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RESOLUTIONS AND STABILITY OF C-GORENSTEIN FLAT MODULES

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ABSTRACT. In this paper, we first investigate the relationship between $\mathcal{W}$-(co)resolutions and $\mathcal{X}$-(co)resolutions for two full subcategories $\mathcal{W}$ and $\mathcal{X}$ of an abelian category with $\mathcal{W} \subseteq \mathcal{X}$. Then some applications are given. In particular, we obtain the stability of the category of C-Gorenstein flat modules under the procedure used to define these entities, which is different from that established by Sather-Wagstaff, Sharif and White.

1. Introduction. For a kind of generalization of Gorenstein projective, injective and flat modules, Holm and Jørgensen introduced [**10**] the notions of C-Gorenstein (G_C- for short) projective, injective and flat modules, where C is a semidualizing module. However, letting $\mathcal{W}$ be a full subcategory of an abelian category, Sather-Wagstaff, Sharif and White introduced [**14**] the Gorenstein category $\mathcal{G}(\mathcal{W})$, which unifies the following ideas: Gorenstein projective and injective modules [**5**]; V-Gorenstein projective and injective modules [**7**], which were defined differently than those in [**10**]. As they play an important role in relative homological algebra, Gorenstein projective, injective and flat modules, and their generalized versions have been studied by many authors since the pioneering work of Auslander and Bridger [**1**].

In [**14**], Sather-Wagstaff, Sharif and White also investigated the stability of Gorenstein categories $\mathcal{G}(\mathcal{W})$. When $\mathcal{W}$ is selforthogonal, they showed that an iteration of the procedure used to define these entities yields exactly the objects in $\mathcal{G}(\mathcal{W})$, that is, $\mathcal{G}(\mathcal{G}(\mathcal{W})) \subseteq \mathcal{G}(\mathcal{W})$. Furthermore, in [**15**], the authors established a similar stability for a

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subcategory of G_C-flat R-modules over a commutative Noetherian ring, i.e.,

$$\mathcal{G}(\mathcal{GF}_C(R) \cap \mathcal{B}_C(R)) \subseteq \mathcal{GF}_C(R) \cap \mathcal{B}_C(R),$$

where $\mathcal{GF}_C(R)$ denotes the category of G_C-flat modules, and $\mathcal{B}_C(R)$ denotes the Bass class associated to C.

It is natural to consider stability for the category of G_C-flat R-modules:

Question 1.1. *Must $\mathcal{G}(\mathcal{GF}_C(R))$ be contained in $\mathcal{GF}_C(R)$?*

In this paper, we will prove that the containment always holds true over a coherent ring.

This paper is organized as follows. In Section 2, we give some necessary notation and definitions. In Section 3, we first investigate the relationship between $\mathcal{W}$-(co)resolutions and $\mathcal{X}$-(co)resolutions with $\mathcal{W}$ a subcategory of $\mathcal{X}$ and show that these unify some known results related to C-Gorenstein modules and Gorenstein categories. Then, as an application, we establish another kind of stability of the G_C-flat modules under the very process used to define these entities. Using this result, we obtain the containment in Question 1.1 over a commutative coherent ring.

2. Preliminaries. Throughout this article, for convenience, we assume that R is a commutative ring with identity and all modules are unitary. Denote the category of R-modules by $\mathcal{M}(R)$, and denote the subcategory of projective, injective and flat R-modules by $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$, respectively.

We first recall some definitions from [**16**].

Definition 2.1. An R-module C is *semidualizing* if it satisfies the following.

(i) C admits a (possibly unbounded) resolution by finitely generated projective R-modules.
(ii) The natural homothety map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism.
(iii) $\mathrm{Ext}_R^{i\geqslant 1}(C, C) = 0$.

Relative to a semidualizing module C, we set:

$$\mathcal{P}_C(R) = \{C \otimes_R P \mid P \text{ is a projective } R\text{-module}\}$$
$$\mathcal{F}_C(R) = \{C \otimes_R F \mid F \text{ is a flat } R\text{-module}\}$$
$$\mathcal{I}_C(R) = \{\mathrm{Hom}_R(C, I) \mid I \text{ is an injective } R\text{-module}\}.$$

The elements in the sets above are called *C-projective, C-flat and C-injective R-modules*, respectively.

