 1 & -t/(1-tx_k) \\ 0 & 1 \end{pmatrix}_k,$$

$$\exp(-tH_k)\exp(X) = \exp\left(\sum_{j\neq k} x_j F_j + x_k e^{2t} F_k\right) \exp\left(-tH_k\right),$$

and

$$\exp(-tF_k)\exp(X) = \exp\left(\sum_{j\neq k} x_j F_j + (x_k - t)F_k\right).$$

This proves the lemma. □

4. Zeta distributions and integrals for functions in $I(s)$. The functions $\overline{\nabla}(X)^t$ and $\nabla(Y)^t$ are locally L^1 functions for $\mathrm{Re}\,(t) > -(e+1)$, and they define tempered distributions by the integrals:

$$\overline{\mathbf{Z}}(f,t) = \int_{\overline{\mathfrak{n}}} f(X)\overline{\nabla}(X)^t dX, \quad \text{for } f \in \mathcal{S}(\overline{\mathfrak{n}})$$

and

$$\mathbf{Z}(h,t) = \int_{\mathfrak{n}} h(Y)\nabla(Y)^t dY, \quad \text{for } h \in \mathcal{S}(\mathfrak{n}).$$

Here, $\mathcal{S}(\overline{\mathfrak{n}})$ (respectively, $\mathcal{S}(\mathfrak{n})$) denotes the space of Schwartz functions on $\overline{\mathfrak{n}}$ (respectively, $\mathfrak{n}$). Note that, in the range $\mathrm{Re}\,(t) > -(e+1)$, both expressions are complex analytic functions of t. We will see that there is a meromorphic continuation to all of $t \in \mathbf{C}$. This can be viewed by a general result of [**2**] and the following statements which are contained in [**13**]. Both $\overline{\nabla}^2$ and ∇^2 are polynomials by [**1,** Lemma 2.10], for example. There is a polynomial $b(t)$ so that

$$\overline{\nabla}(\partial_X)^2\overline{\nabla}(X)^t = b(t)\overline{\nabla}(X)^{t-2}$$

and

$$\nabla(\partial_Y)^2\nabla(Y)^t = b(t)\nabla(Y)^{t-2}.$$

In particular, for $b_k(t) = b(t)b(t-2)b(t-4)\cdots b(t-2(k-1))$,

$$\overline{\nabla}(\partial_X)^{2k}\overline{\nabla}(X)^t = b_k(t)\overline{\nabla}(X)^{t-2k}$$

and

$$\nabla(\partial_Y)^{2k}\nabla(Y)^t = b_k(t)\nabla(Y)^{t-2k}.$$

We integrate by parts to obtain

$$\overline{\mathbf{Z}}(\overline{\nabla}(\partial_X)^{2k}f,t) = b_k(t)\overline{\mathbf{Z}}(f,t-2k)$$

and

$$\mathbf{Z}(\nabla(\partial_Y)^{2k}h,t) = b_k(t)\mathbf{Z}(h,t-2k),$$

for sufficiently large $\mathrm{Re}\,(t)$. Since the left hand side is analytic for $\mathrm{Re}\,(t) > -(e+1)$, both $\overline{\mathbf{Z}}(f,t)$ and $\mathbf{Z}(h,t)$ continue to meromorphic functions on $\mathrm{Re}\,(t) > -(e+1)-2k$ for any k.

We now turn to the Fourier transformation and functional equation. Let B denote the Killing form of $\mathfrak{g}$. Then B gives a nondegenerate pairing between $\overline{\mathfrak{n}}$ and $\mathfrak{n}$. Since the form $-B(\cdot,\,\theta(\cdot))$ is positive definite on $\mathfrak{g}$, we may define inner products

$$\langle X_1, X_2\rangle = -\frac{n}{4m}B(X_1,\theta(X_2)) \quad \text{on } \overline{\mathfrak{n}}$$

and

$$\langle Y_1, Y_2\rangle = -\frac{n}{4m}B(Y_1,\theta(Y_2)) \qquad \text{on } \mathfrak{n}.$$

For example, if $G = \mathrm{GL}(2n,\mathbf{R})$, then $\langle X_1, X_2\rangle$ is the trace form on $\overline{\mathfrak{n}}$. Define the Fourier transforms by

$$\widehat{f}(Y) = \int_{\overline{\mathfrak{n}}} f(X)e^{-2\pi i\langle X,Y\rangle}dX, \quad \text{for } f\in L^1(\overline{\mathfrak{n}})$$

and

$$\widehat{h}(X) = \int_{\mathfrak{n}} h(Y) e^{-2\pi i \langle X,Y\rangle} dY, \quad \text{for } h \in L^1(\mathfrak{n}).$$

We may regard the Fourier transform $\widehat{f}$ on $\overline{\mathfrak{n}}$ (respectively, $\widehat{h}$ on $\mathfrak{n}$) as a function on $\mathfrak{n}$ (respectively, $\overline{\mathfrak{n}}$). The functional equation relates the two distributions ∇ and $\overline{\nabla}$ via the Fourier transform. We let Γ denote the gamma function on $\mathbf{C}$.

Theorem 4.1 ([**11**])**.** *Let $t \in \mathbf{C}$ and $f \in \mathcal{S}(\overline{\mathfrak{n}})$. As meromorphic functions*

$$\frac{\pi^{nt/2}}{\Gamma_n(t)} \overline{\mathbf{Z}}\left(f, t - \frac{m}{n}\right) = \frac{\pi^{n(-t+m/n)/2}}{\Gamma_n(-t+m/n)} \mathbf{Z}(\widehat{f}, -t), \tag{4.1}$$

where

$$\Gamma_n(t) = \prod_{j=0}^{n-1} \Gamma\left(\frac{t-jd}{2}\right).$$

To show that the functional equation (4.1) holds for functions in $I(s)$, we consider the integral

$$\overline{\mathbf{Z}}(F_s, t) = \int_{\overline{\mathfrak{n}}} F_s(X) \overline{\nabla}(X)^t dX, \quad \text{for } F_s \in I(s).$$

Lemma 4.2. *$\overline{\mathbf{Z}}(F_s, t)$ converges absolutely on $-(e+1) < \mathrm{Re}\,(t) < \mathrm{Re}\,(s) - d(n-1)$.*

Proof. Combining Lemma 2.3, Proposition 2.4 and Lemma 3.1, we obtain:

$$\overline{\mathbf{Z}}(F_s, t) = c \int_{K \cap L} \left[\int_{\Omega} \frac{\varphi(k\,\kappa(\overline{n}_Y))}{\prod_{j=1}^{n} (1 + x_j^2)^{(s+m/n)/2}} \prod_{j=1}^{n} x_j^{t+e} \prod_{1 \le i < j \le n} (x_i^2 - x_j^2)^d \prod_{j=1}^{n} dx_j \right] dk, \tag{4.2}$$

for some constant c. Here $Y = \sum_{j=1}^{n} x_j F_j$. By expanding $(x_i^2 - x_j^2)^d$, we can write the integrand as a finite linear combination of terms

$$\varphi(k\,\kappa(\overline{n}_Y)) \prod_{j=1}^{n} \frac{x_j^{t+e+k}}{(1+x_j^2)^{(s+m/n)/2}}, \quad \text{where } k = 0, 2, \ldots, 2d(n-1).$$

