ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 46, Number 3, 2016

C*-ALGEBRAS OF 2-GROUPOIDS

MASSOUD AMINI

ABSTRACT. We define topological 2-groupoids and study locally compact 2-groupoids with 2-Haar systems. We consider quasi-invariant measures on the sets of 1-arrows and unit space and build the corresponding vertical and horizontal modular functions. For a given 2-Haar system, we construct the vertical and horizontal full C*-algebras of a 2-groupoid and show that they are independent of the choice of the 2-Haar system, up to strong Morita equivalence. We make a correspondence between their bounded representations on Hilbert spaces and those of the 2-groupoid on Hilbert bundles. We show that representations of certain closed 2-subgroupoids are induced to representations of the 2-groupoid and use regular representation to build the vertical and horizontal reduced C*-algebras of the 2-groupoid. We establish strong Morita equivalence between C*-algebras of the 2-groupoid of composable pairs and those of the 1-arrows and unit space. We describe the reduced C*-algebras of r-discrete principal 2-groupoids and find their ideals and masa's.

1. Introduction. A *Lie bialgebroid* is a compatible dual pair (A, A^*) of Lie algebroids **[18]**, the basic example being the pair (TM, T^*M) of tangent and cotangent bundles of a Poisson manifold M. *Poisson groupoids* are the infinitesimal objects associated to symplectic double groupoids **[17, 18, 26]**. A *double Lie groupoid* is essentially a groupoid object in the category of Lie groupoids **[5, 18]** and can be presented by a square whose edges are Lie groupoids with certain compatibility conditions (i.e., filling condition). Just as a Lie groupoid induces a simplicial manifold, a double Lie groupoid induces a bisimplicial manifold (called its *nerve*). After applying the Artin and Mazur bar construction **[9]** to the nerve of a double Lie groupoid we get a simplicial manifold which is a (weak) local Lie 2-groupoid in the sense of **[29]** and a Lie

2010 AMS *Mathematics subject classification.* Primary 18D05, 46L05, 46L55.

Keywords and phrases. 2-Category, 2-groupoid, 2-Haar system, C^*-algebras of 2-groupoids, induced representations, strong Morita equivalence.

The author was partly supported by a grant from IPM, No. 92430215.

Received by the editors on March 16, 2014, and in revised form on August 8, 2014.

DOI:10.1216/RMJ-2016-46-3-693

2-groupoid in the presence of the filling condition [**19**]. If $G \rightrightarrows M$ is a Lie groupoid whose source map is a fibration, then the fundamental groupoids of G and M form a double Lie groupoid and the fibration assumption gives the filling condition. This is a Lie 2-groupoid. By truncating this Lie 2-groupoid, one could recover the Haefliger fundamental groupoid [**14**] of G [**19**]. More generally, if (A, A^*) is a Lie bialgebroid, there is a Courant algebroid structure on $A \oplus A^*$ [**16**]. It has been suggested by Ševera and Roytenberg that the global groupoid-like object corresponding to a Courant algebroid, and in particular, the double of a Poisson groupoid, is a symplectic 2-groupoid.

The 2-groupoids which appear in Poisson and symplectic geometry are usually Hausdorff. On the other hand, in noncommutative geometry, certain quotient spaces are described by non-commutative C*-algebras. When the symmetry groups of such quotient spaces are non-Hausdorff, it is more appropriate to describe such symmetry groups and groupoids using crossed modules of groupoids [**7**]. One motivating example is the gauge action on the irrational rotation algebra A_ϑ, which is the universal C*-algebra generated by two unitaries U and V satisfying the commutation relation $UV = \lambda VU$ with $\lambda := \exp(2\pi i\vartheta)$. Since A_ϑ is the crossed product $C(\mathbb{T}) \rtimes_\lambda \mathbb{Z}$, for the canonical action of $\mathbb{Z}$ on $\mathbb{T}$ by $n \cdot z := \lambda^n \cdot z$, it could be viewed as the noncommutative analog of the non Hausdorff quotient space $\mathbb{T}/\lambda^{\mathbb{Z}}$. This latter group acts on itself by translations, thus $\mathbb{T}/\lambda^{\mathbb{Z}}$ is a symmetry group of A_ϑ. More generally, one may define actions of crossed modules on C*-algebras similar to the twisted actions in the sense of Green [**12**], and build crossed products for such actions. The resulting crossed product is functorial: if two actions are equivariantly Morita equivalent in a suitable sense, their crossed products are Morita-Rieffel equivalent C*-algebras [**7**].

Crossed modules of discrete groups are used in homotopy theory to classify 2-connected spaces up to homotopy equivalence. They are equivalent to strict 2-groups [**3, 21**]. The crossed modules of topological groupoids in [**7**] are equivalent to strict topological 2–groupoids. In [**7**], the authors basically discussed crossed modules of groups, but also emphasized the importance of dealing with groupoids, such as holonomy groupoid of foliations or groupoid of germs of a pseudogroup of transformations, which are only locally Hausdorff (see also [**20**]). According to the principles of noncommutative geometry, such non Hausdorff spaces of arrows should be viewed as the orbit space of another groupoid. This is exactly where the higher category theory naturally comes in. Indeed one could write every locally Hausdorff groupoid as the truncation of a Hausdorff topological weak 2–groupoid. Also the crossed modules of groupoids correspond naturally to strict 2–groupoids. For a Hausdorff étale groupoid G and the interior $H \subseteq G$ of the set of loops (arrows with same source and target) in G, the quotient G/H is a locally Hausdorff, étale groupoid, and the pair (G, H) together with the embedding $H \to G$ and the conjugation action of G on H is a crossed module of topological groupoids. The corresponding C*-algebra $C^*(G, H)$ is the C*-algebra of foliations in the sense of Connes [**10**]. The C*-algebra of general (non Hausdorff) groupoids are studied in detail by Renault in [**24**].

In this paper we study the C*-algebra of 2–groupoids. We follow the footsteps of Renault in [**24**]. We study locally compact 2-groupoids and show that the notion of similarity, studied by Ramsay for groupoids [**23**], also applies to 2-groupoids and use it to show that 2-groupoids are similar to groupoid bundles. In 2-groupoids, one has two sets of units (objects and 1-arrows); thus, we should expect 2-Haar systems, consisting of invariant families of (Borel) measures indexed by these sets of units. We show that invariance of 2-Haar systems corresponds to the vertical and horizontal products on the set of 2-arrows, and construct explicit 2-Haar systems for some basic examples.

We consider quasi-invariant measures on 1-arrows and objects and build the corresponding vertical and horizontal modular functions. For a given 2-Haar system, we construct the vertical and horizontal full C*-algebras of a 2-groupoid and show that, moving from one 2-Haar system to another, these C*-algebras remain unchanged up to strong Morita equivalence.

We show that there are two natural classes of vertical and horizontal representations for a 2-groupoid on Hilbert bundles indexed by 1-arrows and objects, respectively, and make a correspondence between these and vertically or horizontally bounded representations of the corresponding C*-algebras on Hilbert spaces.

We show that the induction machinery of Mackey-Green-Rieffel applies to induce representations of certain closed 2-subgroupoids to representations of the 2-groupoid and use regular representation to build the vertical and horizontal reduced C^*-algebras of a 2-groupoid.

We show that C^*-algebras of the 2-groupoid of composable pairs for vertical and horizontal products are Morita equivalent to the C^*-algebras of 1-arrows and objects.

We study the reduced C^*-algebras of r-discrete principal 2-groupoids in more detail and find their ideals and maximal abelian subalgebras.

2. Locally compact 2-groupoids. This section is devoted to the notion of topological 2-groupoids. We adopt the algebraic setting of 2-groupoids, see for instance [**21**], to the topological framework of [**24**].

2.1. Strict 2-categories. We define a strict 2–category and describe it as a category enriched over categories. We adopt the notation and terminology of [**8**] (see also [**3**]). For two objects x and y of the first order category, we have a category of morphisms from x to y, and the composition of morphisms lifts to a bifunctor between these morphism categories (compare to the *pre-additive* category, a category enriched over abelian groups).