Definition 2.2. An R-module M is said to be *G_C-injective* if there exists an exact sequence

$$\mathbb{X} = \cdots \longrightarrow \mathrm{Hom}_R(C, I^1) \longrightarrow \mathrm{Hom}_R(C, I^0) \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

in $\mathcal{M}(R)$ with each I_i and I^i injective, such that $M \cong \mathrm{Im}(\mathrm{Hom}_R(C, I^0) \to I_0)$ and $\mathrm{Hom}_R(\mathrm{Hom}_R(C, I), \mathbb{X})$ is exact for every injective module I. The exact sequence $\mathbb{X}$ is called a *complete $\mathcal{I}_C\mathcal{I}$-resolution* of M.

The G_C-projective module is defined dually.

Definition 2.3. An R-module N is said to be *G_C-flat* if there exists an exact sequence

$$\mathbb{Y} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \cdots$$

in $\mathcal{M}(R)$ with each F_i and F^i flat, such that $N \cong \mathrm{Im}(F_0 \to C \otimes_R F^0)$ and $\mathrm{Hom}_R(C, I) \otimes_R \mathbb{Y}$ is exact for every injective module I. The exact sequence $\mathbb{Y}$ is called a *complete $\mathcal{F}\mathcal{F}_C$-resolution* of N.

We will denote the classes of G_C-injective, G_C-projective and G_C-flat R-modules by $\mathcal{GI}_C(R)$, $\mathcal{GP}_C(R)$ and $\mathcal{GF}_C(R)$, respectively.

Remark 2.4. When $C = R$, these definitions are the same as those of Gorenstein injective, Gorenstein projective and Gorenstein flat R-modules, which are denoted by $\mathcal{GI}(R)$, $\mathcal{GP}(R)$ and $\mathcal{GF}(R)$ respectively.

By using the definition of G_C-flat modules, the proof of the next lemma is a standard argument.

Lemma 2.5. *The following are equivalent for an R-module M:*

(i) *M is G_C-flat.*
(ii) $\mathrm{Tor}^R_{\geqslant 1}(\mathcal{I}_C(R), M) = 0$, *and there exists an exact sequence*

$$0 \longrightarrow M \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \cdots$$

in $\mathcal{M}(R)$ with each F^i flat, such that $\mathrm{Hom}_R(C, I) \otimes_R$ *leaves it exact for every injective module I.*

Definition 2.6. The Bass class $\mathcal{B}_C(R)$ with respect to C consists of all R-modules N satisfying:

(i) $\mathrm{Ext}^{\geqslant 1}_R(C, N) = 0 = \mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_R(C, N))$, and
(ii) the map $C \otimes_R \mathrm{Hom}_R(C, N) \to N$ is an isomorphism.

Definition 2.7 (see [**14**])**.** Let $\mathcal{W}$ be a full subcategory of an abelian category. The Gorenstein category denoted by $\mathcal{G}(\mathcal{W})$ consists of all objects A isomorphic to $\mathrm{Coker}(\delta_1^{\mathbb{X}})$ for some exact complex $\mathbb{X}$ in $\mathcal{W}$, such that the complexes $\mathrm{Hom}_R(W', \mathbb{X})$ and $\mathrm{Hom}_R(\mathbb{X}, W'')$ are exact for each W' and W'' in $\mathcal{W}$. In this case, $\mathbb{X}$ is said to be a complete $\mathcal{W}$-resolution of A.

Definition 2.8 (see [**6**])**.** Let $\mathcal{X}$ be a subcategory of $\mathcal{M}(R)$. An $\mathcal{X}$-preenvelope of an R-module M is an R-module homomorphism $\varphi : M \to X$, where $X \in \mathcal{X}$ such that, for each $X' \in \mathcal{X}$, the homomorphism

$$\mathrm{Hom}_R(\varphi, X') : \mathrm{Hom}_R(X, X') \longrightarrow \mathrm{Hom}_R(M, X')$$

is surjective. $\mathcal{X}$ is said to be a preenveloping class, if every R-module has an $\mathcal{X}$-preenvelope.