Each is bounded by the maximum of $|\varphi|$ and a product of variable integrals of the form:

$$\int_0^\infty \frac{x_j^{\mathrm{Re}\,(t)+e+k}}{(1+x_j^2)^{(\mathrm{Re}\,(s)+m/n)/2}}\, dx_j,$$

which converge if $-1 < \mathrm{Re}\,(t) + e + k < \mathrm{Re}\,(s) + m/n - 1$. This proves Lemma 4.2. □

Corollary 4.3. *$F_s \in L^1(\overline{\mathfrak{n}})$ for $\mathrm{Re}\,(s) > d(n-1)$ and $F_s \in L^2(\overline{\mathfrak{n}})$ for $\mathrm{Re}\,(s) > -(e+1)/2$.*

Proof. If $t = 0$ in Lemma 4.2, we get the condition on s so that $F_s \in L^1(\overline{\mathfrak{n}})$. Note also that $F_s \in L^2(\overline{\mathfrak{n}})$ if and only if $h_s^2 \in L^1(\overline{\mathfrak{n}})$. This is the case when $e+k < 2(s+(m/n))-1$ for all $k = 0, 2, \ldots, 2d(n-1)$. □

The meromorphic continuation of $\overline{\mathbf{Z}}(F_s, t)$ in (s, t) given in the following lemma has a proof similar to the proof that $\overline{\mathbf{Z}}(f, t)$ has a continuation to $\mathbf{C}$, where $f \in \mathcal{S}$, as described in the beginning of this section.

Lemma 4.4. *$\overline{\mathbf{Z}}(F_s, t)$ can be extended meromorphically to the range*

$$\bigcup_{k \in \mathbf{Z}_{\geq 0}} \left\{(s,t) \in \mathbf{C}^2 : -(e+1) - 2k < \mathrm{Re}\,(t) < \mathrm{Re}\,(s) - d(n-1) - 2k\right\}.$$

Proof. Note that $\overline{\nabla}^{\,t}(X)$ vanishes on $\overline{\nabla}(X) = 0$ for $\mathrm{Re}\,(t) > 0$ and $F_s(X)$ vanishes at infinity for $\mathrm{Re}\,(s) > -m/n$. Also, since each ∂^α, $|\alpha| = 1$, acts as an element in the enveloping algebra $\mathcal{U}(\mathfrak{g})$, $\partial^\alpha(F_s) \in I(s)$ for any multi-index α. So is $\overline{\nabla}(\partial_X)^{2k}(F_s)$, $F_s \in I(s)$, for any $k \in \mathbf{Z}_{\geq 0}$. Therefore, for $F_s \in I(s)$,

$$\overline{\mathbf{Z}}(\overline{\nabla}(\partial_X)^{2k} F_s, t) = b_k(t) \overline{\mathbf{Z}}(F_s, t-2k) \tag{4.3}$$

holds as convergent integrals when $2k < \mathrm{Re}\,(t) < \mathrm{Re}\,(s) - d(n-1)$ by Lemma 4.2. Since the left hand side is analytic on $-(e+1) < \mathrm{Re}\,(t) < \mathrm{Re}\,(s) - d(n-1)$, $\overline{\mathbf{Z}}(F_s, t)$ continues to a meromorphic function on $-(e+1) - 2k < \mathrm{Re}\,(t) < \mathrm{Re}\,(s) - d(n-1) - 2k$ for all $k \in \mathbf{Z}_{\geq 0}$. $\square$

5. Proof of the main results. In this section, we prove the following theorems.

Theorem 5.1. *Let $F_s \in I(s)$. Then the family of integrals $\overline{\mathbf{Z}}(F_s, t)$ is complex analytic on $-(e+1) < \mathrm{Re}\,(t) < \mathrm{Re}\,(s) - d(n-1)$ and has a meromorphic continuation to all of $\mathbf{C}^2$.*

Theorem 5.2. *Let $F_s \in I(s)$. Then the functional equation:*

$$\frac{\pi^{nt/2}}{\Gamma_n(t)}\overline{\mathbf{Z}}\left(F_s, t - \frac{m}{n}\right) = \frac{\pi^{n(-t+m/n)/2}}{\Gamma_n(-t+m/n)}\mathbf{Z}(\widehat{F_s}, -t) \tag{5.1}$$

holds as meromorphic functions in $(s,t) \in \mathbf{C}^2$.

We will use the integral formula Proposition 2.4 for the polar coordinates to reduce the integral $\mathbf{Z}(F_s, t)$ to the integrals over a noncompact radial set and a compact set. The main part of the proof of Theorem 5.1 is to have a meromorphic continuation of the integral over the noncompact set. We use the formulas of the action by the Lie subalgebras $\mathfrak{sl}(2, \mathbf{R})_j$. These formulas give us appropriate differential operators that we will use to extend the defining range of s and t. On the other hand, Theorem 5.2 can be proven using [**6,** Lemma 4.34] and Theorem 5.1.

5.1. Proof of Theorem 5.1. $(\mathbf{R}^+)^n$ contains $n!$ disjoint cones of the form,

$$\sigma\Omega = \{\mathbf{x} = (x_1, \ldots, x_n) : x_{\sigma(1)} > \cdots > x_{\sigma(n)} > 0\}.$$

Here σ is in S_n, the symmetric group on n letters. Note also that the disjoint union $\cup_{\sigma \in S_n} \sigma\Omega$ is dense and open in $(\mathbf{R}^+)^n$. Then the integrand of (4.2) is invariant under the action of permuting x_j's except the possible negative sign from the term

$$\prod_{1 \leq i < j \leq n} (x_i^2 - x_j^2)^d.$$

We define the sign function on $(\mathbf{R}^+)^N$ as:

$$\varepsilon(x_1, \cdots, x_N) = \operatorname{sign}\left(\prod_{i<j} (x_i - x_j) \right).$$

Then we can rewrite equation (4.2) as

$$\overline{\mathbf{Z}}(F_s, t) = \frac{c}{n!} \int_{K \cap L} \left[\int_{(\mathbf{R}^+)^n} \frac{(k^{-1} \cdot \varphi)(\kappa(\overline{n}_Y))}{\prod_{j=1}^n (1 + x_j^2)^{(s+m/n)/2}} \prod_{j=1}^n x_j^{t+e} \right.$$
$$\left. \prod_{1 \le i < j \le n} (x_i^2 - x_j^2)^d \, \varepsilon(\mathbf{x})^d \prod_{j=1}^n dx_j \right] dk,$$

where

$$Y = \sum_{j=1}^n x_j F_j.$$

By expanding $(x_i^2 - x_j^2)^d$, the integral $\overline{\mathbf{Z}}(F_s, t)$ can be viewed as a finite linear combination of integrals of the form:

(5.2)
$$\int_{K \cap L} \left[\int_{(\mathbf{R}^+)^n} \frac{(k^{-1} \cdot \varphi)(\kappa(\overline{n}_Y))}{\prod_{j=1}^n (1 + x_j^2)^{(s+m/n)/2}} \prod_{j=1}^n x_j^{t+c_j} \varepsilon(\mathbf{x})^d \prod_{j=1}^n dx_j \right] dk,$$

where $e \le c_j \le e + 2d(n-1)$. As the first step toward proving Theorem 5.1, we assume that $k = I$ in equation (5.2) and consider the integral over the noncompact set $(\mathbf{R}^+)^n$. Note that the meromorphic continuation of the integrals of the form (5.2) implies the meromorphic continuation of the integral $\overline{\mathbf{Z}}(F_s, t)$. To make an induction argument work in the following discussion, we need to consider the more general integral given in equation (5.3), which makes the power in the integrand arbitrary on x_j's and $\sqrt{1 + x_j^2}$. Then, the meromorphic continuation of equation (5.2) can be considered as a special case.

