ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 46, Number 3, 2016

ON THE GRAPH OF MODULES OVER COMMUTATIVE RINGS

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ABSTRACT. Let M be a module over a commutative ring and let $\mathrm{Spec}(M)$ be the collection of all prime submodules of M. We topologize $\mathrm{Spec}(M)$ with quasi-Zariski topology and, for a subset T of $\mathrm{Spec}(M)$, we introduce a new graph $G(\tau^*_T)$, called the *quasi-Zariski topology-graph*. It helps us to study algebraic (respectively, topological) properties of M (respectively, $\mathrm{Spec}(M)$) by using graph theoretical tools. Also, we study the annihilating-submodule graph and investigate the relation between these two graphs.

1. Introduction. Throughout this paper, R is a commutative ring with a non-zero identity and M is a unital R-module. By $N \leq M$ (respectively $N < M$) we mean that N is a submodule (respectively proper submodule) of M and $\Lambda(M)$ is the set of all non-zero submodules of M. For any pair of submodules $N \subseteq L$ of M and any element m of M, we denote L/N and the residue class of m modulo N in M/N by $\overline{L}$ and $\overline{m}$, respectively.

For a submodule N of M, the *colon ideal of M into N* is defined by $(N : M) = \{r \in R \mid rM \subseteq N\} = \mathrm{Ann}(M/N)$. Further if I is an ideal of R, the submodule $(N :_M I)$ is defined by $\{m \in M : \Im \subseteq N\}$. Moreover, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of positive integers, the ring of integers, and the field of rational numbers, respectively.

For a subset T of $\mathrm{Spec}(M)$, $\Im(T)$ is the intersection of all members of T.

A *prime submodule* of M is a submodule $P \neq M$ such that, whenever $re \in P$ for some $r \in R$ and $e \in M$, we have $r \in (P : M)$ or $e \in P$ **[13]**.

2010 AMS *Mathematics subject classification.* Primary 13C13, 13C99.

Keywords and phrases. Prime submodule, top module, quasi-Zariski topology, graph, vertices, annihilating-submodule.

Received by the editors on August 4, 2013, and in revised form on July 11, 2014.

DOI:10.1216/RMJ-2016-46-3-729

The *prime spectrum* (or simply, the *spectrum*) of M is the set of all prime submodules of M and denoted by $\operatorname{Spec}(M)$. Also, the set of all maximal submodules of M is denoted by $\operatorname{Max}(M)$.

The *prime radical* $\sqrt{N}$ is defined to be the intersection of all prime submodules of M containing N, and in the case of N is not contained in any prime submodule, $\sqrt{N}$ is defined to be M. Note that the intersection of all prime submodule M is denoted by $\operatorname{rad}(M)$.

The *quasi-Zariski topology* on $X := \operatorname{Spec}(M)$ is described as follows: put $V^*(N) = \{P \in X : P \supseteq N\}$ and $\xi^*(M) = \{V^*(N) : N$ is a submodule of $M\}$. Then there exists a topology τ^* on X having ξ^* as the set of closed subsets of $\operatorname{Spec}(M)$ if and only if ξ^* is closed under the finite union. When this is the case, τ^*_M is called the *quasi-Zariski topology* on $\operatorname{Spec}(M)$ and M is called a *top module* [**14**].

If $\operatorname{Spec}(M) \neq \emptyset$, the mapping $\psi : \operatorname{Spec}(M) \to \operatorname{Spec}(R/\operatorname{Ann}(M))$ such that $\psi(P) = (P : M)/\operatorname{Ann}(M) = \overline{(P : M)}$ for every $P \in \operatorname{Spec}(M)$, is called the *natural map* of $\operatorname{Spec}(M)$ [**6**].

A topological space X is said to be *connected* if there does not exist a pair U, V of disjoint non-empty open sets of X whose union is X. A topological space X is irreducible if, for any decomposition $X = X_1 \cup X_2$ with closed subsets X_i of X with $i = 1, 2$, we have $X = X_1$ or $X = X_2$. A subset X' of X is connected (respectively irreducible) if it is connected (respectively irreducible) as a subspace of X.

The zero-divisor graph of R, $\Gamma(R)$, is a graph with the vertex set $Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$. The concept of the zero-divisor graph was first introduced by Beck (see [**7**]). Since many properties of a ring are closely tied to the behavior of its ideals, it is valuable to replace the vertices of the zero-divisor graph by the non-zero annihilator ideals. The idea of a graph, whose vertices are a subset of ideals of a ring, was introduced recently in [**10**]. They defined $AG(R)$, the annihilating-ideal graph of R, to be a graph whose vertices are ideals of R with non-zero annihilators and in which two vertices I and J are adjacent if and only if $IJ = 0$.

Let N and L be submodules of M. Then the product of N and L is defined by $(N : M)(L : M)M$ and denoted by NL, and clearly $N^k = (N : M)^k M$ (see [**3**]).

In [**4**], the present authors generalized the above idea, introduced the annihilating-submodule graph $AG(M)$ and investigated some of its related properties. The (undirected) graph $AG(M)$ is a graph with vertices $V(AG(M)) = \{N \leq M\colon$ there exists a non-zero proper submodule L of M with $NL = 0\}$, where distinct vertices N, L are adjacent if and only if $NL = 0$.

As we know, the closed subset $V^*(N)$, where N is a submodule of M, plays an important role in the quasi-Zariski topology on $\mathrm{Spec}(M)$. Our main purpose in this article is to employ these sets and define a new graph $G(\tau_T^*)$, called the *quasi-Zariski topology-graph*. By using this graph, we study algebraic (respectively, topological) properties of M (respectively, $\mathrm{Spec}(M)$). Further, we investigate the relationship between $G(\tau_T^*)$ and $AG(M/\Im(T))$, where T denotes a non-empty subset of $\mathrm{Spec}(M)$ and $\Im(T)$ is the intersection of all members of T.

