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NOTES ON $\log(\zeta(s))''$

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ABSTRACT. Motivated by the connection to the pair correlation of the Riemann zeros, we investigate the second derivative of the logarithm of the Riemann ζ function, in particular, the zeros of this function. Theorem 1.2 gives a zero-free region. Theorem 1.4 gives an asymptotic estimate for the number of nontrivial zeros to height T. Theorem 1.7 is a zero density estimate.

1. Introduction. Bogomolny and Keating [**4**] were the first to observe that the function $(\zeta'(s)/\zeta(s))'$ appears in the pair correlation for the Riemann zeros.[1] In that context, Berry and Keating [**2**] wrote:

> The appearance of $\zeta(s)$ indicates an astonishing resurgence property of the zeros: in the pair correlation of high Riemann zeros, the low Riemann zeros appear as resonances.

There has been extensive investigation into the zeros of $\zeta'(s)$ and their connection to the Riemann hypothesis, via the logarithmic derivative $\zeta'/\zeta(s)$. However, there seems to be nothing in the literature about the zeros of the derivative:

$$\log(\zeta(s))'' = \left(\frac{\zeta'(s)}{\zeta(s)}\right)' = \frac{\zeta(s)\zeta''(s) - \zeta'(s)^2}{\zeta(s)^2}.$$

The connection to the pair correlation of the Riemann zeros is motivation for further study.

Further motivation comes from Montgomery's review of Levinson [**6**], in which he says:

> The author's method can be applied to functions other than $G(s)$, and in particular one may use differential

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operators of higher order. Whether sharper results can be obtained in this manner remains to be seen.

Notation. We let

$$\nu(s) = \zeta(s)\zeta''(s) - \zeta'(s)^2.$$

Elementary facts. Near $s = 1$,

$$\log(\zeta(s))'' = \frac{1}{(s-1)^2} + O(1).$$

Near a zero ρ of $\zeta(s)$ of order n_ρ,

$$\log(\zeta(s))'' = \frac{-n_\rho}{(s-\rho)^2} + O(1),$$

so $\nu(s)$ has a zero of order $2n_\rho - 2$. In particular, for a simple zero of $\zeta(s)$, this tells us that $\nu(\rho) \neq 0$. There are no other poles. The zeros of $\log(\zeta(s))''$ are the zeros of $\nu(s)$, exclusive of any possible multiple zeros of $\zeta(s)$.

For $\mathrm{Re}(s) > 1$, we have that

$$\nu(s) = \sum_n \left(\sum_{d|n} \log(d)^2 - \log(d) \log\left(\frac{n}{d}\right)\right) n^{-s}. \tag{1.1}$$

With $\Lambda(n)$ the Von Mangoldt's function and $\tau(n)$ the divisor function, we have that

$$\log(\zeta(s))'' = \sum_n \Lambda(n) \log(n) n^{-s}, \qquad \zeta(s)^2 = \sum_n \tau(n) n^{-s}.$$

Thus, we also have that

$$\nu(s) = \sum_n \left(\sum_{d|n} \Lambda(d) \log(d)\, \tau\left(\frac{n}{d}\right)\right) n^{-s}. \tag{1.2}$$

We will let $a(n)$ denote the Dirichlet series coefficients of $\nu(s)$, given by either equation (1.1) or equation (1.2). Let

$$A(x) = \sum_{n<x} a(n).$$

We have that, for $c > 1$,

$$A(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \nu(w) \frac{x^w}{w}\, dw.$$

Moving the contour past the pole at $s = 1$, we have that, for $0 < c < 1$,

$$A(x) = x \cdot p\left(\log(x)\right) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \nu(w) \frac{x^w}{w}\, dw, \tag{1.3}$$

where

$$p(t) = \frac{t^3}{6} + \left(C_0 - \frac{1}{2}\right)t^2 + (1 - 4C_1 - 2C_0)t + 4C_2 + 4C_1 + 2C_0 - 1,$$

where C_0 is the Euler constant, and C_1 and C_2 are Stieltjes constants. With $p(t)$ as above, one can show by Euler MacLaurin summation [**7,** Appendix B] and the "method of the hyperbola" [**7,** equation (2.9)] that

$$A(x) = x \cdot p\left(\log(x)\right) + O\big(x^{1/2}\log(x)^2\big), \tag{1.4}$$

i.e., the integral in equation (1.3) is $O(x^{1/2}\log(x)^2)$.