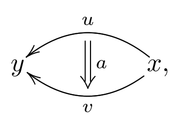
The arrows between objects

$$u : x \longrightarrow y$$

are called 1–morphisms. We write

$$x = d(u) \quad \text{and} \quad y = r(u).$$

The arrows between arrows



are called 2-morphisms (or *bigons*). Note that there are other ways to describe 2-categories using triangles or other shapes as 2-morphisms [**3**]. We write

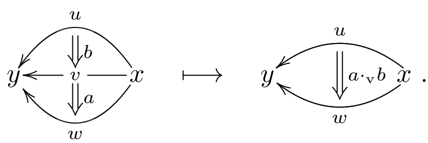
$$u = d(a), \qquad v = r(a),$$

and

$$x = d^2(a), \qquad y = r^2(a).$$

Note that the 2-morphism a is defined if $d^2(a) = dr(a)$ and $r^2(a) = rd(a)$.

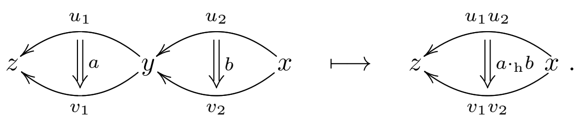
The category structure on the space of arrows $x \to y$ gives a *vertical composition* of 2-morphisms,



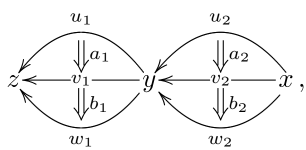
The vertical product $a \cdot_{\mathrm{v}} b$ is defined if $r(b) = d(a)$. The composition functor between the arrow categories gives a composition of 1-morphisms

$$z \xleftarrow{u} y \xleftarrow{v} x \quad \longmapsto \quad z \xleftarrow{uv} x\,,$$

which is defined if $r(v) = d(u)$, and a *horizontal composition* of 2-morphisms



The horizontal product $a \cdot_{\mathrm{h}} b$ is defined if $r^2(b) = d^2(a)$. These three compositions are assumed to be associative and unital, with the same units for the vertical and horizontal products. The horizontal and vertical products commute. Given a diagram:



composing first vertically and then horizontally, or vice versa, produces the same 2-morphism

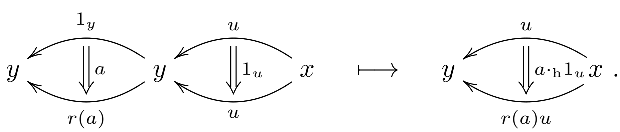
$$u_1 u_2 \Longrightarrow v_1 v_2.$$

We denote the inverse of a 1-morphism u by u^{-1} and vertical and horizontal inverses of a 2-morphism a by $a^{-\mathrm{v}}$ and $a^{-\mathrm{h}}$.

Categories form a strict 2-category with small categories as objects, functors between categories as arrows, and natural transformations between functors as 2-morphisms. The composition of 1-morphisms is the composition of functors and the vertical composition of 2-morphisms is the composition of natural transformations. The horizontal composition of 2-morphisms yields a canonical natural transformation. Another example of a strict 2-category has C*-algebras as objects, nondegenerate *-homomorphisms as 1-morphisms, and unitary intertwiners between such *-homomorphisms as 2-morphisms [**8**].

Definition 2.1. A 2-*groupoid* is a strict 2-category in which all 1-morphisms and 2-morphisms are invertible (both for the vertical and horizontal product).

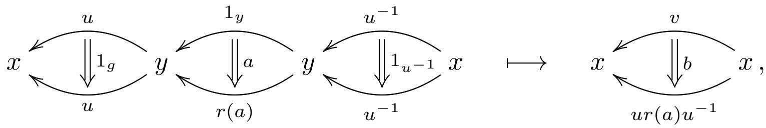
This is also called a strict 2-groupoid by some authors [**8**]. In this paper, all 2-groupoids are assumed to be small 2-categories, namely, the classes of objects and morphisms are sets. A (strict) 2*−group* is a strict 2-groupoid with a single object. Given a 2-groupoid $\mathcal{G}$, its objects $\mathcal{G}^0$ and 1-morphisms $\mathcal{G}^1$ form a groupoid, and so do the 1-morphisms and 2-morphisms $\mathcal{G}^2$ with vertical composition. We usually write $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ and denote the subset of composable elements in $\mathcal{G}^1 \times \mathcal{G}^1$ by $\mathcal{G}^{(1)}$ and the subsets of elements in $\mathcal{G}^2 \times \mathcal{G}^2$ which are vertically or horizontally composable by $\mathcal{G}^{(2\mathrm{v})}$ or $\mathcal{G}^{(2\mathrm{h})}$. We may use horizontal products with unit 2-morphisms to produce any 2-morphism from a 2-morphisms that starts at a unit 1-morphism:



The 2-morphisms starting at the identity 1-morphisms at the object x form a group $\mathcal{G}_x$ with respect to horizontal composition, and the range map is a homomorphism from the set of such 2-morphisms to the isotropy group bundle

$$H = \bigsqcup_{x \in \mathcal{G}^0} \mathcal{G}_x$$

of the groupoid $(\mathcal{G}^0, \mathcal{G}^1)$. This map is onto when $\mathcal{G}$ is 2-transitive (i.e., for each $u, v \in \mathcal{G}^1$ there is $a \in \mathcal{G}^2$ with $d(a) = u$ and $r(a) = v$). Furthermore, the groupoid $\mathcal{G}$ acts on the group bundle H by horizontal conjugation:



where $b = 1_u \cdot_{\mathrm{h}} a \cdot_{\mathrm{h}} 1_{u^{-1}}$. We may consider the map

$$r : \bigsqcup_{x \in \mathcal{G}^0} \mathcal{G}_x \longrightarrow \bigsqcup_{x \in \mathcal{G}^0} \mathcal{G}_x^x$$

and regard $(H, \mathcal{G}^1, r)$ as a *crossed module* of groupoids. Conversely, for each crossed module $(H, \mathcal{G}^1, r)$ where H is a bundle of groups, $\mathcal{G}^1$ is a groupoid and $r : H \to \mathcal{G}^1$ is a groupoid homomorphism. There is a unique 2-groupoid $\mathcal{G}$, whose isotropic group bundle is isomorphic to H, whose set of 1-morphisms is isomorphic to $\mathcal{G}^1$, and its range map realizes (after identification) as r. For crossed modules and their (strict) actions on C*-algebras we refer the reader to [**7**].

As a concrete example, consider the map

$$r_\vartheta : \mathbb{Z} \longrightarrow \mathbb{T}; \qquad n \longmapsto e^{2\pi i n \vartheta},$$

where $\vartheta \in \mathbb{R}$. Then $\mathbb{T}$ acts on $\mathbb{Z}$ by conjugation and the corresponding crossed module is the symmetry of the rotation algebra A_ϑ [**7**]. This gives a 2-groupoid with a single object, 1-morphisms $\mathbb{T}$ and 2-morphisms $\mathbb{Z} \times \mathbb{T}$ [**8**].

2.2. 2-Groupoids. Let $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ be a 2-groupoid, then $\mathcal{G}$ is called 1-principal if the map

$$(r, d) : \mathcal{G}^1 \longrightarrow \mathcal{G}^0 \times \mathcal{G}^0$$

is one-to-one, 2-principal if the map

$$(r, d) : \mathcal{G}^2 \longrightarrow \mathcal{G}^1 \times \mathcal{G}^1$$

is one-to-one, and principal if both are 1-principal and 2-principal. If we replace one-to-one with onto, we get the notions of 1-transitive, 2-transitive, and transitive. Note that 2-transitivity here is different from the property of each two nodes being connected by paths of length 2.

