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SOME FIFTH ROOTS THAT ARE CONSTRUCTIBLE BY MARKED RULER AND COMPASS

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ABSTRACT. We show that there are cubic irrationals whose real fifth roots can be constructed by marked ruler and compass.

1. Introduction. In [**2**], we examined points that can be constructed by compass and marked ruler and, using the results developed there, we proved that the regular hendecagon (11-gon) is constructible by these tools. This was our first result in a program to try to characterize all points that can be constructed with marked ruler and compass.

Related to this program, we observe that Baragar [**1**] has shown that there are quintic irrationalities which are constructible numbers, i.e., the coordinates of points constructible using a compass and marked ruler. But, as far as we know, no one has shown that the fifth root of any constructible number (which is not already a fifth power of a constructible number) can be constructed. For example, at present, it is still an open question as to whether or not $\sqrt[5]{2}$ is constructible.

In our present work, we use the machinery developed in [**2**] to show that there are cubic irrationals whose real fifth roots are constructible by marked ruler and compass. The proof is somewhat more complicated than that of showing that the regular hendecagon is constructible and raises the question of why the proof was so "miraculously" simple in the latter case.

The search for constructible fifth roots was also motivated by the fact that all real fifth roots of rational numbers are "q-constructible," an analog of constructibility by compass and marked ruler. See [**5**] for the details.

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2. Preliminaries. For a detailed description of classical straightedge and compass constructions, as well as marked ruler constructions, see, for example, Martin's text [**4,** Chapters 2, 9].

Now, for convenience, we review our so-called *restricted marked ruler and compass constructions*, cf. [**1, 2**].

We consider a straightedge on which there are two marks, one unit apart, and which allow for what is called verging, cf. [**4,** pages 124, 125]. By verging through a point V, between two curves (lines and circles), we determine two points Q_1 and Q_2, obtained by drawing a line through the point V which intersects the two curves at points Q_1 on one curve and Q_2 on the other such that Q_1 and Q_2 are one unit apart. This can be done by a marked ruler once we are given a verging point and a pair of curves. The points Q_i are thus said to be constructed by verging.

With a marked ruler and compass, we may construct lines and circles which allows us to verge between a pair of lines, a line and a circle, and a pair of circles. In the classical case of straightedge and compass, we allow intersections of two lines, a line and a circle, and two circles. However, nothing is lost by restricting to the first two cases (with an appropriate initial set of constructible points). In a similar manner, as in [**2**], we restrict to verging between pairs of lines and between a line and a circle, but not between two circles. Hence, we use the term *restricted marked ruler and compass constructions.* Here is a formal definition as given in [**2**].

A point is *restricted marked ruler and compass constructible* or an RMC point, for short, if it is the terminal point in a finite sequence of points in $\mathbb{R}^2$ such that each point is in the "starter set" $\{(0,0),(1,0)\}$ or is obtained in one of the following ways:

(i) as the intersection of two distinct lines, each passing through a pair of points earlier in the sequence;

(ii) as the intersection of two distinct circles, each passing through points earlier in the sequence and with earlier points as their centers;

(iii) as the intersection of a line and a circle, as determined in (i) and (ii), respectively;

(iv) as one of two points obtained by verging through an earlier point between two distinct lines as given in (i);

(v) as one of two points obtained by verging through an earlier point between a line and a circle as given in (i) and (ii), respectively. We call the point on the circle the *primary* point.

For convenience, we call a sequence of points as given in the definition an RMC *sequence*.

A *restricted marked ruler and compass number* or RMC *number*, again for brevity, is defined as the x-coordinate of an RMC point on the x-axis.

Now let

$$\mathcal{C} = \{x \in \mathbb{R} : x \text{ is an RMC number}\}.$$

$\mathcal{C}$ is a subfield of $\mathbb{R}$ containing all real 2-3 towers of $\mathbb{Q}$, since $\mathcal{C}$ contains all marked ruler numbers, and marked ruler numbers are characterized as those lying in real 2-3 towers of $\mathbb{Q}$, again see [**4,** Chapter 9], for example. By [**1**], we know that an RMC number in $\mathcal{C}$ lies in a real tower $\{K_j\}$ of $\mathbb{Q}$ with $[K_{j+1} : K_j] \leq 6$. As noted above, Baragar has even shown that there are real quintic irrationalities over $\mathbb{Q}$ which are in $\mathcal{C}$. In fact, when adjoining one of these quintic irrationalities to $\mathbb{Q}$, its Galois closure over $\mathbb{Q}$ has absolute Galois group isomorphic to S_5, the symmetric group on five objects, cf. [**3,** page 282]. This should not be a surprise, as chances are "excellent" that such a Galois group would be S_5.

In contrast to this, we will show that there are real cubic irrationals r over $\mathbb{Q}$, and thus RMC numbers, such that $\sqrt[5]{r} \in \mathcal{C}$ and

$$[\mathbb{Q}(\sqrt[5]{r}) : \mathbb{Q}(r)] = 5.$$

From this it follows that

$$\mathrm{Gal}(\mathbb{Q}(\zeta_5, \sqrt[5]{r})/\mathbb{Q}(r)) \simeq F_{20},$$

the Frobenius group of order 20, where $\zeta_5 = \exp 2\pi i/5$, is one of the primitive fifth roots of unity, cf. [**3,** subsection 13.2]. Notice, too, that $\sqrt[5]{r}$ is solvable over $\mathbb{Q}$.