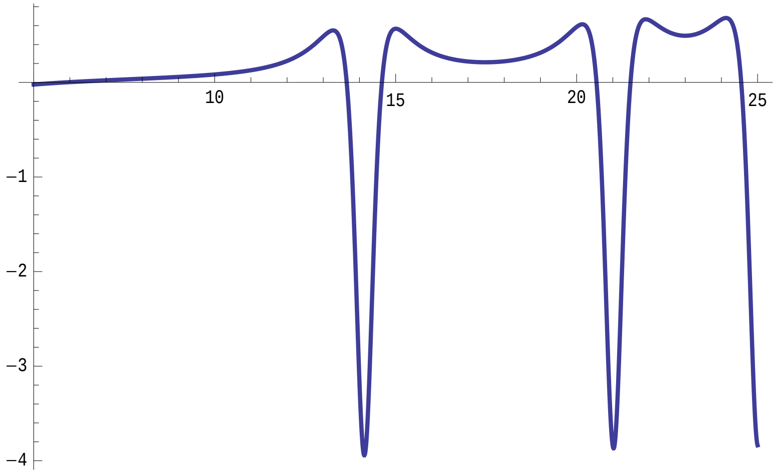

FIGURE 1. $\mathrm{Re}((\zeta'/\zeta)'(1+it))$ is the resurgent contribution of $\zeta(s)$ to pair correlation.

Functional equation. As usual, let

$$\chi(s) = 2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) = \frac{\pi^{(s-1)/2}\Gamma((1-s)/2)}{\pi^{-s/2}\Gamma(s/2)}.$$

Differentiating the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, we deduce that

(1.5)

$$\nu(s) = \chi^2(s)\left(\nu(1-s) + \left(\psi'(1-s) - \left(\frac{\pi}{2}\right)^2 \csc\left(\frac{\pi s}{2}\right)^2\right)\zeta(1-s)^2\right).$$

Here, $\psi'(s)$ denotes the derivative of the Digamma function:

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}.$$

Stirling's formula tells us that, as $s \to \infty$ in the region $|\arg(s)| \le \pi - \delta$,

$$\psi'(s) = \frac{1}{s} + O\left(\frac{1}{s^2}\right).$$

As $t \to \infty$, we have that, for $\sigma > a$ fixed,

$$\chi^2(s) \ll t^{1-2\sigma}, \tag{1.6}$$

$$\chi^2(s)\left(\psi'(1-s) - \left(\frac{\pi}{2}\right)^2 \csc\left(\frac{\pi s}{2}\right)^2\right) \ll t^{-2\sigma}. \tag{1.7}$$

Thus, as $s \to \infty$ in the region $|\arg(s)| \le \pi - \delta$,

$$\nu(s) = \begin{cases} O(1) & \sigma \ge 1 + \delta > 1, \\ O(t^{1-2\sigma}) & \sigma \le -\delta < 0. \end{cases} \tag{1.8}$$

From the functional equation,

$$\zeta(1-s) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s),$$

we deduce

$$\log(\zeta(1-s))'' = -\frac{\pi^2}{4}\sec^2\left(\frac{\pi s}{2}\right) + \psi'(s) + \log(\zeta(s))''. \tag{1.9}$$

Asymptotics. With $a(n) \ll n^{\epsilon}$, we can estimate the sum of the series for $n \geq 3$ to obtain:

$$\log(\zeta(s))'' = \frac{\log(2)^2}{2^s} + O\left(\frac{\exp(-\sigma)}{1+\epsilon-\sigma}\right) \quad \text{for } \sigma > 1+\epsilon.$$

Now, $|\sec^2(\pi s/2)| \ll \exp(-\pi t)$. Thus, we have the next proposition.