$G(\tau_T^*\ )$ is an undirected graph with vertices $V(G(\tau_T^*)) = \{N < M\colon$ there exists $K < M$ such that $V^*(N) \cup V^*(K) = T$ and $V^*(N), V^*(K) \neq T\}$, where T is a non-empty subset of $\mathrm{Spec}(M)$ and distinct vertices N and L are adjacent if and only if $V^*(N) \cup V^*(L) = T$ (see Definition 2.1).

Let M be a top module. In Section 2 of this article, among other results, it is shown that the quasi-Zariski topology-graph $G(\tau_T^*)$ is connected and $\mathrm{diam}(G(\tau_T^*)) \leq 3$. Further if $G(\tau_T^*)$ contains a cycle, then $\mathrm{gr}(G(\tau_T^*)) \leq 4$ (see Theorem 2.6). Also, it is shown that $G(\tau_T^*)$ has a bipartite subgraph (see Theorem 2.14).

In Section 3, we explore more properties of $AG(M)$. In Proposition 3.4, we show that if M is a non-simple semisimple R-module, then every non-zero proper submodule of M is a vertex. In Theorem 3.7, we provide some useful characterizations for those modules M for which $AG(M) = K_\alpha$, where $|\Lambda(M)| = \alpha$.

In Section 4, the relationship between $G(\tau_T^*)$ and $AG(M/\Im(T))$ is investigated. We show that, if N and L are non-zero proper submodules of M which are adjacent in $G(\tau_T^*)$, then $\sqrt{N}/\Im(T)$ and $\sqrt{L}/\Im(T)$ are adjacent in $AG(M/\Im(T))$ (see Proposition 4.5). Also we show that, if M is a finitely generated module and $G(\tau_T^*) \neq \emptyset$, then $AG(M/\Im(T))$ is isomorphic with a subgraph of $G(\tau^*_T)$. Further, we prove that, if M is a fully semiprime module, then $G(\tau^*_T)$ is isomorphic with a subgraph of $AG(M/\Im(T))$ (see Theorem 4.6).

Let us introduce some graphical notation that is used in what follows. A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a non-empty set of vertices, $V(G)$, a set $E(G)$ of edges, and an incident function ψ_G that associates an unordered pair of distinct vertices with each edge. The edge e joins x and y if $\psi_G(e) = \{x, y\}$, and we say x and y are adjacent. The degree $d_G(x)$ of a vertex x is the number of edges incident with x. A path in graph G is a finite sequence of vertices $\{x_0, x_1, \ldots, x_n\}$, where x_{i-1} and x_i are adjacent for each $1 \leq i \leq n$ and we denote $x_{i-1} - x_i$ for an existing edge between x_{i-1} and x_i. The number of edges crossed to get from x to y in a path is called the length of the path. A graph G is connected if a path exists between any two distinct vertices. For distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y and, if there is no such path, then $d(x, y) = \infty$. The diameter of G is $\operatorname{diam}(G) = \sup\{d(x, y) : x, y \in V(G)\}$. The girth of G, denoted by $\operatorname{gr}(G)$, is the length of the shortest cycle in G and, if G contains no cycles, then $\operatorname{gr}(G) = \infty$ (see [**1**]).

A graph H is a subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and ψ_H is the restriction of ψ_G to $E(H)$. We denote the complete graph on n vertices by K_n. A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V; that is, U and V are each independent sets and complete bipartite graphs on n and m vertices, denoted by $K_{n,m}$, where V and U are of size n and m, respectively, and $E(G)$ connects every vertex in V with all vertices in U (see [**16**]).

In the rest of this article, M denotes a top module, T a non-empty subset of $\operatorname{Spec}(M)$, $\Im(T)$ is the intersection of all members of T, $\widehat{M}$ represents the R-module $M/\Im(T)$, and for a submodule N of M, $\widehat{N} = N/\Im(T)$, where $\Im(T) \subseteq N$, is a submodule of $\widehat{M}$.

2. The qausi-Zariski topology-graph.

Definition 2.1. We define a *quasi-Zariski topology-graph* $G(\tau^*_T)$ with vertices $V(G(\tau^*_T))$= $\{N < M$: there exists $K < M$ such that $V^*(N) \cup$

$V^*(K) = T$ and $V^*(N), V^*(K) \neq T\}$, where distinct vertices N and L are adjacent if and only if $V^*(N) \cup V^*(L) = T$.

Notation 2.2. By [**14,** Lemma 2.1], if M is a top module, then for every pair of submodules N and L of M, we have $V^*(N) \cup V^*(L) = V^*(\sqrt{N}) \cup V^*(\sqrt{L}) = V^*(\sqrt{N} \cap \sqrt{L})$.

Proposition 2.3. *The following statements hold.*

(i) $G(\tau_T^*) \neq \emptyset$ *if and only if* T *is closed and is not an irreducible subset of* Spec(M).
(ii) $G(\tau_T^*) \neq \emptyset$ *if and only if* $T = V^*(\Im(T))$ *and* T *is not an irreducible subset of* Spec(M).
(iii) $G(\tau_T^*) \neq \emptyset$ *if and only if* $T = V^*(\Im(T))$ *and* $\Im(T)$ *is not a prime submodule of* M.

Proof.

(i) Straightforward.
(ii) Suppose that $G(\tau_T^*) \neq \emptyset$. By part (i), it is enough to show that $T = V^*(\Im(T))$ which is a closed set. Clearly, $T \subseteq V^*(\Im(T))$. Next, let $V^*(N)$ be any closed subset of Spec(M) containing T. Then $P \supseteq N$ for every $P \in T$ so that $\Im(T) \supseteq N$. Hence, for every $Q \in V^*(\Im(T))$ and $Q \supseteq \Im(T) \supseteq N$, namely, $V^*(\Im(T)) \subseteq V^*(N)$, it follows that $V^*(\Im(T))$ is the smallest closed subset of Spec(M) containing T. Hence, $V^*(\Im(T)) = T$.
(iii) It follows from part (ii) and [**8,** Theorem 3.4]. □

Example 2.4. Set $R := \mathbb{Z}$ and $M := \mathbb{Z} \oplus \mathbb{Z}(p^\infty)$, where p is a prime integer of $\mathbb{Z}$. Then, by [**6,** Examples 3.1], $\text{Max}(M) = \{p_i\mathbb{Z} \oplus \mathbb{Z}(p^\infty) : i \in \mathbb{N}\}$, $\text{Spec}(M) = \text{Max}(M) \cup \{(\mathbf{0}) \oplus \mathbb{Z}(p^\infty)\}$, where p_i is a prime number for every $i \in \mathbb{N}$, and M is a top module. We have $V^*((\mathbf{0}) \oplus \mathbb{Z}(p^\infty)) = \text{Spec}(M)$. Hence Spec$(M)$ is irreducible and $G(\tau^*_{\text{Spec}(M)}) = \emptyset$.