3. Stability of G_C-flat modules. To begin, we prove some results in a more general setting and then apply them to the categories of interest. Let $\mathcal{A}$ be an abelian category, and fix additive full subcategories $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{X}$ of $\mathcal{A}$ such that $\mathcal{V}$, $\mathcal{W} \subseteq \mathcal{X}$. Write $\mathcal{V} \perp \mathcal{W}$ if $\mathrm{Ext}^{i\geqslant 1}_{\mathcal{A}}(V, W) = 0$ for each object V in $\mathcal{V}$ and each object W in $\mathcal{W}$.

Recall from [**14**] that $\mathcal{W}$ is a *cogenerator* for $\mathcal{X}$ if, for each object X in $\mathcal{X}$, there exists an exact sequence in $\mathcal{X}$,

$$0 \longrightarrow X \longrightarrow W \longrightarrow X' \longrightarrow 0,$$

such that W is an object in $\mathcal{W}$. The subcategory $\mathcal{W}$ is an *injective cogenerator* for $\mathcal{X}$, if $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ and $\mathcal{X} \perp \mathcal{W}$.

Generator and *projective generator* are defined dually.

A sequence $\mathbb{X}$ in $\mathcal{A}$ is called $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-*exact* if $\mathrm{Hom}_{\mathcal{A}}(X, \mathbb{X})$ is exact for each object X in $\mathcal{X}$. Dually, it is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$-*exact* if $\mathrm{Hom}_{\mathcal{A}}(\mathbb{X}, X)$ is exact for each object X in $\mathcal{X}$, and $\mathbb{X}$ is $\mathcal{X}\otimes_R$-exact if $X \otimes_R \mathbb{X}$ is exact for each X in $\mathcal{X}$.

Lemma 3.1. *Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ closed under extensions. Suppose that*

$$0 \longrightarrow K \longrightarrow X_1 \stackrel{f}{\longrightarrow} X_0 \longrightarrow A \longrightarrow 0 \tag{3.1}$$

is an exact sequence in $\mathcal{A}$ with X_0, X_1 in $\mathcal{X}$.

(i) *If $\mathcal{W}$ is a cogenerator for $\mathcal{X}$, then we have the following exact sequence*:

$$0 \longrightarrow K \longrightarrow W \longrightarrow X \longrightarrow A \longrightarrow 0 \tag{3.2}$$

in $\mathcal{A}$ with W in $\mathcal{W}$ and X in $\mathcal{X}$. Moreover, if $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$ and (3.1) *is* $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$*-exact, then so is* (3.2).

(ii) *If $\mathcal{V}$ is a generator for $\mathcal{X}$, then we have the following exact sequence:*

$$0 \longrightarrow K \longrightarrow X' \longrightarrow V \longrightarrow A \longrightarrow 0 \tag{3.3}$$

with V in $\mathcal{V}$ and X' in $\mathcal{X}$. Moreover, if $\mathcal{V}$ is a projective generator for $\mathcal{X}$ and (3.1) *is* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, -)$*-exact, then so is* (3.3).

Proof. By a similar argument to the proof of [**12,** Proposition 2.2], we obtain the assertion. □

Theorem 3.2. *Let n be a positive integer, and let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ closed under extensions. Suppose that*

$$0 \longrightarrow K \longrightarrow X_{n-1} \longrightarrow X_{n-2} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow A \longrightarrow 0 \tag{3.4}$$

is an exact sequence in $\mathcal{A}$ with all X_i in $\mathcal{X}$, then:

(i) *if* $\mathcal{W}$ *is a cogenerator for* $\mathcal{X}$*, then we have the following exact sequences*:

$$0 \longrightarrow K \longrightarrow W_{n-1} \longrightarrow W_{n-2} \longrightarrow \cdots \longrightarrow W_0 \longrightarrow B \longrightarrow 0 \tag{3.5}$$

and $0 \to A \to B \to X \to 0$ *with all* W_i *in* $\mathcal{W}$ *and* X *in* $\mathcal{X}$*. Moreover, if* $\mathcal{W}$ *is an injective cogenerator for* $\mathcal{X}$ *and* (3.4) *is* $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$*-exact, then so is* (3.5).