We introduce some notation. Let:

$$\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{N}^n, \quad \mathbf{b} = (b_1, \ldots, b_n), \quad \mathbf{c} = (c_1, \ldots, c_n) \in (\mathbf{R}^+)^n.$$

Let N be a positive integer with $N \leqslant n$. Let:

$$\mathbf{p} = (p_1, \ldots, p_N) \in \mathbf{Z}^N \quad \text{with } 1 \leqslant p_1 < \cdots < p_N \leqslant n.$$

$$\mathbf{q} = (q_1, \dots, q_n) \in \mathbf{Z}^n \quad \text{with } 1 \leqslant q_j \leqslant n \text{ for all } j = 1, \dots, n.$$

For $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbf{R}^+)^n$, we define $\mathbf{x_p} = (x_{p_1}, \dots, x_{p_N}) \in (\mathbf{R}^+)^N$ and $[\mathbf{x_q}] = \kappa(\overline{n}_Y) \in K$, for $Y = \sum_{j=1}^n x_{q_j} F_j$.

Proposition 5.3. *Let $\varphi \in C^\infty(K/K \cap M)$. Then the integral:*

$$\begin{aligned}(5.3)\quad & \mathbf{T}_n\left(s\mathbf{a} + \mathbf{b}, t\mathbf{a} + \mathbf{c}, \mathbf{p}, \varphi[\mathbf{x_q}]\right) \\ &:= \int_{(\mathbf{R}^+)^n} \prod_{j=1}^n \frac{x_j^{a_j t + c_j}}{(1 + x_j^2)^{(a_j s + b_j + m/n)/2}} \cdot \varepsilon(\mathbf{x_p})^d \, \varphi[\mathbf{x_q}] \, dx_n \cdots dx_1,\end{aligned}$$

converges absolutely and is complex analytic on

$$\begin{aligned}(5.4)\quad \mathcal{D}_0 = \Bigg\{ (s,t) \in \mathbf{C}^2 : \max_j \left\{ \frac{-1 - c_j}{a_j} \right\} &< \mathrm{Re}\,(t) \\ &< \mathrm{Re}\,(s) + \min_j \left\{ \frac{b_j + m/n - 1 - c_j}{a_j} \right\} \Bigg\}.\end{aligned}$$

Moreover, it has a meromorphic continuation to all of $(s,t) \in \mathbf{C}^2$.

The proof of the convergence range $\mathcal{D}_0$ is analogous to the proof of Lemma 4.2. The main ingredient for proving Proposition 5.3 is to apply some differential operators from the Lie algebra action. For $j = 1, \dots, n$, we define differential operators $D^j_{s,t}$ by the action of the Lie algebra element:

$$\begin{pmatrix} 0 & t \\ s - t + n - 2 & 0 \end{pmatrix}_j = tE_j + (s - t + n - 2)F_j.$$

Then we have

$$D^j_{s,t} = \left(s + \frac{m}{n}\right) t\, x_j + t\left(1 + x_j^2\right) \frac{\partial}{\partial x_j} - \left(s + \frac{m}{n} - 2\right) \frac{\partial}{\partial x_j},$$

by Lemma 3.2. Let

$${D^j_{s,t}}^\dagger = \left(s + \frac{m}{n}\right) t\, x_j - t\frac{\partial}{\partial x_j} \circ \left(1 + x_j^2\right) + \left(s + \frac{m}{n} - 2\right) \frac{\partial}{\partial x_j}.$$

Then ${D^j_{s,t}}^\dagger$ is the formal adjoint of $D^j_{s,t}$, and we have

$$(5.5)\qquad {D^j_{s,t}}^\dagger\, x_j^t = \left(s + \frac{m}{n} - t - 2\right) t\, x_j^{t-1} \left(1 + x_j^2\right).$$

We shall denote the constant $(s + (m/n) - t - 2)t$ by $(s : t)$. For a function f on an open interval (a, b), we set

$$[f(x)]_{x\to a^+}^{x\to b^-} := \lim_{x\to b^-} f(x) - \lim_{x\to a^+} f(x)$$

and

$$f(x)|_{x\to b^-} := \lim_{x\to b^-} f(x),$$

if the limits exist. By integration of parts, it is easily proven that

$$(5.6)\quad \int_a^b f(x_l) \left(D^l_{s,t}\, g(x_l)\right) dx_l = \int_a^b \left({D^l_{s,t}}^\dagger f(x_l)\right) g(x_l)\, dx_l + \left[\left\{t\left(1 + x_l^2\right) - \left(s + \frac{m}{n} - 2\right)\right\} f(x_l) g(x_l)\right]_{x_l\to a^+}^{x_l\to b^-},$$

for all $(s, t) \in \mathbf{C}^2$ at which each term is defined. Here, f and g are continuous on $[a, b]$ and differentiable on (a, b). We define $E^l(s, t, \mathbf{p}, \varphi[\mathbf{x_q}])$ as 0 if d is even or d is odd and $l \neq p_i$ for all $i = 1, \dots, N$. If d is odd and $l = p_i$ for some $i = 1, \dots, N$, then

$$E^l(s, t, \mathbf{p}, \varphi[\mathbf{x_q}]) = \sum_{\substack{i=1 \\ (p_i\neq l)}}^{N} \frac{2 \cdot \{t(1 + x_{p_i}^2) - (s + m/n - 2)\} x_{p_i}^t}{(1 + x_{p_i}^2)^{(s+m/n)/2}} \left(\varepsilon(\mathbf{x_p})\varphi[\mathbf{x_q}]\right)\Big|_{x_l\to x_{p_i}^-}.$$

Note that E^l does not contain the variable x_l.

Lemma 5.4. *For $(s, t) \in \mathbf{C}^2$ with $0 < \mathrm{Re}\,(t) < \mathrm{Re}\,(s) + (m/n) - 2$,*

$$\int_0^\infty x_l^t\, \varepsilon(\mathbf{x_p})^d D^l_{s,t}\left(\frac{\varphi[\mathbf{x_q}]}{(1 + x_l^2)^{(s+m/n)/2}}\right) dx_l = (s : t) \int_0^\infty x_l^{t-1}\, \varepsilon(\mathbf{x_p})^d \left(\frac{\varphi[\mathbf{x_q}]}{(1 + x_l^2)^{(s-2+m/n)/2}}\right) dx_l + E^l(s, t, \mathbf{p}, \varphi[\mathbf{x_q}]).$$

Remark 5.5. Note that the integrand contains the possibly discontinuous function $\varepsilon(\mathbf{x_p})^d$. This means that the integral must be broken into some of the integrals with end points of the discontinuities.