3. Characterization of RMC numbers. As we have done in our previous paper, [**2**], we make use of the intersection of a conchoid and a circle. It is known, cf. [**1**], that determining one point of the pair of points obtained by verging between a line and some other curve can be given as a point of intersection of a conchoid associated with the line (and the verging point) and the curve. Here is how it works. Let V be a (verging) point and L a line not passing through V. For any point P on L consider the line through V and P. Let Q and Q' be the two points on this line that are of distance 1 to P. The conchoid $\mathrm{Con}_{L,V}$ associated with the verging point V and axis L is the set of all points Q, Q' as given above as P varies over all points of L.

In particular, when $V = (0,0)$ and L, the line with equation $x = a$, where $a > 0$, the conchoid $\mathrm{Con}_{x=a,(0,0)}$ satisfies the relation $(x-a)^2(x^2+y^2) = x^2$, cf. [**1**].

We now consider the coordinates of the points of the intersection of the conchoid

$$\mathrm{Con} : (x-a)^2(x^2+y^2) = x^2,$$

associated with the origin and the vertical line $x = a$, and the circle $C = C_{(b,c),s}$

$$C : (x-b)^2 + (y-c)^2 = s^2.$$

Instead of dealing directly with the coordinates x and y of a point P of an intersection, we found it more convenient to work with $z = x/(x-a)$, the so-called signed distance from the origin to P (as $|z|$ is this distance).

Let

$(**)$

$$\begin{aligned}
a_1 &= -2,\\
a_2 &= 1 - 2(s^2 - b^2 + c^2 + 2ab),\\
a_3 &= 4(s^2 - b^2 + c^2 + ab),\\
a_4 &= (s^2 - b^2 + c^2 + 2ab)^2 - 2(s^2 - b^2 + c^2) - 4c^2\big(s^2 - (a-b)^2\big),\\
a_5 &= 2\,(ab)^2 - 2(s^2 - b^2 - c^2 + ab)^2,\\
a_6 &= (s^2 - b^2 - c^2)^2,
\end{aligned}$$

and define

$$\begin{aligned}
f(X) &= f_{a,b,c,s}(X)\\
&= X^6 + a_1X^5 + a_2X^4 + a_3X^3 + a_4X^2 + a_5X + a_6,
\end{aligned}$$

as the *verging polynomial with parameters* a, b, c, s.

Then we have the following result from [**2**].

Theorem 3.1 (Verging theorem)**.**

(i) *Let* L *be the vertical line in* $\mathbb{R}^2$*, with equation* $x = a$*,* $a > 0$*, and* C *the circle centered at* (b, c)*,* $c > 0$*, with radius* s*. If* $P = (x, y)$ *is a point of intersection of the conchoid* $\mathrm{Con}_{L,(0,0)}$ *and the circle* C*, then the signed distance,*

$$z = \frac{x}{x-a},$$

is a root of the polynomial,

$$f(X) = X^6 + a_1X^5 + a_2X^4 + a_3X^3 + a_4X^2 + a_5X + a_6,$$

satisfying conditions $(**)$ *above.*

(ii) *Now suppose*

$$f(X) = X^6 + a_1X^5 + a_2X^4 + a_3X^3 + a_4X^2 + a_5X + a_6 \in \mathbb{R}[X],$$

and let

$$m = 2 - 2a_2 - a_3,$$
$$B = 2a_4 + a_5 - 3 + 4a_2 - a_2^2 + \frac{5}{2}\,a_3 - \frac{a_2a_3}{2}.$$

Moreover, suppose that the following conditions are satisfied:

$$a_1 = -2, \tag{0}$$

$$(2a_6 + a_5)^2 = a_6\,m^2, \tag{1}$$

$$a_3 > -\varepsilon\sqrt{m^2 - 8a_5} \qquad (\varepsilon = 1,\ \textit{or}\ -1), \tag{2_ε}$$

in which case,

$$c = c_\varepsilon = \sqrt{\frac{1}{8}\left(a_3 + \varepsilon\sqrt{m^2 - 8a_5}\right)},$$

$$m > -\frac{B}{2c_\varepsilon^2} \tag{3_ε}$$

in which case,

$$a = a_\varepsilon = \sqrt{\frac{m}{4} + \frac{B}{8c_\varepsilon^2}},$$

and whence,

$$b = b_\varepsilon = \frac{m}{4a_\varepsilon},$$

$$(4_\varepsilon) \qquad \frac{m^2}{a_\varepsilon^2} > 16c_\varepsilon^2 + 8(1 - a_2 - a_3),$$

and then,

$$s = s_\varepsilon = \sqrt{\frac{m^2}{16a_\varepsilon^2} - \frac{1}{2}(1 - a_2 - a_3) - c_\varepsilon^2}.$$

Finally,

$$(5_\varepsilon) \qquad (s_\varepsilon^2 - b_\varepsilon^2 - c_\varepsilon^2)^2 = a_6.$$

Then $f(X)$ is a verging polynomial with verging parameters $a_\varepsilon, b_\varepsilon, c_\varepsilon$ and s_ε given above.