(ii) *If* $\mathcal{V}$ *is a generator for* $\mathcal{X}$*, then we have the following exact sequences*:

$$0 \longrightarrow L \longrightarrow V_{n-1} \longrightarrow V_{n-2} \longrightarrow \cdots \longrightarrow V_0 \longrightarrow A \longrightarrow 0 \tag{3.6}$$

and $0 \to X' \to L \to K \to 0$ *with all* V_i *in* $\mathcal{V}$ *and* X' *in* $\mathcal{X}$*. Moreover, if* $\mathcal{V}$ *is a projective generator for* $\mathcal{X}$ *and* (3.4) *is* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, -)$*-exact, then so is* (3.6).

Proof. We give the proof of part (i), and part (ii) is proved dually.

We proceed by induction on n. If $n = 1$, the assumption gives rise to an exact sequence $0 \to K \to X_0 \to A \to 0$ with X_0 in $\mathcal{X}$. Since $\mathcal{W}$ is a cogenerator for $\mathcal{X}$, we have an exact sequence $0 \to X_0 \to W_0 \to X \to 0$, where W_0 in $\mathcal{W}$ and X in $\mathcal{X}$. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & X_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & W_0 & \longrightarrow & B & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X & = & X & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The middle row and the last column in the above diagram are the desired two exact sequences. In addition, if $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, then $\mathrm{Ext}^1_{\mathcal{A}}(A, \mathcal{W}) = 0$ since the first row in the above diagram is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. The exactness of the third column implies that $\mathrm{Ext}^1_{\mathcal{A}}(B, \mathcal{W}) = 0$, and hence the middle row is also $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact.

Now assume that $n \geq 2$, and set $M = \mathrm{Coker}(X_{n-1} \to X_{n-2})$. Because the sequence $0 \to K \to X_{n-1} \to X_{n-2} \to M \to 0$ is exact, Lemma 3.1 yields a $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact exact sequence $0 \to K \to W_{n-1} \to X'_{n-2} \to M \to 0$ with W_{n-1} in $\mathcal{W}$ and X'_{n-2} in $\mathcal{X}$. Set $K' = \mathrm{Im}(W_{n-1} \to X'_{n-2})$. Then we obtain the exactness of

$$0 \longrightarrow K' \longrightarrow X'_{n-2} \longrightarrow X_{n-3} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow A \longrightarrow 0,$$

which is also $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})$-exact. So, by the induction hypothesis, we get the assertion. □

Remark 3.3.

(1) From [**9,** Theorem 2.5], we know that $\mathcal{GP}(R)$ is closed under extensions, and $\mathcal{P}(R)$ is both a projective generator and an injective cogenerator for $\mathcal{GP}(R)$. Thus, Theorem 3.2 implies the result of [**12,** Theorem 2.4].
(2) By [**16,** Theorem 2.8 and Proposition 2.9], we know that $\mathcal{GP}_C(R)$ is closed under extensions, $\mathcal{P}(R)$ is a projective generator and $\mathcal{P}_C(R)$ is an injective cogenerator for $\mathcal{GP}_C(R)$. So Theorem 3.2 implies the result of [**13,** Lemma 2.8].

An exact sequence $\cdots \to X_1 \to X_0 \to A \to 0$ in $\mathcal{A}$ with each $X_i \in \mathcal{X}$ is said to be an $\mathcal{X}$*-resolution* of A. An $\mathcal{X}$*-coresolution* of A is defined dually.

Corollary 3.4. *Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ closed under extensions, and let A be an object in $\mathcal{A}$.*

(i) *If $\mathcal{W}$ is a cogenerator for $\mathcal{X}$, then A has an $\mathcal{X}$-coresolution if and only if it has a $\mathcal{W}$-coresolution.*
(ii) *If $\mathcal{V}$ is a generator for $\mathcal{X}$, then A has an $\mathcal{X}$-resolution if and only if it has a $\mathcal{V}$-resolution.*

Proof.

(i) It is enough to show the "only if" part. Let $0 \to A \to X_0 \to X_1 \to \cdots$ be an $\mathcal{X}$-coresolution of A, and set $A_i = \mathrm{Im}(X_i \to X_{i+1})$ for each $i \geq 0$. By Theorem 3.2 (i) for the case $n = 1$, we have the following exact sequences: $0 \to A \to W_0 \to B \to 0$ and $0 \to A_0 \to B \to X \to 0$ with W_0 in $\mathcal{W}$ and X in $\mathcal{X}$. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A_0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X_1 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & A_1 & = & A_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $X_1 \in \mathcal{X}$. Thus, $X' \in \mathcal{X}$ since $\mathcal{X}$ is closed under extensions, and so B has an $\mathcal{X}$-coresolution $0 \to B \to X' \to X_2 \to \cdots$. By repeating the preceding process, we have that A has a $\mathcal{W}$-coresolution.