Example 2.5. Set $R := \mathbb{Z}$ and $M := \mathbb{Q} \oplus (\oplus_{i\in\mathbb{N}}\mathbb{Z}/p_i\mathbb{Z})$. Then by [**6,** Examples 3.1],

$$\text{Max}(M) = \{\mathbb{Q} \oplus (\oplus_{i\in\mathbb{N}, i\neq j}\mathbb{Z}/p_i\mathbb{Z})\},$$
$$\text{Spec}(M) = \text{Max}(M) \cup \{(\mathbf{0}) \oplus (\oplus_{i\in\mathbb{N}}\mathbb{Z}/p_i\mathbb{Z})\},$$

and M is a top module. Now, $\mathbb{Q} \oplus (\mathbf{0})$ and $\{(\mathbf{0}) \oplus (\oplus_{i\in\mathbb{N}}\mathbb{Z}/p_i\mathbb{Z})\}$ are adjacent in $G(\tau^*_{\mathrm{Spec}(M)})$ so that $G(\tau^*_{\mathrm{Spec}(M)}) \neq \emptyset$.

The following theorem illustrates some graphical parameters.

Theorem 2.6. *The quasi-Zariski topology-graph $G(\tau^*_T)$ is connected and $\mathrm{diam}(G(\tau^*_T)) \leq 3$. Moreover, if $G(\tau^*_T)$ contains a cycle, then $\mathrm{gr}(G(\tau^*_T)) \leq 4$.*

Proof. Suppose $N, K \in V(G(\tau^*_T))$ and they are not adjacent. Then $V^*(N) \cup V^*(K) \neq T$, so there exist $L, V \in V(G(\tau^*_T))$ with $V^*(\sqrt{N} \cap \sqrt{L}) = V^*(\sqrt{K} \cap \sqrt{V}) = T$. If $L = V$, then $N - L - K$ is a path of length 2. Thus, we assume that $L \neq V$. If $V^*(\sqrt{L} \cap \sqrt{V}) = T$, then $N - L - V - K$ is a path of length 3. If $V^*(\sqrt{L} \cap \sqrt{V}) \neq T$, then $N - \sqrt{L} \cap \sqrt{V} - K$ is a path of length 2 (if $N = \sqrt{L} \cap \sqrt{V}$, then $V^*(N) \cup V^*(K) = V^*(L) \cup V^*(V) \cup V^*(K)$ so that $T = V^*(\sqrt{V} \cap \sqrt{K}) = V^*(\sqrt{L} \cap \sqrt{V} \cap \sqrt{K})$. Thus, $V^*(\sqrt{N}) \cap V^*(\sqrt{K}) = T$, a contradiction. Similarly, we have $K \neq \sqrt{L} \cap \sqrt{V}$). Now suppose that $\mathrm{gr}(G(\tau^*_T)) > 4$. We can assume that $\mathrm{gr}(G(\tau^*_T)) = k$, where $k > 4$. Then $N_1 - N_2 - N_3 - N_4 - N_5 - \cdots - N_{k-1} - N_k - N_1$ is a cycle of length k. Clearly, $V^*(N_2) \cup V^*(N_{k-1}) \neq T$. Now one can see that $N_1 - \sqrt{N_2} \cap \sqrt{N_{k-1}} - N_k - N_1$ is a 3-cycle, a contradiction. So we have $\mathrm{gr}(G(\tau^*_T)) \leq 4$. Hence, the proof is complete. □

Proposition 2.7. *Let M be an R-module, and let $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ be the natural map. Suppose $\mathrm{Spec}(M)$ is homeomorphic to $\mathrm{Spec}(R/\mathrm{Ann}(M))$ under ψ. Let $(N : M)M$ and $(L : M)M$ be adjacent in $G(\tau^*_T)$, and let $T' = \{\overline{(P : M)} : P \in T\}$. Then $\overline{(N : M)}$ and $\overline{(L : M)}$ are adjacent in $G(\tau^*_{T'})$. Conversely, if $\overline{I}$ and $\overline{J}$ are adjacent in $G(\tau^*_{T'})$, then IM and JM are adjacent in $G(\tau^*_T)$.*

Proof. Since ψ is injective, $\psi^{-1}(T') = T$. Also we have $V^*((N : M)M) \cup V^*((L : M)M) = T$. Hence,

$$\psi(V^*((N : M)M)) \cup \psi(V^*((L : M)M)) = T'.$$

This implies that $V(\overline{N : M}) \cup V(\overline{L : M}) = T'$ (note that $V^*((N : M)M) = T \Leftrightarrow V(\overline{N : M}) = T'$). Conversely, suppose $V(\overline{I}) \cup V(\overline{J}) =$

T'. Then $\psi^{-1}(V(\overline{I})) \cup \psi^{-1}(V(\overline{J})) = T$ so that $V^*(IM) \cup V^*(JM) = T$ (note that $V^*(\overline{I}) = T' \Leftrightarrow V^*(IM) = T$). $\square$

Lemma 2.8. *Let $G(\tau_T^*) \neq \emptyset$ and let $P \in T$. Then P is a vertex if either of the following statements holds.*

(i) *There exists a subset T' of T such that $P \in T'$, $V^*(\cap_{Q \in T'} Q) = T$, and $V^*(\cap_{Q \in T', Q \neq P} Q) \neq T$.*
(ii) *For a submodule N of M, $N \in V(G(\tau_T^*))$ and $\sqrt{N} \cap P \notin V(G(\tau_T^*))$.*

Proof. Straightforward. $\square$

The following theorem shows the situations in which T contains some vertices.

