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KUMMER SURFACES AND K3 SURFACES WITH $(\mathbb{Z}/2\mathbb{Z})^4$ SYMPLECTIC ACTION

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ABSTRACT. In the first part of this paper we give a survey of classical results on Kummer surfaces with Picard number 17 from the point of view of lattice theory. We prove ampleness properties for certain divisors on Kummer surfaces, and we use them to describe projective models of Kummer surfaces of $(1,d)$-polarized abelian surfaces for $d=1,2,3$. As a consequence, we prove that, in these cases, the Néron-Severi group can be generated by lines.

In the second part of the paper we use Kummer surfaces to obtain results on K3 surfaces with a symplectic action of the group $(\mathbb{Z}/2\mathbb{Z})^4$. In particular, we describe the possible Néron-Severi groups of the latter in the case that the Picard number is 16, which is the minimal possible. We also describe the Néron-Severi groups of the minimal resolution of the quotient surfaces which have 15 nodes. We extend certain classical results on Kummer surfaces to these families.

1. Introduction. Kummer surfaces are particular K3 surfaces, obtained as minimal resolutions of the quotient of an abelian surface by an involution. They are algebraic and form a three-dimensional family of K3 surfaces. Kummer surfaces play a central role in the study of K3 surfaces; indeed, certain results on K3 surfaces are easier to prove for Kummer surfaces (due to their relation with abelian surfaces), but can be extended to more general families of K3 surfaces. The most classical example of this is the Torelli theorem, which holds for every K3 surface.

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The aim of this paper is to describe some results on Kummer surfaces, some of which are classical, and to prove that these results extend to four-dimensional families of K3 surfaces. Every Kummer surface has the following properties. It admits the group $(\mathbb{Z}/2\mathbb{Z})^4$ as a group of automorphisms which preserves the period (these automorphisms will be called *symplectic*) and it is also the desingularization of the quotient of a K3 surface by the group $(\mathbb{Z}/2\mathbb{Z})^4$ which acts preserving the period. The families of K3 surfaces with one of these properties are four-dimensional. We study these families using the results on Kummer surfaces, and we prove that several properties of the Kummer surfaces hold more in general for at least one of these families.

The first part of the paper (Sections 2, 3, 4 and 5) is devoted to Kummer surfaces. We first recall their construction and the definition of the Shioda-Inose structure which was introduced by Morrison **[33]**. In particular, we recall that every Kummer surface $Km(A)$ is the quotient of both an abelian surface and a K3 surface by an involution, cf., Proposition 2.16. Since we have these two descriptions of the same surface $Km(A)$, we also obtain two different descriptions of the Néron-Severi group of $Km(A)$, see Proposition 2.6 and Theorem 2.18. In Section 3, we recall that every Kummer surface admits certain automorphisms, and in particular, the group $(\mathbb{Z}/2\mathbb{Z})^4$ as a group of symplectic automorphisms. In Proposition 3.3, we show that the minimal resolution of the quotient of a Kummer surface $Km(A)$ by $(\mathbb{Z}/2\mathbb{Z})^4$ is again $Km(A)$. This gives a third alternative description of a Kummer surface and shows that the family of Kummer surfaces is a subfamily both of the family of K3 surfaces X admitting $(\mathbb{Z}/2\mathbb{Z})^4$ as a group of symplectic automorphisms and of the family of K3 surfaces Y which are quotients of some K3 surfaces by the group $(\mathbb{Z}/2\mathbb{Z})^4$.

The main results on Kummer surfaces are obtained in Section 4 and applied in Section 5. Nikulin **[36]** showed that a non empty set of disjoint smooth rational curves on a K3 surface can be the branch locus of a double cover only if it contains exactly 8 or 16 curves. In the first case, the surface which we obtain by taking the double cover and contracting the (-1)-curves is again a K3 surface. In the second case, the surface obtained in the same way is an abelian surface, and the K3 surface is in fact its Kummer surface.

In the sequel, we call *even sets* the sets of disjoint rational curves in the branch locus of a double cover. In **[13]**, we studied the Néron-

Severi group, the ampleness properties of divisors and the associated projective models of K3 surfaces which admit an even set of 8 rational curves. Here, we prove similar results for K3 surfaces admitting an even set of 16 rational curves i.e., for the Kummer surfaces as well. In Section 4, we prove that certain divisors on Kummer surfaces are nef, big and nef, or ample. In Section 5, we study some maps induced by the divisors considered previously, and we obtain projective models for the Kummer surfaces of the $(1, d)$-polarized abelian surfaces for $d = 1, 2, 3$. As a byproduct, we show that these Kummer surfaces have at least one model such that their Néron-Severi group is generated by lines. Several models described are already well known, but here we suggest a systematic way of producing projective models of Kummer surfaces by using lattice theory.

In the second part of the paper (Sections 6, 7, 8, 9 and 10) we apply the previous results on Kummer surfaces to obtain general results on K3 surfaces X with symplectic action by $(\mathbb{Z}/2\mathbb{Z})^4$ and on the minimal resolutions Y of the quotients $X/(\mathbb{Z}/2\mathbb{Z})^4$. In Theorem 7.1, Proposition 8.1 and Theorem 8.3, we explicitly describe $NS(X)$ and $NS(Y)$, and thus, we describe the families of K3 surfaces X and Y, proving that they are four-dimensional. We further specialize to the family of Kummer surfaces.

In **[25]**, Keum proves that every Kummer surface admits an *Enriques involution*, i.e., a fixed point free involution. Here, we prove that this property extends to every K3 surface X admitting a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$ and with Picard number 16 (the minimal possible). This shows that the presence of a certain group of symplectic automorphisms on a K3 surface implies the presence of a non-symplectic involution as well.

On the other hand, certain results proved for Kummer surfaces also hold for the K3 surfaces Y. In Proposition 8.5, we prove that certain divisors on Y are ample (or nef and big) as we did in Section 4 for Kummer surfaces. The surface Y admits 15 nodes, by construction. We recall that every K3 surface with 16 nodes is in fact a Kummer surface; we prove that similarly every K3 surface which admits 15 nodes is the quotient of a K3 surface by a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$. This result is not trivial; indeed, the analogue for a symplectic action of $\mathbb{Z}/2\mathbb{Z}$ is false, i.e., a K3 surface with eight nodes is not necessarily the quotient of a K3 surface by a symplectic involution. Moreover, we show in Theorem 8.3 that K3 surfaces with 15 nodes exist for polarizations of any degree (some examples are given in Section 10). This answers the question, what is the maximal number of nodes a K3 surface with a given polarization can have (when it does not contain further singularities)? If the polarization L with $L^2 = 2t$ has t even, then the maximal number is 16. It is precisely attained by Kummer surfaces; otherwise, this maximum is 15. Further, in Section 10, we give explicit examples of surfaces X and Y, and we describe their geometry.

2. Generalities on Kummer surfaces.

2.1. Kummer surfaces as quotients of abelian surfaces. Kummer surfaces are K3 surfaces constructed as desingularization of the quotient of an abelian surface A by an involution ι. Equivalently, they are K3 surfaces admitting an even set of 16 disjoint rational curves. We briefly recall the construction. Let A be an abelian surface (here we only consider the case of algebraic Kummer surfaces), and let ι be the involution $\iota : A \to A$, $a \mapsto -a$. Let A/ι be the quotient surface. It has 16 singular points of type A_1, which are the image under the quotient map, of the 16 points of the set

$$A[2] = \{a \in A \text{ such that } 2a = 0\}.$$

Let $\widetilde{A/\iota}$ be the desingularization of A/ι. The smooth surface $Km(A) := \widetilde{A/\iota}$ is a K3 surface. Consider the surface $\widetilde{A}$, obtained from A by blowing up the points in $A[2]$. The automorphism ι on A induces an automorphism $\widetilde{\iota}$ on $\widetilde{A}$ whose fixed locus are the 16 exceptional divisors of the blow up of A. Hence, the quotient $\widetilde{A}/\widetilde{\iota}$ is smooth. It is well known that $\widetilde{A}/\widetilde{\iota}$ is isomorphic to $Km(A)$ and that we have a commutative diagram:

$$\begin{tikzcd} \widetilde{A} \arrow[r, "\gamma"] \arrow[d] & A \arrow[d, "\pi_A"] \\ Km(A) \arrow[r] & A/\iota \end{tikzcd} \tag{2.1}$$

We observe that, on $\widetilde{A}$ there are 16 exceptional curves of the blow up of the 16 points of $A[2] \subset A$. These curves are fixed by the involution $\widetilde{\iota}$ and hence are mapped to 16 rational curves on $Km(A)$.

Each of these curves corresponds uniquely to a point of $A[2]$. Since $A[2] \simeq (\mathbb{Z}/2\mathbb{Z})^4$, we denote these 16 rational curves on $Km(A)$ by K_{a_1,a_2,a_3,a_4}, where $(a_1,a_2,a_3,a_4) \in (\mathbb{Z}/2\mathbb{Z})^4$. Since the points in $A[2]$ are fixed by the involution ι, the exceptional curves on $\widetilde{A}$ are fixed by $\widetilde{\iota}$, and so the curves K_{a_1,a_2,a_3,a_4} are the branch locus of the $2:1$ cyclic cover $\widetilde{A} \to Km(A)$. In particular, the curves K_{a_1,a_2,a_3,a_4} form an even set, i.e.,

$$\frac{1}{2}\left(\sum_{a_i \in \mathbb{Z}/2\mathbb{Z}} K_{a_1,a_2,a_3,a_4}\right) \in NS(Km(A)).$$

Definition 2.1 (cf., [**36**])**.** The *Kummer lattice*, the minimal primitive sublattice of $H^2(Km(A),\mathbb{Z})$ containing the 16 classes of the curves K_{a_1,a_2,a_3,a_4}, is denoted by K.

In [**36**], it is proved that a K3 surface X is a Kummer surface if and only if the Kummer lattice is primitively embedded in $NS(X)$.

Proposition 2.2. [**41,** Section 5 Appendix, Lemma 4]. *The lattice K is a negative definite even lattice of rank* 16. *Its discriminant is* 2^6.

Remark 2.3. Here we briefly recall the properties of K (these are well known and can be found, e.g., in [**4, 33, 41**]):

1) Let W be a hyperplane in the affine four-dimensional space $(\mathbb{Z}/2\mathbb{Z})^4$, i.e., W is defined by an equation of the type

$$\sum_{i=1}^{4} \alpha_i a_i = \epsilon,$$

where $\alpha_i, \epsilon \in \{0,1\}$, and $\alpha_i \neq 0$ for at least one $i \in \{1,2,3,4\}$. The hyperplane W consists of eight points. For every W, the class $(1/2)\sum_{p\in W} K_p$ is in K, and there are 30 classes of this kind.
2) The class $(1/2)\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4} K_p$ is in K.
3) Let $W_i = \{(a_1,a_2,a_3,a_4) \in A[2]$ be such that $a_i = 0\}$, $i = 1,2,3,4$. A set of generators (over $\mathbb{Z}$) of the Kummer lattice is given by the

classes:

$$\frac{1}{2}\Big(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4} K_p\Big), \quad \frac{1}{2}\sum_{p\in W_1} K_p,$$
$$\frac{1}{2}\sum_{p\in W_2} K_p, \qquad \frac{1}{2}\sum_{p\in W_3} K_p,$$
$$\frac{1}{2}\sum_{p\in W_4} K_p, \qquad K_{0,0,0,0}, \quad K_{1,0,0,0}, \quad K_{0,1,0,0},$$
$$K_{0,0,1,0}, \quad K_{0,0,0,1}, \quad K_{0,0,1,1}, \quad K_{0,1,0,1}, \quad K_{1,0,0,1},$$
$$K_{0,1,1,0}, \quad K_{1,0,1,0}, \quad K_{1,1,0,0}.$$

4) The discriminant form of K is isometric to the discriminant form of $U(2)^{\oplus 3}$. In particular, the discriminant group is $(\mathbb{Z}/2\mathbb{Z})^6$, there are 35 non zero elements on which the discriminant form takes value 0 and 28 non zero elements on which the discriminant form takes value 1.
5) With respect to the group of isometries of K, there are 3 orbits in the discriminant group: the orbit of zero, the orbit of the 35 non zero elements on which the discriminant form takes value 0 and the orbit of the 28 elements on which the discriminant form takes value 1.
6) Let V and V' be two-dimensional planes (they are the intersection of two hyperplanes in $(\mathbb{Z}/2\mathbb{Z})^4$ and thus isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$), such that
$$V \cap V' = \{(0,0,0,0)\}.$$
Denote by
$$V * V' := V \cup V' - (V \cap V').$$
Then the classes
$$w_4 := \frac{1}{2}\sum_{p\in V} K_p$$
are 35 classes in $K^\vee/K$, and the discriminant form on them takes value 0; the classes
$$w_6 := \frac{1}{2}\sum_{p\in V*V'} K_p$$
are 28 classes in $K^\vee/K$, and the discriminant form on them takes value 1, (see, e.g., [**10,** Proposition 2.1.13]).

7) Let

$$V_{i,j} = \{(0,0,0,0), \alpha_i, \alpha_j, \alpha_i + \alpha_j\} \subset (\mathbb{Z}/2\mathbb{Z})^4, \quad 1 \leq i, j \leq 4,$$

where

$$\begin{aligned} \alpha_1 &= (1,0,0,0), & \alpha_2 &= (0,1,0,0), \\ \alpha_3 &= (0,0,1,0), & \alpha_4 &= (0,0,0,1). \end{aligned}$$

Then,

$$\begin{aligned} &\frac{1}{2}\Big(\sum_{p\in V_{1,2}} K_p\Big), && \frac{1}{2}\Big(\sum_{p\in V_{1,3}} K_p\Big), \\ &\frac{1}{2}\Big(\sum_{p\in V_{1,4}} K_p\Big), && \frac{1}{2}\Big(\sum_{p\in V_{2,3}} K_p\Big), \\ &\frac{1}{2}\Big(\sum_{p\in V_{2,4}} K_p\Big), && \frac{1}{2}\Big(\sum_{p\in V_{3,4}} K_p\Big), \end{aligned}$$

generate the discriminant group of the Kummer lattice.

Here, we want to relate the Néron-Severi group of the abelian surface A with the Néron-Severi group of its Kummer surface $Km(A)$. Recall that, for an abelian variety A, we have $H^2(A,\mathbb{Z}) = U^{\oplus 3}$ (see, e.g., [**33,** Theorem-Definition 1.5]).

Proposition 2.4. *The isometry ι^* induced by ι is the identity on $H^2(A,\mathbb{Z})$.*

Proof. The harmonic two forms on A are $dx_i \wedge dx_j$, $i \neq j$, $i,j = 1,2,3,4$, where x_i are the local coordinates of A viewed as the real four-dimensional variety $(\mathbb{R}/\mathbb{Z})^4$. By the definition of ι, we have:

$$dx_i \wedge dx_j \stackrel{\iota^*}{\longmapsto} d(-x_i) \wedge d(-x_j) = dx_i \wedge dx_j.$$

So ι induces the identity on $H^2(A,\mathbb{R}) = H^2(A,\mathbb{Z}) \otimes \mathbb{R}$, and hence on $H^2(A,\mathbb{Z})$, since $H^2(A,\mathbb{Z})$ is torsion free. $\square$

Let $\widetilde{A}$ be the blow up of A in the 16 fixed points of the involution ι, and let $\pi_A : A \to A/\iota$ be the $2:1$ cover. With a slight abuse of notation, we denote by π_A also the $2:1$ cover $\widetilde{A} \to Km(A)$.

As in [**33,** Section 3], let $H_{\tilde{A}}$ be the orthogonal complement in $H^2(\widetilde{A},\mathbb{Z})$ of the exceptional curves, and let $H_{Km(A)}$ be the orthogonal complement in $H^2(Km(A),\mathbb{Z})$ of the 16 (-2)-curves on $Km(A)$. Then $H_{\tilde{A}} \cong H^2(A,\mathbb{Z})$, and there are the natural maps (see [**33,** Section 3]):

$$\begin{aligned} \pi_A^* &: H_{Km(A)} \longrightarrow H_{\tilde{A}} \cong H^2(A,\mathbb{Z}); \\ \pi_{A_*} &: H^2(A,\mathbb{Z}) \cong H_{\tilde{A}} \longrightarrow H_{Km(A)} \subset H^2(Km(A),\mathbb{Z}). \end{aligned}$$

Lemma 2.5. *We have*

$$\pi_{A*}(U^{\oplus 3}) = \pi_{A*}(H^2(A,\mathbb{Z})^{\iota^*}) = H^2(A,\mathbb{Z})^{\iota^*}(2) = U^{\oplus 3}(2).$$

Proof. Follows from [**33,** Lemma 3.1] and Proposition 2.4. □

By this lemma, we can write

$$\Lambda_{K3} \otimes \mathbb{Q} \cong H^2(Km(A),\mathbb{Q}) \simeq \left(U(2)^{\oplus 3} \oplus \langle -2\rangle^{\oplus 16}\right) \otimes \mathbb{Q}.$$

The lattice $U(2)^{\oplus 3} \oplus \langle -2\rangle^{\oplus 16}$ has index 2^{11} in $\Lambda_{K3} \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.

Proposition 2.6. *Let $Km(A)$ be the Kummer surface associated to the abelian surface A. Then the Picard number of $Km(A)$ is $\rho(Km(A)) = \rho(A) + 16$, in particular, $\rho(Km(A)) \geq 17$.*

The transcendental lattice of $Km(A)$ is $T_{Km(A)} = T_A(2)$. The Néron-Severi group $NS(Km(A))$ is an overlattice $\mathcal{K}'_{NS(A)}$ of $NS(A)(2) \oplus K$ and

$$[NS(Km(A)) : (NS(A)(2) \oplus K)] = 2^{\rho(A)}.$$

Proof. We have that

$$\pi_{A_*}(NS(A) \oplus T_A) = NS(A)(2) \oplus T_A(2),$$

and this lattice is orthogonal to the 16 (-2)-classes in $H^2(Km(A),\mathbb{Z})$ arising from the desingularization of A/ι.

Since π_{A_*} preserves the Hodge decomposition, we have

$$NS(A)(2) \subset NS(Km(A))$$

and

$$T_A(2) = T_{Km(A)}$$

(cf., [**33,** Proposition 3.2]). Hence, the Néron-Severi group of $Km(A)$ is an overlattice of finite index of $NS(A)(2) \oplus K$. In fact, we have

$$\begin{aligned}\operatorname{rank} NS(Km(A)) &= 22 - \operatorname{rank} T_A = 22 - (6 - \operatorname{rank}(NS(A))) \\ &= 16 + \operatorname{rank}(NS(A)) = \operatorname{rank}(NS(A)(2) \oplus K).\end{aligned}$$

The index of this inclusion is computed comparing the discriminant of these two lattices; indeed,

$$2^{6-\rho(A)} d(T_A) = d(T_{Km(A)}) = d(NS(Km(A))),$$

and

$$d(NS(A)(2) \oplus K) = 2^6 2^{\rho(A)} d(NS(A)) = 2^{6+\rho(A)} d(T_A);$$

thus,

$$\begin{aligned}&d(NS(A)(2) \oplus K)/d(NS(Km(A))) \\ &\qquad = 2^{6+\rho(A)} d(T_A)/2^{6-\rho(A)} d(T_A) = 2^{2\rho(A)},\end{aligned}$$

which is equal to

$$[NS(Km(A)) : (NS(A)(2) \oplus K)]^2$$

(see, e.g., [**4,** Chapter I, Lemma 2.1]). □

Now, we will consider the generic case, i.e., the case of Kummer surfaces with Picard number 17. By Proposition 2.6, if $Km(A)$ has Picard number 17, then its Néron-Severi group is an overlattice, $\mathcal{K}'_{4d}$, of index 2 of

$$NS(A)(2) \oplus K \simeq \mathbb{Z}H \oplus K,$$

where $H^2 = 4d$, $d > 0$. In the next proposition, we describe the possible overlattices of $\mathbb{Z}H \oplus K$ with $H^2 = 4d$, and hence, the possible Néron-Severi groups of the Kummer surfaces with Picard number 17.