In **[2]**, we defined a *real* RMC *tower* of $\mathbb{Q}$ to be a tower of fields $\{K_j\}_{j=0}^n$ for some nonnegative integer n, such that

$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq \mathbb{R},$$

where the degree $[K_{j+1} : K_j] \leq 6$, and such that, if $[K_{j+1} : K_j] = 5$ or 6, then $K_{j+1} = K_j(z_{j+1})$ for some root z_{j+1} of a verging polynomial $f(X) \in K_j[X]$.

Given this, we then proved, see **[2]**, the following characterization theorem of RMC numbers.

Theorem 3.2 (Characterization theorem). *A real number α is an* RMC *number, i.e., $\alpha \in \mathcal{C}$, if and only if there is a real* RMC *tower $\{K_j\}_{j=0}^n$ of $\mathbb{Q}$ such that $\alpha \in K_n$.*

4. Exhibiting some RMC numbers that are fifth roots. We now examine the question of whether or not there are RMC numbers whose (real) fifth roots are also RMC. We start with a proposition. As a convention, for any real number r, $\sqrt[5]{r}$ will always denote the real fifth root of r.

Proposition 4.1 (Separation theorem). *Let r be a number constructible by a marked ruler and such that $\sqrt[5]{r} \notin \mathbb{Q}(r)$. Now suppose* *further that* $\sqrt[5]{r} \in \mathcal{C}$. *Then there exists a real* RMC *tower* $K_0, \ldots, K_n$ *of* $\mathbb{Q}$ *with* $n > 0$, *such that* $[K_n : K_{n-1}] = 5$, *with* $r \in K_{n-1}$ *and* $\sqrt[5]{r} \in K_n - K_{n-1}$, *i.e.*, $K_n = K_{n-1}(\sqrt[5]{r})$.

In other words, we can find an RMC tower that separates r from $\sqrt[5]{r}$.

Proof. Since r is assumed to be a marked ruler number, we know that there is a real 2-3 tower $L_0, \ldots, L_\ell$ of $\mathbb{Q}$ such that $r \in L_\ell$ and thus $[L_j : L_{j-1}] = 1, 2$ or 3 for each $j = 1, \ldots, \ell$.

Now since $\sqrt[5]{r}$ is assumed to be in $\mathcal{C}$, the Characterization theorem implies that there is a real RMC tower $N_0, N_1, \ldots, N_m$ of $\mathbb{Q}$ with $\sqrt[5]{r} \in N_m$. If we combine the two towers, then we get

$$L_0 \subseteq \cdots \subseteq L_\ell \subseteq L_\ell N_1 \subseteq \cdots \subseteq L_\ell N_m,$$

which is an RMC tower, too, as is easily seen. Since $\sqrt[5]{r} \notin \mathbb{Q}(r)$, the prime radical theorem, cf. [**3,** subsection 4.2 D], implies that $[\mathbb{Q}(\sqrt[5]{r}) : \mathbb{Q}(r)] = 5$. Moreover, since $5 \nmid [L_\ell : \mathbb{Q}]$, we see that $[L_\ell(\sqrt[5]{r}) : L_\ell] = 5$, by shifting from $\mathbb{Q}(\sqrt[5]{r})/\mathbb{Q}(r)$ to $L_\ell(\sqrt[5]{r})/L_\ell$. Thus, there is a positive integer $t \leq m$ such that $\sqrt[5]{r} \in L_\ell N_t - L_\ell N_{t-1}$. Again, by the prime radical theorem, we see that

$$[L_\ell N_{t-1}(\sqrt[5]{r}) : L_\ell N_{t-1}] = 5.$$

This implies that

$$5 \mid [L_\ell N_t : L_\ell N_{t-1}].$$

But then, by the definition of an RMC tower,

$$[L_\ell N_t : L_\ell N_{t-1}] \leq 6,$$

and thus,

$$[L_\ell N_t : L_\ell N_{t-1}] = 5,$$

and so,

$$L_\ell N_t = L_\ell N_{t-1}(\sqrt[5]{r}).$$

Therefore,

$$L_0, \ldots, L_\ell, L_\ell N_1, \ldots, L_\ell N_t$$

is the desired RMC tower. $\square$

Here is how we will use this result.

Corollary 4.2. *Suppose that r is a marked ruler number such that $\sqrt[5]{r} \notin \mathbb{Q}(r)$. Then $\sqrt[5]{r}$ is an* RMC *number if and only if there exists a subfield K of $\mathcal{C}$, such that $r \in K$ but $\sqrt[5]{r} \notin K$, numbers $a, b, c, s \in K$, where $a, c, s > 0$, and a point*

$$(x, y) = P \in \mathrm{Con}_{x=a,(0,0)} \cap C_{(b,c),s},$$

such that the signed distance $z = x/(x-a)$ from the origin to P satisfies

$$K(\sqrt[5]{r}) = K(z).$$

Notice that when $K(\sqrt[5]{r}) = K(z)$, we have

$$z = u_0 + u_1\sqrt[5]{r} + u_2\sqrt[5]{r^2} + u_3\sqrt[5]{r^3} + u_4\sqrt[5]{r^4},$$

for uniquely determined $u_j \in K$.