Theorem 2.9. *Suppose T is a finite set and $G(\tau_T^*) \neq \emptyset$. Then*

(i) *$T \cap V(G(\tau_T^*)) \neq \emptyset$.*
(ii) *If $T \subseteq \mathrm{Max}(M)$, then every $P \in T$ is a vertex.*
(iii) *If $P \in T \cap \mathrm{Min}(M)$, then P is a vertex.*

Proof.

(i) Let $P \in T$. Then we have $V^*(P) \cup V^*(\cap_{Q \in T, Q \neq P} Q) = T$. If $V^*(\cap_{Q \in T, Q \neq P} Q) \neq T$, then P is a vertex. Otherwise, we have $V^*(\cap_{Q \in T, Q \neq P} Q) = T$. Since T is not irreducible, there exists a non-empty subset T' of T and $P' \in T'$ such that
$$V^*(\cap_{P \in T \setminus T'} P) \neq T \quad \text{and} \quad V^*(\cap_{P \in (T \setminus T') \cup \{P'\}} P) = T.$$
Thus, $P' \in T \cap V(G(\tau_T^*))$.
(ii) Clearly, $V^*(P) \cup V^*(\cap_{Q \in T, Q \neq P} Q) = T$ and $V^*(\cap_{Q \in T, Q \neq P} Q) \neq T$.
(iii) Clearly, $V^*(P) \cup V^*(\cap_{Q \in T, Q \neq P} Q) = T$ and $V^*(\cap_{Q \in T, Q \neq P} Q) \neq T$.

$\square$

Example 2.10. Consider Example 2.4. If $|T| \geq 2$ and $T \subseteq \{p_1\mathbb{Z} \oplus \mathbb{Z}(p^\infty), \ldots, p_n\mathbb{Z} \oplus \mathbb{Z}(p^\infty)\}$, then every element of T is a vertex. Moreover, in Example 2.5, if $|T| \geq 2$ and

$$T \subseteq \{\mathbb{Q} \oplus (\oplus_{i\in\mathbb{N}, i\neq 1}\mathbb{Z}/p_i\mathbb{Z}), \ldots, \mathbb{Q} \oplus (\oplus_{i\in\mathbb{N}, i\neq n}\mathbb{Z}/p_i\mathbb{Z})\},$$

then every element of T is a vertex.

Definition 2.11. We define a subgraph $G_d(\tau_T^*)$ of $G(\tau_T^*)$ with vertices $V((G_d(\tau_T^*))) = \{N < M$: there exists $L < M$ such that $V^*(N) \cup V^*(L) = T$, $V^*(N), V^*(L) \neq T$ and $V^*(N) \cap V^*(L) = \emptyset\}$, where distinct vertices N and L are adjacent if and only if $V^*(N) \cup V^*(L) = T$ and $V^*(N) \cap V^*(L) = \emptyset$. It is clear that the degree of every $N \in V((G_d(\tau_T^*)))$ is the number of submodules K of M such that $V^*(L) = V^*(K)$, where L is adjacent to N.

We need the following remark.

Remark 2.12. We recall that the *Zariski topology* on $\operatorname{Spec}(M)$ is the topology τ_M described by taking the set $Z(M) = \{V(N) : N \leq M\}$ as the set of closed sets of $\operatorname{Spec}(M)$, where $V(N) = \{P \in \operatorname{Spec}(M) : (P : M) \supseteq (N : M)\}$ [**12**]. If M is a multiplication module, then $\tau_M = \tau_M^*$ by [**14,** Theorem 3.5].

Proposition 2.13. *The following statements hold.*

(i) $G_d(\tau_T^*) \neq \emptyset$ *if and only if* $T = V^*(\Im(T))$ *and* T *is disconnected.*
(ii) *Suppose* $\widehat{M}$ *is an Artinian module and* T *is closed. Then* $G_d(\tau_T^*) = \emptyset$ *if and only if* $R/\operatorname{Ann}(\widehat{M})$ *contains no idempotent other than* $\overline{0}$ *and* $\overline{1}$.

Proof.

(i) Straightforward.

(ii) Since $\widehat{M}$ is an Artinian module, then $\widehat{M}/\operatorname{rad}(\widehat{M})$ is a Noetherian module by [**8,** Corollary 2.30]. As $\widehat{M}/\operatorname{rad}(\widehat{M})$ is a finitely generated top module, it is a multiplication module by [**14,** Theorem 3.5]. It follows that $\tau_{\hat{M}/\operatorname{rad}(\hat{M})} = \tau^*_{\hat{M}/\operatorname{rad}(\hat{M})}$ by Remark 2.12. So $\tau_{\hat{M}} = \tau^*_{\hat{M}}$ because $\widehat{M}$ and $\widehat{M}/\operatorname{rad}(\widehat{M})$ are homeomorphic by Lemma 4.1. Also, the natural map of $\widehat{M}/\operatorname{rad}(\widehat{M})$ is surjective (for, $\widehat{M}/\operatorname{rad}(\widehat{M})$ is finitely generated). Hence, the natural map of $\widehat{M}$ is surjective by the above arguments. Now the result follows from [**12,** Corollary 3.8]. □