Theorem 2.7. *Let $Km(A)$ be a Kummer surface with Picard number 17, and let H be a divisor generating $K^\perp \subset NS(Km(A))$, $H^2 > 0$. Let d be a positive integer such that $H^2 = 4d$, and let $\mathcal{K}_{4d} := \mathbb{Z}H \oplus K$. Then*

$$NS(Km(A)) = \mathcal{K}'_{4d},$$

where $\mathcal{K}'_{4d}$ is generated by $\mathcal{K}_{4d}$ and by a class $(H/2, v_{4d}/2)$, with:

- $v_{4d} \in K$, $v_{4d}/2 \notin K$ *and* $v_{4d}/2 \in K^\vee$*, in particular,* $v_{4d} \cdot K_i \in 2\mathbb{Z}$*);*
- $H^2 \equiv -v_{4d}^2 \mod 8$*, in particular,* $v_{4d}^2 \in 4\mathbb{Z}$*.*

The lattice $\mathcal{K}'_{4d}$ *is the unique even lattice, up to isometry, such that* $[\mathcal{K}'_{4d} : \mathcal{K}_{4d}] = 2$ *and* K *is a primitive sublattice of* $\mathcal{K}'_{4d}$*. Hence, one can assume that,*
if $H^2 \equiv 0 \mod 8$*, then*

$$v_{4d} = \sum_{p \in V_{1,2}} K_p = K_{0,0,0,0} + K_{1,0,0,0} + K_{0,1,0,0} + K_{1,1,0,0};$$

if $H^2 \equiv 4 \mod 8$*, then*

$$\begin{aligned} v_{4d} &= \sum_{p \in (V_{1,2} * V_{3,4})} K_p \\ &= K_{0,0,0,1} + K_{0,0,1,0} + K_{0,0,1,1} + K_{1,0,0,0} + K_{0,1,0,0} + K_{1,1,0,0}. \end{aligned}$$

Proof. The conditions on v_{4d} for constructing the lattice $\mathcal{K}_{4d}$ can be proved as in [**13,** Proposition 2.1]. The uniqueness of $\mathcal{K}'_{4d}$ and the choice of v_{4d} follows from the description of the orbits under the group of isometries of K on the discriminant group $K^\vee/K$, see Remark 2.3. □

Remark 2.8. (cf., [**4, 11**]).

1) Let

$$\omega_{ij} := \pi_{A*}(\gamma^*(dx_i \wedge dx_j)), \quad i < j,\ i,j = 1,2,3,4,$$

(we use the notation of diagram (2.1) and as above π_A denotes both $\widetilde{A} \to Km(A)$ and $A \to A/\iota$). The six vectors $\omega_{i,j}$ form the basis of $U(2)^{\oplus 3}$. The lattice generated by the Kummer lattice K and by the six classes

$$u_{ij} = \frac{1}{2}\left(\omega_{ij} + \sum K_{a_1,a_2,a_3,a_4}\right),$$

where the sum is over $(a_1,a_2,a_3,a_4) \in (\mathbb{Z}/2\mathbb{Z})^4$ such that $a_i = a_j = 0$, $\{i,j,h,k\} = \{1,2,3,4\}$ and $h < k$, is isometric to Λ_{K3}.

2) Observe that since for each $d \in \mathbb{Z}_{>0}$ there exist abelian surfaces with Néron-Severi group isometric to $\langle 2d \rangle$ for each d there exist Kummer surfaces with Néron-Severi group isomorphic to $\mathcal{K}'_{4d}$.

Let $\mathcal{F}_d$, $d \in \mathbb{Z}_{>0}$, denote the family of $\mathcal{K}'_{4d}$-polarized K3 surfaces. Then:

Corollary 2.9. *The moduli space of the Kummer surfaces has a countable number of connected irreducible components, which are the $\mathcal{F}_d$, $d \in \mathbb{Z}_{>0}$.*

Proof. Every Kummer surface is polarized with a lattice $\mathcal{K}'_{4d}$, for some d, by Proposition 2.6 and Theorem 2.7. On the other hand, if a K3 surface is $\mathcal{K}'_{4d}$ polarized, then there exists a primitive embedding of K in its Néron-Severi group and, by **[36,** Theorem 1], it is a Kummer surface. $\square$

Remark 2.10. The classes of type $(H + v_{4d} + \sum_{p\in W} K_p)/2$, where H and v_{4d} are as in Theorem 2.6 and W is a hyperplane of $(\mathbb{Z}/2\mathbb{Z})^4$, are classes in $\mathcal{K}'_{4d}$. We describe these kinds of classes modulo the lattice $\oplus_{p\in(\mathbb{Z}/2\mathbb{Z})^4}\mathbb{Z}K_p$. We use the notation of Theorem 2.7.

If $H^2 = 4d \equiv 0 \mod 8$, the lattice $\mathcal{K}'_{4d}$ contains:

- four classes of type $(H - \sum_{p\in J_4} K_p)/2$ for certain $J_4 \subset (\mathbb{Z}/2\mathbb{Z})^4$ which contain four elements. These classes are $(H + v_{4d})/2$ and the classes $(H + v_{4d} + \sum_{p\in W} K_p)/2$, where
$$W \supset \{(0,0,0,0),\ (1,0,0,0),\ (0,1,0,0),\ (1,1,0,0)\};$$
- 24 classes of type $(H - \sum_{p\in J_8} K_p)/2$ for certain $J_8 \subset (\mathbb{Z}/2\mathbb{Z})^4$ which contain eight elements. These classes are $(H + v_{4d} + \sum_{p\in W} K_p)/2$, where $W \cap \{(0,0,0,0),\ (1,0,0,0),\ (0,1,0,0),\ (1,1,0,0)\}$ contains two elements.
- Four classes of type $(H - \sum_{p\in J_{12}} K_p)/2$ for certain $J_{12} \subset (\mathbb{Z}/2\mathbb{Z})^4$ which contain 12 elements. These classes are $(H + v_{4d} + \sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4} K_p)/2$ and the classes $(H + v_{4d} + \sum_{p\in W} K_p)/2$, where
$$W \cap \{(0,0,0,0),\ (1,0,0,0),\ (0,1,0,0),\ (1,1,0,0)\} = \emptyset.$$

If $H^2 = 4d \equiv 4 \mod 8$, the lattice $\mathcal{K}'_{4d}$ contains:

- 16 classes of type $(H - \sum_{p\in J_6} K_p)/2$ for certain $J_6 \subset (\mathbb{Z}/2\mathbb{Z})^4$ which contain six elements. These classes are $(H + v_{4d})/2$ and the classes $(H+v_{4d}+\sum_{p\in W}K_p)/2$, where

$$W\cap\{(1,0,0,0),\ (0,1,0,0),\ (1,1,0,0),\ (0,0,0,1),\ (0,0,1,0),\ (0,0,1,1)\}$$

contains four elements;

- 16 classes of type $(H-\sum_{p\in J_{10}}K_p)/2$ for certain $J_{10}\subset(\mathbb{Z}/2\mathbb{Z})^4$ which contain 10 elements. These classes are $(H+v_{4d}+\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4}K_p)/2$ and the classes $(H+v_{4d}+\sum_{p\in W}K_p)/2$, where

$$W\cap\{(1,0,0,0),\ (0,1,0,0),\ (1,1,0,0),\ (0,0,0,1),\ (0,0,1,0),\ (0,0,1,1)\}$$

contains 2 elements.

Remark 2.11. The discriminant group of $\mathcal{K}'_{4d}$ is generated by:

$$\begin{aligned}(H/4d)+\frac{1}{2}\Big(\sum_{p\in V_{3,4}}K_p\Big),\quad &\frac{1}{2}\Big(\sum_{p\in V_{1,3}}K_p\Big),\\ \frac{1}{2}\Big(\sum_{p\in V_{1,4}}K_p\Big),\quad &\frac{1}{2}\Big(\sum_{p\in V_{2,3}}K_p\Big),\\ \frac{1}{2}\Big(\sum_{p\in V_{2,4}}K_p\Big),&\end{aligned}$$

if $H^2=4d\equiv 0 \mod 8$, and by

$$\begin{aligned}(H/4d)+\frac{1}{2}\Big(\sum_{p\in V_{1,2}}K_p\Big),\quad &\frac{1}{2}\Big(\sum_{p\in V_{1,3}}K_p\Big),\\ \frac{1}{2}\Big(\sum_{p\in V_{1,4}}K_p\Big),\quad &\frac{1}{2}\Big(\sum_{p\in V_{2,3}}K_p\Big),\\ \frac{1}{2}\Big(\sum_{p\in V_{2,4}}K_p\Big),&\end{aligned}$$

if $H^2=4d\equiv 4 \mod 8$.

2.2. Kummer surfaces as K3 surfaces with 16 nodes. Let S be a surface with n nodes, and let $\widetilde{S}$ be its minimal resolution. On $\widetilde{S}$, there are n disjoint rational curves which arise from the resolution of the nodes of S. If $\widetilde{S}$ is a K3 surface, then $n\leq 16$, [**36,** Corollary 1]. By

[**36,** Theorem 1], if a K3 surface admits 16 disjoint rational curves, then they form an even set and the K3 surface is in fact a Kummer surface. Conversely, as remarked in the previous section, every Kummer surface contains 16 disjoint rational curves. Thus, the Kummer surfaces are the K3 surfaces admitting the maximal numbers of disjoint rational curves or, equivalently, they are the K3 surfaces which admit a singular model with the maximal number of nodes.

2.3. Kummer surfaces as a quotient of K3 surfaces.

Definition 2.12. (cf., [**33,** Definition 5.1]). An involution ι on a K3 surface Y is a *Nikulin involution* if $\iota^*\omega = \omega$ for every $\omega \in H^{2,0}(Y)$.

Every Nikulin involution has eight isolated fixed points and the minimal resolution X of the quotient Y/ι is again a K3 surface ([**38,** Sections 5, 11]). The minimal primitive sublattice of $NS(X)$ containing the eight exceptional curves from the resolution of the singularities of Y/ι is called the *Nikulin lattice*, and it is denoted by N, its discriminant is 2^6.

Definition 2.13. (cf., [**15**]). A Nikulin involution ι on a K3 surface Y is a *Morrison-Nikulin* involution if ι^* switches two orthogonal copies of $E_8(-1)$ embedded in $NS(Y)$.

By definition, if Y admits a Morrison-Nikulin involution, then $E_8(-1) \oplus E_8(-1) \subset NS(Y)$. A Morrison-Nikulin involution has the following properties (cf., [**33,** Theorems 5.7, 6.3]):

- $T_X = T_Y(2)$;
- the lattice $N \oplus E_8(-1)$ is primitively embedded in $NS(X)$;
- the lattice K is primitively embedded in $NS(X)$, and so X is a Kummer surface.

Definition 2.14. (cf., [**33,** Definition 6.1]). Let Y be a K3 surface and ι a Nikulin involution on Y. The pair (Y, ι) is a *Shioda-Inose structure* if the rational quotient map $\pi : Y \dashrightarrow X$ is such that X is a Kummer surface and π_* induces a Hodge isometry $T_Y(2) \cong T_X$.

The situation is shown in the following diagram (A^0 denotes an abelian surface):

$$\begin{tikzcd} & A^0 \arrow[dl] \arrow[dr, dashed] & & Y \arrow[dl, dashed] \arrow[dr] & \\ A^0/i & & X = Km(A^0) \arrow[ll] \arrow[rr] & & Y/\iota \end{tikzcd}$$

We have $T_Y \cong T_{A^0}$ by [**33,** Theorem 6.3].

Let Y be a K3 surface and ι a Nikulin involution on Y. By [**33,** Theorems 5.7, 6.3], we conclude that:

Corollary 2.15. *A pair (Y, ι) is a Shioda-Inose structure if and only if ι is a Morrison-Nikulin involution.*

For the next result, see [**40,** Lemma 2].

Proposition 2.16. *Every Kummer surface is the desingularization of the quotient of a* K3 *surface by a Morrison-Nikulin involution, i.e., it is associated to a Shioda-Inose structure.*

Remark 2.17. In [**40,** Lemma 5], it is proved that, if X is a K3 surface with $\rho(X) = 20$, then each Shioda-Inose structure is induced by the same abelian surface. This means that, if (X, ι_1) and (X, ι_2) are Shioda-Inose structures and $Y_i = Km(B_i)$ is the Kummer surface minimal resolution of X/ι_i, $i = 1, 2$, then $B_1 = B_2$, and so $Y_1 = Y_2$.

By Proposition 2.16, it follows that Kummer surfaces can also be defined as K3 surfaces which are desingularizations of the quotients of K3 surfaces by Morrison-Nikulin involutions. This definition leads to a different description of the Néron-Severi group of a Kummer surface, which we give in the following.

Theorem 2.18. *Let Y be a* K3 *surface admitting a Morrison-Nikulin involution ι. Then $\rho(Y) \geq 17$ and $NS(Y) \simeq R \oplus E_8(-1)^2$, where R is an even lattice with signature $(1, \rho(Y) - 17)$. Let X be the desingularization of Y/ι. Then $NS(X)$ is an overlattice of index $2^{(\mathrm{rank}(R))}$ of $R(2) \oplus N \oplus E_8(-1)$.*

In particular, if $\rho(Y) = 17$*, then*

$$NS(Y) \simeq \langle 2d \rangle \oplus E_8(-1)^2.$$

The surface X *is the Kummer surface of the* $(1,d)$*-polarized abelian surface and the Néron-Severi group of* X *is an overlattice of index* 2 *of* $\langle 4d \rangle \oplus N \oplus E_8(-1)$*.*

Proof. By [**33,** Theorem 6.3], and the fact that $E_8(-1)$ is unimodular, one can write

$$NS(Y) = R \oplus E_8(-1)^{\oplus 2},$$

with R even of signature $(1, \rho(Y) - 17)$. In [**33,** Theorem 5.7], it is proved that $N \oplus E_8(-1)$ is primitively embedded in $NS(X)$. Thus, arguing on the discriminant of the transcendental lattices of Y and X and on the lattice R as in Proposition 2.6, one concludes the first part of the proof. For the last part of the assertion observe that the lattices $NS(Y)$ and T_Y are uniquely determined by their signature and discriminant forms ([**33,** Theorem 2.2]), so $T_Y = \langle -2d \rangle \oplus U^2$. By construction,

$$T_Y(2) = T_X = T_{A^0}(2) \quad \text{so} \quad T_{A^0} = T_Y.$$

This uniquely determines $NS(A^0)$, which is isometric to $\langle 2d \rangle$. Hence, A^0 is a $(1,d)$-polarized abelian surface. $\square$

The overlattices $\mathcal{N}'_{2d}$ of index 2 of $\langle 2d \rangle \oplus N$ are described in [**13**] and, by Theorem 2.18, we conclude that, if $\rho(Y) = 17$, then $NS(X) \simeq \mathcal{N}'_{4d} \oplus E_8(-1)$.

Remark 2.19. Examples of Shioda-Inose structures on K3 surfaces with Picard number 17 are given, e.g., in [**9,** Appendix], [**15, 27, 30, 43**]. In all of these papers, the Morrison-Nikulin involutions of Shioda-Inose structures are induced of a translation by a 2-torsion section on an elliptic fibration. In particular, in [**27**], all the Morrison-Nikulin involutions induced in such a way on elliptic fibrations with a finite Mordell-Weil group are classified.

Remark 2.20. Proposition 2.6 and Theorem 2.18 give two different descriptions of the same lattice (the Néron-Severi group of a Kummer surface of Picard number 17). The first one is associated to the construction of the Kummer surface as a quotient of an abelian surface; the second one is associated to the construction of the same surface as a quotient of another K3 surface. In general, it is an open problem to pass from one description to the other, and hence, to find the relation between these two constructions of a Kummer surface. However, in certain cases, this relation is known. In [**35**], Naruki describes the Néron-Severi group of the Kummer surface of the Jacobian of a curve of genus 2 as in our Proposition 2.6, and he determines a nef divisor that gives a $2:1$ map to $\mathbb{P}^2$ (we describe this map in subsection 5.1). Then, 16 curves on $\mathbb{P}^2$ are constructed, and it is proved that their pull backs on the Kummer surface generate the lattice $N \oplus E_8(-1)$. Similarly, this relation is known if the abelian surface is $E \times E'$, the product of two non isogenous elliptic curves E and E'. In [**39**], the Néron-Severi group of $Km(E \times E')$ is described as in Proposition 2.6. Then the elliptic fibrations on this K3 surface are classified. In particular, there exists an elliptic fibration with a fiber of type II^* and two fibers of type I_0^*. The components of II^* which do not intersect the 0 section generate a lattice isometric to $E_8(-1)$ and are orthogonal to the components of I_0^*. The components with multiplicity 1 of the two fibers of type I_0^* generate a lattice isometric to N and orthogonal to the copy of $E_8(-1)$ that we have described before. Thus, one has an explicit relation between the two descriptions of the Néron-Severi group.

3. Automorphisms on Kummer surfaces. In general, it is a difficult problem to describe the full automorphism group of a given K3 surface. However, for certain Kummer surfaces it is known. For example, the group of automorphisms of the Kummer surface of the Jacobian of a curve of genus 2 is described in [**24, 29**]. Similarly, the group $\mathrm{Aut}(Km(E \times F))$ is determined in [**26**] in the cases: E and F generic and non isogenous, E and F generic and isogenous, E and F isogenous and with complex multiplication.

A different approach to the study of the automorphisms of K3 surfaces is to fix a particular group of automorphisms and to describe the families of K3 surfaces admitting such (sub)groups of automorphisms. For this point of view the following two known results (Propositions 3.1 and 3.3) assure that every Kummer surface admits particular automorphisms. Moreover, we also prove a result (Proposition 3.5), which limits the list of the admissible finite group of symplectic automorphisms on a generic Kummer surface.

3.1. Enriques involutions on Kummer surfaces. We recall that an *Enriques involution* is a fixed point free involution on a K3 surface.

Proposition 3.1. ([**25,** Theorem 2]). *Every Kummer surface admits an Enriques involution.*

To prove the proposition, in [**25**], the following is first shown (see [**19, 37**]).

Proposition 3.2. ([**25,** Theorem 1]). *A* K3 *surface admits an Enriques involution if and only if there exists a primitive embedding of the transcendental lattice of the surface in* $U\oplus U(2)\oplus E_8(-2)$*, such that its orthogonal complement does not contain classes with self-intersection equal to* -2*.*

In [**25**], the author applies the proposition to the transcendental lattice of any Kummer surface. We observe that this does not give an explicit geometric description of the Enriques involution.

3.2. Finite groups of symplectic automorphisms on Kummer surfaces.

Proposition 3.3. (see, e.g., [**11**]). *The group* $G = (\mathbb{Z}/2\mathbb{Z})^4$ *acts symplectically on every Kummer surface* $Km(A)$*. The elements of* G *are induced by the translation by points of order* 2 *on the abelian surface* A *and the desingularization of* $Km(A)/G$ *is isomorphic to* $Km(A)$; *thus, every Kummer surface is also the desingularization of the quotient of a Kummer surface by* $(\mathbb{Z}/2\mathbb{Z})^4$*.*

Proof. Let $A[2]$ be the group generated by 2-torsion points. This is isomorphic with $(\mathbb{Z}/2\mathbb{Z})^4$, it operates on A by translation and commutes with the involution ι. Hence, it induces an action of $G = (\mathbb{Z}/2\mathbb{Z})^4$ on $Km(A)$, and thus on $H^2(Km(A),\mathbb{Z})$. Observe that G leaves the lattice $U(2)^{\oplus 3}\simeq\langle\omega_{ij}\rangle$ invariant; in fact, G, as a group generated by translations on A, does not change the real two forms $dx_i\wedge dx_j$. Since

$T_{Km(A)} \subset U(2)^{\oplus 3}$, the automorphisms induced on $Km(A)$ by G are symplectic. Moreover, since ι and G commute, we obtain that the surfaces $Km(A/A[2])$ and $\widetilde{Km(A)}/G$ are isomorphic. Finally, from the exact sequence,

$$0 \longrightarrow A[2] \longrightarrow A \xrightarrow{\cdot 2} A \longrightarrow 0,$$

we have $A/A[2] \cong A$, and so

$$\widetilde{Km(A)}/G \simeq Km(A/A[2]) \simeq Km(A). \quad \square$$

Remark 3.4. One can also consider the quotient of $Km(A)$ by subgroups of $G = (\mathbb{Z}/2\mathbb{Z})^4$, for example, by one involution. Such an involution is induced by translation by a point of order 2. Take the abelian surface $A \cong \mathbb{R}^4/\Lambda$, where $\Lambda = \langle 2e_1, e_2, e_3, e_4\rangle$ and consider the translation t_{e_1} by e_1. Thus, $A/\langle t_{e_1}\rangle$ is the abelian surface $B := \mathbb{R}^4/\langle e_1, e_2, e_3, e_4\rangle$. So the desingularization of the quotient of $Km(A)$ by the automorphism induced by t_{e_1} is again a Kummer surface and, more precisely, it is $Km(B)$. In particular, if $NS(A) = \langle 2d\rangle$, then $NS(B) = \langle 4d\rangle$, **[6]**. This implies that, if $NS(Km(A)) \simeq \mathcal{K}'_{4d}$, then $NS(Km(B)) \simeq \mathcal{K}'_{8d}$. Analogously, one can consider the subgroups $G_n = (\mathbb{Z}/2\mathbb{Z})^n \subset G$ (generated by translations), $n = 1, 2, 3$: if $NS(Km(A)) \simeq \mathcal{K}'_{4d}$, then $NS(Km(A/G_n)) \simeq \mathcal{K}'_{4\cdot 2^n \cdot d}$.