For each $x \in \mathcal{G}^0$ and $u \in \mathcal{G}^1$ we have the isotropy group

$$\mathcal{G}_x^x = \{u \in \mathcal{G}^1 : d(u) = r(u) = x\}$$

and 2-isotropy groups

$$\mathcal{G}_u^u = \{a \in \mathcal{G}^2 : d(a) = r(a) = u\}$$

and

$$\mathcal{G}_{u,x}^{u,x} = \{a \in \mathcal{G}^2 : d(a) = r(a) = u, d^2(a) = r^2(a) = x\}$$

with respect to the vertical and horizontal multiplication. We also have the isotropy groupoid $\mathcal{G}(x) = (\mathcal{G}^2(x), \mathcal{G}^1(x))$ where

$$\mathcal{G}^2(x) = \{a \in \mathcal{G}^2 : d^2(a) = r^2(a) = x\}$$

and

$$\mathcal{G}^1(x) = \{r(a) : a \in \mathcal{G}^2(x)\}$$

with vertical multiplication. The sets $\mathcal{G}_x$, $\mathcal{G}^y$ and $\mathcal{G}_x^y$ for $x, y \in \mathcal{G}^0$ and $\mathcal{G}_u$, $\mathcal{G}^v$ and $\mathcal{G}_u^v$ for $u, v \in \mathcal{G}^1$ are defined similarly. The equivalence relations $x \sim y$ and $u \sim v$ are defined by $\mathcal{G}_x^y \neq \emptyset$ and $\mathcal{G}_u^v \neq \emptyset$, respectively, with 1-orbits $[x]$, 2-orbits $[u]$ and orbit spaces $\mathcal{G}^0 \backslash \mathcal{G}^1$ and $\mathcal{G}^1 \backslash \mathcal{G}^2$.

Example 2.2. We give three basic examples of 2-groupoids.

(i) (Transformation 2-group.) Let S be an additive group with identity 0 acting from the right on a set U, and put

$$\mathcal{G}^1 = U \times S \quad \text{and} \quad \mathcal{G}^0 = U \times \{o\}.$$

Let T be a multiplicative group with identity 1 acting from the left on S and acting trivially from the right on U. Put $\mathcal{G}^2 = T \times U \times S$ and identify $U \times S$ $\{1\} \times U \times S$. Assume that the left action of T on S is distributive

$$t \cdot (s + s') = t \cdot s + t \cdot s',$$

for $s, s' \in S$ and $t \in T$. Define $r(u, s) = (u, 0)$ and $d(u, s) = (u \cdot s, 0)$ and partial multiplication by $(u, s) \cdot (u \cdot s, s') = (u, s + s')$ with $(u, s)^{-1} = (u \cdot s, -s)$. Also define $r(t, u, s) = (1, u, s)$ and $d(t, u, s) = (1, u, t \cdot s)$ and vertical multiplication by

$$(t, u, t' \cdot s') \cdot_{\mathrm{v}} (t', u, s') = (tt', u, s')$$

with

$$(t,u,s)^{-\mathrm{v}} = (t^{-1}, u, t\cdot s)$$

and horizontal multiplication by

$$(t,u,s)\cdot_{\mathrm{h}} (t, u\cdot s, s') = (t,u,s+s')$$

with

$$(t,u,s)^{-\mathrm{h}} = (t, u\cdot s, -s).$$

(ii) (Principal 2-groupoid.) Let X be a set and put $\mathcal{G}^2 = X^{(5)}$, $\mathcal{G}^1 = X^{(3)}$, $\mathcal{G}^0 = X$. Define

$$r(x,y,z) = z \quad \text{and} \quad d(x,y,z) = x$$

and $(x,y,z)\cdot(z,u,v) = (x,y,v)$ with $(x,y,z)^{-1} = (z,y,x)$. Define

$$r(x,y,z,u,v) = (x,u,v) \quad \text{and} \quad d(x,y,z,u,v) = (x,y,v)$$

and vertical multiplication by $(x,y,z,u,v)\cdot_{\mathrm{v}} (x,u,s,t,v) = (x,y,z,t,v)$ with $(x,y,z,u,v)^{-\mathrm{v}} = (x,u,z,y,v)$ and horizontal multiplication by $(x,y,z,u,v)\cdot_{\mathrm{h}} (v,w,s,t,r) = (x,y,s,u,r)$ with $(x,y,z,u,v)^{-\mathrm{h}} = (v,u,z,y,x)$.

(iii) (Groupoid bundle.) If $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ satisfies $d(u) = r(u)$ for each $u \in \mathcal{G}^1$ then

$$\mathcal{G} = \bigsqcup_{x\in\mathcal{G}^0} \mathcal{G}(x)$$

is a groupoid bundle.

For 2-groupoids $\mathcal{G}$ and $\mathcal{H}$, a vertical morphism $\varphi : \mathcal{G} \to \mathcal{H}$ of 2-groupoids is a pair $\varphi = (\varphi^2, \varphi^1)$ such that $\varphi^2(a\cdot_{\mathrm{v}} b) = \varphi^2(a)\cdot_{\mathrm{v}} \varphi^2(b)$ and $\varphi^1(uv) = \varphi^1(u)\varphi^1(v)$, for $a,b\in\mathcal{G}^2$ and $u,v\in\mathcal{G}^1$, whenever both sides are defined. Two vertical morphisms φ,ψ from $\mathcal{G}$ to $\mathcal{H}$ are called *similar* if there are maps $\vartheta^2 : \mathcal{G}^1 \to \mathcal{H}^2$ and $\vartheta^1 : \mathcal{G}^0 \to \mathcal{H}^1$ such that

$$d(\vartheta^2(u)) = \vartheta^1(d(u)),\ r(\vartheta^2(u)) = \vartheta^1(r(u))$$

and

$$\vartheta^2\circ r(a)\cdot_{\mathrm{v}} \varphi^2(a) = \psi^2(a)\cdot_{\mathrm{v}} \vartheta^2\circ d(a),\ \vartheta^1\circ r(u)\varphi^1(u) = \psi^1(u)\vartheta^1\circ r(u)$$

for $u\in\mathcal{G}^1$ and $a\in\mathcal{G}^2$. We write $\varphi \sim_{\mathrm{v}} \psi$. We say that $\mathcal{G}$ and $\mathcal{H}$ are v-similar if there are vertical morphisms $\varphi : \mathcal{G}\to\mathcal{H}$ and

$\psi : \mathcal{H} \to \mathcal{G}$ such that $\varphi \circ \psi \sim_{\mathrm{v}} id_{\mathcal{H}}$ and $\psi \circ \varphi \sim_{\mathrm{v}} id_{\mathcal{G}}$. The notions of horizontal morphisms and h-similarity are defined similarly and the latter is denoted by $\sim_{\mathrm{h}}$.

Definition 2.3. Let $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ be a 2-groupoid and

$$\mathcal{E} = (\mathcal{E}^1, \mathcal{E}^0) \quad \text{with } \mathcal{E}^0 \subseteq \mathcal{G}^0$$

and

$$\mathcal{E}^1 \subseteq \{u \in \mathcal{G}^1 : d(u), r(u) \in \mathcal{E}^0\}.$$

The 2-groupoid $\mathcal{G}_{\mathcal{E}} = (\mathcal{E}^2, \mathcal{E}^1, \mathcal{E}^0)$, where $\mathcal{E}^2 = \{a \in \mathcal{G}^2 : d(a), r(a) \in \mathcal{E}^1\}$, is called the *restriction* of $\mathcal{G}$ to $\mathcal{E}$. We say that $\mathcal{E}$ is *full* if $\mathcal{E}^0$ meets each equivalence class in $\mathcal{G}^0$ and $\mathcal{E}^1$ meets each equivalence class in $\mathcal{G}^1$.

The next lemma is proved by Ramsay for groupoids [**23,** Theorem 1.7].