Theorem 2.14. *$G_d(\tau_T^*)$ is a bipartite graph.*

Proof. At first we assume that $G_d(\tau_T^*)$ contains a cycle. We show that $\mathrm{gr}(G_d(\tau_T^*)) \leq 4$. Now suppose that $\mathrm{gr}(G_d(\tau_T^*)) > 4$. We can assume that $\mathrm{gr}(G_d(\tau_T^*)) = k$, where $k > 4$. Then $N_1 - N_2 - N_3 - N_4 - N_5 - \cdots - N_{k-1} - N_k - N_1$ is a cycle of length k. Clearly, $V^*(N_{k-1}) = V^*(N_1)$. Hence, one can see that $N_1 - N_2 - N_3 - \cdots - N_{k-2} - N_1$ is a cycle, a contradiction. So we have $\mathrm{gr}(G_d(\tau_T^*)) \leq 4$. Now, by [**16,** Proposition 1.6.1], G is a bipartite graph if and only if it does not contain an odd cycle. Hence, by Theorem 2.6, it is enough to show that $\mathrm{gr}(G_d(\tau_T^*)) \neq 3$. Suppose $N - L - K - N$ is a 3-cycle. Then

$$\begin{aligned}\emptyset &= (V^*(N) \cap V^*(L)) \cup (V^*(N) \cap V^*(K)) \\ &= V^*(N) \cap (V^*(L) \cup V^*(K)) = V^*(N) \cap T = V^*(N).\end{aligned}$$

Hence, $V(N) = \emptyset$, a contradiction. □

Corollary 2.15. *By Theorem* 2.14*, if $G_d(\tau_T^*)$ contains a cycle, then $\mathrm{gr}(G_d(\tau_T^*)) = 4$.*

Example 2.16. Set $R := \mathbb{Z}$ and $M := \mathbb{Z}/12\mathbb{Z}$. So $\mathrm{Spec}(M) = \mathrm{Max}(M) = \{2\mathbb{Z}/12\mathbb{Z}, 3\mathbb{Z}/12\mathbb{Z}\}$. Set $T := \mathrm{Spec}(M)$. Clearly, $G(\tau_T^*) = G_d(\tau_T^*)$ is a bipartite graph and $\mathbb{Z}/(\cap_{P\in T} P : M) \cong \mathbb{Z}/6\mathbb{Z}$ contains idempotents other than $\overline{0}$ and $\overline{1}$.

Example 2.17. Set $R := \mathbb{Z}$ and $M := \mathbb{Z}/30\mathbb{Z}$. So $\mathrm{Spec}(M) = \mathrm{Max}(M) = \{2\mathbb{Z}/30\mathbb{Z}, 3\mathbb{Z}/30\mathbb{Z}, 5\mathbb{Z}/30\mathbb{Z}\}$. Set $T := \mathrm{Spec}(M)$. Clearly, $G_d(\tau_T^*)$ is a bipartite graph and $\mathbb{Z}/(\cap_{P\in T} P : M) \cong \mathbb{Z}/30\mathbb{Z}$ contains idempotents other than $\overline{0}$ and $\overline{1}$.

The above example shows that $G_d(\tau_T^*)$ is not always connected.

Proposition 2.18. *The following statements hold.*

(i) *$G_d(\tau^*_T)$ with two parts U and V is a complete bipartite graph if and only if for every $N, L \in U$ (respectively in V), $V^*(N) = V^*(L)$.*
(ii) *$G_d(\tau^*_T)$ is connected if and only if it is a complete bipartite graph.*

Proof. Use the fact that if N and L are two vertices, then $d(N, L) = 2$ if and only if $V^*(N) = V^*(L)$. □

We end this section with the following question.

Question 2.19. *Let $G(\tau^*_T) \neq \emptyset$, where T is an infinite subset of* Spec(M). *Is $T \cap V(G(\tau^*_T)) \neq \emptyset$?*

3. The annihilating-submodule graph. As we mentioned before, $AG(M)$ is a graph with vertices $V(AG(M)) = \{N \leq M : NL = 0$ for some $0 \neq L < M\}$, where distinct vertices N and L are adjacent if and only if $NL = 0$ (here we recall that the product of N and L is defined by $(N : M)(L : M)M$).

The following results reflect some basic properties of the annihilating-submodule graph of a module.

Proposition A ([4, Proposition 3.2]). Let N be a non-zero proper submodule of M.

(i) N is a vertex in $AG(M)$ if $\mathrm{Ann}(N) \neq \mathrm{Ann}(M)$ or $(0 :_M (N : M)) \neq 0$.
(ii) N is a vertex in $AG(M)$, where M is a multiplication module, if and only if $(0 :_M (N : M)) \neq 0$.

Remark 3.1. In the annihilating-submodule graph $AG(M)$, M itself can be a vertex. In fact M is a vertex if and only if every non-zero submodule is a vertex if and only if there exists a non-zero proper submodule N of M such that $(N : M) = \mathrm{Ann}(M)$. For example, for every submodule N of $\mathbb{Q}$ (as a $\mathbb{Z}$-module), $(N : \mathbb{Q}) = 0$. Hence, $\mathbb{Q}$ is a vertex in $AG(\mathbb{Q})$.

Theorem B ([4, Theorem 3.3]). Assume that M is not a vertex. Then the following hold.

(i) $AG(M)$ is empty if and only if M is a prime module.

(ii) A non-zero submodule N of M is a vertex if and only if $(0 :_M (N : M)) \neq 0$.

Theorem C ([4, Theorem 3.4]). The annihilating-submodule graph $AG(M)$ is connected and $\text{diam}(AG(M)) \leq 3$. Moreover, if $AG(M)$ contains a cycle, then $\text{gr}(AG(M)) \leq 4$.

Lemma 3.2. *Let M be an R-module and $\text{Ann}(M)$ a prime ideal. Then $\text{diam}(AG(M)) \leq 2$.*

Proof. Suppose N and L are adjacent in $AG(M)$. Then $(N : M) = \text{Ann}(M)$ or $(L : M) = \text{Ann}(M)$. Assume that $(N : M) = \text{Ann}(M)$. So every non-zero submodule M is a vertex and adjacent to N. Hence, $\text{diam}(AG(M)) \leq 2$. □

Proposition 3.3. *The following statements hold.*

(i) *Let $M = Rm$ be a cyclic R-module. Then M is not a vertex.*
(ii) *Let $M = M_1 \oplus M_2$, where M_1, M_2 are non-zero R-submodules of M. Then every non-zero submodule of M_1 is adjacent to every non-zero submodule of M_2.*
(iii) *Assume that $AG(M) = \emptyset$. Then module M is an indecomposable module.*

Proof.