Proof. The proof is an immediate consequence of the definition of RMC numbers, Theorem 3.2 and Proposition 4.1, in which we take $K = K_{n-1}$. □

From now on, we assume that z is the signed distance as given in Corollary 4.2. By the Verging theorem, this signed distance, z, is a root of a verging polynomial $f(X) = X^6 + a_1X^5 + \cdots + a_6$, where $a_j \in K$. But notice that z is of degree 5 over K. Let

$$p(X) = X^5 + b_1X^4 + b_2X^3 + b_3X^2 + b_4X + b_5$$

be the minimal polynomial of z over K. Thus, $p(X) \mid f(X)$ in $K[X]$, and consequently, there is an element η of K, such that

$$f(X) = (X - \eta)p(X).$$

Multiplying out the right side and comparing coefficients yield the following relations:

$$\begin{aligned} -2 = a_1 &= b_1 - \eta, \\ a_2 &= b_2 - \eta b_1, \\ a_3 &= b_3 - \eta b_2, \\ a_4 &= b_4 - \eta b_3, \end{aligned}$$

$$\begin{aligned} a_5 &= b_5 - \eta b_4, \\ a_6 &= \quad\; - \eta b_5. \end{aligned}$$

Now, since

$$z = \sum_{j=0}^{4} u_j \sqrt[5]{r^j},$$

we have

$$p(X) = \prod_{i=0}^{4} (X - z_i)\,,$$

where

$$z_i = \sum_{j=0}^{4} u_j \zeta_5^{ij} \sqrt[5]{r^j}.$$

We thus see in particular that

$$b_1 = -\sum_{i=0}^{4} z_i = -\sum_i \sum_j u_j \zeta_5^{ij} \sqrt[5]{r^j} = -\sum_j u_j \sqrt[5]{r^j} \sum_i \zeta_5^{ij} = -5u_0,$$

since

$$\sum_{i=0}^{4} \zeta_5^{ij} = \begin{cases} 5, & \text{if } 5 \mid j, \\ 0, & \text{otherwise,} \end{cases}$$

and consequently,

$$\eta = 2 - 5u_0.$$

To get the b_j and then a_j in terms of the u_j, one may use Newton's identities that relate coefficients of a polynomial with the "power sums" of the roots. Use

$$p(X) = X^5 + b_1 X^4 + \cdots + b_5 = \prod_{j=0}^{4} (X - z_j),$$

for z_j as above and the power sums, given as

$$s_m = \sum_{i=0}^{4} z_i^m,$$

for any positive integer m. In this particular case, Newton's identities can be given as

$$s_m + b_1 s_{m-1} + \cdots + b_{m-1} s_1 + m b_m = 0 \quad \text{for all } m \in \mathbb{N},$$

where we define $b_m = 0$ for $m > 5$. Now, solving for the b_j yields

$$\begin{aligned}
b_1 &= -s_1, \\
b_2 &= \frac{1}{2}\left(s_1^2 - s_2\right), \\
b_3 &= \frac{1}{6}\left(-s_1^3 + 3s_1 s_2 - 2s_3\right), \\
b_4 &= \frac{1}{24}\left(s_1^4 - 6s_1^2 s_2 + 8s_1 s_3 + 3s_2^2 - 6s_4\right), \\
b_5 &= \frac{1}{120}\left(-s_1^5 + 10s_1^3 s_2 - 20s_1^2 s_3 - 15s_1 s_2^2 + 30s_1 s_4 + 20s_2 s_3 - 24s_5\right),
\end{aligned}$$

cf. [**6,** Section 46].

Now we need to get the power sums s_j in terms of the u_j. We have

$$s_m = \sum_{i=0}^{4} z_i^m = \sum_{i=0}^{4} \left(\sum_{j=0}^{4} u_j \, \zeta_5^{ij} \sqrt[5]{r^j} \right)^m.$$

By multiplying out and switching the order of the summations, one obtains (with some effort, but verified by machine) the following relations:

$$\begin{aligned}
s_1 &= 5u_0, \\
s_2 &= 10r(u_1u_4 + u_2u_3) + 5u_0^2, \\
s_3 &= 15r^2(u_2u_4^2 + u_3^2u_4) + 15r(2u_0u_1u_4 + 2u_0u_2u_3 + u_1^2u_3 + u_1u_2^2) + 5u_0^3, \\
s_4 &= 20r^3u_3u_4^3 + 10r^2(6u_0u_2u_4^2 + 6u_0u_3^2u_4 + 3u_1^2u_4^2 \\
&\qquad + 12u_1u_2u_3u_4 + 2u_1u_3^3 + 2u_2^3u_4 + 3u_2^2u_3^2) \\
&\quad + 20r(3u_0^2u_1u_4 + 3u_0^2u_2u_3 + 3u_0u_1^2u_3 + 3u_0u_1u_2^2 + u_1^3u_2) \\
&\quad + 5u_0^4, \\
s_5 &= 5r^4u_4^5 + 5r^3(20u_0u_3u_4^3 + 20u_1u_2u_4^3 + 30u_1u_3^2u_4^2 \\
&\qquad + 30u_2^2u_3u_4^2 + 20u_2u_3^3u_4 + u_3^5) \\
&\quad + 5r^2(30u_0^2u_2u_4^2 + 30u_0^2u_3^2u_4 + 30u_0u_1^2u_4^2 + 120u_0u_1u_2u_3u_4
\end{aligned}$$

$$
\begin{aligned}
&\quad + 20u_0u_1u_3^3 + 20u_0u_2^3u_4 \\
&\quad + 30u_0u_2^2u_3^2 + 20u_1^3u_3u_4 + 30u_1^2u_2^2u_4 \\
&\quad + 30u_1^2u_2u_3^2 + 20u_1u_2^3u_3 + u_2^5) \\
&+ 5r(20u_0^3u_1u_4 + 20u_0^3u_2u_3 + 30u_0^2u_1^2u_3 \\
&\quad + 30u_0^2u_1u_2^2 + 20u_0u_1^3u_2 + u_1^5) + 5u_0^5.
\end{aligned}
$$