Lemma 2.4. *If $\mathcal{E}$ is full, then $\mathcal{G}_{\mathcal{E}} \sim_{\mathrm{v}} \mathcal{G}$.*

Proof. Since $\mathcal{E}^0$ meets each equivalence class in $\mathcal{G}^0$, for each $x \in \mathcal{G}^0$ and $y \in \mathcal{E}^0$ there is $\vartheta^1(x) \in \mathcal{G}^1$ such that $d(\vartheta^1(x)) = x$, $r(\vartheta^1(x)) = y$, and $\vartheta^1(z)) = z$ for $z \in \mathcal{E}^0$. Consider the canonical injection $\varphi^1 : \mathcal{G}^1_{\mathcal{E}} \hookrightarrow \mathcal{G}^1$ and define $\psi^1 : \mathcal{G}^1 \hookrightarrow \mathcal{G}^1_{\mathcal{E}}$ by $\psi^1(u) = \vartheta^1(r(u))u\vartheta^1(d(u))^{-1}$. Also as $\mathcal{E}^1$ meets each equivalence class in $\mathcal{G}^1$, for each $u \in \mathcal{G}^1$, $v = \vartheta^1(r(u))$ and $w = \vartheta^1(d(u))$, there is $\vartheta^2(u) \in \mathcal{G}^2$ such that $d(\vartheta^2(u)) = w$, $r(\vartheta^2(u)) = v$, and $\vartheta^2(e)) = e$ for $e \in \mathcal{E}^1$. Consider the canonical injection $\varphi^2 : \mathcal{G}^2_{\mathcal{E}} \hookrightarrow \mathcal{G}^2$ and define $\psi^2 : \mathcal{G}^2 \hookrightarrow \mathcal{G}^2_{\mathcal{E}}$ by $\psi^2(a) = \vartheta^2(r(a)) \cdot_{\mathrm{v}} a \cdot_{\mathrm{v}} \vartheta^2(d(a))^{-v}$. Then it is straightforward to check that $\varphi = (\varphi^2, \varphi^1)$ and $\psi = (\psi^2, \psi^1)$. We have $\varphi \circ \psi = \psi \sim_{\mathrm{v}} id = \psi \circ \varphi$. □

Corollary 2.5. *Every 2-groupoid is v-similar to a groupoid bundle. A 2-groupoid is v-similar to a groupoid if and only if its set of objects consists of only one equivalence class.*

Proof. Let $\mathcal{E}^0$ contain one element from each equivalence class in $\mathcal{G}^0$ and $\mathcal{E}_x = (\mathcal{G}^x_x, \{x\})$ for $x \in \mathcal{E}^0$. Then

$$\mathcal{G} \sim_{\mathrm{v}} \bigsqcup_{x \in \mathcal{E}^0} \mathcal{G}_{\mathcal{E}_x},$$

and $\mathcal{G}_{\mathcal{E}_x}$ is isomorphic to the isotropy groupoid $\mathcal{G}(x)$. □

2.3. Topological 2-groupoids and 2-Haar systems. In this section, we define locally compact 2-groupoids and introduce the related Borel measures. We associate to each locally compact 2-groupoid a pair of (quasi-invariant) Borel measures on the objects and 1-morphisms, two vertical and horizontal Haar systems and build the corresponding modular functions. Throughout this section, we identify $\mathcal{G}^0$ with a subset of $\mathcal{G}^1$ and $\mathcal{G}^1$ with a subset of $\mathcal{G}^2$ by identifying $x \in \mathcal{G}^0$ with 1_x and $u \in \mathcal{G}^1$ with 1_u.

Definition 2.6. A topological 2-groupoid is a 2-groupoid $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ and a topology on $\mathcal{G}^2$, such that:

(i) The maps $u \mapsto u^{-1}$ and $a \mapsto a^{-\mathrm{v}}$, $a \mapsto a^{-\mathrm{h}}$ are continuous on $\mathcal{G}^1$ and $\mathcal{G}^2$.
(ii) The maps $(u, v) \mapsto uv$ and $(a, b) \mapsto a \cdot_{\mathrm{v}} b$, $(a, b) \mapsto a \cdot_{\mathrm{h}} b$ are continuous on their domains.

Lemma 2.7. *For any topological* 2*-groupoid,* $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$*,*

(i) *the maps* $u \mapsto u^{-1}$ *and* $a \mapsto a^{-\mathrm{v}}$, $a \mapsto a^{-\mathrm{h}}$ *are homeomorphisms on* $\mathcal{G}^1$ *and* $\mathcal{G}^2$*.*
(ii) *The source and range maps* d, r *are continuous on* $\mathcal{G}^1$ *and* $\mathcal{G}^2$*.*
(iii) *If* $\mathcal{G}^1$ *is Hausdorff,* $\mathcal{G}^0 \subseteq \mathcal{G}^1$ *is closed, and if* $\mathcal{G}^2$ *is Hausdorff,* $\mathcal{G}^0 \subseteq \mathcal{G}^1$*,* $\mathcal{G}^1 \subseteq \mathcal{G}^2$ *and* $\mathcal{G}^0 \subseteq \mathcal{G}^2$ *are closed.*
(iv) *If* $\mathcal{G}^0$ *is Hausdorff,* $\mathcal{G}^{(1)} \subseteq \mathcal{G}^1 \times \mathcal{G}^1$ *is closed, and if* $\mathcal{G}^1$ *is Hausdorff,* $\mathcal{G}^{(2\mathrm{v})} \subseteq \mathcal{G}^2 \times \mathcal{G}^2$ *and* $\mathcal{G}^{(2\mathrm{h})} \subseteq \mathcal{G}^2 \times \mathcal{G}^2$ *are closed.*
(v) *For the range equivalence* $a \sim_r b$ *defined by* $r(a) = r(b)$*, the orbit space* $\mathcal{G}^2/{\sim_r}$ *is homeomorphic to* $\mathcal{G}^1$*. Similarly* $\mathcal{G}^1/{\sim_r}$ *is homeomorphic to* $\mathcal{G}^0$*.*

Proof.

(i) These maps are their own inverses.

(ii) We have $d(u) = u^{-1}u$, $r(u) = uu^{-1}$ and $d(a) = a^{-\mathrm{v}} \cdot_{\mathrm{v}} a$, $r(a) = a \cdot_{\mathrm{v}} a^{-\mathrm{v}}$.

(iii), (iv). These follow from (ii) and uniqueness of limit in Hausdorff spaces.

(v) The map $[a] \mapsto r(a)$ is a homeomorphism from $\mathcal{G}^2/\!\sim_r$ onto $\mathcal{G}^1$. It is the same for $\mathcal{G}^1$. □

Definition 2.8. A locally compact 2-groupoid is a topological 2-groupoid $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ such that $\mathcal{G}^0$ and $\mathcal{G}^1$ are Hausdorff Borel subsets of $\mathcal{G}^2$ and every point of $\mathcal{G}^2$ has an open, relatively compact, Hausdorff neighborhood.

For the rest of this paper, $\mathcal{G}$ is a locally compact 2-groupoid. We put

$$C_c(\mathcal{G})=\{f:\mathcal{G}^2 \to \mathbb{C}: f \text{ is continuous and } \operatorname{supp}(f) \subseteq \mathcal{G}^2 \text{ is compact}\},$$

where $\operatorname{supp}(f)$ is the complement of the union of open Hausdorff subsets of $\mathcal{G}^2$ on which f vanishes. By assumption, $\mathcal{G}^2$ is a union of compact Hausdorff sets K, and the algebraic direct limit $C_c(\mathcal{G}) = \lim_{\to} C(K)$ is endowed with an inductive limit topology.