(i) This follows from Remark 3.1 and the fact that every cyclic R-module is multiplication.
(ii) Let $0 \neq N \leq M_1$ and $0 \neq K \leq M_2$. Clearly, $(N \oplus (\mathbf{0}) : M) = (N : M_1) \cap (0 : M_2)$. Hence, $(N \oplus (\mathbf{0}) : M) \subseteq (0 : M_2)$. Similarly, $((\mathbf{0}) \oplus K : M) \subseteq (0 : M_1)$. Therefore, $(N \oplus (\mathbf{0}))((\mathbf{0}) \oplus K) = 0$. This in turn implies that N and K are adjacent in $AG(M)$.
(iii) The proof follows from part (ii). □

We allow α to be infinite cardinal, where $\alpha = |\Lambda(M)|$. (We recall that $\Lambda(M)$ is the set of all non-zero submodules of M.)

Proposition 3.4. *The following statements hold.*

(i) *Let M be a non-simple semisimple R-module. Then every non-zero proper submodule of M is a vertex.*
(ii) *Let M be a non-simple homogeneous semisimple R-module. Then $AG(M) = K_\alpha$.*
(iii) *Let M be a prime module with a non-zero socle. Then $AG(M) = \emptyset$ or $AG(M) = K_\alpha$.*
(iv) *Let M be a non-simple module with a non-zero socle. Then $AG(M) \neq \emptyset$. In particular, $AG(M) \neq \emptyset$ when M is a non-simple Artinian module.*

Proof.

(i) Since M is a semisimple module, we have $M = \oplus_{\alpha \in I} T_\alpha$ where, for each $\alpha \in I$, T_α is a simple submodule of M. Now let N be an arbitrary non-zero submodule of M. Then, by [**2,** Proposition 9.4], there exist a subset $I' \subseteq I$ and a decomposition $N \cong \oplus_{\alpha \in I'} T_\alpha$. Set $K \cong \oplus_{I \setminus I'} T_\alpha$. Then $NK \subseteq N \cap K = 0$. It follows that N is a vertex.

(ii) Since M is a homogeneous semisimple module, it is clear that $\mathrm{Ann}(M)$ is a maximal ideal of R. Hence for every non-zero submodule N of M, we have $(N : M) = (0 : M)$. We conclude that if N and K are two non-zero distinct submodules of M, then $NK = 0$, as desired.

(iii) This follows from part (ii) because every prime module with a non-zero socle is homogeneous semisimple (see [**9,** Corollary 1.9]).

(iv) Suppose that M is not a simple module with $\mathrm{Soc}(M) \neq 0$. Then there exists a minimal submodule Rm of M, where m is a non-zero element of M. Now $(0 : m)$ is a maximal ideal of R and we have $(Rm)((0 : m)M) = 0$. This shows that $AG(M) \neq \emptyset$. □

Example 3.5. Put $R := \mathbb{Z}$ and $M := \oplus_{i \in \mathbb{N}} \mathbb{Z}_2$. Since M is a direct sum of isomorphic simple modules, then M is a homogeneous semisimple module. For every non-zero proper submodule N of M, we have $(N : M) = \mathrm{Ann}(M)$. Hence every non-zero submodule N and K are adjacent in $AG(M)$.

Proposition 3.6. *Let M be a non-simple prime module. Then $AG(M) = K_\alpha$, if and only if every non-zero proper submodule of M is adjacent to M.*

Proof. The sufficiency is clear.

To see the converse, let $N \in V(AG(M))$. Then there exists a non-zero proper submodule L of M such that $NL = 0$. Since $\mathrm{Ann}(M)$ is a prime ideal of R, it follows that $(N : M) = \mathrm{Ann}(M)$ or $(L : M) = \mathrm{Ann}(M)$. So every non-zero submodule M is a vertex by Remark 3.1. Now, since $AG(M)$ is a complete graph, every non-zero proper submodule of M is adjacent to M. □

Theorem 3.7. *Consider the following statements.*

(i) $\mathrm{Ann}(M)$ *is a prime ideal and* M *is a divisible* $R/\mathrm{Ann}(M)$*-module.*
(ii) *Every non-zero proper submodule of* M *is adjacent to* M*.*
(iii) *For each ideal* I *of* R*, we have* $IM = M$ *or* $IM = 0$*.*
(iv) $AG(M) = K_\alpha$*.*
(v) M *is a non-simple homogeneous semisimple module.*

Then (i) → (ii) → (iii) → (iv) → (i). *Moreover, if* M *is a finitely generated module then* (v) ↔ (i)*.*

Proof. (i) → (ii). Let N be a non-zero proper submodule of M. We show that $(N : M) = \mathrm{Ann}(M)$. Suppose $r \in (N : M)$ and $rM \neq 0$. Since M is divisible by $R/\mathrm{Ann}(M)$, we have $rM = M$. This implies that $N = M$, a contradiction. Hence, N is adjacent to M, as desired.

(ii) → (i) and (ii) → (iii) are clear.

(iii) → (ii). Let N be a non-zero proper submodule of M and I an ideal of R. Then $(IM : M) = \mathrm{Ann}(M)$ by hypothesis, where $IM \neq M$. Now we have $(N : M) = ((N : M)M : M) = \mathrm{Ann}(M)$. This shows that N is adjacent to M, as required.

(ii) ↔ (iv). Straightforward.

(ii) → (v). Let M be a finitely generated R-module and let $(N : M) = \mathrm{Ann}(M)$ for every proper submodule N of M. Then M is a divisible $R/\mathrm{Ann}(M)$-module. We show that $R/\mathrm{Ann}(M)$ is a field. Suppose not. Then M has a maximal submodule, say N. So $(N : M)$ is a maximal ideal R. Hence there exists $0 \neq r \in (N : M)$. But $rM = M$ is a contradiction. So $\mathrm{Ann}(M)$ is a maximal ideal and hence M is a homogeneous semisimple module.

(v) → (ii). It is clear by Proposition 3.4 (ii). □

Note that an R-module M is fully prime (respectively fully semiprime) if each proper submodule of M is prime (respectively semiprime). In [**9,** Corollary 1.9], it is shown that M is fully prime (respectively, fully semiprime) if and only if is homogeneous semisimple (respectively, co-semisimple module).