(ii) is proved dually. □

As an immediate consequence of Corollary 3.4, we obtain two results, in which the second assertion generalizes [**8,** Proposition 2.10].

Corollary 3.5.

(i) *An R-module M has a $\mathcal{GP}_C(R)$-resolution (respectively coresolution) if and only if M has a $\mathcal{P}(R)$-resolution (respectively $\mathcal{P}_C(R)$-coresolution).*

(ii) *If $\mathcal{W}\perp\mathcal{W}$, then an object A in $\mathcal{A}$ has a $\mathcal{G}(\mathcal{W})$-(co)resolution if and only if A has a $\mathcal{W}$-(co)resolution.*

Proof.

(i) follows from Remark 3.3 (2) and Corollary 3.4.

(ii) Since $\mathcal{W}\perp\mathcal{W}$, from [**14,** Corollaries 4.5, 4.7], we know that the Gorenstein category $\mathcal{G}(\mathcal{W})$ is closed under extensions, and $\mathcal{W}$ is both a projective generator and an injective cogenerator for $\mathcal{G}(\mathcal{W})$. The assertion follows from Corollary 3.4. □

Using Theorem 3.2 and Corollary 3.5, we give a simpler proof of the stability of Gorenstein categories $\mathcal{G}(\mathcal{W})$.

Corollary 3.6 ([**14,** Corollary 4.10]). *If $\mathcal{W}\perp\mathcal{W}$, then $\mathcal{G}(\mathcal{G}(\mathcal{W})) = \mathcal{G}(\mathcal{W})$.*

Proof. From [**14,** Remark 4.2], it is easy to see that $\mathcal{G}(\mathcal{W}) \subseteq \mathcal{G}(\mathcal{G}(\mathcal{W}))$.

For the reverse containment, let A be an object in $\mathcal{G}(\mathcal{G}(\mathcal{W}))$ and

$$\cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

a complete $\mathcal{G}(\mathcal{W})$-resolution of A. Since $\mathcal{W}$ is a projective generator for $\mathcal{G}(\mathcal{W})$, by Theorem 3.2 (ii) and Corollary 3.5 (ii), A has a $\mathcal{W}$-resolution:

$$\cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow A \longrightarrow 0,$$

which is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact. We claim that it is also $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact. Indeed, put $A_i = \mathrm{Im}(W_{i+1} \to W_i)$ for any $i \geq 0$. Since $\mathcal{W}$ is an injective cogenerator for $\mathcal{G}(\mathcal{W})$, it is not difficult to show that $\mathrm{Ext}_{\mathcal{A}}^{\geqslant 1}(A, \mathcal{W}) = 0$ from the definition of $\mathcal{G}(\mathcal{G}\mathcal{W})$. Thus $\mathrm{Ext}_{\mathcal{A}}^{\geqslant 1}(A_i, \mathcal{W}) = 0$ for each $i \geq 0$ by dimension shifting, and so the sequence above is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact.

By a dual argument, A has a $\mathcal{W}$-coresolution: $0 \to A \to W^0 \to W^1 \to \cdots$, which is both $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})$-exact and $\mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)$-exact. Thus, $A \in \mathcal{G}(\mathcal{W})$. □

In the following, we use M^+ to denote the character module $\mathrm{Hom}_Z(M, Q/Z)$ of M. The following lemma is contained in [**15,** Lemma 4.1] over a Noetherian ring, but it is also valid in the following situation by a similar argument.

Lemma 3.7.

(i) *Let R be a ring. Then an R-module M is C-flat if and only if M^+ is C-injective.*
(ii) *Assume that R is a coherent ring. If N is a C-injective R-module, then N^+ is C-flat.*

The following result investigates the relations between G_C-flat and G_C-injective modules, which generalizes [**9,** Theorem 3.6].