Definition 2.9. Let $\mathcal{G}$ be a locally compact 2-groupoid. A continuous left 2-Haar system on $\mathcal{G}$ consists of two families of positive Borel measures $\{\lambda_{\mathrm{v}}^u\}$and $\{\lambda_{\mathrm{h}}^x\}$on $\mathcal{G}^2$, where u ranges over $\mathcal{G}^1$ and x ranges over $\mathcal{G}^0$, such that

(i) $\operatorname{supp}(\lambda_{\mathrm{v}}^u) = \mathcal{G}^u$ and $\operatorname{supp}(\lambda_{\mathrm{h}}^x) = \mathcal{G}^x$, for each $u \in \mathcal{G}^1$ and $x \in \mathcal{G}^0$.
(ii) For any $f \in C_c(\mathcal{G})$, the map $u \mapsto \int f d\lambda_{\mathrm{v}}^u$ is continuous on $\mathcal{G}^1$, and the map $x \mapsto \int f d\lambda_{\mathrm{h}}^x$ is continuous on $\mathcal{G}^0$.
(iii) For any $f \in C_c(\mathcal{G})$,

$$\int f(a \cdot_{\mathrm{v}} b)\, d\lambda_{\mathrm{v}}^{d(a)}(b) = \int f(b) d\lambda_{\mathrm{v}}^{r(a)}(b)$$

and

$$\int f(a \cdot_{\mathrm{h}} b)\, d\lambda_{\mathrm{h}}^{d^2(a)}(b) = \int f(b) d\lambda_{\mathrm{h}}^{r^2(a)}(b).$$

Note that, identifying $1_u, 1_v \in \mathcal{G}^2$ with $u, v \in \mathcal{G}^1$, it follows from the first equality in (iii) that

$$\int f(uv)\, d\lambda_{\mathrm{v}}^{d(u)}(v) = \int f(v)\, d\lambda_{\mathrm{v}}^{r(u)}(v).$$

Proposition 2.10. *If $\mathcal{G}$ has a continuous 2-Haar system, we have continuous surjections*

$$\lambda_{\mathrm{v}} : C_c(\mathcal{G}^2) \longrightarrow C_c(\mathcal{G}^1); \quad f \longmapsto \lambda_{\mathrm{v}}(f),\, \lambda_{\mathrm{v}}(f)(u) = \int f\, d\lambda_{\mathrm{v}}^u,$$

and

$$\lambda_{\mathrm{h}} : C_c(\mathcal{G}^2) \longrightarrow C_c(\mathcal{G}^0); \quad f \longmapsto \lambda_{\mathrm{h}}(f),\, \lambda_{\mathrm{h}}(f)(x) = \int f\, d\lambda_{\mathrm{h}}^x.$$

Moreover, the maps $r : \mathcal{G}^2 \to \mathcal{G}^1$, $r : \mathcal{G}^1 \to \mathcal{G}^0$ and $r^2 : \mathcal{G}^2 \to \mathcal{G}^0$ are open, and the associated equivalence relations on $\mathcal{G}^1$ and $\mathcal{G}^0$ are open.

Example 2.11. The 2-Haar systems of the above examples are as follows:

(i) (Transformation 2-group.) Let S and T be locally compact groups with Haar measures λ_S and λ_T acting continuously on a locally compact Hausdorff space, U, as in Example 2.2 (i) and $\mathcal{G}^2 = T \times U \times S$. Then the vertical and horizontal left Haar systems on $\mathcal{G}$ are given by:

$$\lambda_{\mathrm{v}}^{(1,u,s)} = \lambda_T \times \delta_u \times \lambda_1,$$
$$\lambda_{\mathrm{h}}^{(1,u,0)} = \lambda_2 \times \delta_u \times \lambda_S$$
$$(u \in U, s \in S),$$

where λ_1 and λ_2 are arbitrary Borel measures with full support on S and T, respectively.

(ii) (Principal 2-groupoid.) Let X be a locally compact Hausdorff space and $\mathcal{G}^2 = X^{(5)}$. Consider the homeomorphism

$$d : \mathcal{G}^{(x,u,v)} \longrightarrow X^{(2)}; \quad (x,y,z,u,v) \longmapsto (y,z).$$

Let α be any Borel measure on $X^{(2)}$ with full support such that, for each $f \in C_c(\mathcal{G})$, the map,

$$(x,u,v) \longmapsto \int f(x,y,z,u,v)\, d\alpha(y,z),$$

is continuous on $X^{(3)}$. Then

$$\int f\,d\lambda_{\rm v}^{(x,u,v)} = \int f(x,y,z,u,v)\,d\alpha(y,z)$$

defines a vertical left Haar system. The horizontal case is treated similarly.

(iii) (Groupoid bundle.) Let

$$\mathcal{G} = \bigsqcup_{x\in\mathcal{G}^0} \mathcal{G}(x)$$

be a locally compact groupoid bundle. The 2-Haar system is essentially unique (if it exists), that is, any two systems $\{\lambda_{\rm v}^u, \lambda_{\rm h}^x\}$ and $\{\sigma_{\rm v}^u, \sigma_{\rm h}^x\}$ are related via $\lambda_{\rm v}^u = h(u)\sigma_{\rm v}^u$ and $\lambda_{\rm h}^x = k(x)\sigma_{\rm h}^x$, where $h \in C(\mathcal{G}^1)_+$ and $k \in C(\mathcal{G}^0)_+$.

Given a locally compact groupoid $\mathcal{G}$ which has (and all of its locally compact subgroupoids have) a left Haar system, let $\mathfrak{S}$ be the set of all locally compact subgroupoids of $\mathcal{G}$. Then $\mathfrak{S}$ is a locally compact, Hausdorff space in the Fell topology [**11**], and

$$\underline{\mathcal{G}} = \{(\mathcal{H}, x) : \mathcal{H} \in \mathfrak{S}, x \in \mathcal{H}\} \subseteq \mathfrak{S} \times \mathcal{G}$$

is a groupoid bundle with $\underline{\mathcal{G}}(x) = \mathcal{H}$, for $x \in \mathcal{H}$. For the vertical product,

$$(\mathcal{H}, x) \cdot_{\rm v} (\mathcal{H}, y) = (\mathcal{H}, xy) \qquad (\mathcal{H}, x)^{-{\rm v}} = (\mathcal{H}, x^{-1})$$
$$(x \in \mathcal{H}, (x,y) \in \mathcal{H}^{(2)}),$$

and left Haar system $\lambda^{\mathcal{H}}$ on $\mathcal{H}$, $\lambda_{\rm v}^{(\mathcal{H},u)} = \lambda^{\mathcal{H}}$, $u \in \mathcal{H}^{(0)}$, defines a vertical left Haar system on $\underline{\mathcal{G}}$. Similarly, for the horizontal product,

$$(\mathcal{H}, u) \cdot_{\rm v} (\mathcal{H}, u) = (\mathcal{H}, u) \qquad (\mathcal{H}, u)^{-{\rm h}} = (\mathcal{H}, u) \quad (u \in \mathcal{H}^{(0)}),$$

$\lambda_{\rm h}^{(\mathcal{H},u)} = \lambda^{\mathcal{H}}$, $u \in \mathcal{H}^{(0)}$, is a horizontal left Haar system on $\underline{\mathcal{G}}$.

Definition 2.12. A locally compact 2-groupoid $\mathcal{G}$ is called r-discrete if $\mathcal{G}^0 \subseteq \mathcal{G}^1$ and $\mathcal{G}^1 \subseteq \mathcal{G}^2$ are open.

Lemma 2.13. *If $\mathcal{G}$ is r-discrete, then*:

(i) *for each $u \in \mathcal{G}^1$ and $x \in \mathcal{G}^0$, $\mathcal{G}^u$ and $\mathcal{G}^x$ are open in $\mathcal{G}^2$*;

(ii) *if a continuous* 2*-Haar system exists, it is essentially the system of counting measures. In this case,* $d, r : \mathcal{G}^2 \to \mathcal{G}^1$, $d, r : \mathcal{G}^1 \to \mathcal{G}^0$ *and* $d^2, r^2 : \mathcal{G}^2 \to \mathcal{G}^0$ *are local homeomorphisms.*

Proof.

(i) If $a \in \mathcal{G}^u_v$, then $b \mapsto a \cdot_{\mathrm{v}} b$ is a homeomorphism from $\mathcal{G}^v$ to $\mathcal{G}^u$ and $\{v\} = \mathcal{G}^v \cap \mathcal{G}^1$ is open in $\mathcal{G}^v$; hence, $\{a\} = \{a \cdot_{\mathrm{v}} 1_v\}$ is open in $\mathcal{G}^u$. The same argument works for $\mathcal{G}^x$.