Proposition 3.5. *Let G be a finite group of symplectic automorphisms of a Kummer surface $Km(A)$, where A is a $(1, d)$-polarized abelian surface and $\rho(A) = 1$. Then G is either a subgroup of $(\mathbb{Z}/2\mathbb{Z})^4$, $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$.*

Proof. Let G be a finite group acting symplectically on a K3 surface, and denote by Ω_G the orthogonal complement of the G-invariant sublattice of the K3 lattice Λ_{K3}. An algebraic K3 surface admits the group G of symplectic automorphisms if and only if Ω_G is primitively embedded in the Néron-Severi group of the K3 surface, cf., **[18, 38]**; hence, the Picard number is greater than or equal to $\operatorname{rank}(\Omega_G) + 1$. The list of the finite groups acting symplectically on a K3 surface and the values of $\operatorname{rank}(\Omega_G)$ can be found in [**47,** Table 2] (observe that Xiao considers the lattice generated by the exceptional curves in the minimal resolution of the quotient; he denotes its rank by c. This is the same as $\operatorname{rank}(\Omega_G)$ by [**23,** Corollary 1.2]).

Since we are considering Kummer surfaces such that $\rho(Km(A)) = 17$, if G acts symplectically, then $\mathrm{rank}(\Omega_G) \leq 16$. This gives the following list of admissible groups G:

$$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^i \qquad \text{for } i = 1, 2, 3, 4;$$
$$\frac{\mathbb{Z}}{n\mathbb{Z}} \qquad \text{for } n = 3, 4, 5, 6;$$

$\mathcal{D}_m$ for $m = 3, 4, 5, 6$ ($\mathcal{D}_m$ is the dihedral group of order $2m$);

$$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{4\mathbb{Z}}; \quad \left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^2; \quad \frac{\mathbb{Z}}{2\mathbb{Z}} \times \mathcal{D}_4;$$
$$\mathfrak{A}_{3,3}, \text{ (see [34] for the definition); } \mathfrak{A}_4.$$

We can exclude that G acts symplectically on a Kummer surface for all the listed cases except $(\mathbb{Z}/2\mathbb{Z})^i$, $i = 1, 2, 3, 4$, $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ by considering the rank and the *length* of the lattice Ω_G, which is the minimal number of generators of the discriminant group. For example, let us consider the case $G = \mathcal{D}_3$. The lattice $\Omega_{\mathcal{D}_3}$ is an even negative definite lattice of rank 14. Since the group $\mathcal{D}_3$ can be generated by two involutions, $\Omega_{\mathcal{D}_3}$ is the sum of two (non orthognal) copies of $\Omega_{\mathbb{Z}/2\mathbb{Z}} \simeq E_8(-2)$ and admits $\mathcal{D}_3$ as a group of isometries (cf., [**12,** Remark 7.9]). In fact, $\Omega_{\mathcal{D}_3} \simeq DIH_6(14)$, where $DIH_6(14)$ is the lattice described in [**16,** Section 6]. The discriminant group of $\Omega_{\mathcal{D}_3} \simeq DIH_6(14)$ is $(\mathbb{Z}/3\mathbb{Z})^3 \times (\mathbb{Z}/6\mathbb{Z})^2$ [**16,** Table 8]. If $\mathcal{D}_3$ acts symplectically on $Km(A)$, $NS(Km(A))$ is an overlattice of finite index of $\Omega_{\mathcal{D}_3} \oplus R$ where R is a lattice of rank 3. But there are no overlattices of finite index of $\Omega_{\mathcal{D}_3} \oplus R$ with discriminant group $(\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/2d\mathbb{Z}$, which is the discriminant group of $NS(Km(A))$. Indeed, for every overlattice of finite index of $\Omega_{\mathcal{D}_3} \oplus R$, since the rank of R is 3, the discriminant group contains at least two copies of $\mathbb{Z}/3\mathbb{Z}$.

In order to exclude all the other groups G listed before, one must know the rank and the discriminant group of Ω_G. This can be found in [**14,** Proposition 5.1] if G is abelian; in [**12,** Propositions 7.6, 8.1] if $G = \mathcal{D}_m$, $m = 4, 5, 6$, $G = \mathbb{Z}/2\mathbb{Z} \times \mathcal{D}_4$ and $G = \mathfrak{A}_{3,3}$; in [**5,** subsection 4.1.1] if $G = \mathfrak{A}_4$. □

Remark 3.6. We cannot exclude the presence of symplectic automorphisms of orders 3 or 4 on a Kummer surface with Picard number 17, but we have no explicit examples of such an automorphism. It is known that there are no automorphisms of such a type on $Km(A)$, if A is principally polarized, cf., **[24, 29]**. If $Km(A)$ admits a symplectic action of $\mathbb{Z}/3\mathbb{Z}$, then $d \equiv 0 \mod 3$. This follows comparing the lengths of $\Omega_{\mathbb{Z}/3\mathbb{Z}}$ and of $NS(Km(A))$ as in the proof of Proposition 3.5. Moreover, the automorphism of order 3 generates an infinite group of automorphisms with any symplectic involution on $Km(A)$. Otherwise, if it generates a finite group, it must be one of the groups listed in Proposition 3.5, but there are no groups in this list containing both an element of order 2 and one of order 3.

3.3. Morrison-Nikulin involutions on Kummer surfaces. Examples of certain symplectic automorphisms on a Kummer surface (the Morrison-Nikulin involutions) come from the Shioda-Inose structure. We recall that every K3 surface with Picard number at least 19 admits a Morrison-Nikulin involution. In particular, this holds true for Kummer surfaces of Picard number at least 19. This is false for Kummer surfaces with a lower Picard number. In fact, since a Kummer surface with a Morrison-Nikulin involution also admits a Shioda-Inose structure as shown in subsection 2.3, it suffices to prove the following.

Corollary 3.7. *Letting $Y \cong Km(B)$ be a Kummer surface of Picard number* 17 *or* 18, *then Y does not admit a Shioda-Inose structure.*

Proof. If a K3 surface Y admits a Shioda-Inose structure, then by Theorem 2.18 we can write $NS(Y) = R \oplus E_8(-1)^2$ with R, an even lattice of rank 1 or 2. Hence, the length of $NS(Y)$ satisfies $l(A_{NS(Y)}) \leq 2$. It follows immediately that we also have $l(A_{T_Y}) \leq 2$. Let $e_1, \ldots, e_i$, $i = 5$, respectively 4, be the generators of T_Y. Since $Y \cong Km(B)$, we have that $T_Y = T_B(2)$ and so the classes $e_i/2$ are independent elements of $T_Y^\vee/T_Y$; thus, we have $2 \geq l(A_{T_Y}) \geq 4$, which is a contradiction. $\square$

In the case where the Picard number is 19 we can give a more precise description of the Shioda-Inose structure.

Proposition 3.8. *Let $Y \simeq Km(B)$ be a Kummer surface, $\rho(Y) = 19$ (so Y admits a Morrison-Nikulin involution ι). Let $Km(A^0)$ be the Kummer surface which is the desingularization of Y/ι. Then A^0 is not a product of two elliptic curves.*

Proof. If $A^0 = E_1 \times E_2$, E_i, $i = 1, 2$, an elliptic curve, then the classes of E_1 and E_2 in $NS(A^0)$ span a lattice isometric to U. To prove that A^0 is not such a product it suffices to prove that there is no primitive embedding of U in $NS(A^0)$. Assume the contrary. Then $NS(A^0) = U \oplus \mathbb{Z}h$, so $\ell(NS(A^0)) = 1$. Since $Y \simeq Km(B)$ is a Kummer surface, $T_Y \simeq T_B(2)$, and thus $T_{A^0} \simeq T_B(2)$. This implies that $1 = \ell(NS(A^0)) = \ell(T_{A^0}) = 3$, which is a contradiction. $\square$

4. Ampleness of divisors on Kummer surfaces. In this section, we consider projective models of Kummer surfaces with Picard number 17. The main idea is that we can check whether a divisor is ample, nef, or big and nef (which is equivalent to pseudo ample) because we have a complete description of the Néron-Severi group and so of the (-2)-curves. Hence, we can apply the following criterion (see [**4,** Proposition 3.7]).

Let L be a divisor on a K3 *surface such that $L^2 \geq 0$. Then it is* nef *if and only if $L \cdot D \geq 0$, for all effective divisors D, such that $D^2 = -2$.*

This idea was used in [**13,** Proposition 3.2], where it was proven that, if there exists a divisor with a negative intersection with L, then this divisor has self-intersection strictly less than -2. We refer to the description of the Néron-Severi group given in Proposition 2.6, where the Néron-Severi group is generated, over $\mathbb{Q}$, by an ample class and by 16 disjoint rational curves, which form an even set over $\mathbb{Z}$. Since the proofs of the next propositions are very similar to the ones given in [**13,** Section 3] (where the Néron-Severi groups of the K3 surfaces considered are generated over $\mathbb{Q}$ by an ample class and by eight disjoint rational curves forming an even set) we omit them. We denote by ϕ_L the map induced by the ample (or nef, or big and nef) divisor L on $Km(A)$.

Proposition 4.1. (*cf.*, [**13,** Proposition 3.1]). *Let $Km(A)$ be a Kummer surface such that $NS(Km(A)) \simeq \mathcal{K}'_{4d}$. Let H be as in*

Theorem 2.7*. Then we may assume that H is pseudo ample and $|H|$ has no fixed components.*

Remark 4.2. The divisor H is orthogonal to all curves of the Kummer lattice, so ϕ_H contracts them. The projective model associated to this divisor is an algebraic K3 surface with 16 nodes forming an even set. More precisely, $\phi_H(Km(A))$ is a model of A/ι.

Proposition 4.3. (*cf.*, [**13,** Propositions 3.2, 3.3])*. Let $Km(A)$ be a Kummer surface such that $NS(Km(A)) \simeq \mathcal{K}'_{4d}$.*

- *If $d \geq 3$, i.e., $H^2 \geq 12$, then the class*

$$H - \frac{1}{2}\Bigg(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4} K_p\Bigg) \subset NS(Km(A)),$$

  is an ample class. For $m \in \mathbb{Z}_{>0}$, the classes

$$m\Bigg(H - \frac{1}{2}\Bigg(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4} K_p\Bigg)\Bigg)$$

  and

$$mH - \frac{1}{2}\Bigg(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4} K_p\Bigg)$$

  are ample.
- *If $d = 2$, i.e.,*

$$\Bigg(H - \frac{1}{2}\Bigg(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4} K_p\Bigg)\Bigg)^2 = 0,$$

  then

$$m\Bigg(H - \frac{1}{2}\Bigg(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4} K_p\Bigg)\Bigg),$$

  is nef *for $m \geq 1$ and*

$$mH - \frac{1}{2}\Bigg(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4} K_p\Bigg)$$

  is ample for $m \geq 2$.

Proposition 4.4. (*cf.*, [**13,** Proposition 3.4]). *The divisors*

$$H-\frac{1}{2}\Bigg(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4}K_p\Bigg),$$

$$mH-\frac{1}{2}\Bigg(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4}K_p\Bigg)$$

and

$$m\Bigg(H-\frac{1}{2}\Bigg(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4}K_p\Bigg)\Bigg),\quad m\in\mathbb{Z}_{>0},$$

do not have fixed components for $d\geq 2$.

Lemma 4.5. (*cf.*, [**13,** Lemma 3.1]). *The map* $\phi_{H-(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4}K_p)/2}$ *is an embedding if* $H^2\geq 12$.

Proposition 4.6. (*cf.* [**13,** Proposition 3.5]).

1) *Let* D *be the divisor* $D=H-(K_1+\cdots+K_r)$, *up to relabelling of the indices,* $1\leq r\leq 16$. *Then* D *is pseudo ample for* $2d>r$.
2) *Let* $\overline{D}=(H-K_1-\ldots-K_r)/2$ *with* $r=4,8,12$ *if* $d\equiv 0 \mod 2$ *and* $r=6,10$ *if* $d\equiv 1 \mod 2$. *Then*:
   - *the divisor* $\overline{D}$ *is pseudo ample whenever it has positive self-intersection,*
   - *if* $\overline{D}$ *is pseudo ample, then it does not have fixed components,*
   - *if* $\overline{D}^2=0$, *then the generic element in* $|\overline{D}|$ *is an elliptic curve.*

Remark 4.7. In the assumptions of Lemma 4.5 the divisor

$$H-\frac{1}{2}\Bigg(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4}K_p\Bigg)$$

defines an embedding of the surface $Km(A)$ into a projective space which sends the curves of the Kummer lattice to lines. A divisor D as in Proposition 4.6 defines a map from the surface $Km(A)$ to a projective space, which contracts some rational curves of the even set and sends the others to conics on the image. Similarly, $\overline{D}$ defines a map from the surface $Km(A)$ to a projective space which contracts some rational curves of the even set and sends the others to lines on the image.

5. Projective models of Kummer surfaces with Picard number 17. Here we consider certain Kummer surfaces with Picard number 17, and we describe projective models determined by the divisors presented in the previous section. Some of these models are very classical.

5.1. Kummer of the Jacobian of a genus 2 curve. Let C be a general curve of genus 2. It is well known that the Jacobian $J(C)$ is an abelian surface such that $NS(J(C)) = \mathbb{Z}L$, with $L^2 = 2$ and $T_{J(C)} \simeq \langle -2 \rangle \oplus U \oplus U$. Hence,

$$NS(Km(J(C))) \simeq \mathcal{K}_4' \quad \text{and} \quad T_{Km(J(C))} \simeq \langle -4 \rangle \oplus U(2) \oplus U(2),$$

see Proposition 2.6 and Theorem 2.7.

Here, we want to reconsider some known projective models of $Km(J(C))$, see [**17,** Chapter 6], using the description of the classes in the Néron-Severi group introduced in the previous section.

The singular quotient surface $J(C)/\iota$ is a quartic in $\mathbb{P}^3$ with 16 nodes. For each of these nodes there exist six planes which pass through that node, and each plane contains five more nodes. Each of these planes cuts the singular quartic surface in a conic with multiplicity 2. In this way, we obtain 16 hyperplane sections which are double conics. These 16 conics are called *tropes*. They are the images, under the quotient map $J(C) \to J(C)/\iota$, of different embeddings of C in $J(C)$.

We have seen that every Kummer surface admits an Enriques involution, cf., Proposition 3.1. If the Kummer surface is associated to the Jacobian of a curve of genus 2, an explicit equation of this involution on the singular model of $Km(J(C))$ in $\mathbb{P}^3$ is given in [**25,** subsection 3.3].

5.1.1. The polarization H. The map ϕ_H contracts all the curves in the Kummer lattices, and hence, $\phi_H(Km(J(C)))$ is the singular quotient $J(C)/\iota$ in $\mathbb{P}^3$. The class H is the image in $NS(Km(J(C)))$ of the class generating $NS(J(C))$ (Proposition 2.6). The 16 classes (described in Remark 2.10, case $4d \equiv 4 \mod 8$) of the form

$$u_{J_6} := \frac{1}{2}\left(H - \sum_{p \in J_6} K_p\right)$$

correspond to the tropes. Indeed,

$$2u_{J_6} + \sum_{p \in J_6} K_p = H,$$

so they correspond to a curve in a hyperplane section with multiplicity 2; $u_{J_6}^2 = -2$, thus they are rational curves; $u_{J_6} \cdot H = 2$, thus they have degree 2. In particular, the trope corresponding to the class u_{J_6} passes through the nodes obtained by contracting the six curves K_p, where $p \in J_6$. It is a classical result, cf., [**20,** Chapter I, Section 3], that the rational curves of the Kummer lattice and the rational curves corresponding to the tropes in this projective model form a 16_6 configuration of rational curves on $Km(J(C))$. The intersections between the curves K_p, $p \in (\mathbb{Z}/2\mathbb{Z})^4$ and the classes u_{J_6} can be directly checked.

5.1.2. The polarization $H - K_{0,0,0,0}$. Another well-known model is obtained by projecting the quartic surface in $\mathbb{P}^3$ from a node. This gives a $2:1$ cover of $\mathbb{P}^2$, branched along six lines, which are the image of the tropes passing through the node from which we are projecting. The lines are all tangent to a conic, cf., [**35,** Section 1]. Taking the node associated to the contraction of the curve $K_{0,0,0,0}$, then the linear system associated to the projection of $J(C)/\iota$ from this node is $|H - K_{0,0,0,0}|$. The classes u_{J_6} such that $(0,0,0,0) \in J_6$ are sent to lines and the curve $K_{0,0,0,0}$ is sent to a conic by the map,

$$\phi_{H-K_{0,0,0,0}} : Km(J(C)) \longrightarrow \mathbb{P}^2.$$

This conic is tangent to the lines which are images of the tropes u_{J_6}. So the map $\phi_{H-K_{0,0,0,0}} : Km(J(C)) \to \mathbb{P}^2$ exhibits $Km(A)$ as a double cover of $\mathbb{P}^2$, branched along six lines tangent to the conic $\mathcal{C} := \phi_{H-K_{0,0,0,0}}(K_{0,0,0,0})$. The singular points of the quartic $J(C)/\iota$, which are not the center of this projection, are singular points of the double cover of $\mathbb{P}^2$. So the classes K_{a_1,a_2,a_3,a_4} of the Kummer lattice, such that $(a_1,a_2,a_3,a_4) \neq (0,0,0,0)$ are singular points for $\phi_{H-K_{0,0,0,0}}(Km(J(C)))$, and in fact, correspond to the 15 intersection points of the 6 lines in the branch locus. Observe that, if we fix three of the six lines, the conic $\mathcal{C}$ is tangent to the edges of this triangle. The remaining three lines form a triangle as well and the edges are tangent to the conic $\mathcal{C}$. By a classical theorem of projective plane geometry (a consequence of Steiner's theorem on generation of conics) the six vertices of the triangles are contained in another conic $\mathcal{D}$, and in fact, this conic is the image of one of the tropes which do not pass through the singular point corresponding to $K_{0,0,0,0}$. This can be checked directly on $NS(Km(J(C)))$. Observe that we have in total 10 such conics.

5.1.3. Deformation. We observe that this model of $Km(J(C))$ exhibits the surface as a special member of the four-dimensional family of K3 surfaces which are a $2:1$ cover of $\mathbb{P}^2$ branched along six lines in general position. The covering involution induces a non-symplectic involution on $Km(J(C))$ which fixes six rational curves. By Nikulin's classification of non-symplectic involutions, cf., e.g., [**1,** Section 2.3], the general member of the family has a Néron-Severi group isometric to $\langle 2\rangle \oplus A_1 \oplus D_4 \oplus D_{10}$ and a transcendental lattice isometric to $U(2)^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$, which clearly contains

$$T_{Km(J(C))} \simeq U(2)^{\oplus 2} \oplus \langle -4\rangle.$$

This is a specific case of Proposition 7.13.

5.1.4. The polarization $2H - [(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4})/2]K_p$. We denote this polarization by D. The divisor D is ample by Proposition 4.3. Since $D^2 = 8$, the map ϕ_D gives a smooth projective model of $Km(J(C))$ as an intersection of three quadrics in $\mathbb{P}^5$. Using suitable coordinates, we can write C as

$$y^2 = \prod_{i=0}^{5}(x - s_i),$$

with $s_i \in \mathbb{C}$, $s_i \neq s_j$ for $i \neq j$ (it is the double cover of $\mathbb{P}^1$ ramified on six points). Then, by [**46,** Theorem 2.5], $\phi_D(Km(J(C)))$ has equation

$$\begin{cases} z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0, \\ s_0 z_0^2 + s_1 z_1^2 + s_2 z_2^2 + s_3 z_3^2 + s_4 z_4^2 + s_5 z_5^2 = 0, \\ s_0^2 z_0^2 + s_1^2 z_1^2 + s_2^2 z_2^2 + s_3^2 z_3^2 + s_4^2 z_4^2 + s_5^2 z_5^2 = 0, \end{cases} \tag{5.1}$$

in $\mathbb{P}^5$. The curves of the Kummer lattice are sent to lines by the map ϕ_D; indeed, $D \cdot K_p = 1$ for each $p \in (\mathbb{Z}/2\mathbb{Z})^4$. The image of the rational curves associated to a divisor of type u_{J_6}, i.e., the curves which are tropes on the surface $\phi_H(Km(J(C)))$, are lines. In fact, we compute $D \cdot u_{J_6} = 1$. So on the surface $\phi_D(Km(J(C)))$ we have 32 lines which admit a 16_6 configuration. Keum [**24,** Lemma 3.1] proves that the set of the tropes and the curves K_p, $p \in (\mathbb{Z}/2\mathbb{Z})^4$ generate the Néron-Severi group, over $\mathbb{Z}$. Here, we find the same result as a trivial application of Theorem 2.7. Moreover, we can give a geometric interpretation of this fact; indeed, this implies that the Néron-Severi group of the surface $\phi_D(Km(J(C)))$ is generated by lines (other results about the Néron-Severi group of K3 surfaces generated by lines can be found e.g., in [**7**]). More precisely, the following hold.