Notice that the s_m are given as

$$
s_m = 5\sum_{k=0}^{m-1} r^k \sum_{\substack{\mu_0,\ldots,\mu_4\geq 0\\ \sum_j \mu_j = m\\ \sum_j j\mu_j = 5k}} \frac{m!}{\mu_0!\mu_1!\mu_2!\mu_3!\mu_4!} u_0^{\mu_0}u_1^{\mu_1}u_2^{\mu_2}u_3^{\mu_3}u_4^{\mu_4}.
$$

From this, we combine the two sets of expressions to obtain the b_j in terms of u_j and r:

$$
\begin{aligned}
b_1 &= -5u_0,\\
b_2 &= 5[-r(u_1u_4 + u_2u_3) + 2u_0^2],\\
b_3 &= 5[-r^2(u_2u_4^2 + u_3^2u_4) + r(3u_0u_1u_4 + 3u_0u_2u_3 - u_1^2u_3 - u_1u_2^2) - 2u_0^3],\\
b_4 &= 5[-r^3u_3u_4^3 + r^2(2u_0u_2u_4^2 + 2u_0u_3^2u_4 + u_1^2u_4^2 - u_1u_2u_3u_4 - u_1u_3^3 \\
&\qquad - u_2^3u_4 + u_2^2u_3^2) \\
&\quad + r(-3u_0^2u_1u_4 - 3u_0^2u_2u_3 + 2u_0u_1^2u_3 + 2u_0u_1u_2^2 - u_1^3u_2) + u_0^4],\\
b_5 &= -r^4u_4^5 + r^3(5u_0u_3u_4^3 + 5u_1u_2u_4^3 - 5u_1u_3^2u_4^2 - 5u_2^2u_3u_4^2 + 5u_2u_3^3u_4 \\
&\qquad - u_3^5) \\
&\quad + r^2(-5u_0^2u_2u_4^2 - 5u_0^2u_3^2u_4 - 5u_0u_1^2u_4^2 + 5u_0u_1u_2u_3u_4 + 5u_0u_1u_3^3 \\
&\qquad + 5u_0u_2^3u_4 - 5u_0u_2^2u_3^2 + 5u_1^3u_3u_4 - 5u_1^2u_2^2u_4 \\
&\qquad - 5u_1^2u_2u_3^2 + 5u_1u_2^3u_3 - u_2^5) \\
&\quad + r(5u_0^3u_1u_4 + 5u_0^3u_2u_3 - 5u_0^2u_1^2u_3 - 5u_0^2u_1u_2^2 + 5u_0u_1^3u_2 - u_1^5) \\
&\quad - u_0^5.
\end{aligned}
$$

Hence, the b_m have the same form as the s_m, but with different coefficients:

$$b_m = \sum_{k=0}^{m-1} r^k \sum_{\substack{\mu_0,\ldots,\mu_4 \geq 0 \\ \sum_j \mu_j = m \\ \sum_j j\mu_j = 5k}} A_{\mu_0,\ldots,\mu_4}\, u_0^{\mu_0} u_1^{\mu_1} u_2^{\mu_2} u_3^{\mu_3} u_4^{\mu_4},$$

where $A_{\mu_0,\ldots,\mu_4}$ are the integers determined above.

Now we can write down the a_j in terms of the u_j and r:

$$\begin{aligned}
\eta &= 2 - 5u_0, \\
a_2 &= -5r(u_1u_4 + u_2u_3) - 15u_0^2 + 10u_0, \\
a_3 &= -5r^2(u_2u_4^2 + u_3^2u_4) \\
&\quad + r(-10u_0u_1u_4 - 10u_0u_2u_3 - 5u_1^2u_3 + 10u_1u_4 - 5u_1u_2^2 + 10u_2u_3) \\
&\quad + 40u_0^3 - 20u_0^2, \\
a_4 &= -5r^3u_3u_4^3 \\
&\quad + r^2(-15u_0u_2u_4^2 - 15u_0u_3^2u_4 + 5u_1^2u_4^2 - 5u_1u_2u_3u_4 - 5u_1u_3^3 \\
&\qquad - 5u_2^3u_4 + 5u_2^2u_3^2 + 10u_2u_4^2 + 10u_3^2u_4) \\
&\quad + r(60u_0^2u_1u_4 + 60u_0^2u_2u_3 - 15u_0u_1^2u_3 - 15u_0u_1u_2^2 - 30u_0u_1u_4 \\
&\qquad - 30u_0u_2u_3 - 5u_1^3u_2 + 10u_1^2u_3 + 10u_1u_2^2) - 45u_0^4 + 20u_0^3, \\
a_5 &= -r^4u_4^5 + r^3(-20u_0u_3u_4^3 + 5u_1u_2u_4^3 - 5u_1u_3^2u_4^2 - 5u_2^2u_3u_4^2 \\
&\qquad + 5u_2u_3^3u_4 - u_3^5 + 10u_3u_4^3) \\
&\quad + r^2(45u_0^2u_2u_4^2 + 45u_0^2u_3^2u_4 + 20u_0u_1^2u_4^2 - 20u_0u_1u_2u_3u_4 \\
&\qquad - 20u_0u_1u_3^3 - 20u_0u_2^3u_4 + 20u_0u_2^2u_3^2 - 20u_0u_2u_4^2 \\
&\qquad - 20u_0u_3^2u_4 + 5u_1^3u_3u_4 - 5u_1^2u_2^2u_4 - 5u_1^2u_2u_3^2 \\
&\qquad - 10u_1^2u_4^2 + 5u_1u_2^3u_3 + 10u_1u_2u_3u_4 + 10u_1u_3^3 - u_2^5 \\
&\qquad + 10u_2^3u_4 - 10u_2^2u_3^2) \\
&\quad + r(-70u_0^3u_1u_4 - 70u_0^3u_2u_3 + 45u_0^2u_1^2u_3 + 45u_0^2u_1u_2^2 + 30u_0^2u_1u_4 \\
&\qquad + 30u_0^2u_2u_3 - 20u_0u_1^3u_2 - 20u_0u_1^2u_3 - 20u_0u_1u_2^2 \\
&\qquad - u_1^5 + 10u_1^3u_2) \\
&\quad + 24u_0^5 - 10u_0^4, \\
a_6 &= r^4u_4^5(2 - 5u_0) + r^3(5u_0 - 2)(5u_0u_3u_4^3 + 5u_1u_2u_4^3 - 5u_1u_3^2u_4^2
\end{aligned}$$

$$
\begin{aligned}
&\quad - 5u_2^2u_3u_4^2 + 5u_2u_3^3u_4 - u_3^5) \\
&+ r^2(5u_0 - 2)(-5u_0^2u_2u_4^2 - 5u_0^2u_3^2u_4 - 5u_0u_1^2u_4^2 \\
&\quad + 5u_0u_1u_2u_3u_4 + 5u_0u_1u_3^3 + 5u_0u_2^3u_4 - 5u_0u_2^2u_3^2 + 5u_1^3u_3u_4 \\
&\quad - 5u_1^2u_2^2u_4 - 5u_1^2u_2u_3^2 + 5u_1u_2^3u_3 - u_2^5) \\
&+ r(5u_0 - 2)(5u_0^3u_1u_4 + 5u_0^3u_2u_3 - 5u_0^2u_1^2u_3 \\
&\quad - 5u_0^2u_1u_2^2 + 5u_0u_1^3u_2 - u_1^5) \\
&+ (2 - 5u_0)u_0^5.
\end{aligned}
$$

Example 4.3. As a simple example, we show a negative result. Suppose we verge through a point V between a line L and a circle C, where V has coordinates which are marked ruler numbers and L and C have equations whose coefficients are marked ruler numbers. For all marked ruler numbers r such that $\sqrt[5]{r} \notin \mathbb{Q}(r)$, the distance from V to the primary point P is never $\sqrt[5]{r}$.

Suppose otherwise. Then, by applying an appropriate rigid motion, we may assume V, L, C and P are as given in Corollary 4.2, and the signed distance z from V to P is $\sqrt[5]{r}$ for some marked ruler number r such that $\sqrt[5]{r} \notin \mathbb{Q}(r)$.

Now let K be the field of marked ruler numbers. By the characterization of marked ruler numbers in terms of 2-3 towers we see that, for any subfield F of K of finite degree over $\mathbb{Q}$, $[F : \mathbb{Q}] = 2^\alpha 3^\beta$, for some nonnegative integers α and β.

Now notice that K satisfies the conditions in Corollary 4.2, for clearly all the verging parameters are in K; r is in K, since r is a marked ruler number. But, $\sqrt[5]{r} \notin K$, since otherwise $5 = [\mathbb{Q}(\sqrt[5]{r}) : \mathbb{Q}(r)]$ (by the prime radical theorem) would be a divisor of $[\mathbb{Q}(\sqrt[5]{r}) : \mathbb{Q}] = 2^k 3^\ell$ by the comment above. Thus, by Corollary 4.2, we have

$$z = \sqrt[5]{r} = u_0 + u_1\sqrt[5]{r} + u_2\sqrt[5]{r^2} + u_3\sqrt[5]{r^3} + u_4\sqrt[5]{r^4},$$

and so, by the uniqueness of the u_j, $u_1 = 1$ and $u_0 = u_2 = u_3 = u_4 = 0$. Also we see that $z = \sqrt[5]{r}$ is a root of the verging polynomial,

$$f(X) = X^6 + a_1X^5 + \cdots + a_6,$$

associated with the verging parameters above. By our previous calculations, we see that

$$\eta = 2, \qquad b_1 = b_2 = b_3 = b_4 = 0, \qquad b_5 = -r,$$

as can be verified independently, since $X^5 - r$ is the minimal polynomial of $\sqrt[5]{r}$ over $\mathbb{Q}(r)$. Also notice that

$$a_1 = -2, \qquad a_2 = a_3 = a_4 = 0, \qquad a_5 = -r, \qquad a_6 = 2r,$$

which again can be verified directly.