Proof. We treat the following two cases.

Case 1. d is even or $l \neq p_i$ for all $i = 1, \ldots, N$. Then $\varepsilon(\mathbf{x_p})^d$ is continuous with respect to the variable x_l. We apply the integration by parts equation (5.6), along with equation (5.5) on $(0, \infty)$. We note that

$$\left[\left\{t\left(1+x_l^2\right) - \left(s+\frac{m}{n}-2\right)\right\} x_l^t\, \varepsilon(\mathbf{x_p})^d \left(\frac{\varphi[\mathbf{x_q}]}{(1+x_l^2)^{(s+m/n)/2}}\right)\right]_{x_l \to 0^+}^{x_l \to \infty} = 0, \tag{5.7}$$

on $0 < \mathrm{Re}\,(t) < \mathrm{Re}\,(s) + m/n - 2$. This proves Case 1. $\square$

Case 2. d is odd and $l = p_i$ for some $i = 1, \ldots, N$. Then $\varepsilon(\mathbf{x_p})^d$ is not continuous. Without loss of generality, we may assume that $0 < x_1 < \cdots < \widehat{x_l} < \cdots < x_n$ in the following computations. Then, we consider the partition of the interval $(0, \infty)$ defined by x_{p_i}'s. Those $N-1$ points can be used to divide $(0, \infty)$ into N non-overlapping open subintervals:

$$(x_{p_0}, x_{p_1}), (x_{p_1}, x_{p_2}), \ldots, (x_{p_{i-1}}, x_{p_{i+1}}), \ldots, (x_{p_N}, x_{p_{N+1}}),$$

where $x_{p_0} = 0$ and $x_{p_{N+1}} = \infty$. Therefore, we can write

$$\int_0^\infty x_l^t\, \varepsilon(\mathbf{x_p})^d D_{s,t}^l \left(\frac{\varphi[\mathbf{x_q}]}{(1+x_l^2)^{(s+m/n)/2}}\right) dx_l = \sum_{\substack{k=0\\(k\neq i)}}^{N} \int_{x_{p_k}}^{x_{p_{k+1}}} x_l^t\, \varepsilon(\mathbf{x_p})^d D_{s,t}^l \left(\frac{\varphi[\mathbf{x_q}]}{(1+x_l^2)^{(s+m/n)/2}}\right) dx_l.$$

Since $\varepsilon(\mathbf{x_p})^d$ is a constant (± 1) on each subinterval, by equations (5.6) and (5.5), we have

$$\int_{x_{p_k}}^{x_{p_{k+1}}} x_l^t\, \varepsilon(\mathbf{x_p})^d D_{s,t}^l \left(\frac{\varphi[\mathbf{x_q}]}{(1+x_l^2)^{(s+m/n)/2}}\right) dx_l$$

$$= (s:t)\int_{x_{p_k}}^{x_{p_{k+1}}} x_l^{t-1}\,\varepsilon(\mathbf{x_p})^d\left(\frac{\varphi[\mathbf{x_q}]}{(1+x_l^2)^{(s-2+m/n)/2}}\right)dx_l$$

$$+\left[\left\{t(1+x_l^2)-(s+n-2)\right\}x_l^t\,\varepsilon(\mathbf{x_p})^d\left(\frac{\varphi[\mathbf{x_q}]}{(1+x_l^2)^{(s+m/n)/2}}\right)\right]_{x_l\to x_{p_k}^+}^{x_l\to x_{p_{k+1}}^-}.$$

Case 2 follows from equations (5.7) and the fact that

$$\varepsilon(\mathbf{x_p})|_{x_l\to x_{p_k}^-} = -\varepsilon(\mathbf{x_p})|_{x_l\to x_{p_k}^+}. \qquad \square$$

The equation in Lemma 5.4 can be used to compute an integral whose integrand contains differential operators of the form $D_{s,t}^l$. Let a be a positive integer. We define:

$$\widetilde{D}_{s,t}^l = D_{s,t}^l \circ \frac{1}{(1+x_l^2)}.$$

Define $\psi_a[\mathbf{x}] = \varphi[\mathbf{x_q}]$ and $\psi_{a-1},\ldots,\psi_0$ inductively by the equation:

$$\frac{\psi_k[\mathbf{x}]}{(1+x_l^2)^{(s-2k+m/n)/2}} = D_{s-2k,t-k}^l\frac{\psi_{k+1}[\mathbf{x}]}{(1+x_l^2)^{(s-2k+m/n)/2}}$$

for $k=0,1,\ldots,a-1$. Then we have

$$\frac{\psi_0[\mathbf{x}]}{(1+x_l^2)^{(s+m/n)/2}} = \left(\prod_{k=0}^{a-1}\widetilde{D}_{s-2k,t-k}^l\right)\frac{\varphi[\mathbf{x_q}]}{(1+x_l^2)^{(s-2a+m/n)/2}}.$$

Since $D_{s-2k,t-k}$ acts as an element in the Lie algebra on $I(s-2k)$, it preserves the representation space $I(s-2k)$. Therefore, the condition that $\varphi\in C^\infty(K/K\cap M)$ implies $\psi_k\in C^\infty(K/K\cap M)$ for all $k=0,\ldots,a-1$. We let $\gamma_0(s:t)=1$. Define

$$\gamma_k(s:t) = \prod_{r=0}^{k-1}(s-2r:t-r)\quad\text{for } k=1,\ldots,a-1$$

and

$$\mathbf{E}^{l,a}(s,t,\mathbf{p},\varphi[\mathbf{x_q}]) = \sum_{k=0}^{a-1}\gamma_k(s:t)E^l(s-2k,t-k,\mathbf{p},\psi_{k+1}[\mathbf{x}]).$$

Corollary 5.6. *For $(s,t) \in \mathbf{C}^2$ with $a < \mathrm{Re}\,(t) < \mathrm{Re}\,(s) - a + (m/n) - 2$,*

$$\int_0^\infty x_l^t\,\varepsilon(\mathbf{x_p})^d \frac{\psi_0[\mathbf{x}]}{(1+x_l^2)^{(s+m/n)/2}}\,dx_l = \gamma_a(s:t)\int_0^\infty x_l^{t-a}\,\varepsilon(\mathbf{x_p})^d \frac{\varphi[\mathbf{x_q}]}{(1+x_l^2)^{(s-2a+m/n)/2}}\,dx_l + \mathbf{E}^{l,a}(s,t,\mathbf{p},\varphi[\mathbf{x_q}]).$$

Proof. We show the following equation by induction on k (for $k = 0,\dots,a$):

$$\int_0^\infty x_l^t\,\varepsilon(\mathbf{x_p})^d \frac{\psi_0[\mathbf{x}]}{(1+x_l^2)^{(s+m/n)/2}}\,dx_l = \gamma_k(s:t)\int_0^\infty x_l^{t-k}\,\varepsilon(\mathbf{x_p})^d \frac{\psi_k[\mathbf{x}]}{(1+x_l^2)^{(s-2k+m/n)/2}}\,dx_l + \sum_{r=0}^{k-1}\gamma_r(s:t)E^l(s-2r,t-r,\mathbf{p},\psi_{r+1}[\mathbf{x}]). \tag{5.8}$$

Then the corollary follows when $k = a$.