Proposition 1.1. *As $s \to \infty$ in a vertical strip $1+\epsilon < \sigma < \sigma_0$,*

$$\log(\zeta(1-s))'' = \frac{\log(2)^2}{2^s} + O\left(\frac{\exp(-\sigma)}{1+\epsilon-\sigma}\right) + O\left(\frac{1}{s}\right). \tag{1.10}$$

On the other hand, if $t \to \infty$ with $|s|^2 < 2^{\sigma}$, then

$$\log(\zeta(1-s))'' = \frac{1}{s} + O\left(\frac{1}{s^2}\right). \tag{1.11}$$

On the border of these two asymptotic regimes, we will see a cancelation where

$$\frac{1}{s} \approx \frac{-\log(2)^2}{2^s},$$

creating zeros of $\nu(s)$, which we refer to as *asymptotically trivial of the first kind.* Equating modulus and argument, this occurs when

$$2^{\sigma} \approx \log(2)^2(\sigma^2+t^2)^{1/2} \quad \text{or} \quad \sigma \approx \frac{\log(t)}{\log(2)},$$

and also,

$$\tan(t\log(2)) \approx \frac{t}{\sigma}.$$

With σ and t positive, both $\cos(t\log(2))$ and $\sin(t\log(2))$ need to be negative. Since σ is very small compared to t, we deduce that $t\log(2)$ is slightly larger than $2\pi n + 3\pi/2$ for integer n, i.e., the imaginary part is approximately $9.1n + 6.8$. The real part is near $1 - \log(t)/\log(2\pi)$. One sees 11 examples of these asymptotically trivial zeros to the left of the critical line on the right side of Figure 2.

There is a double pole of

$$-\frac{\pi^2}{4}\sec^2\frac{\pi(1-s)}{2} + \psi'(1-s)$$

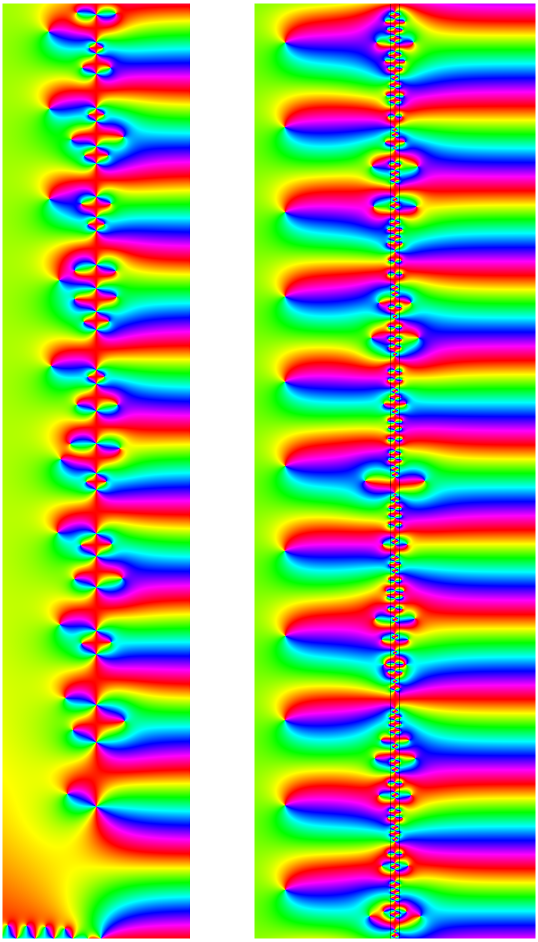

FIGURE 2. Argument of $\log(\zeta(s))''$. On the left, the vertical strip $-9.5 \leq \sigma \leq 10.5$, and $0 \leq t \leq 100$. On the right, $-14.5 \leq \sigma \leq 15.5$ and $10^4 \leq t \leq 10^4 + 100$. The dotted lines denote $\sigma = 0$ and $\sigma = 1$.

at the negative even integers. Equation (1.11) implies that, as $s \to \infty$ with $\arg(s)$ a constant $(\pi/2) - \delta$, $\arg(\log(\zeta(s))'')$ is asymptotically constant (in fact, asymptotic to δ). For each double pole arising from a negative even integer, $\nu(s)$ will have, by the argument principle, a pair of complex conjugate zeros inside of the rays $\arg(s) = \pi \pm \delta$. We refer to these zeros as *asymptotically trivial of the second kind.* Examples in the upper half plane can be seen on the bottom left of Figure 2; more examples can be seen in Figure 3. It would be interesting to understand the asymptotic behavior of the imaginary part of these zeros.