Now, by condition (i) in the verging theorem, we have $9r^2 = 8r$ and thus $r = 8/9$. Hence $\sqrt[5]{8/9}$ is the only candidate for a signed distance as described above. In this case, notice that $a_5 = -8/9$ and $a_6 = 16/9$.

With respect to condition (2_ε), we must have $0 > -\varepsilon \cdot 10/3$, in which case we see that $\varepsilon = +1$, and thus $c_+ = \sqrt{5/12}$.

Next, notice that $m = 2$ and $B = -35/9$.

Finally, we check condition (3_+). This condition does not hold, since $2 \not> 14/3$. Thus, $\sqrt[5]{r}$ is not a signed distance.

The reason this example is computationally easy is that condition (i) is simple to work with in this case. This is what happened in constructing a hendecagon, too, which made the computations fairly simple, cf. **[2]**. Clearly, however, if more of the u_j do not vanish, then (i) becomes seemingly unmanageable.

Example 4.4. Our next example involves determining signed distances of the form $z = \sqrt[5]{r} + u\sqrt[5]{r^2}$ (cf. Corollary 4.2), where again r is a marked ruler number with $\sqrt[5]{r} \notin \mathbb{Q}(r)$, and $u \in K$ with K is again the field of marked ruler numbers. Hence, we assume that $u_1 = 1$, without loss of generality, $u_2 = u$, and that the rest of the u_j vanish. Notice that, if z is an RMC number (implying that $K(z) \subseteq \mathcal{C}$), then so is $\sqrt[5]{r}$, since

$$K \subsetneq K(z) \subseteq K(\sqrt[5]{r}),$$

which implies that $K(z) = K(\sqrt[5]{r})$, since $[K(\sqrt[5]{r}) : K] = 5$ by the prime radical theorem. We now plug these values into the above expressions for the a_j and find that $\eta = 2$,

$$a_1 = -2,$$
$$a_2 = 0,$$

$$\begin{aligned}
a_3 &= -5u^2r,\\
a_4 &= 5(2u^2-u)r,\\
a_5 &= -u^5r^2+(10u-1)r,\\
a_6 &= 2u^5r^2+2r.
\end{aligned}$$

Hence, if z is a signed distance and a root of the verging polynomial $f(X)$, then by the verging theorem, condition (i) must be satisfied, i.e.,

$$(2a_6+a_5)^2 = a_6m^2,$$

where in our present case $m = 2+5u^2r$. Therefore, condition (i) becomes

$$A_0r^4+A_1r^3+A_2r^2+A_3r = 0,$$

where the $A_j = A_j(u)$ are given by

$$\begin{aligned}
A_0 &= u^9(9u-50),\\
A_1 &= -2u^4(20u^3-30u^2-9u+25),\\
A_2 &= -8u^5+60u^2+60u+9,\\
A_3 &= -8.
\end{aligned} \tag{4.1}$$

Since $r \neq 0$, equation (4.1) thus becomes the cubic equation in r and tenth degree in u,

$$g(u,r)=0,$$

where

$$g(u,r) = A_0r^3+A_1r^2+A_2r+A_3.$$

If we are given some specific marked ruler number r, say 2, for example, for which $\sqrt[5]{r} \notin \mathbb{Q}(r)$, then we would need to solve a tenth degree polynomial equation in u, check to see if u is a marked ruler number, and then check the other conditions of the verging theorem to see if z is an RMC number. Needless to say, this seems like a daunting task.

The trick for us is to consider a convenient value for u, say some integer or even a rational number, and then to solve for r which will be of degree at most 3 over $\mathbb{Q}$ and thus a marked ruler number. We then would like to check the conditions of the verging theorem to see if $\sqrt[5]{r}+u\sqrt[5]{r^2}$ is an RMC number.

One obvious choice of u is $50/9$, which yields a quadratic equation in r, since we then have $A_0 = 0$. Unfortunately, the zeros of this equation do not satisfy (3_ε) of the verging theorem.

After some trial and error, which we will say more about below, we chose $u = 7$. For $u = 7$, the cubic equation becomes

$$(7^9 \cdot 13)r^3 - (2 \cdot 7^4 \cdot 5352)r^2 - 131087r - 8 = 0,$$

or equivalently,

$$524596891r^3 - 25700304r^2 - 131087r - 8 = 0.$$

The rational root theorem, for example, shows that this equation is irreducible over $\mathbb{Q}$. A quick check yields three real solutions for r, one positive and the other two negative. Their approximate values are

$$r \doteq 0.05365\ldots \qquad -0.00006\ldots \qquad -0.00460\ldots.$$

We consider the unique positive value for r (the other two values do not satisfy all the conditions of the verging theorem). For convenience, we estimate the values of the a_j, m and B.

$$\begin{aligned}
a_1 &= -2,\\
a_2 &= 0,\\
a_3 &\doteq -13.14503\ldots\\
a_4 &\doteq 24.41221\ldots\\
a_5 &\doteq -44.67969\ldots\\
a_6 &\doteq 96.87083\ldots\\
m &\doteq 15.14503\ldots\\
B &\doteq -31.71786\ldots.
\end{aligned}$$