If $k = 0$, then it is trivial. We suppose equation (5.8) is true (for k). Then, by the definition of ψ_k and Lemma 5.4, we have

$$\begin{aligned}\int_0^\infty x_l^{t-k}\,\varepsilon(\mathbf{x_p})^d \frac{\psi_k[\mathbf{x}]}{(1+x_l^2)^{(s-2k+m/n)/2}}\,dx_l &= \int_0^\infty x_l^{t-k}\,\varepsilon(\mathbf{x_p})^d\,\widetilde{D}^l_{s-2k,t-k}\frac{\psi_{k+1}[\mathbf{x}]}{(1+x_l^2)^{(s-2(k+1)+m/n)/2}}\,dx_l\\ &= (s-2k:t-k)\int_0^\infty x_l^{t-(k+1)}\,\varepsilon(\mathbf{x_p})^d \frac{\psi_{k+1}[\mathbf{x}]}{(1+x_l^2)^{(s-2(k+1)+m/n)/2}}\,dx_l\\ &\quad + E^l(s-2k,t-k,\mathbf{p},\psi_{k+1}[\mathbf{x}]).\end{aligned}$$

We substitute the above computations into (5.8) to obtain the induction statement for $k+1$.

By equation (5.7), the equation (5.8) is valid on

$$0 < \mathrm{Re}\,(t) - k < s - 2k + \frac{m}{n} - 2, \quad \text{for } k = 0,\dots,a.$$

This gives the explicit convergence range $a < \mathrm{Re}\,(t) < \mathrm{Re}\,(s) - a + m/n - 2$. □

Corollary 5.6 is used to compute an integral T_n over $(\mathbf{R}^+)^n$ whose integrand contains differential operators of the form $D^l_{s,t}$ for $l = 1, 2, \ldots, n$. Recall that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, N, $\mathbf{p}$ and $\varphi[\mathbf{x_q}]$ are as in Proposition 5.3. Let $\mathbf{s} = s\mathbf{a} + \mathbf{b} = (s_1, \ldots, s_n)$ and $\mathbf{t} = t\mathbf{a} + \mathbf{c} = (t_1, \ldots, t_n)$. Define $\eta_n[\mathbf{x}] = \varphi[\mathbf{x_q}]$ and $\eta_{n-1}, \ldots, \eta_0$ by the equations, inductively,

$$\frac{\eta_{k-1}[\mathbf{x}]}{(1+x_k^2)^{(s_k+m/n)/2}} = \left(\prod_{r=0}^{a_k-1} \widetilde{D}^k_{s_k-2r,t_k-r}\right) \frac{\eta_k[\mathbf{x}]}{(1+x_k^2)^{(s_k-2a_k+m/n)/2}}$$

for $n \geqslant k \geqslant 1$. Then, we have

$$\frac{\eta_0[\mathbf{x}]}{\prod_{k=1}^n (1+x_k^2)^{(s_k+m/n)/2}} = \prod_{k=1}^n \left(\prod_{r=0}^{a_k-1} \widetilde{D}^k_{s_k-2r,t_k-r}\right) \frac{\varphi[\mathbf{x_q}]}{\prod_{k=1}^n (1+x_k^2)^{(s_k-2a_k+m/n)/2}}.$$

Since $\varphi \in C^\infty(K/K\cap M)$, $\eta_k \in C^\infty(K/K\cap M)$ for all $k = 0, \ldots, n-1$. We let $\gamma_0(\mathbf{s}:\mathbf{t}) = 1$. Define

$$\gamma_k(\mathbf{s}:\mathbf{t}) = \prod_{r=1}^k \gamma_{a_r}(s_r : t_r) \quad \text{for } k = 1, \ldots, n-1$$

and

$$\begin{aligned}\mathbf{E}_{n-1}&(\mathbf{s}, \mathbf{t}, \mathbf{p}, \varphi[\mathbf{x_q}]) \\ &= \sum_{k=0}^{n-1} \gamma_k(\mathbf{s}:\mathbf{t}) \int_{(\mathbf{R}^+)^{n-1}} \prod_{j<k} \frac{x_j^{t_j-a_j}}{(1+x_j^2)^{(s_j-2a_j+m/n)/2}} \\ &\qquad \prod_{j>k} \frac{x_j^{t_j}}{(1+x_j^2)^{(s_j+m/n)/2}} \mathbf{E}^{k,a_k}(s_k, t_k, \mathbf{p}, \eta_k[\mathbf{x}]) \prod_{j\neq k} dx_j.\end{aligned}$$

Corollary 5.7. *For sufficiently large α and small β, we have*

$$\begin{aligned}(5.9)\quad \mathbf{T}_n&(\mathbf{s}, \mathbf{t}, \mathbf{p}, \eta_0[\mathbf{x}]) \\ &= \gamma_n(\mathbf{s}:\mathbf{t})\mathbf{T}_n(\mathbf{s}-2\mathbf{a}, \mathbf{t}-\mathbf{a}, \mathbf{p}, \varphi[\mathbf{x_q}]) + \mathbf{E}_{n-1}(\mathbf{s}, \mathbf{t}, \mathbf{p}, \varphi[\mathbf{x_q}]),\end{aligned}$$

on $\alpha < \mathrm{Re}(t) < \mathrm{Re}(s) + \beta$. In particular, if d is even, then we have

$$\mathbf{T}_n(\mathbf{s}, \mathbf{t}, \mathbf{p}, \eta_0[\mathbf{x}]) = \gamma_n(\mathbf{s}:\mathbf{t})\mathbf{T}_n(\mathbf{s}-2\mathbf{a}, \mathbf{t}-\mathbf{a}, \mathbf{p}, \varphi[\mathbf{x_q}]).$$

Proof. We use induction on k (for $k = 0, \dots, n$) to obtain the following statement:

(5.10)

$$\mathbf{T}_n(\mathbf{s},\mathbf{t};\mathbf{p};\eta_0(\mathbf{x})) = \gamma_k(\mathbf{s}:\mathbf{t}) \int_{(\mathbf{R}^+)^n} \prod_{j\leqslant k} \frac{x_j^{t_j-a_j}}{(1+x_j^2)^{(s_j-2a_j+m/n)/2}} \prod_{j>k} \frac{x_j^{t_j}}{(1+x_j^2)^{(s_j+m/n)/2}} \cdot \varepsilon(\mathbf{x_p})^d \cdot \eta_k(\mathbf{x}) \prod_{j=1}^n dx_j + \sum_{r=0}^{k-1} \gamma_r(\mathbf{s}:\mathbf{t}) \int_{(\mathbf{R}^+)^{n-1}} \prod_{j<r} \frac{x_j^{t_j-a_j}}{(1+x_j^2)^{(s_j-2a_j+m/n)/2}} \prod_{j>r} \frac{x_j^{t_j}}{(1+x_j^2)^{(s_j+m/n)/2}} \cdot \mathbf{E}^{r,a_r}(s_r,t_r,\mathbf{p},\eta_r[\mathbf{x}]) \prod_{j\neq r} dx_j.$$

Then the lemma follows when $k = n$.