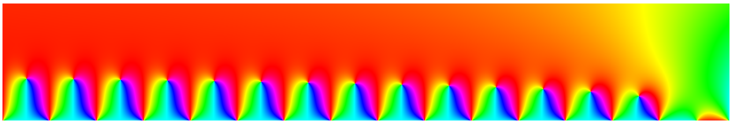

FIGURE 3. Argument of $\log(\zeta(s))''$ in the region $-30 \leq \sigma \leq 1$, and $0 \leq t \leq 5$.

Zero free region. From the general theory of Dirichlet series, $\nu(s)$ has a right half plane free of zeros.

Theorem 1.2. *For* $\mathrm{Re}(s) \geq 4.25$, *we have that* $\nu(s) \neq 0$.

Remark 1.3. Mathematica shows that there is a zero near $s = 3.494 + 23.285i$.

Proof. We have, by the triangle inequality,

$$|\nu(s)| \geq \frac{a(2)}{2^\sigma} - \sum_{n=3}^{\infty} \frac{a(n)}{n^\sigma}.$$

From summation by parts and the fact that

$$\lim_{y\to\infty} A(y)y^{-\sigma} = 0$$

we deduce that, with parameter x to be determined,

$$|\nu(s)| \geq \frac{a(2)}{2^\sigma} - \sum_{n=3}^{x} \frac{a(n)}{n^\sigma} + \frac{A(x)}{x^\sigma} - \sigma \int_x^\infty A(t)t^{-\sigma-1}\,dt.$$

From equation (1.4), it will suffice that we satisfy the two inequalities:

$$\frac{a(2)}{2^\sigma} - \sum_{n=3}^{x} \frac{a(n)}{n^\sigma} > \frac{1.5}{x^{\sigma/2}}$$

and

$$\frac{A(x)}{x^\sigma} - \sigma \int_x^\infty p(\log(t))t^{-\sigma}\,dt - \left|10\cdot\sigma\int_x^\infty \log(t)^2 t^{-\sigma-1/2}\,dt\right| > -\frac{1}{x^{\sigma/2}}.$$

Once $x > 4$ is fixed,

$$a(2) - \sum_{n=3}^{x} a(n)\left(\frac{2}{n}\right)^\sigma$$

is an increasing function of σ, bounded above by $a(2)$, and $(2/\sqrt{x})^\sigma$ is decreasing to 0. Thus, if the first inequality holds at σ_0, it will hold on the interval $[\sigma_0, \infty)$.

Next, observe

$$\sigma\int_x^\infty p(\log(t))t^{-\sigma}\,dt$$
$$= x^{-\sigma}\left(x\cdot p(\log(x)) + \frac{q_1}{\sigma-1} + \frac{q_2}{(\sigma-1)^2} + \frac{q_3}{(\sigma-1)^3} + \frac{q_4}{(\sigma-1)^4}\right),$$

where the q_j are certain polynomials in x and $\log(x)$ in terms of the Stieltjes constants, positive for $x \geq 4$. Meanwhile,

$$10\cdot\sigma\int_x^\infty \log(t)^2 t^{-\sigma-1/2}\,dt$$
$$= x^{1/2-\sigma}\left(10\log(x)^2 + \frac{r_1}{\sigma-1/2} + \frac{r_2}{(\sigma-1/2)^2} + \frac{r_3}{(\sigma-1/2)^3}\right),$$

for certain r_i, polynomials in $\log(x)$ with positive coefficients. Thus, our second inequality is equivalent to:

$$x^{\sigma/2} > x\cdot p(\log(x)) + 10x^{1/2}\log(x)^2 - A(x)$$
$$+ x^{1/2}\left(\sum_{j=1}^{4}\frac{q_j}{(\sigma-1)^j} + \sum_{i=1}^{3}\frac{r_i}{(\sigma-1/2)^i}\right).$$

For fixed $x \geq 4$, the left side increases in σ, and the right side decreases in σ, so, again, this will hold on an interval $[\sigma_0, \infty)$. With $x = 40$, a calculation verifies that $\sigma_0 = 4.25$ suffices. Furthermore, we deduce that, for $\sigma > 4.25$,

$$\frac{a(2)}{2^\sigma} - \sum_{n=3}^{\infty} \frac{a(n)}{n^\sigma} > \frac{0.5}{40^{\sigma/2}}. \tag{1.12}$$

□

The number of zeros for $\nu(s)$. Let

$$N_\nu(T) = \sharp\{\rho \mid \nu(\rho) = 0,\ 0 < \mathrm{Im}(\rho) < T,\ -4 < \mathrm{Re}(\rho)\}.$$

This count excludes the two flavors of asymptotically trivial zeros described above, except for an $O(1)$ error.