Theorem 3.8. *For an R-module M, we consider the following conditions*:

(i) *M is G_C-flat.*
(ii) *M^+ is G_C-injective.*

Then (i) *implies* (ii). *If R is coherent, then the converse holds true.*

Proof.

(i) $\Rightarrow$ (ii). Let

$$\mathbb{Z} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \cdots$$

be a complete $\mathcal{F}\mathcal{F}_C$-resolution of M. Then $\mathbb{Z}^+$ is exact with F_i^+ injective and $(C \otimes_R F^i)^+$ C-injective by Lemma 3.7 (i). Because $\mathrm{Hom}_R(\mathcal{I}_C(R), \mathbb{Z}^+) \cong (\mathcal{I}_C(R) \otimes_R \mathbb{Z})^+$ is exact, $\mathbb{Z}^+$ is a complete $\mathcal{I}_C\mathcal{I}$-resolution of M^+, and so M^+ is G_C-injective.

(ii) $\Rightarrow$ (i). Assume that M^+ is G_C-injective. Firstly, for any $i > 0$, $(\mathrm{Tor}_i^R(\mathcal{I}_C(R), M))^+ \cong \mathrm{Ext}_R^i(\mathcal{I}_C(R), M^+) = 0$ by [**3,** Chapter VI, Proposition 5.1]. Thus, $\mathrm{Tor}_i^R(\mathcal{I}_C(R), M) = 0$ for any $i > 0$.

By Lemma 2.5, it suffices to show that M has an $\mathcal{F}_C(R)$-coresolution:

$$0 \longrightarrow M \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \cdots,$$

and $\mathcal{I}_C(R) \otimes_R -$ leaves it exact. Because M^+ is G_C-injective, there is an exact sequence $0 \to K \to \mathrm{Hom}_R(C, I_0) \overset{f}{\to} M^+ \to 0$ with I_0 injective. Then $0 \to M^{++} \overset{f^+}{\to} (\mathrm{Hom}_R(C, I_0))^+ \to K^+ \to 0$ is exact and $(\mathrm{Hom}_R(C, I_0))^+$ is C-flat by Lemma 3.7 (ii). Since $\sigma : M \to M^{++}$ is injective, we have a monomorphism $M \overset{f^+\sigma}{\to} (\mathrm{Hom}_R(C, I_0))^+$. On the other hand, $\mathcal{F}_C(R)$ is a preenveloping class by [**11,** Proposition 5.10], and suppose that $M \overset{g}{\to} C \otimes_R F^0$ is a C-flat preenvelope of M. It is easy to see that g is injective. So we get an exact sequence

$$(*) \qquad 0 \to M \to C \otimes_R F^0 \to M_1 \to 0,$$

with $M_1 = \mathrm{Coker} g$. Thus, $0 \to M_1^+ \to (C \otimes_R F^0)^+ \to M^+ \to 0$ is exact. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\operatorname{Hom}_R(C\otimes_R F^0,\mathcal{I}_C(R)^+) & \xrightarrow{\operatorname{Hom}_R(g,\mathcal{I}_C(R)^+)} & \operatorname{Hom}_R(M,\mathcal{I}_C(R)^+) & \longrightarrow & 0 & & \\
\cong\downarrow & & \cong\downarrow & & & & \\
\operatorname{Hom}_R(\mathcal{I}_C(R),(C\otimes_R F^0)^+) & \longrightarrow & \operatorname{Hom}_R(\mathcal{I}_C(R),M^+) & \longrightarrow & \operatorname{Ext}^1_R(\mathcal{I}_C(R),M_1^+) & \longrightarrow & 0
\end{array}$$

The first row is exact since g is a C-flat preenvelope, so $\operatorname{Ext}^1_R(\mathcal{I}_C(R), M_1^+) = 0$. Therefore, M_1^+ is G_C-injective by the dual version of [**16,** Corollary 3.8]. Because $(\operatorname{Tor}_1^R(\mathcal{I}_C(R), M_1))^+ \cong \operatorname{Ext}^1_R(\mathcal{I}_C(R), M_1^+) = 0$, the sequence $(*)$ is $\mathcal{I}_C(R)\otimes_R -$ exact.

By a similar argument to M_1, repeating the process, we obtain the desired exact sequence. Therefore, M is G_C-flat. $\square$

From Theorem 3.8, we conclude that the category of G_C-flat modules has nice properties when the ring in question is coherent.