If $k = 0$, then it is trivial. We suppose the above statement is true (for k). By the Fubini theorem and Corollary 5.6, we have the following equalities on the convergence range of the integrals.

$$\int_{(\mathbf{R}^+)^n} \prod_{j\leqslant k} \frac{x_j^{t_j-a_j}}{(1+x_j^2)^{(s_j-2a_j+m/n)/2}} \prod_{j>k} \frac{x_j^{t_j}}{(1+x_j^2)^{(s_j+m/n)/2}} \cdot \varepsilon(\mathbf{x_p})^d \cdot \eta_k[\mathbf{x}] \prod_{j=1}^n dx_j$$

$$= \int_{(\mathbf{R}^+)^{n-1}} \prod_{j\leqslant k} \frac{x_j^{t_j-a_j}}{(1+x_j^2)^{(s_j-2a_j+m/n)/2}} \prod_{j>k+1} \frac{x_j^{t_j}}{(1+x_j^2)^{(s_j+m/n)/2}} \cdot \left(\int_0^\infty x_{k+1}^{t_{k+1}} \cdot \frac{\varepsilon(\mathbf{x_p})^d \cdot \eta_k[\mathbf{x}]}{(1+x_{k+1}^2)^{(s_{k+1}+m/n)/2}} \, dx_{k+1} \right) \prod_{j\neq k+1} dx_j$$

$$= \int_{(\mathbf{R}^+)^{n-1}} \prod_{j\leqslant k} \frac{x_j^{t_j-a_j}}{(1+x_j^2)^{(s_j-2a_j+m/n)/2}} \prod_{j>k+1} \frac{x_j^{t_j}}{(1+x_j^2)^{(s_j+m/n)/2}} \cdot \left(\int_0^\infty x_{k+1}^{t_{k+1}} \prod_{r=0}^{a_{k+1}-1} \widetilde{D}^{k+1}_{s_{k+1}-2r,t_{k+1}-r} \frac{\varepsilon(\mathbf{x_p})^d \cdot \eta_{k+1}[\mathbf{x}]}{(1+x_{k+1}^2)^{(s_{k+1}-2a_{k+1}+m/n)/2}} \, dx_{k+1} \right)$$

$$\prod_{j\neq k+1} dx_j$$

$$= \gamma_{a_{k+1}}(s_{k+1} : t_{k+1}) \int_{(\mathbf{R}^+)^n} \prod_{j\leqslant k+1} \frac{x_j^{t_j-a_j}}{(1+x_j^2)^{(s_j-2a_j+m/n)/2}}$$

$$\prod_{j>k+1} \frac{x_j^{t_j}}{(1+x_j^2)^{(s_j+m/n)/2}} \cdot \varepsilon(\mathbf{x_p})^d \cdot \eta_{k+1}[\mathbf{x}] \prod_{j=1}^{n} dx_j$$

$$+ \int_{(\mathbf{R}^+)^{n-1}} \prod_{j<k+1} \frac{x_j^{t_j-a_j}}{(1+x_j^2)^{(s_j-2a_j+m/n)/2}}$$

$$\prod_{j>k+1} \frac{x_j^{t_j}}{(1+x_j^2)^{(s_j+m/n)/2}} \cdot \mathbf{E}^{k+1,a_{k+1}}(s_{k+1}, t_{k+1}, \mathbf{p}, \eta_{k+1}[\mathbf{x}]) \prod_{j\neq k+1} dx_j.$$

We substitute the above computation for equation (5.10) to obtain the induction statement for $k+1$. □

Corollary 5.7 is used to prove Proposition 5.3.

Proof of Proposition 5.3. The analyticity of the integral $\mathbf{T}_n$ is a standard application of Morera's theorem as follows. The continuity follows from the Lebesgue dominated convergence theorem. For any simple closed curve $C \in \mathcal{D}_0$ in $t \in \mathbf{C}$ (respectively, $s \in \mathbf{C}$) for fixed s (respectively, t), the integral over C of the integrand of $\mathbf{T}_n$ is 0 by Cauchy's integral formula. By Fubini's theorem, we show the integral over C of the integral $\mathbf{T}_n$ is 0.

We use the induction on n to prove the meromorphic continuation part. For $n = 1$,

$$\mathbf{E}_{n-1}(\mathbf{s}, \mathbf{t}, \mathbf{p}, \varphi[\mathbf{x_q}]) = 0$$

because $\varepsilon(x) = 1$. This case can be proven by equation (5.9). We suppose that $\mathbf{T}_{n-1}(\mathbf{s}, \mathbf{t}, \mathbf{p}, \varphi[\mathbf{x_q}])$ has a meromorphic continuation to all of (s,t) in $\mathbf{C}^2$ for all choices of $\mathbf{s}$, $\mathbf{t}$, $\mathbf{p}$, $\mathbf{q}$, and $\varphi \in C^\infty(K/K\cap M)$. Then, $\mathbf{E}_{n-1}(\mathbf{s}, \mathbf{t}, \mathbf{p}, \varphi[\mathbf{x_q}])$ can be extended to a meromorphic function in $(s,t) \in \mathbf{C}^2$ because it is a finite sum of integrals of the form $\mathbf{T}_{n-1}$. We rewrite equation (5.9) in the following form:

$$\begin{aligned}(5.11)\quad &\mathbf{T}_n\left(\mathbf{s},\mathbf{t},\mathbf{p},\varphi[\mathbf{x_q}]\right)\\ &= \frac{\mathbf{T}_n\left(\mathbf{s}+2\mathbf{a},\mathbf{t}+\mathbf{a},\mathbf{p},\eta_0[\mathbf{x}]\right)-\mathbf{E}_{n-1}(\mathbf{s}+2\mathbf{a},\mathbf{t}+\mathbf{a},\mathbf{p},\varphi[\mathbf{x_q}])}{\gamma_n(\mathbf{s}+2\mathbf{a}:\mathbf{t}+\mathbf{a})}.\end{aligned}$$

Then the right hand side of equation (5.11) can be defined as

$$\begin{aligned}(5.12)\quad \mathcal{D}_1 = \Bigg\{(s,t)\in\mathbf{C}^2 : &\max_j\left\{\frac{-1-c_j}{a_j}\right\}-1<\mathrm{Re}\,(t)\\ &<\mathrm{Re}\,(s)+\min_j\left\{\frac{b_j+m/n-1-c_j}{a_j}\right\}+1\Bigg\},\end{aligned}$$

which contains $\mathcal{D}_0$. Therefore, the left hand side of equation (5.11) can be extended to $\mathcal{D}_1$ as a meromorphic function in (s,t). We apply equation (5.11) repeatedly to extend the defining range of $\mathbf{T}_n$ to all of $(s,t)\in\mathbf{C}^2$ meromorphically. □

Proof of Theorem 5.1. We can regard $\overline{\mathbf{Z}}(F_s,t)$ as a finite linear combination of integrals of the form (5.2), which is

$$(5.13)\qquad \int_{K\cap L}\mathbf{T}_n\left(s\mathbf{1},t\mathbf{1}+\mathbf{c},\mathbf{x},(k^{-1}\cdot\varphi)[\mathbf{x}]\right)dk.$$

Here, $K\cap L$ is compact. Therefore, we may apply an analogous argument with the analytic part of the proof of Proposition 5.3. Furthermore, the meromorphic part follows from the fact that the integrand of equation (5.13) has a meromorphic continuation to all of (s,t) in $\mathbf{C}^2$. □

5.2. Proof of Theorem 5.2. We begin with the definition of $\nu(M,N)(F)$ on $C^N(\overline{\mathfrak{n}})$:

$$\nu(M,N)(F)=\sup_{X\in\overline{\mathfrak{n}}}\left\{(1+\|X\|^2)^M\cdot\sum_{\alpha,|\alpha|\leqslant N}|\partial^\alpha F(X)|\right\}.$$

It is known that the functional equation (4.1) still holds for functions satisfying certain decay condition. Precisely, we have the following proposition.