(ii) Every point in $\mathcal{G}^u$ has a positive λ^u_{v} measure and, replacing λ^u_{v} with $\alpha_u \lambda^u_{\mathrm{v}}$, where $\alpha_u = (\lambda_{\mathrm{v}}(\xi_{\mathcal{G}^1})(u))^{-1}$, we may assume that $\lambda^u_{\mathrm{v}}(\{u\}) = 1$. Therefore $\lambda^u_{\mathrm{v}}(\{a\}) = \lambda^v_{\mathrm{v}}(\{v\}) = 1$, for each $a \in \mathcal{G}^u_v$. Also, a compact neighborhood V of a meets $\mathcal{G}^u$ in finitely many points, and we may assume that $\mathcal{G}^u \cap V = \{a\}$, that gives $\lambda^{r(a)}_{\mathrm{v}}(V) = 1$. By continuity, we may assume that $\lambda^u_{\mathrm{v}}(V) = 1$, for each $u \in r(V)$, that is, $r : V \to \mathcal{G}^1$ is injective. The same argument works for r^2 using λ^x_{h}. □

Definition 2.14. Let $\mathcal{G}$ be a locally compact 2-groupoid. A subset s of $\mathcal{G}^2$ is called a $\mathcal{G}^1$-set if the restrictions of d and r to s are one-to-one. This is equivalent to $s \cdot_{\mathrm{v}} s^{-1}$ and $s^{-1} \cdot_{\mathrm{v}} s$ being contained in $\mathcal{G}^1$. A subset s of $\mathcal{G}^2$ is called a $\mathcal{G}^0$-set if the restrictions of d^2 and r^2 to s are one-to-one, or equivalently, $s \cdot_{\mathrm{h}} s^{-1}$ and $s^{-1} \cdot_{\mathrm{h}} s$ are contained in $\mathcal{G}^0$.

In the above definition, the products are considered as products of sets. Note that both $\mathcal{G}^1$-sets and $\mathcal{G}^0$-sets form an inverse semigroup and, for each $a \in \mathcal{G}^2$ and $\mathcal{G}^1$-set s, if $d(a) \in r(s)$ (respectively $r(a) \in d(s)$) then the set $a \cdot_{\mathrm{v}} s$ (respectively $s \cdot_{\mathrm{v}} a$) is a singleton, and so defines an element of $\mathcal{G}^2$ denoted again by $a \cdot_{\mathrm{v}} s$ (respectively $s \cdot_{\mathrm{v}} a$). Also, there is a map $r(s) \to d(s)$; $u \mapsto u \cdot s := d(u \cdot_{\mathrm{v}} s)$, satisfying $u \cdot (s \cdot_{\mathrm{v}} t) = (u \cdot s) \cdot_{\mathrm{v}} t$, for $\mathcal{G}^1$-sets s, t. Similarly, for $a \in \mathcal{G}^2$ and $\mathcal{G}^0$-set s with $d^2(a) \in r^2(s)$ (respectively $r^2(a) \in d^2(s)$) the element $a \cdot_{\mathrm{h}} s$ (respectively $s \cdot_{\mathrm{v}} a$) of $\mathcal{G}^2$ is defined, and the map $r^2(s) \to d^2(s)$; $x \mapsto x \cdot s := d^2(x \cdot_{\mathrm{h}} s)$, satisfies $x \cdot (s \cdot_{\mathrm{h}} t) = (x \cdot s) \cdot_{\mathrm{h}} t$, for $\mathcal{G}^0$-sets s, t.

Proposition 2.15. *For a locally compact* 2*-groupoid* $\mathcal{G}$*, the following are equivalent*:

(i) $\mathcal{G}$ *is* r*-discrete and has a continuous left* 2*-Haar system,*

(ii) *the maps* $r : \mathcal{G}^2 \to \mathcal{G}^1$ *and* $r^2 : \mathcal{G}^2 \to \mathcal{G}^0$ *are local homeomorphisms,*

(iii) *the product maps* $\mathcal{G}^{(1)} \to \mathcal{G}^1$, $\mathcal{G}^{(2\mathrm{v})} \to \mathcal{G}^1$ *and* $G^{(2\mathrm{h})} \to \mathcal{G}^0$ *are local homeomorphisms,*

(iv) $\mathcal{G}^2$ *has an open basis consisting of open* G^1*-sets and one consisting of open* $\mathcal{G}^0$*-sets.*

Proof.

(i) $\Rightarrow$ (ii). Lemma 4.8 (ii).

(ii) $\Rightarrow$ (iii). Given $(a, b) \in \mathcal{G}^{(2\mathrm{v})}$, choose compact neighborhoods U and V of a and b on which, respectively, r and d are homeomorphisms. Then the vertical product is one-to-one on the compact neighborhood $U \times V$ of (a, b). The product map on $\mathcal{G}^{(1)}$ is just the restriction of the vertical product. The argument for the horizontal product is similar.

(iii) $\Rightarrow$ (iv). For an open neighborhood U of $a \in \mathcal{G}^2$, find open sets V, W with $a \in V \subseteq U$ and $a^{-\mathrm{v}} \in W \subseteq U^{-\mathrm{v}}$ such that the vertical product is one-to-one on $V \times W$. Then $V \cap W^{-\mathrm{v}}$ is an open $\mathcal{G}^1$-set inside U. The same argument works for $\mathcal{G}^0$-sets.

(iv) $\Rightarrow$ (i). For $u \in \mathcal{G}^1$, choose an open $\mathcal{G}^1$-set s such that $u \in r(s)$ and $s \cdot_{\mathrm{v}} s^{-\mathrm{v}}$ are open in $\mathcal{G}^2$. Let λ_{v}^u be the counting measure on $\mathcal{G}^u$. Write any $f \in C_c(\mathcal{G})$ as a finite sum of continuous functions supported on open $\mathcal{G}^1$-sets, and observe that, for a continuous function g supported on an open $\mathcal{G}^1$-set s,

$$\lambda_{\mathrm{v}}(g)(u) = \lambda_{\mathrm{v}}^u(g) = \sum_{a \in \mathcal{G}^u \cap s} g(a) = g(u \cdot_{\mathrm{v}} s);$$

thus, $\lambda_{\mathrm{v}}(g)$ is continuous. The same argument works for $\lambda_{\mathrm{h}}(g)$. □

3. C^*-algebras of 2-groupoids. This is the main section of the paper. Here, we construct full and reduced C^*-algebras of a locally compact 2-groupoid. Our construction follows **[24]** closely.

3.1. Quasi-invariant measures. Let $\mathcal{G}$ be a locally compact 2-groupoid with continuous left 2-Haar system $\{\lambda_{\mathrm{v}}^u\}$and $\{\lambda_{\mathrm{h}}^x\}$, and let $\{\lambda_{\mathrm{v}u}\}$ and $\{\lambda_{\mathrm{h}x}\}$ be the images of this system under the inverse maps $a \mapsto a^{-\mathrm{v}}$ and $a \mapsto a^{-\mathrm{h}}$. Then the latter is a continuous right 2-Haar system.

Borel measures μ^1 and μ^0 on $\mathcal{G}^1$ and $\mathcal{G}^0$ induce measures

$$\nu_{\rm v} = \int \lambda_{\rm v}^u d\mu^1(u), \qquad \nu_{\rm h} = \int \lambda_{\rm h}^x d\mu^0(x)$$

with images

$$\nu_{\rm v}^{-1} = \int \lambda_{{\rm v}u} d\mu^1(u), \qquad \nu_{\rm h}^{-1} = \int \lambda_{{\rm h}x}\, d\mu^0(x)$$

and induced measures

$$\nu_{\rm v}^2 = \int \lambda_{\rm v}^u \times \lambda_{{\rm v}u} d\mu^1(u), \qquad \nu_{\rm h}^2 = \int \lambda_{\rm h}^x \times \lambda_{{\rm h}x}\, d\mu^0(x).$$

Definition 3.1. A Borel measure μ^1 on $\mathcal{G}^1$ is called quasi-invariant if $\nu_{\rm v} \sim \nu_{\rm v}^{-1}$. A Borel measure μ^0 on $\mathcal{G}^0$ is called quasi-invariant if $\nu_{\rm h} \sim \nu_{\rm h}^{-1}$.