By construction, (0) and (1) of the verging theorem are certainly satisfied. For (2_ε), we must have $\varepsilon = +1$, since $a_3 < 0$. In this case, we get

$$c = c_+ \doteq 1.17681\ldots.$$

But then (3_+) is satisfied and yields

$$a = a_+ \doteq 0.96094\ldots$$

in which case

$$b = b_+ \doteq 3.94015\ldots .$$

Now, (4_+) is also satisfied and gives

$$s = s_+ \doteq 2.65846\ldots .$$

Finally, to double check (5_+), notice that

$$(s^2 - b^2 - c^2)^2 \doteq 96.87083\ldots \doteq a_6.$$

The other possible solution of (1), above, would give a value for a_6 between 5 and 6. Thus, by the verging theorem, $z = \sqrt[5]{r} + 7\sqrt[5]{r^2}$ is an RMC number and therefore so is $\sqrt[5]{r}$. This example provides a nontrivial case of a fifth root of a marked ruler number r which is constructible by marked ruler and compass.

We now summarize this result as a theorem.

Theorem 4.5. *Let r be the unique positive real root of the irreducible polynomial*

$$524596891X^3 - 25700304X^2 - 131087X - 8.$$

Hence, r is constructible by marked ruler. Then the real fifth root, $\sqrt[5]{r}$, is constructible by marked ruler and compass.

By numerical experimentation, it appears that, for any real number u in the interval $[6.5,\ 34]$, the cubic equation $g(u, r) = 0$ above has exactly one positive solution $r = r_u$, and the corresponding value of z satisfies the conditions of the verging theorem. We found that, for every integer in the interval $[7,\ 34]$, the number $\sqrt[5]{r_u}$ is an RMC number by arguing as we did above for $u = 7$. Moreover, as u increases over all real numbers from 6.5 to 34, r_u appears to decrease from about $1/11.033$ to $1/4329$. With this in mind and with the goal of finding rational numbers whose real fifth roots are RMC numbers, we tried to pick rational numbers in this range for r in the hope of finding a solution of $g(u, r) = 0$ with $u \in [6.5,\ 34]$ such that u would be a marked ruler number. Given this observation, we tried taking $r = 1/64$ and solving for u in the equation:

$$g\left(u,\ \frac{1}{64}\right) = 0.$$

The strategy was that perhaps $g(u,\ 1/64)$ has a splitting field whose Galois group could show that u is marked ruler constructible, in which case, $\sqrt[5]{1/64}$ and thus also $\sqrt[5]{2}$ would be shown to be constructible by marked ruler and compass. Using `Mathematica` we found that g factors into the linear factor $u+2$ and a ninth degree irreducible polynomial over $\mathbb{Q}$, namely,

$$g\left(x,\ \frac{1}{64}\right) = \frac{1}{2^{18}}\,(x+2)h(x),$$

where

$$\begin{aligned} h(x) = 3^2\,x^9 - 2^2\cdot 17\,x^8 + 2^3\cdot 17\,x^7 - 2^4\cdot 3 \\ \cdot 59\,x^6 + 2^5\cdot 3^3\cdot 11\,x^5 - 2^6\cdot 7\cdot 113\,x^4 + 2^8 \\ \cdot 383\,x^3 - 2^9\cdot 383\,x^2 + 2^{10}\cdot 7\cdot 89\,x - 2^{11}\cdot 503. \end{aligned}$$

Now let $y = x/2$ and $k(y) = 2^9 h(x)$. Then

$$\begin{aligned} k(y) = 9\,y^9 - 34\,y^8 - 34\,y^7 - 354\,y^6 + 594\,y^5 \\ - 1582\,y^4 + 1532\,y^3 - 1532\,y^2 + 2492\,y - 2012, \end{aligned}$$

which is irreducible over $\mathbb{Z}$.

Moreover, modulo 5,

$$k(y) \equiv 4(y^2+2y+3)(y^7+2y^6+2y^5+4y^4+2y^3+y^2+4) \bmod 5\mathbb{Z}[y],$$

where the two factors are irreducible in $\mathbb{F}_5[y]$, again by using `Mathematica`. Now, by a theorem of Dedekind, cf. [**3,** page 398], we see that if we embed the Galois group, G, of the splitting field of $k(y)$ over $\mathbb{Q}$ into S_9, then G contains an element σ such that

$$\sigma = (a_1\,a_2)(b_1\,b_2,\dots,b_7),$$

where

$$\{a_1,a_2\}\cup\{b_1,\dots,b_7\} = \{1,2,\dots,9\}.$$

Thus, G contains an element of order 14, implying that the order of G is a multiple of 7. Now, by this, we see that u cannot be a marked ruler number, since otherwise, G would have order $2^a 3^b$, cf. [**3,** Theorems 10.3.6, 10.3.11].

5. Conclusion. As we have shown above there are nontrivial examples of real fifth roots of cubic irrationals which are constructible by marked ruler and compass. However, this still leaves open the question of whether or not fifth roots of all rational numbers are constructible by marked ruler and compass.

We leave the reader with a specific question. Is $\sqrt[5]{2}$, for example, constructible by marked ruler and compass? It may be possible to answer this question with some further analysis of the results given in this paper–at least that is our hope.

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