Proposition 5.8 ([**6**, Lemma 4.34]). *Suppose $F \in C^\infty(\overline{\mathfrak{n}})$ satisfies $\nu(M_0+1, M_0)(F) < \infty$ for sufficiently large M_0. Then the functional equation* (4.1) *holds as meromorphic functions in t*:

$$\frac{\pi^{nt/2}}{\Gamma_n(t)}\overline{\mathbf{Z}}\left(F, t-\frac{m}{n}\right) = \frac{\pi^{n(-t+m/n)/2}}{\Gamma_n(-t+m/n)}\mathbf{Z}(\widehat{F}, -t).$$

Note also that functions in $I(s)$ have a decay condition for sufficiently large $\operatorname{Re}(s)$ by Lemma 3.1.

Lemma 5.9. *Let $\operatorname{Re}(s) \geqslant 2M - m/n$. If $F_s \in I(s)$, then $\nu(M,N)(F_s) < \infty$ for all $N \geqslant 0$.*

Proof. For $X \in \overline{\mathfrak{n}}$, we write

$$X = m \cdot \sum_{j=1}^{n} x_j F_j, \quad m \in M.$$

Then we have

$$\begin{aligned}
\|X\|^2 &= \left\langle \sum_{j=1}^{n} x_j F_j, \sum_{j=1}^{n} x_j F_j \right\rangle \\
&= \frac{n}{4m} \operatorname{Tr}\left(\operatorname{ad}\left(\sum_{j=1}^{n} x_j F_j\right) \operatorname{ad}\left(\sum_{j=1}^{n} x_j E_j\right)\right) \\
&= \frac{n}{4m} \cdot 4(d(n-1)+(e+1)) \sum_{j=1}^{n} x_j^2 \\
&= \sum_{j=1}^{n} x_j^2.
\end{aligned}$$

On the other hand, since each ∂^α, $|\alpha| = 1$, acts as an element in the enveloping algebra $\mathcal{U}(\mathfrak{g})$, so $\partial^\alpha(F_s) \in I(s)$ for any multi-index α. Then $\partial^\alpha(F_s)(X)$ is of the form

$$\frac{\varphi(\kappa(\overline{n}_Y))}{\prod_{j=1}^{n}(1+x_j^2)^{(s+m/n)/2}}$$

for some $\varphi \in C^\infty(K/M)$. Here, $Y = \sum_{j=1}^{n} x_j F_j$.

Therefore, for any nonnegative integer N, we have

$$\nu(M,N)(F_s) = \sup_X \left\{ \left(1+\|X\|^2\right)^M \cdot \sum_{\alpha,|\alpha|\leqslant N} |\partial^\alpha F_s| \right\}$$
$$\leqslant c \sup_{(\mathbf{R}^+)^n} \left\{ \frac{\left(1+\sum_{j=1}^n x_j^2\right)^M}{\prod_{j=1}^n (1+x_j^2)^{(\mathrm{Re}\,(s)+m/n)/2}} \right\},$$

which is finite because $M \leqslant (\mathrm{Re}\,(s)+m/n)/2$. □

Corollary 5.10. *For sufficiently large* $\mathrm{Re}\,(s)$, *the functional equation*

$$\frac{\pi^{nt/2}}{\Gamma_n(t)} \overline{\mathbf{Z}}\left(F_s, t-\frac{m}{n}\right) = \frac{\pi^{n(-t+m/n)/2}}{\Gamma_n(-t+m/n)} \mathbf{Z}(\widehat{F_s}, -t). \tag{5.14}$$

holds for functions F_s *in* $I(s)$ *as meromorphic functions of* $t \in \mathbf{C}$.

We combine Theorem 5.1 and Corollary 5.10 to conclude Theorem 5.2.

6. Some comments on representation theory. The standard intertwining map

$$\widetilde{B}_s : \mathrm{Ind}_P^G(s) \longrightarrow \mathrm{Ind}_P^G(-s)$$

is defined by:

$$\widetilde{B}_s(f)(X) = \int_{\overline{\mathfrak{n}}} f(\overline{n}_X w \overline{n}_{X_1})\, dX_1$$

for those values of s for which the integral converges. See, for example, [**7,** page 174]. We may define $\widetilde{A}_s$ on $I(s)$ by

$$\widetilde{A}_s(F_s)(X) = \overline{\mathbf{Z}}\left(\tau_X F_s, s-\frac{m}{n}\right)$$
$$= \int_{\overline{\mathfrak{n}}} F_s(X_1)\overline{\nabla}(X_1-X)^{s-m/n}\, dX_1,$$

where $\tau_X F_s = F_s(\cdot + X)$. Then the integral $\widetilde{A}_s(F_s)(X)$ converges absolutely for $\mathrm{Re}\,(s) > d(n-1)$ by Lemma 4.2. The next lemma says that it equals $\widetilde{B}_s(f)(X)$.

Lemma 6.1. *Let $F_s \in I(s)$ be the function corresponding to $f \in \mathrm{Ind}_P^G(s)$. Then*

$$\widetilde{A}_s(F_s) = \widetilde{B}_s(f) \quad \text{for } \mathrm{Re}\,(s) > d(n-1).$$

A short calculation along with [**7,** pages 183, 200] proves Lemma 6.1. Therefore, $\widetilde{A}_s$ is a G-intertwining operator from $I(s)$ to $I(-s)$ for $\mathrm{Re}\,(s) > d(n-1)$. By Lemma 6.1, Proposition 6.2 is a slight variation of a special case of the main result in [**15**]. The proof here is different than that in [**15**].

Proposition 6.2. *For $f \in \mathrm{Ind}_P^G(s)$, let F_s be the corresponding element of $I(s)$. Then $\widetilde{A}_s(F_s)$ is complex analytic in s for $\mathrm{Re}\,(s) > d(n-1)$ and has meromorphic continuation to all of $\mathbf{C}$.*

Proof. The meromorphic continuation part follows by setting $t = s - (m/n)$ in Theorem 5.1. □

For $s \in \mathbf{R}$, the induced picture $\mathrm{Ind}_P^G(s)$ has a 'standard' Hermitian form. See [**7,** Proposition 14.23], for example. This is given by

$$\langle f_1, f_2 \rangle = \int_{\overline{\mathfrak{n}}} f_1(\overline{n}_Y) \overline{\frac{\pi^{ns/2}}{\Gamma_n(s)} \widetilde{B}_s(f_2)(\overline{n}_Y)}\, dY.$$

By Lemma 6.1, this becomes a 'standard' Hermitian form on $I(s)$ defined by

$$\langle F_s, G_s \rangle = \int_{\overline{\mathfrak{n}}} F_s(Y) \overline{\frac{\pi^{ns/2}}{\Gamma_n(s)} \widetilde{A}_s(G_s)(Y)}\, dY, \tag{6.1}$$

as convergent integrals for $\mathrm{Re}\,(s) > d(n-1)$.