Corollary 3.9. *Let R be a coherent ring. Then the category of G_C-flat R-modules contains all flat and C-flat R-modules.*

Proof. Assume that M is a flat (respectively C-flat) R-module. It follows from Lemma 3.7 (i) that M^+ is injective (respectively C-injective), and hence G_C-injective by the dual version of [**16,** Proposition 2.6]. Theorem 3.8 implies that M is G_C-flat. $\square$

Corollary 3.10. *Let R be a coherent ring, and $0\to M_1\to M\to M_2\to 0$ an exact sequence of R-modules.*

(i) *Assuming that M_2 is G_C-flat, one has M_1 is G_C-flat if and only if M is G_C-flat.*
(ii) *Assuming that M_1 and M are G_C-flat and that $\operatorname{Tor}_1^R(\mathcal{I}_C(R), M_2) = 0$, then M_2 is G_C-flat.*

Proof.

(i) By Theorem 3.8 and the injective version of [**16,** Theorem 2.8].
(ii) We have an exact sequence $0\to M_2^+\to M^+\to M_1^+\to 0$ with M_1^+ and M^+ G_C-injective by Theorem 3.8. Because

$$\operatorname{Ext}^1_R(\mathcal{I}_C(R), M_2^+) \cong (\operatorname{Tor}_1^R(\mathcal{I}_C(R), M_2))^+ = 0,$$

M_2^+ is G_C-injective by the injective version of [**16,** Corollary 3.8]. Thus, M_2 is G_C-flat by Theorem 3.8 again. □

Combining Theorems 3.2 and 3.8, we get the following result for the category of G_C-flat modules. The first assertion generalizes [**4,** Lemma 2.19].

Proposition 3.11. *Let R be a coherent ring and n a positive integer. If*

$$0 \longrightarrow K \longrightarrow G_{n-1} \longrightarrow G_{n-2} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

is an exact sequence of R-modules with all G_i G_C-flat, then:

(i) *There exist exact sequences*

$$0 \longrightarrow K \longrightarrow C \otimes F_{n-1} \longrightarrow C \otimes F_{n-2} \longrightarrow \cdots \longrightarrow C \otimes F_0 \longrightarrow N \longrightarrow 0$$

and $0 \to M \to N \to G \to 0$ of R-modules with all F_i flat and G G_C-flat.

(ii) *There exist exact sequences*

$$0 \longrightarrow L \longrightarrow F'_{n-1} \longrightarrow F'_{n-2} \longrightarrow \cdots \longrightarrow F'_0 \longrightarrow M \longrightarrow 0$$

and $0 \to H \to L \to K \to 0$ of R-modules with all F'_i flat and H G_C-flat.

Proof. From Theorem 3.8 and the dual version of [**16,** Theorem 2.8 and Proposition 2.9], we know that $\mathcal{GF}_C(R)$ is closed under extensions, $\mathcal{F}(R)$ is a generator and $\mathcal{F}_C(R)$ is a cogenerator for $\mathcal{GF}_C(R)$. The assertion follows from Theorem 3.2. □

We denote $\mathcal{G}^2\mathcal{F}_C(R) = \{A \in \mathcal{M}(R) \mid$ there exists an exact sequence $\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$ in $\mathcal{M}(R)$ with all G_i and G^i in $\mathcal{GF}_C(R)$, such that $A \cong \mathrm{Im}(G_0 \to G^0)$ and $\mathcal{I}_C(R)\otimes_R$-leaves it exact$\}$.

Theorem 3.12. *When R is a coherent ring, $\mathcal{G}^2\mathcal{F}_C(R) = \mathcal{GF}_C(R)$.*

Proof. Because every flat and C-flat R-module is G_C-flat by Corollary 3.9, it is clear that $\mathcal{GF}_C(R) \subseteq \mathcal{G}^2\mathcal{F}_C(R)$.