Proposition 5.1. *The Néron-Severi group of the* K3 *surfaces which are smooth complete intersections of the three quadrics in* $\mathbb{P}^5$ *defined by* (5.1) *is generated by lines.*

Proof. By applying Theorem 2.7 we find a set of classes generating $NS(Km(J(C)))$ which corresponds to lines in the projective model of the Kummer surface $\phi_D(Km(J(C)))$. This set of classes is:

$$\begin{aligned}\mathcal{S} := \Big\{ e_1 &:= \frac{1}{2}(H - v_4),\\
e_2 &:= \frac{1}{2}(H - K_{0,0,0,0} - K_{1,0,0,0} - K_{0,1,0,1} - K_{0,1,1,0}\\
&\quad - K_{1,1,0,0} - K_{0,1,1,1}),\\
e_3 &:= \frac{1}{2}(H - K_{0,0,0,0} - K_{0,1,0,0} - K_{1,1,0,0} - K_{1,0,1,0}\\
&\quad - K_{1,0,0,1} - K_{1,0,1,1}),\\
e_4 &:= \frac{1}{2}(H - K_{0,0,0,0} - K_{0,0,1,0} - K_{0,0,1,1} - K_{1,0,0,1}\\
&\quad - K_{0,1,0,1} - K_{1,1,0,1}),\\
e_5 &:= \frac{1}{2}(H - K_{0,0,0,0} - K_{0,0,0,1} - K_{0,0,1,1} - K_{1,0,1,0}\\
&\quad - K_{1,1,1,0} - K_{0,1,1,0}),\\
e_6 &:= \frac{1}{2}(H - K_{0,0,0,0} - K_{1,0,0,0} - K_{0,1,0,0} - K_{1,1,0,1}\end{aligned}$$

$$- K_{1,1,1,0} - K_{1,1,1,1}), K_{0,0,0,0}, K_{1,0,0,0}, K_{0,1,0,0},$$
$$K_{0,0,1,0}, K_{0,0,0,1}, K_{0,0,1,1}, K_{0,1,0,1}, K_{1,0,0,1},$$
$$K_{0,1,1,0}, K_{1,0,1,0}, K_{1,1,0,0}\Big\}.$$

Indeed, by Theorem 2.7, a set of generators of $NS(Km(J(C)))$ is given by e_1 and a set of generators of the Kummer lattice K (a set of generators of K is described in Remark 2.10). Since, for $j = 2, 3, 4, 5$,

$$e_j - e_1 \equiv \frac{1}{2} \sum_{p \in W_{j-1}} K_p \mod (\oplus_{p \in (\mathbb{Z}/2\mathbb{Z})^4} \mathbb{Z} K_p)$$

and

$$e_1 - e_2 + e_3 - e_6 \equiv \frac{1}{2} \sum_{p \in (\mathbb{Z}/2\mathbb{Z})^4} K_p \mod (\oplus_{p \in (\mathbb{Z}/2\mathbb{Z})^4} \mathbb{Z} K_p),$$

$\mathcal{S}$ is a $\mathbb{Z}$-basis of $NS(Km(J(C)))$. It is immediate to check that every element of this basis has intersection 1 with D and thus is sent to a line by ϕ_D. $\square$

5.1.5. The nef class $H - (H - \sum_{p \in W_i} K_p)/2$. Without loss of generality, we consider $i = 1$, and we call this class $\overline{D}$. By Proposition 4.6, it defines an elliptic fibration on $Km(J(C))$, and the eight (-2)-classes contained in $\overline{D}$ are sections of the Mordell-Weil group; the other eight (-2)-classes are components of the reducible fibers. Observe that the class

$$\frac{1}{2}(H - K_{1,0,0,0} - K_{1,1,0,0} - K_{0,1,0,1} - K_{0,1,1,0} - K_{0,1,1,1} - K_{0,0,0,0})$$

has self intersection -2, intersection 0 with $\overline{D}$ and meets the classes $K_{1,0,0,0}$ and $K_{1,1,0,0}$ in one point. One can easily find three more classes than the previous one, so that the fibration contains four fibers I_4. Checking in **[31,** Table, page 9] we see that this is the fibration number 7 so it has no other reducible fibers, and the rank of the Mordell-Weil group is 3.

5.1.6. Shioda-Inose structure. We now describe the three-dimensional family of K3 surfaces which admit a Shioda-Inose structure associated to $Km(J(C))$ as described in Theorem 2.18. This is obtained by considering K3 surfaces X with $\rho(X) = 17$, with an elliptic fibration with reducible fibers $I_{10}^* + I_2$ and Mordell-Weil group equal to $\mathbb{Z}/2\mathbb{Z}$ (see Shimada's list of elliptic K3 surfaces [**44,** Table 1, Case 1343] for the `arXiv` version of the paper). By using the Shioda-Tate formula, cf, e.g., [**45,** Corollary 1.7], the discriminant of the Néron-Severi group of such a surface is $(2^2 \cdot 2)/2^2$.

The translation t by the section of order 2 on X is a Morrison-Nikulin involution; indeed, it switches two orthogonal copies of $E_8(-1) \subset NS(X)$. Thus, the Néron-Severi group is $\langle 2d \rangle \oplus E_8(-1) \oplus E_8(-1)$, and $d = 1$ because the discriminant is 2. Hence, X has a Shioda-Inose structure associated to the abelian surface $J(C)$. The desingularization of the quotient X/t is the Kummer surface $Km(J(C))$ and has an elliptic fibration induced by the one on X, with reducible fibers $I_5^* + 6I_2$ (this is number 23 of [**31**]) and $\mathbb{Z}/2\mathbb{Z}$ as a Mordell-Weil group. This Shioda-Inose structure was described in [**30,** subsection 5.3].

In Theorem 2.18, we gave a description of the Néron-Severi group of $Km(J(C))$ related to the Shioda-Inose structure. In particular, we showed that $NS(Km(J(C))$ is an overlattice of index 2 of $\langle 4 \rangle \oplus N \oplus E_8(-1)$. The generator of $\langle 4 \rangle$ is denoted by Q; the classes of the rational curves in the Nikulin lattice N by N_i, $i = 1, \ldots, 8$; and the generators of $E_8(-1)$ (we assume that E_j, $j = 1, \ldots, 7$ generate a copy of $A_7(-1)$ and $E_3 \cdot E_8 = 1$) by E_j, $j = 1, \ldots, 8$.

Then a $\mathbb{Z}$-basis of $NS(Km(J(C)))$ is

$$\left\{ \frac{Q + N_1 + N_2}{2}, N_1, \ldots, N_7, \sum_{i=1}^{8} \frac{N_i}{2}, E_1, \ldots, E_8 \right\}.$$

It is easy to identify a copy of N and an orthogonal copy of $E_8(-1)$ in the previous elliptic fibration (the one with reducible fibers $I_5^* + 6I_2$); in particular, we remark that the curves N_i and E_j, $j = 2, \ldots, 8$, are components of the reducible fibers and the curve E_1 can be chosen to be the zero section. This immediately gives the class of the fiber in terms of the previous basis of the Néron-Severi group, $F := Q - 4E_1 - 7E_2 - 10E_3 - 8E_4 - 6E_5 - 4E_6 - 2E_7 - 5E_8$.

5.2. Kummer surface of a $(1, 2)$-polarized abelian surface. In this section, A will always denote a $(1, 2)$ polarized abelian surface, and $NS(A) = \mathbb{Z}L$ where $L^2 = 4$.

5.2.1. The polarization H. By Proposition 4.1, the divisor H is pseudo-ample and the singular model $\phi_H(Km(A))$ has 16 singular points (it is in fact A/ι). Since $H^2 = 8$ and since, by [**42,** Theorem 5.2], H is not hyperelliptic, the K3 surface $\phi_H(Km(A))$ is a complete intersection of three quadrics in $\mathbb{P}^5$. This model is described by Barth in [**2**]:

Proposition 5.2. ([**2,** Proposition 4.6]). *Let us consider the following quadrics*:

$$\begin{aligned} Q_1 &= \{(\mu_1^2+\lambda_1^2)(x_1^2+x_2^2) - 2\mu_1\lambda_1(x_3^2+x_4^2) \\ &\qquad + (\mu_1^2-\lambda_1^2)(x_5^2+x_6^2) = 0\}, \\ Q_2 &= \{(\mu_2^2+\lambda_2^2)(x_1^2-x_2^2) - 2\mu_2\lambda_2(x_3^2-x_4^2) \\ &\qquad + (\mu_2^2-\lambda_2^2)(x_5^2-x_6^2) = 0\}, \\ Q_3 &= \{(\mu_3^2+\lambda_3^2)x_1x_2 - 2\mu_3\lambda_3x_3x_4 + (\mu_3^2-\lambda_3^2)x_5x_6 = 0\}. \end{aligned}$$

Let $r = r_{1,2}r_{2,3}r_{3,1}$ where $r_{k,j} = (\lambda_j^2\mu_k^2 - \lambda_k^2\mu_j^2)(\lambda_j^2\lambda_k^2 - \mu_k^2\mu_j^2)$. If $r \neq 0$, the quadrics Q_1, Q_2, Q_3, generate the ideal of an irreducible surface $Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^5$ of degree 8, which is smooth except for 16 ordinary double points and which is isomorphic to A/ι.

The surface A/ι is then contained in each quadric of the net, $\alpha_1Q_1+\alpha_2Q_2+\alpha_3Q_3$, $\alpha_i \in \mathbb{C}$. We observe that the matrix M associated to this net of quadrics is a block matrix:

$$M = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{bmatrix},$$

where

$$\begin{aligned} B_1 &= \begin{bmatrix} \alpha_1(\mu_1^2+\lambda_1^2)+\alpha_2(\mu_2^2+\lambda_2^2) & \alpha_3(\mu_3^2+\lambda_3^2) \\ \alpha_3(\mu_3^2+\lambda_3^2) & \alpha_1(\mu_1^2+\lambda_1^2)-\alpha_2(\mu_2^2+\lambda_2^2) \end{bmatrix}, \\ B_2 &= \begin{bmatrix} -2\alpha_1\mu_1\lambda_1-2\alpha_2\mu_2\lambda_2 & -2\alpha_3\mu_3\lambda_3 \\ -2\alpha_3\mu_3\lambda_3 & -2\alpha_1\mu_1\lambda_1+2\alpha_2\mu_2\lambda_2 \end{bmatrix}, \\ B_3 &= \begin{bmatrix} \alpha_1(\mu_1^2-\lambda_1^2)+\alpha_2(\mu_2^2-\lambda_2^2) & \alpha_3(\mu_3^2-\lambda_3^2) \\ \alpha_3(\mu_3^2-\lambda_3^2) & \alpha_1(\mu_1^2-\lambda_1^2)-\alpha_2(\mu_2^2-\lambda_2^2) \end{bmatrix}. \end{aligned}$$

A singular quadric of the net is such that

$$\det(M) = \det(B_1)\det(B_2)\det(B_3) = 0.$$

One can easily check that $\det(B_1) = \det(B_2) + \det(B_3)$. So, if α_1, α_2, α_3 are such that $\det(B_i) = \det(B_j) = 0$, $i \neq j$, then also for the third block B_h, $h \neq i$, $h \neq j$, one has $\det(B_h) = 0$. Hence, such a choice corresponds to a quadric of rank 3. There are only four possible choices of $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$ which satisfy the condition $\det(B_i) = 0$ for $i = 1, 2, 3$. Putting $\lambda_i = 1$, $i = 1, 2, 3$, and

$$\begin{aligned} w_1 &= \sqrt{(\mu_2^2 - \mu_3^2)(\mu_2^2\mu_3^2 - 1)}, \\ w_2 &= \sqrt{(\mu_1^2 - \mu_3^2)(\mu_1^2\mu_3^2 - 1)}, \\ w_3 &= \sqrt{(\mu_2^2 - \mu_1^2)(\mu_1^2\mu_2^2 - 1)}, \end{aligned}$$

the rank 3 quadrics S_i correspond to the following choices of $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$:

$$\begin{aligned} &S_1 \colon (\alpha_1, \alpha_2, \alpha_3) = (w_1, w_2, w_3), \\ &S_2 \colon (\alpha_1, \alpha_2, \alpha_3) = (w_1, w_2, -w_3), \\ &S_3 \colon (\alpha_1, \alpha_2, \alpha_3) = (w_1, -w_2, w_3), \\ &S_4 \colon (\alpha_1, \alpha_2, \alpha_3) = (w_1, -w_2, -w_3). \end{aligned}$$

Since for these choices $\det(B_i) = 0$, for $i = 1, 2, 3$, the quadrics S_1, S_2, S_3, S_4 are of type

$$(\beta_1x_1 + \beta_2x_2)^2 + (\beta_3x_3 + \beta_4x_4)^2 + (\beta_5x_5 + \beta_6x_6)^2 = 0,$$

the singular locus of such a quadric is the plane of $\mathbb{P}^5$:

$$\begin{cases} \beta_1x_1 + \beta_2x_2 = 0, \\ \beta_3x_3 + \beta_4x_4 = 0, \\ \beta_5x_5 + \beta_6x_6 = 0. \end{cases}$$

We observe that the singular planes of S_1 and S_2 are complementary planes in $\mathbb{P}^5$, and the same is true for the singular planes of S_3 and S_4. Then, up to a change of coordinates, we can assume that:

$$\begin{aligned} S_1 &= y_1^2 + y_2^2 + y_3^2, \\ S_2 &= z_1^2 + z_2^2 + z_3^2, \end{aligned}$$

$$S_3 = (l_1y_1 + m_1z_1)^2 + (l_2y_2 + m_2z_2)^2 + (l_3y_3 + m_3z_3)^2,$$
$$\frac{A}{\iota} = S_1 \cap S_2 \cap S_3.$$

The intersection between $\mathrm{Sing}(S_1)$ and S_2 is a conic C_2. The intersection of this conic with the hypersurface S_3 is made up of four points. Therefore,

$$\mathrm{Sing}(S_1) \cap (A/\iota) = \mathrm{Sing}(S_1) \cap (S_1 \cap S_2 \cap S_3) = \mathrm{Sing}(S_1) \cap S_2 \cap S_3$$

is made up of four points which must be singular on A/ι (since A/ι is the complete intersection between S_1, S_2 and S_3 and the points are in $\mathrm{Sing}(S_1)$). These four points are nodes of the surface A/ι. There is a complete symmetry between the four quadrics S_1, S_2, S_3 and S_4, so we have:

Lemma 5.3. *On each plane* $\mathrm{Sing}(S_i)$ *there are exactly four singular points of the surface* A/ι.

Let us now consider the classes of Remark 2.10 described by the set $J_8 \subset (\mathbb{Z}/2\mathbb{Z})^4$. We call any of them u_{J_8}. These classes have self intersection -2, and they are effective. Since $u_{J_8} \cdot H = 4$, they correspond to rational quartics on A/ι passing through eight nodes of the surface. Moreover, they correspond to curves with multiplicity 2; indeed,

$$2u_{J_8} + \sum_{\in J_8} K_p$$

is linearly equivalent to H, which is the class of the hyperplane section. The classes of these rational curves and the classes in the Kummer lattice generate the Néron-Severi group of $Km(A)$. These curves are in a certain sense the analogue of the tropes of $Km(J(C))$. Similarly to the tropes of $Km(J(C))$, these curves are rational and obtained as special hyperplane sections of $Km(A)$. They generate the Néron-Severi group of the Kummer surface together with the curves of the Kummer lattice.

5.2.2. The polarization $H - K_{p_1} - K_{p_2} - K_{p_3}$. Let us choose three singular points p_i, $i = 1, 2, 3$, such that p_1, p_2 are contained in $\mathrm{Sing}(S_1)$ and $p_3 \notin \mathrm{Sing}(S_1)$. These three points generate a plane in $\mathbb{P}^5$. The projection of $\phi_H(Km(A))$ from this plane is associated to the linear system $H - K_{p_1} - K_{p_2} - K_{p_3}$. The map

$$\phi_{H-K_{p_1}-K_{p_2}-K_{p_3}} : Km(A) \longrightarrow \mathbb{P}^2,$$

is a $2:1$ cover of $\mathbb{P}^2$ ramified along the union of two conics and two lines. The lines are the images of two of the rational curves with classes of type u_{J_8}, where J_8 contains $p_1, p_2, p_3 \in J_8$. This description of $Km(A)$ was presented in [**10**].

5.2.3. Deformation. This model exhibits $Km(A)$ as a special member of the six-dimensional family of K3 surfaces which are a double cover of $\mathbb{P}^2$ branched along two conics and two lines. The covering involution is a non-symplectic involution fixing four rational curves. By Nikulin's classification of non-symplectic involutions, see e.g., [**1,** subsection 2.3], it turns out that the generic member of this family of K3 surfaces has Néron-Severi group isometric to $\langle 2\rangle \oplus A_1 \oplus D_4^{\oplus 3}$ and transcendental lattice $U(2)^{\oplus 2} \oplus \langle -2\rangle^{\oplus 4}$ (this family is studied in detail in [**28**]). The transcendental lattice $U(2)^{\oplus 2} \oplus \langle -8\rangle$ of $Km(A)$ clearly embeds in the previous lattice.

5.2.4. The polarization $2H - [(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4})/2]K_p$. We call this divisor D. It is ample by Proposition 4.3. The projective model $\phi_D(Km(A))$ is a smooth K3 surface in $\mathbb{P}^{13}$. The curves of the Kummer lattice and those associated to classes of type u_{J_8} are sent to lines, and hence, the Néron-Severi group of $\phi_D(Km(A))$ is generated by lines, cf., Proposition 5.1.

5.2.5. The nef class $(H - \sum_{p\in J_4} K_p)/2$. We call it F. By Proposition 4.6, it defines a map $\phi_F : Km(A) \to \mathbb{P}^1$ which exhibits $Km(A)$ as an elliptic fibration with 12 fibers of type I_2 and Mordell-Weil group isomorphic to $\mathbb{Z}^3 \oplus (\mathbb{Z}/2\mathbb{Z})^2$. Indeed the zero section and three independent sections of infinite order are the curves $K_{a,b,c,d}$ such that $F \cdot K_{a,b,c,d} = 1$. The non trivial components of the 12 fibers of type I_2 are $K_{e,f,g,h}$, such that $F \cdot K_{e,f,g,h} = 0$. The curves

$$F + 2K_{0,0,0,0} - \Big(\sum_{p\in W_3} K_p\Big)/2$$

and

$$F + 2K_{0,0,0,0} - \Big(\sum_{p\in W_4} K_p\Big)/2$$

are two 2-torsion sections. This description of an elliptic fibration on $Km(A)$ follows immediately by the properties of the divisors of the Néron-Severi group. However, a geometrical construction giving the same result is obtained by considering the projection of the model of $\phi_H(Km(A)) \subset \mathbb{P}^5$ from the plane $\mathrm{Sing}(S_1)$. The image of this projection lies in the complementary plane $\mathrm{Sing}(S_2)$ and is a conic C. Let p be a point of C, and let $\mathbb{P}^3_p$ be the space generated by $\mathrm{Sing}(S_1)$ and by p. The fiber over p is $S_2 \cap S_3 \cap \mathbb{P}^3_p$. The fiber over a generic point of C is an elliptic curve (the intersection of two quadrics in $\mathbb{P}^3$). There are 12 points in C, corresponding to the 12 singular points of $\phi_H(Km(A))$ which are not on the plane $\mathrm{Sing}(S_1)$, such that the fibers over these points are singular and in fact of type I_2. A geometrical description of this elliptic fibration is also provided in [**32**], where it is obtained as a double cover of an elliptic fibration on $Km(J(C))$.

5.2.6. Shioda-Inose structure. We now describe the three-dimensional family of K3 surfaces which admit a Shioda-Inose structure associated to $Km(A)$ as described in Theorem 2.18. It is obtained by using results of [**15,** Section 4.6]. Consider the K3 surface X with $\rho(X) = 17$, and admit an elliptic fibration with fibers $I_{16} + 8I_1$ and Mordell-Weil group isometric to $\mathbb{Z}/2\mathbb{Z}$. By [**15,** Proposition 4.7], the discriminant of $NS(X)$ is 4 and the translation t by the 2-torsion section is a Morrison-Nikulin involution. Thus, the desingularization of X/t is a Kummer surface, which is in fact $Km(A)$ by Theorem 2.18. The elliptic fibration induced on $Km(A)$ has $I_8 + 8I_2$ singular fibers and Mordell-Weil group $(\mathbb{Z}/2\mathbb{Z})^2$. Using the curves contained in the elliptic fibration, we can easily identify the sublattice

$$N \oplus E_8(-1) \quad \text{of } NS(Km(A)).$$

The lattice N contains eight non trivial components of the eight fibers of type I_2, and the lattice $E_8(-1)$ is generated by seven components of the fiber of type I_8 and by the zero section.