A formal argument says that we have the following string of equalities:

$$\begin{aligned} \langle F_s, F_s \rangle &= \int_{\overline{\mathfrak{n}}} F_s(Y) \overline{\frac{\pi^{ns/2}}{\Gamma_n(s)} \widetilde{A}_s(F_s)(Y)}\, dY \\ &= \int_{\overline{\mathfrak{n}}} F_s(Y) \overline{\frac{\pi^{ns/2}}{\Gamma_n(s)} \overline{\mathbf{Z}}(\tau_Y F_s, s-(m/n))}\, dY \end{aligned} \tag{6.2}$$

$$= \int_{\bar{\mathfrak{n}}} F_s(Y) \overline{\frac{\pi^{n(-s+m/n)/2}}{\Gamma_n(-s+m/n)} \mathbf{Z}\left(\widehat{\tau_y F_s}, -s\right)} \, dY$$

$$= \frac{\pi^{n(-s+m/n)/2}}{\Gamma_n(-s+m/n)}$$

$$\cdot \int_{\bar{\mathfrak{n}}} \int_{\mathfrak{n}} F_s(Y) e^{-2\pi i \langle X,Y \rangle} \overline{\widehat{F}_s(X)} \nabla(X)^{-s} \, dX \, dY$$

$$= \frac{\pi^{n(-s+m/n)/2}}{\Gamma_n(-s+m/n)} \int_{\mathfrak{n}} \widehat{F}_s(X) \overline{\widehat{F}_s(x)} \nabla(X)^{-s} \, dX$$

$$= \frac{\pi^{n(-s+m/n)/2}}{\Gamma_n(-s+m/n)} \int_{\mathfrak{n}} |\widehat{F}_s(X)|^2 \nabla(X)^{-s} \, dX.$$

However, the first two equalities hold as convergent integrals for $\mathrm{Re}\,(s) > d(n-1)$. The third equality is functional equation (1.2). Note that, in general, there is no range on which all the above integrals converge.

In the case of $\mathrm{SL}(2, \mathbf{R})$, for $0 < s < 1$, the functional equation (1.2) holds as *convergent integrals*, and all integrals in equation (6.2) converge for all functions in $I(s)$. Therefore, we have

$$\langle F_s, F_s \rangle = \frac{\pi^{(1-s)/2}}{\Gamma((1-s)/2)} \int_{\mathbf{R}} |\widehat{F}_s(x)|^2 |x|^{-s} \, dx \quad \text{for } 0 < s < 1.$$

Therefore, for $\mathrm{SL}(2, \mathbf{R})$, the functional equation (1.2) for functions in $I(s)$ is used to show that $\langle \cdot, \cdot \rangle$ is positive definite. Therefore, for $0 < s < 1$, $\langle \cdot, \cdot \rangle$ is an invariant inner product on $I(s)$ and so, $I(s)$ is unitarizable.

The representations $I(s)$ for our class of groups have been given unitary realizations for certain values of s in **[4, 1]**. Understanding the string of equalities (6.2) may serve as a more direct alternative for studying the unitary structure of these representations.

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APPENDIX

A. Tables. The next two tables give information on the groups under consideration in this paper.

TABLE 1.

	G	$n = \text{rank}(\mathfrak{n})$	$m = \dim(\mathfrak{n})$	d	e
1	$\text{GL}(2n, \mathbf{R}), n \geq 2$	n	n^2	1	0
2	$O(2n, 2n)$	n	$n(2n-1)$	2	0
3	$E_7(7)$	3	27	4	0
4	$O(p, q)$, $p, q \geq 3$	2	$p + q - 2$	$(p + q - 4)/2$	0
5	$\text{Sp}(n, \mathbf{C})$	n	$n(n + 1)$	1	1
6	$\text{SL}(2n, \mathbf{C})$	n	$2n^2$	2	1
7	$\text{SO}(4n, \mathbf{C})$	n	$2n(2n - 1)$	4	1
8	$E_{7,\mathbf{C}}$	3	54	8	1
9	$\text{SO}(p, \mathbf{C})$, $p \geq 3$	2	$2(p - 2)$	$p - 4$	1
10	$\text{Sp}(n, n)$	n	$n(2n + 1)$	2	2
11	$\text{GL}(2n, \mathbf{H})$	n	$4n^2$	4	3
12	$\text{SO}(p, 1)$	1	p	0	$p - 1$

TABLE 2. Jordan algebras for the groups in Table 1.

	$V \simeq \mathfrak{n}$	L	∇
1	$M(n \times n, \mathbf{R})$	$\text{GL}(n, \mathbf{R}) \times \text{GL}(n, \mathbf{R})$	$\lvert \det \rvert$
2	$\text{Skew}(2n : \mathbf{R})$	$\text{GL}(2n, \mathbf{R})$	Pfaffian
3	$\text{Herm}(3, \mathbf{O}_{\text{split}})$	$E_6(6) \times \mathbf{R}^\times$	deg. 3 poly
4	$\mathbf{R}^{p-1,q-1}$	$\mathbf{R}^\times O(p - 1, q - 1)$	(X, X)
5	$\text{Sym}(n, \mathbf{C})$	$\text{GL}(n : \mathbf{C})$	$\lvert \det \rvert$
6	$M(n \times n, \mathbf{C})$	$S(\text{GL}(n, \mathbf{C}) \times \text{GL}(n, \mathbf{C}))$	$\lvert \det \rvert$
7	$\text{Skew}(2n, \mathbf{C})$	$\text{GL}(2n, \mathbf{C})$	|Pfaffian|
8	$\text{Herm}(3, \mathbf{O})_\mathbf{C}$	$E_{6,\mathbf{C}}\mathbf{C}^\times$	|deg. 3 poly|
9	$\mathbf{C}^{p-1}$	$SO(p - 2 : \mathbf{C}) \times \mathbf{C}^\times$	$\lvert (Z, Z) \rvert$
10	$\text{Sym}(2n, \mathbf{C}) \cap M(n \times n, \mathbf{H})$	$\text{GL}(n, \mathbf{H})$	$\lvert \det_\mathbf{C}(Z) \rvert^{1/2}$
11	$M(n \times n, \mathbf{H})$	$\text{GL}(n, \mathbf{H}) \times \text{GL}(n, \mathbf{H})$	$\lvert \det_\mathbf{C}(Z) \rvert^{1/2}$
12	$\mathbf{R}^{p-1}$	$\text{SO}(p - 1) \times \mathbf{R}^\times$	$\lVert \cdot \rVert$

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