Theorem 1.4.

$$N_\nu(T) = 2\left(\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}\right) - \frac{\log(2)}{\pi}T + O(\log(T)).$$

Proof. Let C be the boundary (described positively) of the rectangle with vertices $5 + i10$, $5 + iT$, $-4 + iT$ and $-4 + i10$. There are no asymptotically trivial zeros inside of C. By the functional equation and the zero free region, the nontrivial zeros are inside of C. By the argument principle, we need to estimate

$$\begin{aligned}
&\frac{1}{2\pi i}\int_C \frac{d}{ds}\log(\nu(s))\,ds \\
&\quad = \frac{1}{2\pi i}\left\{\int_{-4+i10}^{5+i10} + \int_{5+i10}^{5+iT} + \int_{5+iT}^{-4+iT} + \int_{-4+iT}^{-4+i10}\right\}\frac{d}{ds}\log(\nu(s))\,ds \\
&\quad = \frac{1}{2\pi i}(I_1 + I_2 + I_3 + I_4).
\end{aligned}$$

The integral I_1 is $O(1)$. Next, I_2 equals

$$\log\left(\frac{a(2)}{2^s}\right)\Bigg|_{5+i10}^{5+iT} + \log\left(1 + \sum_{n=3}^{\infty}\frac{a(n)}{a(2)}\left(\frac{2}{n}\right)^s\right)\Bigg|_{5+i10}^{5+iT}. \tag{1.13}$$

From equation (1.12), we see that

$$1 - \sum_{n=3}^{\infty}\frac{a(n)}{a(2)}\left(\frac{2}{n}\right)^{-5} > 0.0025. \tag{1.14}$$

Thus,

$$\text{Re}\left(1+\sum_{n=3}^{\infty}\frac{a(n)}{a(2)}\left(\frac{2}{n}\right)^{5+it}\right)>0, \tag{1.15}$$

and the argument of the expression inside of the second logarithm in equation (1.13) is bounded by $\pm\pi/2$. From the contribution of the first logarithm in equation (1.13), we deduce that $I_2 = -i\log(2)T + O(1)$. Via a fairly routine argument based on Jensen's theorem,[2] one sees that $I_3 = O(\log(T))$.

Finally, for

$$I_4 = \int_{-4+iT}^{-4+i10} \frac{d}{ds}\log(\nu(s))\,ds = \int_{-4+iT}^{-4+i150} \frac{d}{ds}\log(\nu(s))\,ds + O(1),$$

we will use functional equation (1.5) in the form:

(1.16)

$$\nu(s) = \chi^2(s)\nu(1-s)\left(1+\left(\psi'(1-s)-\left(\frac{\pi}{2}\right)^2\csc\left(\frac{\pi s}{2}\right)^2\right)\frac{\zeta(1-s)^2}{\nu(1-s)}\right).$$

We observe that, for $t \geq 150$,

$$\left|\psi'(5-it)-\left(\frac{\pi}{2}\right)^2\csc\left(\frac{\pi(4+it)}{2}\right)^2\right| < \frac{1}{140}, \tag{1.17}$$

by the exponential decay of the cosecant and Stirling's formula of asymptotes for $\psi'(5-it)$. Also,

$$|\log(\zeta(5-it))''| \geq \frac{\log(2)^2}{2^5} - \sum_{n=3}^{\infty}\frac{\Lambda(n)\log(n)}{n^5} \geq 0.0075, \tag{1.18}$$

$$\left|\frac{\zeta(5-it)^2}{\nu(5-it)}\right| \leq \frac{1}{0.0075} < 135.$$

The product of equations (1.17) and (1.18) is < 1 in absolute value, and thus,

$$\text{Re}\left(1+\left(\psi'(5-it)-\left(\frac{\pi}{2}\right)^2\csc\left(\frac{\pi(4+it)}{2}\right)^2\right)\frac{\zeta(5-it)^2}{\nu(5-it)}\right)>0,$$

and the argument of this expression is bounded between $-\pi/2$ and $\pi/2$. This implies that, on the vertical line $-4+it$, $T \geq t \geq 150$,

$$\operatorname{Im}(\log(\nu(s))) = \operatorname{Im}(\log(\chi^2(s)\nu(1-s))) + O(1).$$