Conversely, let $A \in \mathcal{G}^2\mathcal{F}_C(R)$ and

$$\mathbb{X} = \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

be an exact sequence in $\mathcal{M}(R)$ with all G_i and G^i in $\mathcal{GF}_C(R)$ and $A \cong \operatorname{Im}(G_0 \to G^0)$. Put $A_i = \operatorname{Im}(G_{i+1} \to G_i)$ and $A^i = \operatorname{Im}(G^i \to G^{i+1})$ for any $i \geq 0$. Consider the short exact sequences $0 \to A_0 \to G_0 \to A \to 0$ and $0 \to A_{i+1} \to G_{i+1} \to A_i \to 0$ for any $i \geq 0$. Since $\mathcal{I}_C(R) \otimes_R \mathbb{X}$ is exact, $\operatorname{Tor}^R_{\geq 1}(\mathcal{I}_C(R), A) = 0$ and $\operatorname{Tor}^R_{\geq 1}(\mathcal{I}_C(R), A_i) = 0$ for any $i \geq 0$ by dimension shifting. Similarly, $\operatorname{Tor}^R_{\geq 1}(\mathcal{I}_C(R), A^i) = 0$ for any $i \geq 0$.

To prove $A \in \mathcal{GF}_C(R)$, by Lemma 2.5, it suffices to show that A has an $\mathcal{F}_C(R)$-coresolution:

$$0 \longrightarrow A \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \cdots,$$

and $\mathcal{I}_C(R) \otimes_R -$ leaves it exact. Because $0 \to A \to G^0 \to A^0 \to 0$ is exact, by Proposition 3.11, there exist exact sequences

$$0 \longrightarrow A \longrightarrow C \otimes F^0 \longrightarrow N^0 \longrightarrow 0$$

and

$$0 \longrightarrow A^0 \longrightarrow N^0 \longrightarrow G \longrightarrow 0$$

with F^0 flat and G G_C-flat. Form the exactness of the second sequence, $\operatorname{Tor}^R_{j\geq 1}(\mathcal{I}_C(R), N^0) = 0$, and hence, the first one is $\mathcal{I}_C(R) \otimes_R -$ exact. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A^0 & \longrightarrow & N^0 & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & G^1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & A^1 & = & A^1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Because both G^1 and G are G_C-flat, so is G' by Corollary 3.10. Since $\operatorname{Tor}^R_{j\geq 1}(\mathcal{I}_C(R), A^i) = 0$ for any $i \geq 1$, we get an exact sequence

$$0 \longrightarrow N^0 \longrightarrow G' \longrightarrow G^2 \longrightarrow G^3 \longrightarrow \cdots$$

in $\mathcal{M}(R)$, which is $\mathcal{I}_C(R) \otimes_R -$ exact. Repeating the process, we obtain the desired exact sequence. Thus, $A \in \mathcal{GF}_C(R)$. $\square$

In the special case $C = R$, we obtain the main theorem of [2] and [17, Theorem 4.3].

Corollary 3.13. *$\mathcal{G}^2\mathcal{F}(R) = \mathcal{G}\mathcal{F}(R)$.*

As a consequence of Theorem 3.12, we give an affirmative answer to Question 1.1 over a coherent ring.

Corollary 3.14. *For a coherent ring R, $\mathcal{G}(\mathcal{G}\mathcal{F}_C(R)) \subseteq \mathcal{G}\mathcal{F}_C(R)$.*

Proof. Let $M \in \mathcal{G}(\mathcal{G}\mathcal{F}_C(R))$, and let

$$\mathbb{G} = \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

be a complete $\mathcal{G}\mathcal{F}_C(R)$-resolution of M. From Lemma 3.7 (ii), we know that $(\mathcal{I}_C(R))^+ \subseteq \mathcal{F}_C(R)$, are included in $\mathcal{G}\mathcal{F}_C(R)$ by Corollary 3.9. Thus, the complex

$$(\mathcal{I}_C(R) \otimes_R \mathbb{G})^+ \cong \mathrm{Hom}_R(\mathbb{G}, (\mathcal{I}_C(R))^+),$$

is exact since $\mathrm{Hom}_R(\mathbb{G}, \mathcal{G}\mathcal{F}_C(R))$ is exact, and so $\mathcal{I}_C(R) \otimes_R \mathbb{G}$ is exact. Theorem 3.12 yields that $M \in \mathcal{G}\mathcal{F}_C(R)$. □

Remark 3.15. It is convenient to mention that all the results from Lemma 3.7 to Corollary 3.14 also hold true for non-commutative coherent rings.

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