By the uniqueness of the Radon-Nikodym derivative we have the following result which defines vertical and horizontal modular functions. We put $\nu_{{\rm v}0} = D_{\rm v}^{1/2}\nu_{\rm v}$ and $\nu_{{\rm h}0} = D_{\rm h}^{1/2}\nu_{\rm h}$.

Proposition 3.2 (Modular functions)**.** *For quasi-invariant measure μ^1 on $\mathcal{G}^1$ there is a locally $\nu_{\rm v}$-integrable positive function $D_{\rm v}$ such that $\nu_{\rm v} = D_{\rm v}\nu_{\rm v}^{-1}$ and*

(i) $D_{\rm v}(a \cdot_{\rm v} b) = D_{\rm v}(a)D_{\rm v}(b)$ *($\nu_{\rm v}^2$-almost everywhere), $D_{\rm v}(a^{-{\rm v}}) = D_{\rm v}(a)^{-1}$ ($\nu_{\rm v}$-almost everywhere),*

(ii) *if $\mu'^1 = g^1\mu^1$ where g^1 is positive and locally μ^1-integrable, then $D'_{\rm v} = (g^1 \circ r)D_{\rm v}(g^1 \circ d)^{-1}$ satisfies $\nu'_{\rm v} = D'_{\rm v}\nu'^{-1}_{\rm v}$.*

Similarly, for quasi-invariant measure μ^0 on $\mathcal{G}^0$, there is a locally $\nu_{\rm v}$-integrable positive function $D_{\rm h}$ such that $\nu_{\rm h} = D_{\rm h}\nu_{\rm h}^{-1}$ and relations similar to (i) *and* (ii) *above hold.*

For locally compact topological spaces X and Y and surjective map $p : X \to Y$, a measure class $\mathcal{C}$ on X and (probability) measure $\mu \in \mathbb{C}$, $p_*\mathbb{C}$ is the measure class of $p_*\mu := \mu \circ p^{-1}$. A pseudo-image of $\mu \in \mathbb{C}$ is a measure in $p_*\mathbb{C}$. If (X, μ) and (Y, ν) are measure spaces and $s : X \to Y; x \mapsto x \cdot s$ is a bi-measurable bijection, then μ lifts to a measure $\mu \cdot s$ on Y defined by

$$\int f(y)\, d\mu \cdot s(y) = \int f(x \cdot s)\, d\mu(x) \quad (f \in C_c(Y))$$

and, when $\mu \cdot s \ll \nu$, we denote the corresponding Radon-Nikodym derivative by $d\mu \cdot s/d\nu$ and say that s is non singular if it induces an isomorphism of the corresponding measure algebras [**24,** page 29].

For quasi-invariant measures μ^1 and μ^0 subsets $A^1 \subseteq \mathcal{G}^1$ and $A^0 \subseteq \mathcal{G}^0$ are called *almost invariant* if $r(a) \in A^1$ is equivalent to $d(a) \in A^1$ ($\nu_{\rm v}$-almost everywhere) and $r^2(a) \in A^0$ is equivalent to $d^2(a) \in A^0$ ($\nu_{\rm h}$-almost everywhere). The measures μ^1 and μ^0 are called *ergodic* if every almost invariant set is null or co-null. For arbitrary Borel measures μ^1 and μ^0 the pseudo-images $[\mu^1]$ and $[\mu^0]$ of $\nu_{\rm v}$ and $\nu_{\rm h}$ under d and d^2 are quasi-invariant and in the same measure class as μ^1 and μ^0 if and only if μ^1 and μ^0 are quasi-invariant [**24,** 1.3.6]. If $\alpha_{\rm v}^u$ and $\alpha_{\rm h}^x$ are pseudo-images of $\lambda_{\rm v}^u$ and $\lambda_{\rm h}^x$, then the measure class of $\alpha_{\rm v}^u$ and $\alpha_{\rm h}^x$ depends only on the orbits of u and x in $\mathcal{G}^1$ and $\mathcal{G}^0$, $\alpha_{\rm v}^u$ and $\alpha_{\rm h}^x$ are ergodic, and every quasi-invariant pair carried by the orbits of u and x are equivalent to $\alpha_{\rm v}^u$ and $\alpha_{\rm h}^x$ [**24,** 1.3.8].

Let μ^1 be a Borel measure on $\mathcal{G}^1$ with induced measure $\nu_{\rm v}$, and let s be a $\nu_{\rm v}$-measurable $\mathcal{G}^1$-set. The measure $\nu_{\rm v}$ is called quasi-invariant under s if the map $a \mapsto a \cdot_{\rm v} s^{-{\rm v}}$ is non singular from $(d^{-1}(d(s)), \nu_{\rm v})$ to $(d^{-1}(r(s)), \nu_{\rm v})$. Let $\delta_{\rm v}(\cdot, s) = d(\nu_{\rm v} \cdot s^{-{\rm v}})/d\nu_{\rm v}$ be the corresponding Radon-Nikodym derivative. The measure μ^1 is called quasi-invariant under s if the map $u \mapsto u \cdot s^{-{\rm v}}$ is non singular from $(d(s), \mu^1)$ to $(r(s), \mu^1)$ and $\Delta_{\rm v}(\cdot, s) = d(\mu^1 \cdot s^{-{\rm v}})/d\mu^1$ is the corresponding Radon-Nikodym derivative. For a Borel measure μ^0 on $\mathcal{G}^0$ the horizontal functions $\delta_{\rm h}$ and $\Delta_{\rm h}$ are defined similarly. An argument similar to [**24,** 1.3.19, 1.3.20] proves the following.

Lemma 3.3. *Under the above quasi-invariance properties,*

(i) $\delta_{\rm v}(s(a), s) = \delta_{\rm v}(a, s)$, ($\nu_{\rm v}$-*almost everywhere* $a \in d^{-1}(r(s))$),
(ii) $\delta_{\rm v}(u, s) = D_{\rm v}(u \cdot s)\Delta_{\rm v}(u, s)$ (μ^1-*almost everywhere* $u \in r(s)$),

and the same for $\delta_{\rm h}$ *and* $\Delta_{\rm h}$.

A $\mathcal{G}^1$-set s is said to be Borel (continuous) if the restrictions of d and r to s are Borel isomorphisms (homeomorphisms) onto a Borel (open) subset of G^1. It is called non singular if there is a Borel (continuous) positive function $\delta_{\rm v}(\cdot, s)$ on $r(s)$, bounded above and below on compact subsets of $\mathcal{G}^1$, such that $\delta_{\rm v}(d(a), s) = d(\lambda_{\rm v}^u \cdot s^{-{\rm v}})/d\lambda_{\rm v}^u(a)$ for every $u \in \mathcal{G}^1$ and $\lambda_{\rm v}^u$-almost everywhere $a \in d^{-1}(r(s))$. A non singular Borel $\mathcal{G}^1$-set s is also non singular with respect to the induced measure $\nu_{\rm v}$ of any Borel measure μ^1 on $\mathcal{G}^1$ and $\delta_{\rm v}(d(a), s) = d(\nu_{\rm v} \cdot s^{-{\rm v}})/d\nu_{\rm v}(a)$ for $\nu_{\rm v}$–almost everywhere $a \in d^{-1}(r(s))$. The set of non singular Borel G^1-sets also form an inverse semigroup,

$$\delta_{\rm v}(u, s \cdot_{\rm v} t) = \delta_{\rm v}(u, s)\delta_{\rm v}(u \cdot s, t) \quad (u \in r(s \cdot_{\rm v} t)),$$

and

$$\delta_{\rm v}(u, s^{-{\rm v}}) = \delta_{\rm v}(u \cdot s^{-{\rm v}}, s)^{-1} \quad (u \in d(s)).$$

3.2. Full C^*-algebras. In this section, we study representation theory of 2-groupoids and the corresponding C^*-algebras. Representation theory of topological groupoids is well studied **[22, 24]** and is shown to be much more involved than that of topological groups, but also resembling some similarities **[1, 2]**.