Corollary 3.8. *Let R be an integral domain with $\dim(R) = 1$, and let M be an R-module. Then every non-zero proper submodule of M is adjacent to M if and only if one of the following statements hold*:

(i) *M is a homogeneous semisimple module.*
(ii) *M is a divisible module.*

Proof. Suppose that every non-zero proper submodule of M is adjacent to M. Then $\operatorname{Ann}(M)$ is a prime ideal of R and M is a divisible $R/\operatorname{Ann}(M)$-module by Theorem 3.7. If $\operatorname{Ann}(M) = 0$, then M is a divisible R-module. Otherwise, since $\dim(R) = 1$, it follows that $\operatorname{Ann}(M)$ is a maximal ideal of R so that $(N : M) = \operatorname{Ann}(M)$ for every proper submodule N of M. Thus every proper submodule of M is prime by [**14,** Corollary 1.2]. This means that M is a homogeneous semisimple. Conversely, first we assume that M is a homogeneous semisimple module. Then $\operatorname{Ann}(M)$ is a maximal ideal of R so that every non-zero proper submodule M is adjacent to M. In case M is a divisible module, the claim follows from Theorem 3.7. □

4. The relationship between $G(\tau^*_T)$ and $AG(M)$. A proper submodule N of M is said to be semiprime in M if, for every ideal I of R and every submodule K of M, $I^2K \subseteq N$ implies that $IK \subseteq N$. Further, M is called a semiprime module if $(0) \subseteq M$ is a semiprime submodule. Every intersection of prime submodules is a semiprime submodule. A proper ideal I of R is semiprime if, for every ideal J and K of R, $J^2K \subseteq I$ implies that $JK \subseteq I$ [**17**].

Lemma 4.1. *Suppose T is a closed subset of $\operatorname{Spec}(M)$ equipped with the natural topology induced from of $\operatorname{Spec}(M)$. Then T and $\operatorname{Spec}(\widehat{M})$ are homeomorphic.*

Proof. Let $\phi : \operatorname{Spec}(\widehat{M}) \to T = V^*(\Im(T))$ be defined by $\phi(\widehat{Q}) = Q$, where $Q \in \operatorname{Spec}(M)$. Clearly ϕ is a bijection map. We show that ϕ is a continuous map. Let $U = T \cap V^*(N)$ be a closed subset of T, where N is a proper subset of M. Then we have $\phi^{-1}(U) = V^*(\widehat{N + \Im(T)})$. We show that ϕ is closed. Suppose U is a closed subset of $\mathrm{Spec}(\widehat{M})$. Then $U = V^*(\widehat{N})$, where $N \leq M$. It is easy to see that $\phi(U) = V^*(N)$. □

One may think that since T and $\mathrm{Spec}(\widehat{M})$ are homeomorphic, studying $G(\tau_T^*)$ can be reduced to studying $G(\tau^*_{\mathrm{Spec}(L)})$, where L is a semiprime module. But the following example shows that this is not true.

Example 4.2. Set $R := \mathbb{Z}$, $M := \mathbb{Z}/12\mathbb{Z}$, and $T := \mathrm{Spec}(M)$. Then $G(\tau_T^*) = K_{1,2}$ but $G(\tau^*_{\mathrm{Spec}(M/\,\mathrm{rad}(M))}) = K_2$.

Remark 4.3. In fact $G(\tau_T^*)$ is a non-empty graph if and only if $|E(G(\tau_T^*))| \geq 1$. The following lemma shows that the graph $AG(M)$ also has this property (i.e., $|E(AG(M))| \geq 1$) if M is a semiprime module such that it is not a vertex in $AG(M)$.

Lemma 4.4. *Assume that M is not a vertex in $AG(M)$. Then M is a semiprime module if and only if for every non-zero submodule N of M and each positive integer k, $N^k \neq 0$.*

Proof. The necessity is clear.

To see the converse, let N be a submodule of M and let I be an ideal of R. Let $I^2N = 0$ and $IN \neq 0$. Then we have $(IN)^2 = (IN : M)^2M \subseteq I^2N = 0$, a contradiction. Hence, M is a semiprime module. □

Proposition 4.5. *The following statements hold.*

(i) *Suppose N and L are adjacent in $G(\tau_T^*)$. Then $\sqrt{N}$ and $\sqrt{L}$ are adjacent in $AG(\widehat{M})$.*
(ii) *$G(\tau_T^*)$ is isomorphic with a subgraph of $AG(\widehat{M})$ or $|E(G(\tau_T^*))| \geq 2$.*

Proof.

(i) Straightforward.
(ii) Assume that $G(\tau_T^*)$ is not isomorphic with a subgraph of $AG(\widehat{M})$. Hence there exist $N, L \in V(G(\tau_T^*))$ such that N and L are

adjacent and $N \neq \sqrt{N}$. It follows that $N - L - \sqrt{N}$ is a path of length 2. □

Theorem 4.6. *The following statements hold.*

(i) *Let M be a finitely generated module and $G(\tau_T^*) \neq \emptyset$. Then $AG(\widehat{M})$ is isomorphic with a subgraph of $G(\tau_T^*)$.*
(ii) *Let M be a fully semiprime module. Then $G(\tau_T^*)$ is isomorphic with a subgraph of $AG(\widehat{M})$.*
(iii) *Let M be a semisimple module and suppose M is not a vertex in $AG(M)$. Then $G(\tau_T^*)$ and $AG(\widehat{M})$ are isomorphic.*
(iv) *Let M be a homogeneous semisimple module. Then $AG(\widehat{M}) = K_\alpha$, where $\alpha = |\Lambda(\widehat{M})|$ and $G(\tau_T^*) = \emptyset$.*

Proof.

(i) By **[14,** Theorem 3.5], every finitely generated top module is multiplication. One can see that if $\widehat{N}$ and $\widehat{L}$ are adjacent in $AG(\widehat{M})$, then N and L are adjacent in $G(\tau_T^*)$.