As in the case of the Jacobian of a curve of genus 2, we give a $\mathbb{Z}$-basis of the Néron-Severi group of $Km(A)$ related to the Shioda-Inose structure, and we identify the class of the fiber of this fibration. With the previous notation, a $\mathbb{Z}$-basis is given by

$$\left\{\frac{\langle Q+N_1+N_2+N_3+N_4\rangle}{2}, N_1,\ldots,N_7,\sum_{i=1}^{8}\frac{N_i}{2}, E_1,\ldots,E_8\right\},$$

where $Q^2=8$ and Q is orthogonal to $N\oplus E_8(-1)$. The class of the fiber in terms of the previous basis of the Néron-Severi group is

$$F:=Q-5E_1-10E_2-15E_3-12E_4-9E_5-6E_6-3E_7-8E_8.$$

5.3. Kummer surface of a $(1,3)$ **polarized abelian surface.** Let A be a $(1,3)$ polarized abelian surface. Then, $NS(A)=\mathbb{Z}L$, $L^2=6$.

5.3.1. The polarization H**.** The model of the singular quotient A/ι is associated to the divisor H in $NS(Km(A))$ with $H^2=12$. By Proposition 4.1 and [**42,** Theorem 5.2] this model is a singular K3 surface in $\mathbb{P}^7$.

Let us now consider the 16 classes of Remark 2.10 associated to the set $J_{10}\subset(\mathbb{Z}/2\mathbb{Z})^4$. We call any of them $u_{J_{10}}$. They are (-2)-classes (see Remark 2.10) and are sent to rational curves of degree 6 on $\phi_H(Km(A))$.

5.3.2. The polarization $H-(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4}K_p)/2$**.** We call it D. It is ample by Proposition 4.3 and, since $D^2=4$, the surface $\phi_D(Km(A))$ is a smooth quartic in $\mathbb{P}^3$. The curves of the Kummer lattice and the curves associated to $u_{J_{10}}$ are sent to lines. Since the classes of the curves in the Kummer lattice and the classes $u_{J_{10}}$ generate the Néron-Severi group of $Km(A)$, the Néron-Severi group of $\phi_D(Km(A))$ is generated by lines, cf., Proposition 5.1.

5.3.3. The polarization $H-K_{0,0,1,0}-K_{0,0,1,1}-K_{1,0,0,0}-K_{0,1,0,0}-K_{0,0,1,1}$**.** This defines a $2:1$ map from $Km(A)$ to $\mathbb{P}^2$. Since 11 curves K_p are contracted, the branch locus is a reducible sextic with 11 nodes.

5.3.4. Deformation. The generic K3 surface double cover of $\mathbb{P}^2$ branched on a reducible sextic with 11 nodes lies in an eight-dimensional family and has transcendental lattice equal to $U(2)^{\oplus 2}\oplus\langle -2\rangle^{\oplus 6}$, see [**1,** subsection 2.3]. Clearly, the transcendental lattice $U(2)^{\oplus 2}\oplus\langle -12\rangle$ can be primitively embedded in $U(2)^{\oplus 2}\oplus\langle -2\rangle^{\oplus 6}$, so the family of

Kummer surfaces of a $(1,3)$-polarized abelian surface is a special three-dimensional subfamily.

5.3.5. The nef class $(H-\sum_{p\in J_6}K_p)/2$**.** We call it F. By Proposition 4.6, it defines an elliptic fibration $Km(A)\to\mathbb{P}^1$ with 10 fibers of type I_2. The components of these fibers not meeting the zero section are the curves $K_{a,b,c,d}$ of the Kummer lattice such that $F\cdot K_{a,b,c,d}=0$. The Mordell-Weil group is $\mathbb{Z}^5$, and the curves $K_{e,f,g,h}$ such that $F\cdot K_{e,f,g,h}=1$ are the zero section and five sections of infinite order (but they are not the $\mathbb{Z}$-generators of the Mordell-Weil group).

5.3.6. Shioda-Inose structure. We now describe the three-dimensional family of K3 surfaces which admits a Shioda-Inose structure associated to $Km(A)$ as described in Theorem 2.18. It was already described independently in [**10,** Remark 3.3.1] and [**27,** subsection 3.1]. Let us consider the K3 surfaces X with $\rho(X)=17$, with an elliptic fibration with reducible fibers $I_6^*+I_6$ and Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$ (as in the `arXiv` version [**44,** Table 1, line 1357]). The translation t by the 2-torsion section is a Morrison-Nikulin involution (in fact, it is immediate to check that it switches two orthogonal copies of $E_8(-1)\subset NS(X)$), and hence, the desingularization of the quotient X/t is a Kummer surface. The latter admits an elliptic fibration induced by the one on X, with reducible fibers $I_3^*+I_3+6I_2$ and a 2-torsion section. By the Shioda-Tate formula, see e.g., [**45,** Corollary 1.7], the discriminant of the Néron-Severi group of such an elliptic fibration is $(4\cdot 3\cdot 2^6)/2^2$, and thus this is the Kummer surface of a $(1,3)$-polarized abelian surface. As in the case of the Jacobian of a curve of genus 2, we give a $\mathbb{Z}$-basis of the Néron-Severi group of $Km(A)$ related to the Shioda-Inose structure, and we can identify the class of the fiber of this fibration. The eight curves N_i are the six non trivial components of each fiber of type I_2 and two non trivial components of I_3^* with multiplicity 1; the curves E_i are the zero section, two components of I_3 and five components of I_3^*. With the previous notation, a $\mathbb{Z}$-basis is given by

$$\left\{\frac{\langle Q+N_1+N_2\rangle}{2},N_1,\dots,N_7,\sum_{i=1}^{8}\frac{N_i}{2},E_1,\dots,E_8\right\},$$

where $Q^2 = 12$ and Q is orthogonal to $N \oplus E_8(-1)$. The class of the fiber in terms of this basis of the Néron-Severi group is

$$\begin{aligned} F := Q - 6E_1 - 12E_2 - 18E_3 - 15E_4 \\ - 12E_5 - 8E_6 - 4E_7 - 9E_8. \end{aligned}$$

6. K3 surfaces with symplectic action of the group $(\mathbb{Z}/2\mathbb{Z})^4$ **and their quotients.** In the following sections we study two four-dimensional families of K3 surfaces that contain subfamilies of Kummer surfaces. Indeed, we have seen that every Kummer surface admits a symplectic action of the group $(\mathbb{Z}/2\mathbb{Z})^4$ (Proposition 3.3), but the moduli space of K3 surfaces with symplectic action by $(\mathbb{Z}/2\mathbb{Z})^4$ has dimension 4, and thus the Kummer surfaces are a three-dimensional subfamily. We will also study the family of K3 surfaces obtained as desingularization of the quotient of a K3 surface by the group $(\mathbb{Z}/2\mathbb{Z})^4$ acting symplectically on it. By Proposition 3.3, this family also contains the three-dimensional family of Kummer surfaces.

Let $G = (\mathbb{Z}/2\mathbb{Z})^4$ be a group of symplectic automorphisms on a K3 surface X. We observe that G contains $(2^4 - 1) = 15$ symplectic involutions, so we have $8 \cdot 15 = 120$ distinct points with non trivial stabilizer group on X, and these are all the points with a non trivial stabilizer on X, cf., [**38,** Section 5]. Moreover, we have a commutative diagram:

$$\begin{array}{ccc} \widetilde{X} & \stackrel{\beta}{\longrightarrow} & X \\ \pi\downarrow & & \downarrow\pi' \\ Y & \stackrel{\tilde{\beta}}{\longrightarrow} & \overline{Y}, \end{array} \tag{6.1}$$

where $\overline{Y}$ is the quotient of X by G, $\widetilde{X}$ is the blow up of X at 120 points with non trivial stabilizer (hence, it contains 120 (-1)-curves) and Y is the minimal resolution of the quotient $\overline{Y}$ and simultaneously the quotient of $\widetilde{X}$ by induced action. Observe that Y contains 15 (-2)-curves coming from the resolution of the singularities. In fact, each fixed point on X has a G-orbit of length 8. In particular, the rank of the Néron-Severi group of Y is at least 15 and in fact 16 if X, and so also Y, is algebraic. In particular, since by [**23,** Corollary 1.2], $\operatorname{rank} NS(X)=\operatorname{rank} NS(Y)$, a K3 surface with a symplectic action of

$(\mathbb{Z}/2\mathbb{Z})^4$ has at least Picard number 15 (16 if it is algebraic). Finally, π is $16:1$ outside the branch locus.

7. K3 surfaces with symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$**.** In this section, we analyze the K3 surface X admitting a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$; in particular, we identify the possible Néron-Severi groups of such a K3 surface if the Picard number is 16, which is the minimum possible for an algebraic K3 surface with this property. This allows us to describe the families of such K3 surfaces, cf., Corollary 7.11, and to prove that every K3 surface admitting $(\mathbb{Z}/2\mathbb{Z})^4$ as a group of symplectic automorphisms also admits an Enriques involution. This generalizes a similar result for Kummer surfaces given in Proposition 3.1.

7.1. The Néron-Severi group of X.

Theorem 7.1 (cf., **[10]**)**.** *Let X be an algebraic* K3 *surface with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$, and let $\Omega^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4} = \langle -8\rangle \oplus U(2)^{\oplus 3}$ be the invariant lattice $H^2(X,\mathbb{Z})^{(\mathbb{Z}/2\mathbb{Z})^4}$. We have $\rho(X) \geq 16$ and, if $\rho(X) = 16$, then denote by L a generator of $\Omega^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4} \cap NS(X)$ with $L^2 = 2d > 0$. Let*

$$\mathcal{L}^{2d}_{(\mathbb{Z}/2\mathbb{Z})^4} := \mathbb{Z}L \oplus \Omega_{(\mathbb{Z}/2\mathbb{Z})^4} \subset NS(X).$$

Denote an overlattice of $\mathcal{L}^{2d}_{(\mathbb{Z}/2\mathbb{Z})^4}$ of index r by $\mathcal{L}'^{2d}_{(\mathbb{Z}/2\mathbb{Z})^4,r}$. Then there are the following possibilities for d, L and r.

1) *If $d \equiv 0 \mod 2$ and $d \not\equiv 4 \mod 8$, then $r = 2$, $L = w_1 := (0,1,t,0,0,0,0) \in \Omega^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4}$ and $L^2 = w_1^2 = 4t$.*
2) *If $d \equiv 4 \mod 8$ and $d \not\equiv -4 \mod 32$, then either $r = 2$, $L = w_1 := (0,1,t,0,0,0,0) \in \Omega^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4}$ and $L^2 = w_1^2 = 4t$, or $r = 4$, $L = w_2 := (1,2,2s,0,0,0,0) \in \Omega^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4}$ and $L^2 = w_2^2 = 8(2s-1)$.*
3) *If $d \equiv -4 \mod 32$, then either $r = 2$, $L = w_1 := (0,1,t,0,0,0,0) \in \Omega^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4}$ and $L^2 = w_1^2 = 4t$, or $r = 4$, $L = w_2 := (1,2,2s,0,0,0,0) \in \Omega^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4}$ and $L^2 = w_2^2 = 8(2s-1)$, or $r = 8$, $L = w_3 := (1,4,4u,0,0,0,0) \in \Omega^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4}$ and $L^2 = w_3^2 = 8(8u-1)$.*

If $NS(X)$ is an overlattice of $\mathbb{Z}w_1 \oplus \Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$, then $T_X \simeq \langle -8\rangle \oplus \langle -4t\rangle \oplus U(2)^{\oplus 2}$.

If $NS(X)$ is an overlattice of $\mathbb{Z}w_2 \oplus \Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$, then

$$T_X \simeq \begin{bmatrix} -8 & 4 \\ 4 & -4s \end{bmatrix} \oplus U(2)^{\oplus 2}.$$

If $NS(X)$ is an overlattice of $\mathbb{Z}w_3 \oplus \Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$, then

$$T_X \simeq \begin{bmatrix} -8 & 2 \\ 2 & -4u \end{bmatrix} \oplus U(2)^{\oplus 2}.$$

Proof. Since $\Omega_{(\mathbb{Z}/2\mathbb{Z})^4} \subset NS(X)$, and X is algebraic, we have $\rho(X) \geq 16$. The proof of the unicity of the possible overlattices of $\mathcal{L}^{2d}_{(\mathbb{Z}/2\mathbb{Z})^4}$ is based on the following idea. Let us consider the lattice orthogonal to $\Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$ in Λ_{K3}. For each element $s(=L) \in \Omega^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4}$ in a different orbit under isometries of $\Omega^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4}$, we can consider the lattice $\mathbb{Z}s \oplus \Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$. To compute the index of the overlattice $R(=NS(X))$ of $\mathbb{Z}s \oplus \Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$ which is primitively embedded in Λ_{K3}, we consider the lattice

$$R^{\perp} = s^{\perp} \cap \Omega^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4} = (\mathbb{Z}s \oplus \Omega_{(\mathbb{Z}/2\mathbb{Z})^4)})^{\perp} \subset \Lambda_{K3}$$

(which is isometric to T_X). Then, we compute the discriminant group of $R^{\perp}$ to get the discriminant group of R, and so we get the index r of $\mathbb{Z}s \oplus \Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$ in $R(=NS(X))$. Recall that

$$\Omega^{\perp}_{(\mathbb{Z}/2\mathbb{Z})^4} \simeq \langle -8 \rangle \oplus U(2)^3 \simeq (\langle -4 \rangle \oplus U^3)(2).$$

The orbits of elements by isometries of this lattice are determined by the orbits of elements by isometries of the lattice $\langle -4 \rangle \oplus U^3$. In the next sections we investigate them. Then the proof of the theorem follows from the results of subsection 7.2.

Moreover, we remark that under our assumptions two overlattices $R_i \supset \mathbb{Z}w_i \oplus \Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$ and $R_j \supset \mathbb{Z}w_j \oplus \Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$, $i \neq j$, cannot be isometric in Λ_{K3} since their orthogonal complements $R_i^{\perp}$ and $R_j^{\perp}$ are different. These are determined in Proposition 7.8 below, and they are the transcendental lattices T_X in our statement. □

7.2. The lattice $\langle -2d \rangle \oplus U \oplus U$.

Lemma 7.2. *Let (a_1, a_2, a_3, a_4) be a vector in the lattice $U \oplus U$. There exists an isometry which sends the vector (a_1, a_2, a_3, a_4) to the* *vector* $(d, de, 0, 0)$. *In particular, the vector* $(a_1, a_2, 0, 0)$ *can be sent to* $(d, de, 0, 0)$ *where* $d = \gcd(a_1, a_2)$ *and* $d^2 e = a_1 a_2$.

Proof. The lattice $U \oplus U$ is isometric to the lattice $\{M(2, \mathbb{Z}), 2\det\}$ of the square matrices of dimension 2 with bilinear form induced by the quadratic form given by the determinant multiplied by 2. Explicitly, the isometry can be written as:

$$U \oplus U \longrightarrow M(2, \mathbb{Z}), \qquad \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} \right) \longmapsto \begin{bmatrix} a_1 & -a_3 \\ a_4 & a_2 \end{bmatrix}.$$

It is well known that, under the action of the orthogonal group $O(M(2, \mathbb{Z}))$, each matrix of $M(2, \mathbb{Z})$ can be sent in a diagonal matrix with diagonal (d_1, d_2), $d_1 \mid d_2$ (this is the Smith normal form). Thus, the lemma follows. □

Lemma 7.3. *There exists an isometry which sends the primitive vector*

$$(a_0, a_1, a_2, a_3, a_4) \in T_{2d} := \langle -2d \rangle \oplus U \oplus U,$$

to a primitive vector $(a, d, de, 0, 0) \in \langle -2d \rangle \oplus U \oplus U$.

Proof. The primitive vector $(a_0, a_1, a_2, a_3, a_4)$ is sent to a primitive vector by any isometry. By Lemma 7.2, there exists an isometry sending $(a_1, a_2, a_3, a_4) \in U \oplus U$ to $(d, de, 0, 0) \in U \oplus U$; thus, there exists an isometry sending $(a_0, a_1, a_2, a_3, a_4)$ to $(a_0, d, de, 0, 0)$, and $(a_0, d, de, 0, 0)$ is primitive. □

The previous lemma allows us to restrict our attention to the vectors in the lattice $\mathcal{A}_{2d} := \langle -2d \rangle \oplus U$.

Lemma 7.4. *There exists an isometry of* $\mathcal{A}_{2d}$ *which sends the vector* $(a, 1, c)$, *to the vector* $(0, 1, r)$, *where* $2c - 2da^2 = 2r$.

Proof. First we observe that $(a, 1, c) \cdot (a, 1, c) = (0, 1, r) \cdot (0, 1, r) = 2r$. Let R_v denote the reflection with respect to $v = (1, 0, d)$. Then, for $w = (x, y, z)$, we have

$$R_v(w) = w - 2\frac{w \cdot v}{v \cdot v} v = \begin{pmatrix} -x + y \\ y \\ -2dx + dy + z \end{pmatrix}.$$

If $a > 0$, we apply the reflection R_v to $(a, 1, c)$, $(v = (1, 0, d))$. Then we obtain the following.

$$R_v \begin{pmatrix} a \\ 1 \\ c \end{pmatrix} = \begin{pmatrix} 1-a \\ 1 \\ -2da+d+c \end{pmatrix}.$$

Let D be the isometry of $\mathcal{A}_{2d}$,

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then,

$$D \circ R_v \begin{pmatrix} a \\ 1 \\ c \end{pmatrix} = \begin{pmatrix} a-1 \\ 1 \\ -2da+d+c \end{pmatrix}.$$

Applying a times the isometry $D \circ R_v$, we obtain

$$(D \circ R_v)^a \begin{pmatrix} a \\ 1 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2r \end{pmatrix}. \quad \square$$

Lemma 7.5. *There exists an isometry of $\mathcal{A}_{2d}$ which sends a vector $q_2 := (wh \pm j, w, wt)$, with $t, h \in \mathbb{Z}$, $w, j \in \mathbb{N}$, $0 < j \leq \llcorner d/2 \lrcorner$, to the vector $p_2 := (j, w, s)$, where $s = -dwh^2 \mp 2dhj + wt$.*

Proof. Without loss of generality, we can assume $h > 0$ (if $h \leq 0$, it is sufficient to consider the action of D). Let us apply the isometry $D \circ R_v$ to the vector q_2:

$$(D \circ R_v) \begin{pmatrix} wh \pm j \\ w \\ wt \end{pmatrix} = \begin{pmatrix} w(h-1) \pm j \\ w \\ -2d(wh \pm j) + dw + wt \end{pmatrix}.$$

As in the previous proof, applying $D \circ R_v$ decreases the first component, and the second remains the same. Applying h-times the isometry to q_2, we obtain that the first component is j or $-j$. In the second case, we again apply the isometry D, and so in both situations, we obtain p_2. $\square$

Lemma 7.6. *Let p be a prime number. Let us consider the lattice $T_{2p} = \langle -2p\rangle \oplus U \oplus U$. There exists an isometry of T_{2p} which sends the vector $q := (n, b, bf, 0, 0)$, $b \in \mathbb{Z}_{>0}$, $n \in \mathbb{N}$, $\gcd(n, b) = 1$, to one of the following vectors*:

- $v_1 = (0, 1, r, 0, 0)$, *where* $2b^2f - 2pn^2 = 2r$;
- $v_2 = (1, 2, 2s, 0, 0)$, *where* $2b^2f - 2pn^2 = 8s - 2p$;
- $v_p = (l, p, pt, 0, 0)$, *where* $2b^2f - 2pn^2 = 2p^2t - 2pl^2$, $0 < l \leq \llcorner p/2 \lrcorner$;
- $v_{2p} = (j, 2p, 2pu, 0, 0)$, *where* $2b^2f - 2pn^2 = 8p^2u - 2pj^2$, $0 < j < p$, $j \equiv 1 \mod 2$.