Similarly, from equations (1.14) and (1.15), we deduce that, on this line,

$$\operatorname{Im}(\log(\nu(s))) = \operatorname{Im}\big(\log\big(\chi^2(s)\log(2)^2 2^{s-1}\big)\big) + O(1).$$

Via Stirling's formula,

$$\arg\big(\chi^2(s)\big)\Big|_{-4+iT}^{-4+i150} = 2T\log\left(\frac{T}{2\pi}\right) - 2T + O(1),$$

while

$$\arg\big(\log(2)^2 2^{s-1}\big)\big|_{-4+iT}^{-4+i150} = -\log(2)T,$$

so

$$\operatorname{Im}(I_4) = 2T\log\left(\frac{T}{2\pi}\right) - 2T - \log(2)T + O(1). \quad \square$$

Zero density results.

Proposition 1.5. *As before, for $p(t)$,*

$$A(x) = x \cdot p(\log(x)) + O_\epsilon\big(x^{1/3+\epsilon}\big). \tag{1.19}$$

Proof. Starting with equations (1.5) and (1.8), the proof very closely follows the $k=2$ case of the error estimates for the divisor function [**11,** Theorem 12.2]. $\square$

Proposition 1.6. *Let*

$$\phi(s) = \big(1 - 2^{1-s}\big)^4 \nu(s).$$

The abscissa of convergence σ_c for the series defining $\phi(s)$ is $\leq 1/3$.

Proof. The Dirichlet series expansion of $\phi(s)$ is $\sum_n b(n)n^{-s}$, where, if $2^j || n$,

$$b(n) = \sum_{m=0}^{\min(4,j)} \binom{4}{m}(-2)^m a\left(\frac{n}{2^m}\right).$$

With $B(x) = \sum_{n \le x} b(n)$, we have that

$$\begin{aligned} B(x) &= \sum_{m=0}^{4} \binom{4}{m} (-2)^m \sum_{\substack{k \le x \\ 2^m | k}} a\left(\frac{k}{2^m}\right) \\ &= \sum_{m=0}^{4} \binom{4}{m} (-2)^m \sum_{n \le x/2^m} a(n). \end{aligned}$$

From equation (1.19), we see that

$$\begin{aligned} B(x) &= \sum_{m=0}^{4} \binom{4}{m} (-2)^m \left(\frac{x}{2^m} \cdot p\left(\log(x) - m\log(2)\right) + O_\epsilon\left(x^{1/3+\epsilon}\right)\right) \\ &= x \cdot \sum_{m=0}^{4} \binom{4}{m} (-1)^m p\left(\log(x) - m\log(2)\right) + O_\epsilon\left(x^{1/3+\epsilon}\right). \end{aligned}$$

With shift operator $Ep(t) = p(t - \log(2))$ and difference operator $\Delta p = (I - E)p$, the main term is $x \cdot \Delta^4 p(\log(x)) = 0$, as p has degree 3 and Δ reduces the degree. Thus,

$$B(x) = O_\epsilon\left(x^{1/3+\epsilon}\right).$$

Therefore, for every $\epsilon > 0$,

$$\limsup_{x\to\infty} \frac{\log|B(x)|}{\log(x)} \le \limsup_{x\to\infty} \frac{(1/3+\epsilon)\log(x) + \log(C(\epsilon))}{\log(x)} \le \frac{1}{3} + \epsilon,$$

and, by [**7,** Theorem 1.3], we obtain $\sigma_c \le 1/3$. □

Theorem 1.7. *If, for positive δ, we denote by $N_{5/6+\delta}(T)$ the number of zeros of $\nu(s)$ in the region $|\mathrm{Im}(s)| \le T$, $5/6 + \delta \le \mathrm{Re}(s)$, then*