(ii) By [**9,** Theorem 2.3], M is a co-semisimple module. So

$$N = \bigcap_{P \in V^*(N)} P,$$

where $N < M$. Hence, by Proposition 4.5 (i), it is easy to see that $G(\tau_T^*)$ is isomorphic with a subgraph of $AG(\widehat{M})$.

(iii) Let M be a semisimple module and suppose M is not a vertex in $AG(M)$. We show that M is a multiplication module. To see this, let N be a proper submodule of M. Then there exists a family $\{T_i, i \in I\}$ of minimal submodules of M such that $N = \oplus_{i \in I} T_i$. Now for each $i \in I$, we have $(T_i : M)M = M$ (note that $(T_i : M)M \neq 0$ because M is not a vertex in $AG(M)$). Hence,

$$N = \bigoplus_{i \in I} (T_i : M)M = \left(\bigoplus_{i \in I} (T_i : M) \right) M.$$

Thus, M is a multiplication module. It follows that, if $\widehat{N}$ and $\widehat{L}$ are adjacent in $AG(\widehat{M})$, then N and L are adjacent in $G(\tau_T^*)$. Since M is a co-semisimple module, by using part (ii), we see that $G(\tau_T^*)$ is isomorphic with a subgraph of $AG(\widehat{M})$. Hence $G(\tau_T^*)$ and $AG(\widehat{M})$ are isomorphic.

(iv) The first assertion follows from Proposition 3.4 (ii). To see the second assertion, $\Im(T)$ is a prime submodule of M (see [**9,** Corollary 1.9]), thus $G(\tau_T^*) = \emptyset$ by Proposition 2.3 (iii). $\square$

Example 4.7. Put $R := \mathbb{Z}$ and $M := \oplus_{i\in\mathbb{N}}\mathbb{Z}/p_i\mathbb{Z}$. Then, by [**6,** Examples 3.1], $\mathrm{Max}(M) = \mathrm{Spec}(M) = \{p_jM\} = \{\oplus_{i\in\mathbb{N},i\neq j}\mathbb{Z}/p_i\mathbb{Z}\}$, and M is a top module. $G(\tau^*_{\mathrm{Spec}(M)})$ is an infinite graph, because every element $\oplus_{i\in\mathbb{N},i\neq j}\mathbb{Z}/p_i\mathbb{Z}$ of $\mathrm{Spec}(M)$ is adjacent to $\mathbb{Z}/p_j\mathbb{Z}$. Hence, by Theorem 4.6 (ii), $AG(M)$ is an infinite graph.

Lemma 4.8. *Assume that $\emptyset \neq V(AG(\widehat{M}) \subseteq \mathrm{Max}(\widehat{M})$. Then $|T| = 2$, $AG(\widehat{M}) = K_2$, and it is isomorphic with a subgraph of $G(\tau_T^*)$.*

Proof. Suppose that $\widehat{P}$ is a vertex in $AG(\widehat{M})$ such that $P \in \mathrm{Max}(M)$. Then there exists a non-zero proper submodule $\widehat{Q}$ of $\widehat{M}$ such that it is adjacent to $\widehat{P}$, where, $Q \in \mathrm{Max}(M)$. One can see that $(P : M) \subseteq (P' : M)$ or $(Q : M) \subseteq (P' : M)$ for every $P' \in T$. Now since $\widehat{M}$ is a top module, by [**14,** Theorem 3.5] $P = P'$ or $Q = P'$. Hence, $V^*(P) \cup V^*(Q) = T$. It follows that $|T| = 2$, $AG(\widehat{M})$ has only one edge and it is isomorphic with a subgraph of $G(\tau_T^*)$. $\square$

Proposition 4.9. *Assume that $G(\tau_T^*) \neq \emptyset$.*

(i) *If $\widehat{M}$ is a Noetherian R-module, then $T = V^*(P_1 \cap \cdots \cap P_n)$, where for each i $(1 \leq i \leq n)$, P_i is a vertex.*
(ii) *If $\widehat{M}$ is an Artinian R-module, then $T = V^*(P_1 \cap \cdots \cap P_n)$, where for each $(1 \leq i \leq n)$, P_i is a vertex. In particular, $|T| = n$.*

Proof.

(i) Since $\widehat{M}$ is a Noetherian module, $\widehat{M}$ has a finite number of minimal prime submodules by [**15,** Theorem 4.2]. Hence

$$\mathrm{Spec}(\widehat{M}) = V^*(\widehat{P_1}) \cup \cdots \cup V^*(\widehat{P_n}),$$

where each i $(1 \leq i \leq n)$, $\widehat{P_i}$ is a minimal prime submodule of $\widehat{M}$ and P_i is a prime submodule of M. So, by Lemma 4.1, we have $T = V^*(P_1) \cup \cdots \cup V^*(P_n)$. Now the result follows from Lemma 2.8 (i).

(ii) As in the proof of Proposition 2.13 (ii), $\widehat{M}/\operatorname{rad}(\widehat{M})$ is a Noetherian module. So $\widehat{M}/\operatorname{rad}(\hat{M})$ has a finite number of minimal prime submodules. Hence, $\widehat{M}$ has a finite number of minimal prime submodules. So we have $T = V^*(P_1) \cup \cdots \cup V^*(P_n)$ by part (i). To see the second assertion, we note that, since $\widehat{M}/\operatorname{rad}(\widehat{M})$ is a finitely generated top module, it is a multiplication module by **[14,** Theorem 3.5]. It follows that $\widehat{M}/\operatorname{rad}(\widehat{M})$ is a cyclic Artinian module by **[11,** Corollary 2.9], and hence, $\operatorname{Spec}(\widehat{M}/\operatorname{rad}(\widehat{M})) = \operatorname{Max}(\widehat{M}/\operatorname{rad}(\widehat{M}))$. So $\operatorname{Spec}(\widehat{M}) = \operatorname{Max}(\widehat{M})$. Hence, by the above arguments, we have $|T| = n$, and the proof is completed. $\square$

Acknowledgments. We would like to thank the referee for valuable comments and a careful reading of our manuscript.

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