Proof. We can assume $n \in \mathbb{N}$ and $b > 0$ (if this is not the case, it suffices to consider the action of $-\mathrm{id}$ and of D). Let us consider the reflection R_v, associated to the vector $v = (1, 0, p, 0, 0)$. We have

$$R_v \begin{pmatrix} n \\ b \\ bf \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -n+b \\ b \\ -2pn + pb + bf \\ 0 \\ 0 \end{pmatrix}.$$

By changing the sign of the first component we obtain $(b-n, b, -2pn + pb + bf, 0, 0)$. By Lemma 7.2, this vector can be transformed to $(b-n, b_1, b_1f_1, 0, 0)$, where $\gcd(b, -2pn + pb + bf) = b_1$. Then $b_1 \leq b := b_0$. We now apply Lemma 7.2 to the vector $(n_1, b_1, b_1f_1, 0, 0)$, with $n_1 := |b-n| > 0$ (eventually changing the sign of b_n by using the matrix D). The second component of the vector b_1 is a positive number, so after a finite number of transformations there exists η such that $b_\eta = b_{\eta+1}$ and $\gcd(n_\eta, b_\eta) = 1$. Since $b_\eta \mid (pb_\eta + b_\eta f)$ and $\gcd(n_\eta, b_\eta) = 1$ (recall that the image of a primitive vector by an isometry is again primitive) $b_\eta = b_{\eta+1}$ if and only if b_η divides $2p$, i.e., if $b_\eta = 1, 2, p, 2p$. Moreover, $\gcd(b_\eta - n_\eta, b_\eta) = 1$. With the use of Lemma 7.3, by applying the transformation D to obtain a positive vector for the first component, after a finite number of transformations, we obtain that q is isometric to one of the vectors $(a, 1, f', 0, 0)$, $(2k+1, 2, 2f', 0, 0)$, $(ph \pm l, p, pf', 0, 0)$ or $(2pk \pm j, 2p, 2pf', 0, 0)$. Applying Lemmas 7.4 and 7.5, we obtain that these vectors are isometric to $(0, 1, r, 0, 0)$, $(1, 2, 2s, 0, 0)$, $(l, p, pt, 0, 0)$ or $(j, 2p, 2pu, 0, 0)$, respectively. $\square$

Remark 7.7. The vector $(ts, t, f, 0, 0)$ is isometric to $(0, t, *, 0, 0)$ by applying $D \circ R_v$ s-times.

Proposition 7.8. *Let p be a prime number. The orbits of the following vectors of T_{2p} under isometries of T_{2p} are all disjoint*:

- $v_0 = (1, 0, 0, 0, 0)$;
- $v_1 = (0, 1, r, 0, 0)$;
- $v_2 = (1, 2, 2s, 0, 0)$;
- $v_p = (l, p, pt, 0, 0)$, *where* $0 < l \leq \llcorner p/2 \lrcorner$;
- $v_{2p} = (j, 2p, 2pu, 0, 0)$, *where* $0 < j < p$, $j \equiv 1 \mod 2$.

Proof. If two vectors x and y of T_{2p} are isometric, then $x^2 = y^2$, and the discriminants of the lattices orthogonal to x and y are equal: $d(x^\perp) = d(y^\perp)$. We list the properties of the vectors v_i in the following table:

v	v_0	v_1	v_2	v_p	v_{2p}
v^2	$-2p$	$2r$	$-2p+8s$	$-2pl^2+2p^2t$	$-2pj^2+8p^2u$
$v^\perp$	$U \oplus U$	$\begin{bmatrix} -2p & 0 \\ 0 & -2r \end{bmatrix} \oplus U$	$\begin{bmatrix} -2p & p \\ p & -2s \end{bmatrix} \oplus U$	$\begin{bmatrix} -2p & 2l \\ 2l & -2t \end{bmatrix} \oplus U$	$\begin{bmatrix} -2p & j \\ j & -2u \end{bmatrix} \oplus U$
$d(v^\perp)$	1	$-4pr$	$-p(4s-p)$	$-4(pt-l^2)$	$-4pu+j^2$

For each copy of vectors x and y chosen from v_0, v_1, v_2, v_p, v_{2p}, the conditions $x^2 = y^2$ and $d(x^\perp) = d(y^\perp)$ are incompatible. For example, let us analyze the cases of v_p and v_{2p}, the other cases being similar. We have:

$$-2pl^2 + 2p^2t = -2pj^2 + 8p^2u \quad \text{and} \quad -4(pt - l^2) = -4pu + j^2.$$

By the first equation, $-l^2 + pt = -j^2 + 4pu$. Substituting in the second equation we obtain $3(pt - l^2) = 0$ and so $pt = l^2$. This implies $p \mid l^2$ and so $p \mid l$. Since $l \leq \llcorner p/2 \lrcorner$, this is impossible. $\square$

The previous results imply the following proposition.

Proposition 7.9. *A primitive vector $(a_0, a_1, a_2, a_3, a_4)$ of the lattice $\langle -2p \rangle \oplus U \oplus U$ is isometric to exactly one of the vectors*:

- $v_0 = (1, 0, 0, 0, 0)$;

- $v_1 = (0, 1, r, 0, 0)$, *where* $2a_1a_2 + 2a_3a_4 - 2pa_0^2 = 2r$;
- $v_2 = (1, 2, 2s, 0, 0)$, *where* $2a_1a_2 + 2a_3a_4 - 2pa_0^2 = -2p + 8s$;
- $v_p = (l, p, pt, 0, 0)$, *where* $0 < l \leq \llcorner p/2 \lrcorner$ *and* $2a_1a_2 + 2a_3a_4 - 2pa_0^2 = -2pl^2 + 2p^2t$;
- $v_{2p} = (j, 2p, 2pu, 0, 0)$, *where* $0 < j < p$, $j \equiv 1 \mod 2$ *and* $2a_1a_2 + 2a_3a_4 - 2pa_0^2 = -2pj^2 + 8p^2u$.

Remark 7.10. In particular, in the case where $p = 2$, the only possibilities are the vectors $(1, 0, 0, 0, 0)$, $(0, 1, r, 0, 0)$, $(1, 2, 2s, 0, 0)$ and $(1, 4, 4u, 0, 0)$.

7.3. The family. Let us denote by $\mathcal{L}^{2d}_{r,w_i}$, the overlattice of index r of $\mathbb{Z}w_i \oplus \Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$, with $w_i^2 = 2d$ described as in Theorem 7.1. If X is a K3 surface such that $NS(X) \simeq \mathcal{L}^{2d}_{r,w_i}$ for certain $r = 2, 4, 8$ and $i = 1, 2, 3$, then $\Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$ is clearly primitively embedded in $NS(X)$, and thus X admits $(\mathbb{Z}/2\mathbb{Z})^4$ as a group of symplectic automorphisms, cf., [**38,** Theorem 4.15]. Hence, the lattices $\mathcal{L}^{2d}_{r,w_i}$ determine the family of algebraic K3 surfaces admitting a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$. More precisely,

Corollary 7.11. *The families of algebraic* K3 *surfaces admitting a symplectic action of* $(\mathbb{Z}/2\mathbb{Z})^4$ *are the families of* $(\mathcal{L}^{2d}_{r,w_i})$*-polarized* K3 *surfaces, for certain* $r = 2, 4, 8$, $i = 1, 2, 3$, $d \in 2\mathbb{N}_{>0}$. *In particular, the moduli space has countable numbers of connected irreducible components of dimension* 4.

Remark 7.12. If one fixes the value of d, then there is a finite number of possibilities for r and w_i. For example, if $d = 2$, then $r = 2$ and $i = 1$, $w_1 = (0, 1, 1, 0, 0, 0, 0)$. This implies that the family of quartic surfaces in $\mathbb{P}^3$ admitting a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$ has only one connected irreducible component of dimension 4. In [**8**], the family of quartics invariant for the Heisenberg group($\simeq (\mathbb{Z}/2\mathbb{Z})^4$) is described. Since it is a four-dimensional family of K3 surfaces admitting $(\mathbb{Z}/2\mathbb{Z})^4$ as a group of symplectic automorphisms, we conclude that the family presented in [**8**] is the family of the $(\mathcal{L}^4_{2,w_1})$-polarized K3 surfaces. The Néron-Severi group of such K3 surfaces are generated by conics as proved in [**8,** Corollary 7.4].

7.4. The subfamily of Kummer surfaces. By Corollary 2.9, for every non negative integer d, there exists a connected irreducible component of the moduli space of Kummer surfaces, which we call $\mathcal{F}_d$ and is the family of the $\mathcal{K}'_{4d}$-polarized K3 surfaces. For every d, the component $\mathcal{F}_d$ is three-dimensional. By Proposition 3.3, it is contained in a connected component of the moduli space of K3 surfaces X, admitting G as group of symplectic automorphisms. Proposition 7.13 identifies the components of the moduli space of K3 surfaces with a symplectic action of G which contains $\mathcal{F}_d$.

Proposition 7.13. *The family of the $\mathcal{K}'_{4d}$-polarized Kummer surfaces is a codimension 1 subfamily of the following families: the $(\mathcal{L}^{4d}_{2,w_1})$-polarized $K3$ surfaces, the $(\mathcal{L}^{8(2d-1)}_{4,w_2})$-polarized $K3$ surfaces and the $(\mathcal{L}^{8(8d-1)}_{8,w_3})$-polarized $K3$ surfaces.*

Proof. It suffices to show that there exists a primitive embedding $\mathcal{L}^h_{i,w_j} \subset \mathcal{K}'_{4d}$ or equivalently a primitive embedding $(\mathcal{K}'_{4d})^\perp \subset (\mathcal{L}^h_{i,w_j})^\perp$, for $(i,j,h) = (2,1,4d)$, $(4,2,8(2d-1))$, $(8,3,8(8d-1))$. We recall that

$$(\mathcal{K}'_{4d})^\perp \simeq \langle -4d \rangle \oplus U(2) \oplus U(2),$$

and $(\mathcal{L}^h_{i,w_j})^\perp$ is the transcendental lattice of the generic K3 surface X described in Theorem 7.1. With the notation of Theorem 7.1 we send a basis of $\langle -4d \rangle \oplus U(2) \oplus U(2)$ to the basis vectors $(0,1,0,0,0,0)$, $(0,0,1,0,0,0)$, $(0,0,0,1,0,0)$, $(0,0,0,0,1,0)$ and $(0,0,0,0,0,1)$ of $(\mathcal{L}^h_{i,w_j})^\perp$, with $t = d$, $s = d$, $u = d$, if $i = 2,4,8$ respectively. We obtain an explicit primitive embedding of $(\mathcal{K}'_{4d})^\perp$ in $(\mathcal{L}^h_{i,w_j})^\perp$. $\square$

We observe that the sublattice of $NS(Km(A))$ is invariant for the action induced by the translation of the two torsion points on A, i.e., invariant for the action of G defined in Proposition 3.3, and is generated by H and

$$\frac{1}{2}\Bigg(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4} K_p\Bigg).$$

Indeed, H is the image of the generator of $NS(A)$ by the map π_{A*}, with the notation of diagram (2.1). Thus, the lattice $\Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$ is isometric to

$$\left\langle H, \frac{1}{2}\Big(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4 K_p}\Big)\right\rangle^{\perp} \cap NS(Km(A)),$$

and in fact, the lattice $\mathcal{L}^{2d}_{2,w_1}$ (which contains $\Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$ and an ample class) is isometric to

$$\left\langle \frac{1}{2}\Big(\sum_{p\in(\mathbb{Z}/2\mathbb{Z})^4} K_p\Big)\right\rangle^{\perp} \cap NS(Km(A)).$$

Remark 7.14. The previous proposition implies that the family of the Kummer surfaces of a $(1, d)$-polarized abelian surface is contained in at least three distinct connected irreducible components of the family of K3 surfaces admitting a symplectic action of G. In particular, the intersection among the connected irreducible components of such a family of K3 surfaces is non empty and of dimension 3.

7.5. Enriques involution. In Section 3, we have seen the result of Keum [**25**]. Every Kummer surface admits an Enriques involution. We now prove that, in general, this property holds for the K3 surfaces admitting $(\mathbb{Z}/2\mathbb{Z})^4$ as a group of symplectic automorphisms and minimal Picard number.

Theorem 7.15. *Let X be a* K3 *surface admitting $(\mathbb{Z}/2\mathbb{Z})^4$ as a group of symplectic automorphisms such that $\rho(X) = 16$. Then X admits an Enriques involution.*

Proof. By Proposition 3.2, it suffices to prove that the transcendental lattice of X admits a primitive embedding in $U \oplus U(2) \oplus E_8(-2)$ whose orthogonal does not contain vectors of length -2. The existence of this embedding can be proved as in [**25**]. We briefly sketch the proof. Let Q be one of the following lattices:

$$\langle -4\rangle \oplus \langle -2t\rangle, \begin{bmatrix} -4 & 2 \\ 2 & -2s \end{bmatrix}, \begin{bmatrix} -4 & 1 \\ 1 & -2u \end{bmatrix}.$$

The transcendental lattice of X is $(U^2 \oplus Q)(2)$. It suffices to prove that there exists a primitive embedding of $U(2) \oplus Q(2)$ in $U \oplus E_8(-2)$.

The lattice $\langle -2\rangle \oplus Q$ is an even lattice with signature $(0,3)$. By [**37,** Theorem 14.4], there exists a primitive embedding of $\langle -2\rangle \oplus Q$ in $E_8(-1)$, which induces a primitive embedding of $\langle -4\rangle \oplus Q(2)$ in $E_8(-2)$. Let b_1, b_2 and b_3 be the bases of $\langle -4\rangle \oplus Q(2)$ in $E_8(-2)$. Let e and f be standard bases of U, i.e., $e^2 = f^2 = 0$, $ef = 1$. Then the vectors e, $e + 2f + b_1$, b_2 and b_3 give a primitive embedding of $U(2) \oplus Q(2)$ in $U \oplus E_8(-2)$ whose orthogonal complement does not contain vectors of length -2, cf., [**25,** Section 2, Proof of Theorem 2]. $\square$

8. The quotient K3 surface. The surface Y obtained as a desingularization of the quotient $X/(\mathbb{Z}/2\mathbb{Z})^4$ contains 15 rational curves M_i, which are the resolution of the 15 singular points of type A_1 on $X/(\mathbb{Z}/2\mathbb{Z})^4$. The minimal primitive sublattice of $NS(Y)$ containing these curves is denoted by $M_{(\mathbb{Z}/2\mathbb{Z})^4}$. It is described in [**38,** Section 7] as an overlattice of the lattice $\langle M_i\rangle_{i=1,\ldots,15}$ of index 2^4.

Proposition 8.1. *Let Y be a* K3 *surface such that there exists a projective* K3 *surface X and a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$ on X with $Y = \widetilde{X/(\mathbb{Z}/2\mathbb{Z})^4}$. Then $\rho(Y) \geq 16$.*

Moreover, if $\rho(Y) = 16$, let $L = M_{(\mathbb{Z}/2\mathbb{Z})^4}^{\perp_{NS(Y)}}$. Then $NS(Y)$ is an overlattice of index 2 of $\mathbb{Z}L \oplus M_{(\mathbb{Z}/2\mathbb{Z})^4}$, where $L^2 = 2d > 0$. In particular, $NS(Y)$ is generated by $\mathbb{Z}L \oplus M_{(\mathbb{Z}/2\mathbb{Z})^4}$ and by a class $(L/2, v/2)$, $v/2 \in M_{(\mathbb{Z}/2\mathbb{Z})^4}^{\vee}/M_{(\mathbb{Z}/2\mathbb{Z})^4}$, which is not trivial in $M_{(\mathbb{Z}/2\mathbb{Z})^4}^{\vee}/M_{(\mathbb{Z}/2\mathbb{Z})^4}$, $L^2 \equiv -v^2 \mod 8$.

Proof. A K3 surface Y obtained as a desingularization of the quotient of a K3 surface X by the symplectic group of automorphisms $(\mathbb{Z}/2\mathbb{Z})^4$ has $M_{(\mathbb{Z}/2\mathbb{Z})^4} \subset NS(Y)$. Since $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ is negative definite and Y is projective (it is the quotient of X which is projective), there is at least one class in $NS(Y)$ which is not in $M_{(\mathbb{Z}/2\mathbb{Z})^4}$, so $\rho(Y) \geq 1 + \operatorname{rank} M_{(\mathbb{Z}/2\mathbb{Z})^4} = 16$. In particular, if $\rho(Y) = 16$, then the orthogonal complement of $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ in $NS(Y)$ is generated by a class with a positive self-intersection; hence, $NS(Y)$ is either $\mathbb{Z}L \oplus M_{(\mathbb{Z}/2\mathbb{Z})^4}$ or an overlattice of $\mathbb{Z}L \oplus M_{(\mathbb{Z}/2\mathbb{Z})^4}$ with a finite index. The discriminant group of $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ is $(\mathbb{Z}/2\mathbb{Z})^7$ by [**38,** Section 7] and so the discriminant group of the lattice $\mathbb{Z}L \oplus M_{(\mathbb{Z}/2\mathbb{Z})^4}$ is $(\mathbb{Z}/2d\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^7$. It has eight generators. If the lattice $\mathbb{Z}L \oplus M_{(\mathbb{Z}/2\mathbb{Z})^4}$ is the Néron-Severi group of a K3 surface Y, then the discriminant group of T_Y also has eight generators, but T_Y has rank $22-\rho(Y)=6$, so this is impossible. Hence, $NS(Y)$ is an overlattice of $\mathbb{Z}L\oplus M_{(\mathbb{Z}/2\mathbb{Z})^4}$. The index of the inclusion and the construction of the overlattice can be computed as in [**13,** Proposition 2.1] or as in Theorem 2.7. $\square$

The Kummer surfaces are also examples of K3 surfaces obtained as a desingularization of the quotient of K3 surfaces by the action of $(\mathbb{Z}/2\mathbb{Z})^4$ as a group of symplectic automorphisms, see Proposition 3.3.

In [**11,** subsections 4.2, 4.3], the action of G on the Kummer lattice and the construction of the surface $\widetilde{Km(A)/G}$ are described. A single curve is obtained by the images of the curves $K_{a,b,c,d}$, $(a,b,c,d)\in(\mathbb{Z}/2\mathbb{Z})^4$ on $Km(A)$ under the quotient map $Km(A)\to Km(A)/G$. This curve can be naturally identified with the curve $K_{0,0,0,0}$ on the minimal resolution $\widetilde{Km(A)/G}\cong Km(A)$, see [**11**]. The minimal resolution also contains 15 (-2)-curves which come from blowing up the nodes on $Km(A)/G$ and can be identified with $K_{e,f,g,h}$, $(e,f,g,h)\in(\mathbb{Z}/2\mathbb{Z})^4\backslash\{(0,0,0,0)\}$. These are the 15 (-2)-curves in $M_{(\mathbb{Z}/2\mathbb{Z})^4}$; hence, $M_{(\mathbb{Z}/2\mathbb{Z})^4}=K_{(0,0,0,0)}^{\perp}\cap K$. This allows us to identify the curves of $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ with points of the space $(\mathbb{Z}/2\mathbb{Z})^4\backslash\{(0,0,0,0)\}$; hence, we denote them by $M_{a,b,c,d}$, $(a,b,c,d)\in(\mathbb{Z}/2\mathbb{Z})^4\backslash\{(0,0,0,0)\}$. More explicitly, we are identifying the curve $K_{a,b,c,d}$ with the curve $M_{a,b,c,d}$ for any $(a,b,c,d)\in(\mathbb{Z}/2\mathbb{Z})^4\backslash\{(0,0,0,0)\}$. By [**38**], the lattice $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ contains the 15 curves $M_{a,b,c,d}$, $(a,b,c,d)\in(\mathbb{Z}/2\mathbb{Z})^4\backslash\{(0,0,0,0)\}$. It is generated by 11 of these curves and by 4 other classes which are linear combinations of these curves with rational coefficients. These four classes must also be contained in K (because $M_{(\mathbb{Z}/2\mathbb{Z})^4}\subset K$), and hence, they correspond to hyperplanes in $(\mathbb{Z}/2\mathbb{Z})^4$ which do not contain the point $(0,0,0,0)$ (because $K_{(0,0,0,0)}\notin M_{(\mathbb{Z}/2\mathbb{Z})^4}$).

From now on, $\overline{K}_W$, respectively $\overline{M}_W$, denotes $1/2\sum_{p\in W}K_p$, respectively $1/2\sum_{p\in W}M_p$, for a subset W of $(\mathbb{Z}/2\mathbb{Z})^4$, respectively W a subset of $(\mathbb{Z}/2\mathbb{Z})^4\backslash\{(0,0,0,0)\}$. We determine the orbits of elements in the discriminant group of $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ and their isometries using those of K.

Proposition 8.2. *Regarding the group of isometries of $M_{(\mathbb{Z}/2\mathbb{Z})^4}$, there are exactly six distinct orbits in discriminant group $M^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^4}/M_{(\mathbb{Z}/2\mathbb{Z})^4}$.*

Proof. Let W be one of the following subspaces.

1) $W = (\mathbb{Z}/2\mathbb{Z})^4$;
2) W is a hyperplane in $(\mathbb{Z}/2\mathbb{Z})^4$;
3) W is a two-dimensional plane in $(\mathbb{Z}/2\mathbb{Z})^4$;
4) $W = V * V'$, where V and V' are two-dimensional planes and $V \cap V'$ is a point.