$$N_{5/6+\delta}(T) \ll_\delta T.$$

Proof. The zeros of $\nu(s)$ coincide with the zeros of $\phi(s)$. We will imitate the proof of [**8,** Theorem 6.18]. For $x_0 > 4.25$, and any integer m, set $K_{r,m}$ to be the circle with center $s_0 = x_0 + (1/2+m)i$ and radius $r = |x_0 - 5/6 - \delta + i/2|$. The circle passes through $5/6 + \delta + mi$ and $5/6+\delta+(m+1)i$. Increasing x_0, if necessary, the circle lies to the right of the line $\mathrm{Re}(s) = 5/6 + \delta/2$. Set $K_{R,m}$ to be the circle with center

$s_0 = x_0 + (1/2 + m)i$ and radius $R = x_0 - 5/6 - \delta/2$. Finally, let

$$A = A(x_0) = 2 \inf_{\mathrm{Re}(s)=x_0} |\phi(s)|.$$

The proof of Theorem 1.2 implies that $A > 0$. Now, [**8,** page 260, Corollary 2] a corollary to Jensen's theorem implies that there exists $C = C(r, R, A)$ such that the number of zeros of $\phi(s)$ in the rectangle

$$\frac{5}{6} + \delta \le \mathrm{Re}(s) \le x_0, \qquad m < \mathrm{Im}(s) \le m + 1,$$

does not exceed

$$C \cdot \iint_{K_{R,m}} |\phi(x+iy)|^2 dx\, dy \le C \cdot \int_{5/6+\delta/2}^{x_0+R} \int_{m+1/2-R}^{m+1/2+R} \left|\phi(x+iy)\right|^2 dy\, dx.$$

Summing over integers $m \in [-T, T]$, we deduce that

$$N_{5/6+\delta}(T) = O\bigg(\int_{5/6+\delta/2}^{x_0+R} \int_{-T+1/2-R}^{T+1/2+R} \left|\phi(x+iy)\right|^2 dy\, dx \bigg).$$

From [**8,** page 315, Corollary], we deduce that

$$\int_{5/6+\delta/2}^{x_0+R} \int_{-T+1/2-R}^{T+1/2+R} \left|\phi(x+iy)\right|^2 dy\, dx \ll_\delta T. \qquad \square$$

Remark 1.8. The referee pointed out a mistake in the proof of [**8,** page 315, Corollary] and supplied a correction. In the notation of that source for $x \ge 1/2 + \epsilon$, we have $2x - \epsilon > 1 + \epsilon$ so that $g(2x - \epsilon + it)$ converges absolutely. This is all the proof requires, not the reference to Bohr and uniform convergence.

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APPENDIX

A. Numerical methods. The graphics in Figures 2 and 3 require the numerical computation of $\zeta(s)\zeta''(s) - \zeta'(s)^2$ on a large grid of points in the complex plane. Numerical computation of derivatives of a function $f(x)$ is often done by a method called Richardson extrapolation [**9,**

subsection 5.7]. One has that

$$\frac{f(x+h)-f(x-h)}{2h} = f'(x) + \tfrac{1}{6}f^{(3)}(x)h^2 + O\bigl(h^4\bigr),$$
$$\frac{f(x+2h)-f(x-2h)}{4h} = f'(x) + \tfrac{2}{3}f^{(3)}(x)h^2 + O\bigl(h^4\bigr),$$

so an appropriate linear combination of the left sides of the two equations computes $f'(x)$ up to an error $O(h^4)$. This can readily be generalized to computation of each value on a rectangular grid of points of $\zeta(s)\zeta'' - \zeta'(s)^2$, up to an error $O(h^8)$, with (asymptotically) a single evaluation of $\zeta(s)$. One uses the saved function values at $\zeta(s \pm h)$, $\zeta(s \pm ih)$ and $\zeta(s + (\pm h \pm ih))$, as well as $\zeta(s)$, and the solution to a linear system of nine equations in nine unknowns.

ENDNOTES

1. See also the recent work of Rodgers [**10**], as well as Ford and Zaharescu [**5**].

2. For example, [**3**].

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