By Remark 2.3, the classes $\overline{K}_W$ are in $K^\vee$ and, if W is as in 1) or 2), the classes $\overline{K}_W \in K$ are trivial in $K^\vee/K$. If W is such that $(0,0,0,0) \notin W$, then the class $\overline{M}_W = \overline{K}_W$ is contained in $M^\vee_{(\mathbb{Z}/2\mathbb{Z})^4}$. Indeed, it is a linear combination with rational coefficients of the curves $M_{(a,b,c,d)}$ with $(a,b,c,d) \in (\mathbb{Z}/2\mathbb{Z})^4 \backslash \{(0,0,0,0)\}$, i.e., it is in $M_{(\mathbb{Z}/2\mathbb{Z})^4} \otimes \mathbb{Q}$. Moreover, the class $\overline{M}_W = \overline{K}_W$ has an integer intersection with all the classes in K, and so, in particular, with all the classes in $M_{(\mathbb{Z}/2\mathbb{Z})^4} \subset K$, they are in $M^\vee_{(\mathbb{Z}/2\mathbb{Z})^4}$. We observe that, if W is a hyperplane, as in case 2), and is such that $(0,0,0,0) \notin W$, then the class $\overline{M}_W$ is a class in $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ (and hence trivial in the discriminant group, see Remark 2.3).

If $(0,0,0,0) \in W$, let W' be $W' = W - \{(0,0,0,0)\}$. The class $\overline{M}_{W'}$ is a class in $M^\vee_{(\mathbb{Z}/2\mathbb{Z})^4}$. Indeed, it is clear that $\overline{M}_{W'} \in M_{(\mathbb{Z}/2\mathbb{Z})^4} \otimes \mathbb{Q}$ has an integer intersection with all the classes $M_{(a,b,c,d)} \in M_{(\mathbb{Z}/2\mathbb{Z})^4}$, $(a,b,c,d) \in (\mathbb{Z}/2\mathbb{Z})^4 \backslash \{(0,0,0,0)\}$. Let Z be a hyperplane of $(\mathbb{Z}/2\mathbb{Z})^4$ which does not contain $(0,0,0,0)$. Since $\overline{M}_Z \in M_{(\mathbb{Z}/2\mathbb{Z})^4}$, we must check that $\overline{M}_{W'} \cdot \overline{M}_Z \in \mathbb{Z}$. We recall that $\overline{K}_W$ is in $K^\vee$, and so it has an integer intersection with all the classes $\overline{K}_Z$. This means that $W \cap Z$ is made up of an even number of points. Since $(0,0,0,0) \notin Z$, $(0,0,0,0) \notin W \cap Z$, and hence $W' \cap Z$, is an even number of points, this implies that $\overline{M}_{W'} \cdot \overline{M}_Z \in \mathbb{Z}$.

If $\overline{M}_W \in M^\vee_{(\mathbb{Z}/2\mathbb{Z})^4}$, then either $\overline{K}_W$ or $\overline{K}_{W \cup \{(0,0,0,0)\}}$ is in $K^\vee$. Indeed by Remark 2.3 the Kummer lattice is generated by the curves $K_{(a,b,c,d)}$, $(a,b,c,d) \in (\mathbb{Z}/2\mathbb{Z})^4$, by four classes of type $\overline{K}_{W_i}$ where W_i is the hyperplane $a_i = 0$, $i = 0,1,2,3$ (see the notation of Remark 2.3) and by the class $\overline{K}_{(\mathbb{Z}/2\mathbb{Z})^4}$. This is clearly equivalent to saying that K is generated by the curves $K_{(a,b,c,d)}$, by four classes of type $\overline{K}_{W_i'}$ where W_i' is the hyperplane $a_i = 1$ and by the class $\overline{K}_{(\mathbb{Z}/2\mathbb{Z})^4}$. If $\overline{M}_W \in M^\vee_{(\mathbb{Z}/2\mathbb{Z})^4}$, then $\overline{M}_W \cdot \overline{M}_{W_i'} = \overline{K}_W \cdot \overline{K}_{W_i'} \in \mathbb{Z}$. Moreover, since $(0,0,0,0) \notin W_i'$, we also have $\overline{K}_{W \cup \{(0,0,0,0)\}} \cdot \overline{K}_{W_i'} \in \mathbb{Z}$.

To conclude that either $\overline{K}_W$ or $\overline{K}_{W\cup\{(0,0,0,0)\}}$ is in $K^\vee$, it suffices to prove either that $\overline{K}_W \cdot \overline{K}_{(\mathbb{Z}/2\mathbb{Z})^4} \in \mathbb{Z}$ or $\overline{K}_{W\cup\{(0,0,0,0)\}} \cdot \overline{K}_{(\mathbb{Z}/2\mathbb{Z})^4} \in \mathbb{Z}$. This is clear, indeed $\overline{K}_W \cdot \overline{K}_{(\mathbb{Z}/2\mathbb{Z})^4} \in \mathbb{Z}$, if and only if W consists of an even number of points. If it is not, clearly $W \cup \{(0,0,0,0)\}$ consists of an even number of points. Thus, the classes $\overline{M}_W$ are in $M^\vee_{(\mathbb{Z}/2\mathbb{Z})^4}$ for the following subspaces.

1) $W = (\mathbb{Z}/2\mathbb{Z})^4 \backslash \{(0,0,0,0)\}$;
2a) W is a hyperplane in $(\mathbb{Z}/2\mathbb{Z})^4$, $(0,0,0,0) \notin W$;
2b) $W\backslash\{(0,0,0,0)\}$ where W is a hyperplane in $(\mathbb{Z}/2\mathbb{Z})^4$, $(0,0,0,0) \in W$;
3a) W is a two-dimensional plane in $(\mathbb{Z}/2\mathbb{Z})^4$ and $(0,0,0,0) \notin W$;
3b) $W\backslash\{(0,0,0,0)\}$ where W is a two-dimensional plane in $(\mathbb{Z}/2\mathbb{Z})^4$ and $(0,0,0,0) \in W$;
4a) $W = V * V'$ where V and V' are two-dimensional planes and $V \cap V'$ is a point, $(0,0,0,0) \notin V * V'$;
4b) $W\backslash\{(0,0,0,0)\}$ where $W = V * V'$, V and V' are two-dimensional planes and $V \cap V'$ is a point, $(0,0,0,0) \in V * V'$.

In the quotient $M^\vee_{(\mathbb{Z}/2\mathbb{Z})^4}/M_{(\mathbb{Z}/2\mathbb{Z})^4}$, each of the above cases corresponds to a class of equivalence. Here we study them. We will denote by H, a hyperplane of $(\mathbb{Z}/2\mathbb{Z})^4$ such that $(0,0,0,0) \notin H$. We observe that $\overline{M}_{W*H} \equiv \overline{M}_W + \overline{M}_H \mod \oplus_p \mathbb{Z}M_p$. Clearly, the two classes $\overline{M}_W$ and $\overline{M}_{W*H}$ coincide in $M^\vee_{(\mathbb{Z}/2\mathbb{Z})^4}/M_{(\mathbb{Z}/2\mathbb{Z})^4}$ if $\overline{M}_H \in M_{(\mathbb{Z}/2\mathbb{Z})^4}$. Let n be the cardinality of $W \cap H$, and let m be the number of curves $M_{(a,b,c,d)}$ appearing in $\overline{M}_{W*H}$ with a rational, non integer coefficient.

In the following table, we list the classes of $M^\vee_{(\mathbb{Z}/2\mathbb{Z})^4}$ which coincide modulo $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ and, for each, we give the value discr of the discriminant form on it. The first value of m in the table is the number of curves in $\overline{M}_W$, and we put a 0 for n.

Case	n	m	discr
1); 2b)	$0; 0, 8; 3$	$15; 7, 7; 7$	$1/2$
3a)	$0, 4, 2, 0$	$4, 4, 8, 12$	0
3b)	$0, 0, 2$	$3, 7, 11$	$1/2$
4a)	$4, 2$	$6, 10$	1
4b)	$4, 2$	$5, 9$	$-1/2$

By Remark 2.3, the orbit of elements in the discriminant group A_K of K is 3 up to isometry. To prove that the orbit of $A_{M_{(\mathbb{Z}/2\mathbb{Z})^4}}$ is 6 up to isometry, one considers the action of the group $GL(4,\mathbb{Z}/2\mathbb{Z})$ on $(\mathbb{Z}/2\mathbb{Z})^4$, which in fact we can identify with a subgroup of $O(A_K)$. Since $GL(4,\mathbb{Z}/2\mathbb{Z})$ fixes $(0,0,0,0)$, it also acts on $(\mathbb{Z}/2\mathbb{Z})^4\backslash\{(0,0,0,0)\}$, and so we can identify it with a subgroup of $O(A_{M_{(\mathbb{Z}/2\mathbb{Z})^4}})$. This means that, under the action of $GL(4,\mathbb{Z}/2\mathbb{Z})$, we have at most six orbits associated to the cases 2a), 1; 2b), 3a), 3b), 4a), 4b). We observe that the orbit of 2a) is one of class $0 \in A_{M_{(\mathbb{Z}/2\mathbb{Z})^4}}$. We now show that all these orbits are disjoint, so we have exactly six (five non trivial) orbits in $M^\vee_{(\mathbb{Z}/2\mathbb{Z})^4}/M_{(\mathbb{Z}/2\mathbb{Z})^4}$. One can check by direct computation that the classes of cases 1) and 2b) coincide in the quotient. The classes in $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ with self-intersection -2 are only $\pm M_{(a,b,c,d)}$, $(a,b,c,d) \in (\mathbb{Z}/2\mathbb{Z})^4\backslash\{(0,0,0,0)\}$. Indeed, each class in $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ is a linear combination,

$$D = \sum_{(a,b,c,d)\in(\mathbb{Z}/2\mathbb{Z})^4-\{(0,0,0,0)\}} \alpha_{(a,b,c,d)} M_{a,b,c,d},$$

with

$$\alpha_{(a,b,c,d)} \in \frac{1}{2}\mathbb{Z}.$$

The condition

$$-2 = D^2 = -2 \sum_{(a,b,c,d)} \alpha^2_{(a,b,c,d)}$$

implies that either there is one $\alpha_{(a,b,c,d)} = \pm 1$ and the others are zero, or there are four $\alpha_{(a,b,c,d)}$ equal to $\pm 1/2$ and the others are zero. Since there are no classes in $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ which are linear combinations with rational coefficients of only four classes, we have $D = \pm M_{(a,b,c,d)}$ for a certain $(a,b,c,d) \in (\mathbb{Z}/2\mathbb{Z})^4\backslash\{(0,0,0,0)\}$. Since the isometries of $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ preserve the intersection product, they send the classes of the curves $M_{(a,b,c,d)}$ either to the class of a curve or to the opposite of the class of a curve. In particular, there are no isometries of $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ which identify classes associated to the six cases 1); 2b), 2a), 3a), 3b), 4a), 4b); indeed, in each class, there is some linear combination with non integer coefficients of a different number of curves $M_{(a,b,c,d)}$. $\square$

Theorem 8.3. *Let Y be a projective* K3 *surface such that there exists a* K3 *surface X and a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$ on X with $Y = \widetilde{X/(\mathbb{Z}/2\mathbb{Z})^4}$, and let $\rho(Y) = 16$.*

Then $NS(Y)$ is generated by $\mathbb{Z}L \oplus M_{(\mathbb{Z}/2\mathbb{Z})^4}$ with $L^2 = 2d > 0$ and by a class

$$\left(\frac{L}{2}, \frac{v}{2}\right), \quad 0 \neq \frac{v}{2} \in \frac{M^\vee_{(\mathbb{Z}/2\mathbb{Z})^4}}{M_{(\mathbb{Z}/2\mathbb{Z})^4}},$$

with $L^2 \equiv -v^2 \mod 8$. Up to isometry, there are only the following possibilities.

(i) *If $d \equiv 1 \mod 4$, then $v/2 = \overline{M}_W$, where*

$$W = \{(1,1,0,1), (1,1,1,0), (1,1,1,1), (1,0,0,0), (0,1,0,0)\}$$

(*case* 4b) *of proof of Proposition* 8.2);

(ii) *if $d \equiv 2 \mod 4$, then $v/2 = \overline{M}_W$, where*

$$W = \{(0,0,0,1), (0,0,1,0), (0,0,1,1), (1,0,0,0), (0,1,0,0), (1,1,0,0)\}$$

(*case* 4a) *of proof of Proposition* 8.2);

(iii) *if $d \equiv 3 \mod 4$, then, either*

(a) *$v/2 = \overline{M}_W$, where*

$$W = \{(0,0,0,1), (0,0,1,0), (0,0,1,1)\}$$

(*case* 3b) *of proof of Proposition* 8.2), *or*

(b) *$v/2 = \overline{M}_W$, where*

$$W = (\mathbb{Z}/2\mathbb{Z})^4 - \{(0,0,0,0)\}$$

(*cases* 1–2b) *of proof of Proposition* 8.2);

(iv) *if $d \equiv 0 \mod 4$, then $v/2 = \overline{M}_W$, where*

$$W = \{(1,1,0,0), (1,1,1,0), (1,1,0,1), (1,1,1,1)\}$$

(*case* 3a) *of proof of Proposition* 8.2).

Moreover, for each $d \in \mathbb{N}$, there exists a K3 *surface S such that $NS(S)$ is an overlattice of index* 2 *of the lattice $\langle 2d \rangle \oplus M_{(\mathbb{Z}/2\mathbb{Z})^4}$.*

In cases (i), (ii), (iii) and (iv),

$$T_Y \simeq U(2) \oplus U(2) \oplus \langle -2 \rangle \oplus \langle -2d \rangle.$$

In case iv), *denote by q_2 the discriminant form of $U(2)$. Then the discriminant group of T_Y is $(\mathbb{Z}/2\mathbb{Z})^5 \times \mathbb{Z}/2d\mathbb{Z}$, with discriminant form*

$$q_2 \oplus q_2 \oplus \begin{pmatrix} 0 & 1/2 \\ 1/2 & (-d-1)/2d \end{pmatrix}.$$

Proof. In Proposition 8.1, we proved that the lattice $NS(Y)$ must be an overlattice of index 2 of $\mathbb{Z}L \oplus M_{(\mathbb{Z}/2\mathbb{Z})^4}$. The unicity of the choice of v up to isometry depends on the description of the orbit of the group of isometries of $M^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^4}/M_{(\mathbb{Z}/2\mathbb{Z})^4}$ given in Proposition 8.2.

By an explicit computation, one can show that the discriminant group of the overlattices described in cases (i), (ii), (iii) and (iv) is $(\mathbb{Z}/2\mathbb{Z})^5 \times (\mathbb{Z}/2d\mathbb{Z})$, and the discriminant form in all cases except (iii) is $q_2 \oplus q_2 \oplus \langle 1/2 \rangle \oplus \langle 1/2d \rangle$. In case iv), the discriminant form is described in the statement. In any case, by [**37,** Theorem 1.14.4 and Remark 1.14.5], the overlattices have a unique primitive embedding in the K3 lattice Λ_{K3}; hence, by the surjectivity of the period map, there exists a K3 surface S as in the statement of the theorem. Moreover, by [**37,** Theorems 1.13.2, 1.14.2] the transcendental lattice is uniquely determined by signature and discriminant form. This concludes the proof. $\square$

Remark 8.4. The Kummer surfaces appear as specializations of the surfaces Y as in Proposition 8.3 such that $d \equiv 0 \mod 2$. Indeed, let us consider the surface Y such that $d = 2d'$. The transcendental lattice of a generic Kummer of a $(1, d')$-polarized abelian surface is

$$T_{Km(A)} \simeq U(2) \oplus U(2) \oplus \langle -4d' \rangle,$$

and it is clearly primitively embedded in

$$T_Y \simeq U(2) \oplus U(2) \oplus \langle -2 \rangle \oplus \langle -4d' \rangle.$$

8.1. Ampleness properties. As in Section 4, we can prove that certain divisors on Y are ample (or nef or nef and big) using the description of the Néron-Severi group of Y given in Theorem 8.3. The ample (or nef or nef and big) divisors define projective models, which can be described in the same way as in Section 5, where we described projective models of the Kummer surfaces.

Proposition 8.5. *With the notation of Theorem* 8.3, *the following properties for divisors on* Y *hold*:

- L *is pseudo ample and it has no fixed components*;
- *the divisor* $D := L - (M_1 + \cdots + M_r)$, $1 \le r \le 15$ *is pseudo ample if* $d > r$;

- *let* $\overline{D} := (L - M_1 - \cdots - M_r)/2 \in NS(Y) \otimes \mathbb{Q}$; *if* $\overline{D} \in NS(Y)$, *then it is pseudo ample if* $d > r$.

8.2. K3 surfaces with 15 nodes. Here we show that a K3 surface with 15 nodes (respectively with 15 disjoint irreducible rational curves) is in fact the quotient (respectively the desigularization of the quotient) of a K3 surface by a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$. This is in a certain sense the generalization of a similar result for Kummer surfaces (cf., subsection 2.2).

Theorem 8.6. *Let* Y *be a projective* K3 *surface with* 15 *disjoint smooth rational curves* M_i, $i = 1, \dots, 15$, *or equivalently, a* K3 *surface admitting a singular model with* 15 *nodes. Then,*

1) $NS(Y)$ *contains the lattice* $M_{(\mathbb{Z}/2\mathbb{Z})^4}$.
2) *There exists a* K3 *surface* X *with a* $G = (\mathbb{Z}/2\mathbb{Z})^4$ *symplectic action, such that* Y *is the minimal resolution of the quotient* X/G.

Proof.

1) Let Q be the orthogonal complement in $NS(Y)$ to $\oplus_{i=1}^{15} \mathbb{Z}M_i$ and R the lattice $Q \oplus (\oplus_{i=1}^{15} \mathbb{Z}M_i)$. Observe that $NS(Y)$ is an overlattice of finite index of R and $R^\vee/R \cong Q^\vee/Q \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 15}$ so $l(R) = l(Q) + 15$. Let us denote by k the index of R in $NS(Y)$; thus, $l(NS(Y)) = l(Q)+15-2k$. On the other hand, the rank of the transcendental lattice is $22 - \mathrm{rank}(R) = 7 - \mathrm{rank}(Q)$. Hence, $l(Q) + 15 - 2k \leq 7 - \mathrm{rank}(Q)$. Thus, $k \geq (8 + l(Q) + \mathrm{rank}(Q))/2$. We observe that k is the minimum number of divisible classes we must add to R in order to obtain $NS(Y)$.

By definition, the lattice Q is primitive in $NS(Y)$; thus, the non trivial classes that we can add to R in order to obtain overlattices are either classes in $(\oplus_i \mathbb{Z}M_i)^\vee/(\oplus_i \mathbb{Z}M_i)$ or classes like $v + v'$, where $v' \in Q^\vee/Q$ and $v \in (\oplus_i \mathbb{Z}M_i)^\vee/(\oplus_i \mathbb{Z}M_i)$ is non trivial. By construction, the independent classes of the second type are at most $l(Q)$, and thus, there are at least

$$\left(\frac{8 + l(Q) + \mathrm{rank}(Q)}{2}\right) - l(Q) = \frac{8 + \mathrm{rank}(Q) - l(Q)}{2}$$

classes which are in $(\oplus_i \mathbb{Z}M_i)^\vee/(\oplus_i \mathbb{Z}M_i)$. We recall that $\mathrm{rank}(Q) - l(Q) \geq 0$, and hence, there are at least four classes which are rational linear combinations of the curves M_i. By [**36,** Lemma 3], such a class in $NS(Y)$ can only contain 16 or 8 classes. Since 16 is not possible in this case, all these classes contain 8 (-2)-curves. Let u_j, $j=1,2,3,4$, be four independent classes in $(\oplus_i \mathbb{Z}M_i)^\vee/(\oplus \mathbb{Z}M_i)$ such that the u_j are contained in $NS(Y)$. For each $j \neq h$, $j,h=1,2,3,4$, there are exactly four rational curves which are summands of both u_i and u_j; otherwise, the sum $u_i+u_j \in NS(Y)$ contains half the sum of k' disjoint rational curves for $k' \neq 8$, which is absurd. It is now a trivial computation to show, as required in $(\oplus_i \mathbb{Z}M_i)^\vee/(\oplus \mathbb{Z}M_i)$, that there are at most four independent classes (and thus exactly four) and that, for each choice of these four classes u_i, the lattice obtained adding the classes u_i, $i=1,2,3,4$, to $\oplus_i \mathbb{Z}M_i$ is exactly $M_{(\mathbb{Z}/2\mathbb{Z})^4}$. Indeed, without loss of generality, the first class can be chosen to be

$$u_1 = \sum_{i=1}^{8}(M_i/2);$$

thus, the second class can be chosen to be

$$u_2 = \sum_{i=1}^{4}(M_i/2) + \sum_{j=9}^{12}(M_j/2).$$

The third class has four curves in common with u_1 and u_2 and thus can be chosen to be $u_3 = (M_1+M_2+M_5+M_6+M_9+M_{10}+M_{13}+M_{14})/2$. Similarly, one determines the class $u_4 = (M_1+M_3+M_5+M_7+M_9+M_{11}+M_{13}+M_{15})/2$.

2) We consider the double cover $\pi_1 : Z_1 \to Y$ ramified on $2u_1$. Since $2u_1$ contains eight disjoint rational curves, Z_1 is smooth. Moreover, the pullback E_i of the curves M_i, $i=1,\dots,8$, have self-intersection -1; hence, these can be contracted to smooth points on a variety Y_1, and the covering involution that determines π_1 descends to a symplectic involution ι_1 on Y_1 with eight isolated fixed points, cf., [**33,** Section 3]. The divisors $2u_i$, $i=2,3,4$, each contain four curves which are also in support of $2u_1$.

We study the pull back of $2u_2$; the study is similar for the other classes. We have

$$\begin{aligned} 2\pi_1^*(u_2) &= \pi_1^*(2u_2) \\ &= 2(E_1+\cdots+E_4) + M_5^1 + M_5^2 + M_6^1 + M_6^2 \end{aligned}$$

$$+ M_7^1 + M_7^2 + M_8^1 + M_8^2,$$

where $\pi_1(M_j^i) = M_j$ for $i = 1, 2$, and $j = 5, 6, 7, 8$. Hence, the divisor $M_5^1 + M_5^2 + M_6^1 + M_6^2 + M_7^1 + M_7^2 + M_8^1 + M_8^2$ is divisible by 2 in $NS(Z_1)$, and so its image is divisible by 2 on $NS(Y_1)$. With the same construction as before, using this class we obtain a K3 surface Y_2 with an action by a symplectic involution ι_2. Observe that ι_1 preserves the divisor $M_5^1 + M_5^2 + M_6^1 + M_6^2 + M_7^1 + M_7^2 + M_8^1 + M_8^2$, and so ι_1 and ι_2 commute on $NS(Y_2)$. Now, considering the pull-back of $2u_3$ and $2u_4$ on Y_2, one can repeat the construction, arriving at a K3 surface $X := Y_4$ with an action by $(\mathbb{Z}/2\mathbb{Z})^4$ and such that the quotient is Y. We observe that the smooth model of a K3 surface admitting a singular model with 15 nodes contains 15 disjoint rational curves, and we prove that such a K3 surface is a $(\mathbb{Z}/2\mathbb{Z})^4$ quotient of a K3 surface. $\square$

Remark 8.7. Now assume that a K3 surface S either has a lattice isometric to $M_{(\mathbb{Z}/2\mathbb{Z})^4}$ primitively embedded in the Néron-Severi group or its Néron-Severi group is an overlattice of $Q \oplus \langle -2 \rangle^{15}$ for a certain lattice Q. Then, Theorems 8.3 and 8.6 do not imply that S is a $(\mathbb{Z}/2\mathbb{Z})^4$ quotient of a K3 surface. Indeed, in the proof of Theorem 8.6 part 2) we used that the lattice $\langle -2 \rangle^{15}$ (contained with index 2^4 in $M_{(\mathbb{Z}/2\mathbb{Z})^4}$) is generated by irreducible rational curves. In other words, the description of the Néron-Severi group from a lattice theoretic point of view is not enough to obtain our geometric characterization. Thus, we cannot conclude that the family of the K3 surfaces, which are $(\mathbb{Z}/2\mathbb{Z})^4$ quotients of K3 surfaces, coincides with the family of K3 surfaces polarized with certain lattices.

Remark 8.8. In the proof of Theorem 8.6 we proved that, if a K3 surface contains 15 disjoint rational curves M_i, then there are 15 subsets $\mathcal{S}_i$, $i = 1, \ldots, 15$, which contain exactly 8 of these curves and which form an even set. Similarly, if a K3 surface has 15 nodes, there are 15 subsets of 8 of these nodes which form an even set. In **[3, 13]**, some geometric properties of the even set of curves and nodes on K3 surfaces are described. For example, if a quartic in $\mathbb{P}^3$ contains eight nodes which form an even set, then the eight nodes are contained in an elliptic curve, and there are three quadrics in $\mathbb{P}^3$ containing these nodes. Hence, if a quartic in $\mathbb{P}^3$ has 15 nodes, each even set $\mathcal{S}_i$ has the previous properties.

Corollary 8.9. *Let Y be a projective* K3 *surface with* 14 *disjoint smooth rational curves M_i, $i = 1, \dots, 14$. Then,*

1) *$NS(Y)$ contains the lattice $M_{(\mathbb{Z}/2\mathbb{Z})^3}$, which is the minimal primitive sublattice of the* K3 *lattice Λ_{K3} that contains the* 14 *rational curves.*
2) *There exists a* K3 *surface with a $(\mathbb{Z}/2\mathbb{Z})^3$ symplectic action, such that Y is the minimal resolution of the quotient of X by this group.*

Proof. The lattice $M_{(\mathbb{Z}/2\mathbb{Z})^3}$ is described in [**38,** Section 7]. The proof of 1) and 2) is essentially the same as the proof of 1) and 2) of Theorem 8.6. □

Remark 8.10. An analogous result to those of Theorem 8.6 and Corollary 8.9 does not hold considering 8 (respectively 12) disjoint rational curves, i.e., considering the group $\mathbb{Z}/2\mathbb{Z}$ (respectively $(\mathbb{Z}/2\mathbb{Z})^2$):

1) If a K3 surface is the minimal resolution of the quotient of a K3 surface by the group $\mathbb{Z}/2\mathbb{Z}$, then it admits a set of eight disjoint rational curves, but if a K3 surface admits a set of eight disjoint rational curves, then it is not necessarily the quotient of a K3 surface by the group $\mathbb{Z}/2\mathbb{Z}$ acting symplectically. An example is given by the K3 surface with an elliptic fibration containing eight fibers of type I_2 and trivial Mordell-Weil group, cf., [**44,** Table 1, Case 99]. It contains eight disjoint rational curves (a component for each reducible fiber), which are not an even set (otherwise the fibration admits a 2-torsion section).

2) If a K3 surface is the minimal resolution of the quotient of a K3 surface by the group $(\mathbb{Z}/2\mathbb{Z})^2$, then it admits a set of 12 disjoint rational curves, but if a K3 surface admits a set of 12 disjoint rational curves, then it is not necessarily the quotient of a K3 surface by a group $(\mathbb{Z}/2\mathbb{Z})^2$ acting symplectically. Thus, it is surely the quotient of a K3 surface by $\mathbb{Z}/2\mathbb{Z}$ (the proof is again similar to that of Theorem 8.6). An example is given by the elliptic K3 surface with singular fibers $2I_0^* + 4I_2$ which is number 466 in Shimada's list [**44**]. The components of multiplicity 1 of fibers of type I_0^* and a component for each fiber of type I_2 are 12 disjoint rational curves. There is exactly one set of eight of these curves which is a 2-divisible class (the sum of the components of the I_0^* fibers of multiplicity 1). By using the Shioda-Tate formula, one can easily show that there are no other divisible classes, and hence, the surface cannot be the quotient of a K3 surface by $(\mathbb{Z}/2\mathbb{Z})^2$.

9. The maps π_* and π^*. In the previous two sections we described the family of the K3 surfaces X admitting a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^4$ and the family of the K3 surfaces Y obtained as desingularizations of the quotients of K3 surfaces by the group $(\mathbb{Z}/2\mathbb{Z})^4$. Here we explicitly describe the relation among these two families. More precisely, in Section 6, we described the quotient map $\pi : \widetilde{X} \to Y$, which of course induces the maps π_* and π^* among the cohomology groups of the surfaces. Here we describe these maps (similar results can be found in [**15**] if the map π is the quotient map by a symplectic involution). With the notation of diagram (6.1) we have the following.

Proposition 9.1. *The map $\pi_* : H^2(\widetilde{X}, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is induced by the following map*

$$\begin{array}{l}\langle -2\rangle^{\oplus 16} \oplus U(2)^{\oplus 3} \oplus \left(\langle -1\rangle^{\oplus 8}\right)^{\oplus 15} \xrightarrow{\pi_*} \langle -2\rangle \oplus U(32)^{\oplus 3} \oplus \langle -2\rangle^{\oplus 15}\\ (k_1, \ldots, k_{16}, u, \{n_{1j}\}_{1\le j\le 8}, \ldots, \{n_{15j}\}_{1\le j\le 8}) \longmapsto (k, u, m_1, \ldots, m_{15}),\end{array}$$

where $\pi_*(k_i) = k$, for all $i = 1, \ldots, 16$; $\pi_*(n_{ij}) = m_i$, for all $j = 1, \ldots, 8$, $i = 1, \ldots, 15$.

The map $\pi^* : H^2(Y, \mathbb{Z}) \to H^2(\widetilde{X}, \mathbb{Z})$ is induced by the following map

$$\begin{array}{l}\langle -2\rangle \oplus U(32)^{\oplus 3} \oplus \langle -2\rangle^{\oplus 15} \xhookrightarrow{\pi^*} \langle -2\rangle^{\oplus 16} \oplus U(2)^{\oplus 3} \oplus \left(\langle -1\rangle^{\oplus 8}\right)^{\oplus 15}\\ (k, u, m_1, \ldots, m_{15}) \mapsto \left(k_1 = k, \ldots, k_{16} = k, 16u, \sum_{j=1}^{8} 2n_{1j}, \ldots, \sum_{j=1}^{8} 2n_{15j}\right).\end{array}$$

Proof. By [**38,** Theorem 4.7], the action of G on Λ_{K3} does not depend on the K3 surface we have chosen; hence, we can consider $X = Km(A)$ and G realized as in Section 3 (i.e., it is induced on $Km(A)$ by the translation of the 2-torsion points of the abelian surface A).

1) π_*. We have seen that G leaves $U(2)^{\oplus 3}$ invariant and in fact $H^2(X, \mathbb{Z})^G \supset U(2)^{\oplus 3}$; however, the map π_* multiplies the intersection form by 16. In fact, for $x_1, x_2 \in U(2)^{\oplus 3}$, we have:

$$\pi^*\pi_*(x_1) = 16x_1$$

so using the projection formula

$$(\pi_* x_1, \pi_* x_2)_Y = (\pi^*\pi_* x_1, x_2)_{\tilde{X}} = 16(x_1, x_2).$$

Due to taking $X = Km(A)$, the classes in $\langle -2\rangle^{\oplus 16}$ correspond to classes permuted by G; their image by π_* is a single (-2)-class in $H^2(Y,\mathbb{Z})$. Finally, the 120 (-1)-classes which are the blow up of the points with a non trivial stabilizer on X are divided in orbits of length eight and each orbit is mapped to the same curve m_i on Y. By using the projection formula and the fact that the stabilizer group of a curve n_{ij} has order 2, we have

$$\begin{aligned}(m_i, m_i)_Y &= (\pi_*(n_{ij}), \pi_*(n_{ij}))_Y \\ &= (\pi^*\pi_*(n_{ij}), n_{ij})_{\tilde{X}} \\ &= (2(n_{i1} + \cdots + n_{i8}), n_{ij})_{\tilde{X}} = -2.\end{aligned}$$

2) π^*. Let $x \in U(32)^{\oplus 3}$ and $y \in U(2)^{\oplus 3}$. Then

$$(\pi^* x, y)_{\tilde{X}} = (x, \pi_* y)_Y = (x, y)_Y = 16(x, y)_{\tilde{X}},$$

so $\pi^*(x) = 16x$. Then we have $\pi^*(u) = 16u$ since u is not a class in the branch locus. Finally,

$$(\pi^*(m_i), n_{hj})_{\tilde{X}} = (m_i, \pi_*(n_{hj}))_Y = (m_i, m_h)_Y = -2\delta_{ih},$$

and $(\pi^*(m_i), k)_{\tilde{X}} = (\pi^*(m_i), u)_{\tilde{X}} = 0$ for $u \in U(32)^{\oplus 3}$. Hence, $\pi^*(m_i)$ is given as in the statement. $\square$

Remark 9.2. The lattice $R := \langle -2\rangle^{\oplus 16} \oplus U(32)^{\oplus 3}$ (which is an overlattice of index 2^5 of $K \oplus U(32)^{\oplus 3}$) has index 2^{23} in Λ_{K3}. Here we want to consider the divisible classes that we must add to $\langle -2\rangle^{\oplus 16} \oplus U(32)^{\oplus 3}$ to obtain the lattice Λ_{K3}. Consider the $\mathbb{Z}$ basis $\{\omega_{ij}\}_{i\neq j}$ of $U(2)^3$ in $H^2(Km(A),\mathbb{Z})$. Recall that we have an exact sequence $0 \to A[2] \to A \xrightarrow{\cdot 2} A \to 0$, which corresponds to the multiplication by 2 on each real coordinate of A. Thus, the copy of $U(32)^{\oplus 3} \subset H^2(Km(A/A[2]),\mathbb{Z})$ is generated by $4\omega_{ij}$. Hence, let e_i, f_i, $i = 1, 2, 3$, be the standard basis of each copy of $U(32)$. Then the elements:

$$\frac{e_i}{4}, \qquad \frac{f_i}{4}$$

are contained in $H^2(Y,\mathbb{Z})$. Adding these classes to R, we find $\langle -2\rangle^{\oplus 16} \oplus U(2)^3$ as an overlattice of index 2^{12} of R.

In Remark 2.8, we describe the construction of the even unimodular overlattice Λ_{K3} of $\langle -2\rangle^{\oplus 16} \oplus U(2)^{\oplus 3}$ (we observe that the index is 2^{11}).

In conclusion, we can explicitly construct the overlattice Λ_{K3} of R and extend the maps, π_*, π^* to this lattice.

10. Some explicit examples. In this section, we provide geometrical examples of K3 surfaces X with Picard number 16 admitting a symplectic action of $G = (\mathbb{Z}/2\mathbb{Z})^4$ and of the quotient X/G, whose desingularization is Y. We follow the notation of diagram (6.1), and we denote by L the polarization on X orthogonal to the lattice $\Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$ and by M the polarization on Y orthogonal to the lattice $M_{(\mathbb{Z}/2\mathbb{Z})^4}$.

10.1. The polarization $L^2 = 4$, $M^2 = L^2$. We consider the projective space $\mathbb{P}^3$ and the group of transformations generated by:

$$\begin{aligned}
(x_0 : x_1 : x_2 : x_3) &\longmapsto (x_0 : -x_1 : x_2 : -x_3)\\
(x_0 : x_1 : x_2 : x_3) &\longmapsto (x_0 : -x_1 : -x_2 : x_3)\\
(x_0 : x_1 : x_2 : x_3) &\longmapsto (x_1 : x_0 : x_3 : x_2)\\
(x_0 : x_1 : x_2 : x_3) &\longmapsto (x_3 : x_2 : x_1 : x_0);
\end{aligned}$$

these transformations generate a group isomorphic to $G = (\mathbb{Z}/2\mathbb{Z})^4$. The invariant polynomials are

$$\begin{aligned}
p_0 &= x_0^4 + x_1^4 + x_2^4 + x_3^4,\\
p_1 &= x_0^2x_1^2 + x_2^2x_3^2,\\
p_2 &= x_0^2x_2^2 + x_1^2x_3^2,\\
p_3 &= x_0^2x_3^2 + x_1^2x_2^2,\\
p_4 &= x_0x_1x_2x_3.
\end{aligned}$$

Hence the generic G-invariant quartic K3 surface is a linear combination.

$$\begin{aligned}
a_0(x_0^4 + x_1^4 + x_2^4 + x_3^4) + a_1(x_0^2x_1^2 + x_2^2x_3^2) + a_2(x_0^2x_2^2 + x_1^2x_3^2)\\
+ a_3(x_0^2x_3^2 + x_1^2x_2^2) + a_4x_0x_1x_2x_3 = 0.
\end{aligned}$$

Since the identity is the only automorphism commuting with all elements of the group G, the number of parameters in the equation is 4, which is also the dimension of the moduli space of the K3 surfaces with symplectic automorphism group G and polarization L with $L^2 = 4$.

We now study the quotient surface. Observe that the quotient of $\mathbb{P}^3$ by G is the Igusa quartic, cf., [**22,** subsection 3.3], which is an order 4 relation between the p_i's, that is,

$$\mathcal{I}_4\colon 16p_4^4+p_0^2p_4^2+p_1^2p_2^2+p_1^2p_3^2+p_2^2p_3^2-4(p_1^2+p_2^2+p_3^2)p_4^2-p_0p_1p_2p_3=0.$$

Hence, the quotient is a quartic K3 surface which is a section of the Igusa quartic by the hyperplane:

$$a_0p_0+a_1p_1+a_2p_2+a_3p_3+a_4p_4=0.$$

The quartics in $\mathbb{P}^3$ admitting $(\mathbb{Z}/2\mathbb{Z})^4$ as a symplectic group of automorphisms are described in a very detailed way in [**8**] (cf., also Remark 7.12). We observe that the subfamily with $a_0=0$ is also a subfamily of the family of quartics considered by Keum [**24,** Example 3.3]. In this subfamily, it is easy to identify an Enriques involution. This is the standard Cremona transformation

$$(x_0:x_1:x_2:x_3)\longrightarrow\left(\frac{1}{x_0}:\frac{1}{x_1}:\frac{1}{x_2}:\frac{1}{x_3}\right).$$

10.2. The polarization $L^2=8$, $M^2=L^2/4=2$. Let X be a K3 surface with a symplectic action of G and $L^2=8$. There are two connected irreducible components of the moduli space on K3 surfaces with these properties, cf., Theorem 7.1 and Corollary 7.11. One is realized as follows. Let us consider the complete intersection of three quadrics in $\mathbb{P}^5$:

$$\begin{cases}\sum_{i=0}^{5}a_ix_i^2=0,\\ \sum_{i=0}^{5}b_ix_i^2=0,\\ \sum_{i=0}^{5}c_ix_i^2=0,\end{cases}$$

with complex parameters a_i, b_i, c_i, $i=0,\ldots,5$. Group G is realized as transformations of $\mathbb{P}^5$ changing an even number of signs in the coordinates. In order to compute the dimension of the moduli space of these K3 surfaces we must choose three independent quadrics in a six-dimensional space. Hence, we must compute the dimension of the Grassmannian of the subspaces of dimension 3 in a space of dimension 6. This is $3(6-3)=9$. Now the automorphisms of $\mathbb{P}^5$ commuting with the automorphisms generating G are the diagonal 6×6-matrices; hence, we find the dimension $9-(6-1)=4$ as expected.

To determine the quotient, one sees that the invariant polynomials under the action of G are $z_0^2, z_1^2, z_2^2, z_3^2, z_4^2, z_5^2$ and the product $z_0z_1z_2z_3z_4z_5$. Denoting them by $y_0, \dots, y_5, t$, then there is a relation

$$t^2 = \prod_{i=0}^{5} y_i,$$

and so we obtain a K3 surface which is the double cover of the plane given by the intersection of the planes of $\mathbb{P}^5$:

$$\begin{cases} \sum_{i=0}^{5} a_i y_i = 0, \\ \sum_{i=0}^{5} b_i y_i = 0, \\ \sum_{i=0}^{5} c_i y_i = 0. \end{cases}$$

The branch locuses are 6 lines meeting at 15 points, whose preimages under the double cover are the 15 nodes of the K3 surface.

We get a special subfamily of K3 surfaces, considering as in subsection 5.1, a curve Γ of genus 2 with:

$$y^2 = \prod_{i=0}^{5}(x - s_i), \quad s_i \in \mathbb{C}, \ s_i \neq s_j,$$

for $i \neq j$. This determines a family of Kummer surfaces with $(\mathbb{Z}/2\mathbb{Z})^4$ action and equations in $\mathbb{P}^5$:

$$\begin{cases} z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0, \\ s_0z_0^2 + s_1z_1^2 + s_2z_2^2 + s_3z_3^2 + s_4z_4^2 + s_5z_5^2 = 0, \\ s_0^2z_0^2 + s_1^2z_1^2 + s_2^2z_2^2 + s_3^2z_3^2 + s_4^2z_4^2 + s_5^2z_5^2 = 0. \end{cases}$$

The quotient surface also specializes to the double cover $t^2 = \prod_i y_i$ of the plane obtained as the intersection of the planes of $\mathbb{P}^5$:

$$\begin{cases} y_0 + y_1 + y_2 + y_3 + y_4 + y_5 = 0, \\ s_0y_0 + s_1y_1 + s_2y_2 + s_3y_3 + s_4y_4 + s_5y_5 = 0, \\ s_0^2y_0 + s_1^2y_1 + s_2^2y_2 + s_3^2y_3 + s_4^2y_4 + s_5^2y_5 = 0. \end{cases}$$

As before, the branch locuses are 6 lines meeting at 15 points, but in this case, there is a conic tangent to the 6 lines.

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