Let $\mathcal{G}$ be a locally compact 2-groupoid with a fixed continuous left 2-Haar system $\{\lambda_{\rm v}^u\}$and $\{\lambda_{\rm h}^x\}$ for $f, g \in C_c(\mathcal{G})$ put

$$f *_{\rm v} g(a) = \int f(a \cdot_{\rm v} b) g(b^{-{\rm v}}) d\lambda_{\rm v}^{d(a)}(b), \quad f_{\rm v}^*(a) = \overline{f}(a^{-{\rm v}}),$$

and

$$f *_{\rm h} g(a) = \int f(a \cdot_{\rm h} b) g(b^{-{\rm h}}) d\lambda_{\rm h}^{d^2(a)}(b), \quad f^{*_{\rm h}}(a) = \overline{f}(a^{-{\rm h}}),$$

for each $a \in \mathcal{G}^2$.

Lemma 3.4. *$C_c(\mathcal{G})$ is a topological $*$-algebra with respect to both vertical and horizontal convolutions and involutions, denoted by $C_{c{\rm v}}(\mathcal{G})$ and $C_{c{\rm h}}(\mathcal{G})$, respectively.*

Proof. An argument like **[24,** 2.1.1] shows that the above operations are well defined and continuous in the inductive limit topology and

$\operatorname{supp}(f*_{\mathrm{v}}g)=\operatorname{supp}(f)\cdot_{\mathrm{v}}\operatorname{supp}(g)$ and $\operatorname{supp}(f^*_{\mathrm{v}})=\operatorname{supp}(f)^{-\mathrm{v}}$. Also,

$$\begin{aligned}
f*_{\mathrm{v}}(g*_{\mathrm{v}}h)(a) &= \int f(a\cdot_{\mathrm{v}} b)(g*_{\mathrm{v}}h)(b^{-\mathrm{v}})\,d\lambda_{\mathrm{v}}^{d(a)}(b)\\
&= \int\int f(a\cdot_{\mathrm{v}} b)g(b^{-\mathrm{v}}\cdot_{\mathrm{v}} c)h(c^{-\mathrm{v}})\,d\lambda_{\mathrm{v}}^{r(b)}(c)\,d\lambda_{\mathrm{v}}^{d(a)}(b)\\
&= \int\int f(a\cdot_{\mathrm{v}} c\cdot_{\mathrm{v}} b)g(b^{-\mathrm{v}})h(c^{-\mathrm{v}})\,d\lambda_{\mathrm{v}}^{d(c)}(b)\,d\lambda_{\mathrm{v}}^{d(a)}(c)\\
&= \int (f*_{\mathrm{v}}g)(a\cdot_{\mathrm{v}} c)h(c^{-\mathrm{v}})\,d\lambda_{\mathrm{v}}^{d(a)}(c)\\
&= (f*_{\mathrm{v}}g)*_{\mathrm{v}}h(a),
\end{aligned}$$

and $f^{**_{\mathrm{v}}}=f$, $(f*_{\mathrm{v}}g)^*_{\mathrm{v}}=f^*_{\mathrm{v}}*_{\mathrm{v}}g^*_{\mathrm{v}}$, for each $a\in\mathcal{G}^2$ and $f,g,h\in C_c(\mathcal{G})$. The same equalities hold for the horizontal operations. $\square$

A representation of $C_{c\mathrm{v}}(\mathcal{G})$ on a Hilbert space H is a $*$-homomorphism $L:C_{c\mathrm{v}}(\mathcal{G})\to B(H)$ which is continuous in the inductive limit topology on the domain and weak operator topology on the range. We have the same definition for representations of $C_{c\mathrm{h}}(\mathcal{G})$. In this section, we only work with non-degenerate representations.

For $f\in C_{c\mathrm{v}}(\mathcal{G})$, put

$$\|f\|_{\mathrm{v,r}}=\sup_{u\in\mathcal{G}^1}\int|f|\,d\lambda_{\mathrm{v}}^u,\qquad \|f\|_{\mathrm{v,d}}=\sup_{u\in\mathcal{G}^1}\int|f|\,d\lambda_{\mathrm{v}u}$$

and $\|f\|_{\mathrm{v}}=\max\{\|f\|_{\mathrm{v,r}},\|f\|_{\mathrm{v,d}}\}$. This is a norm on $C_{c\mathrm{v}}(\mathcal{G})$ defining a topology coarser than the inductive limit topology. We say that a representation L is v-bounded if there is a constant $M>0$ such that $\|L(f)\|\le M\|f\|_{\mathrm{v}}$, for each $f\in C_{c\mathrm{v}}(\mathcal{G})$. We put $\|f\|^{\mathrm{v}}=\sup_L\|L(f)\|$, where the supremum is taken over all v-bounded non-degenerate representations. This is a C^*-seminorm on $C_{c\mathrm{v}}(\mathcal{G})$ and $\|f\|^{\mathrm{v}}\le\|f\|_{\mathrm{v}}$, for each $f\in C_{c\mathrm{v}}(\mathcal{G})$. The norms $\|f\|_{\mathrm{h}}$ and $\|f\|^{\mathrm{h}}$ are defined similarly on $C_{c\mathrm{h}}(\mathcal{G})$ using h-bounded representations.

Definition 3.5. A vertical representation of $\mathcal{G}$ (abbreviated as v–representation) consists of a quasi-invariant Borel measure μ^1 on $\mathcal{G}^1$, a $\mathcal{G}^1$-Hilbert bundle $\mathcal{H}$ over $(\mathcal{G}^1,\mu^1)$, and a map $\pi:\mathcal{G}^2\to\operatorname{Iso}(\mathcal{H})$ such that:

(i) $\pi(a)$ is a map from $\mathcal{H}_{d(a)}$ to $\mathcal{H}_{d(a)}$ and $\pi(u)=id_{\mathcal{H}_u}$, for all $a\in\mathcal{G}^2$ and $u\in\mathcal{G}^1$,

(ii) $\pi(a \cdot_{\mathrm{v}} b) = \pi(a)\pi(b)$ for ν_{v}^2-almost everywhere (a, b),
(iii) $\pi(a^{-\mathrm{v}}) = \pi(a)^{-1}$ for ν_{v}-almost everywhere a,
(iv) $a \mapsto \langle \pi(a)\xi \circ d(a), \eta \circ r(a)\rangle$ is measurable on $\mathcal{G}^2$ for all measurable sections ξ and η.

Using Hilbert bundles over $(\mathcal{G}^0, \mu^0)$, h-representations are defined similarly.

Two v-representations $(\pi_1, \mathcal{H}_1, \mu_1^1)$ and $(\pi_2, \mathcal{H}_2, \mu_2^1)$ are equivalent if $\mu_1^1 \sim \mu_2^1$ and there is an isomorphism ϕ of Hilbert bundles from $\mathcal{H}_1$ onto $\mathcal{H}_2$ which intertwines π_1 and π_2, that is,

$$\pi_2(a)\phi \circ d(a) = \phi \circ r(a)\pi_1(a)$$
$$\text{for } \nu_{\mathrm{v}}\text{-almost everywhere, } a \in \mathcal{G}^2.$$

Let $(\pi, \mathcal{H}, \mu^1)$ be a v-representation and $\Gamma_{\mathrm{v}}(\mathcal{H})$ the Hilbert space of square integrable sections with respect to μ^1. The following lemma is proved as in [**24,** 2.1.7].

Lemma 3.6. *Let $(\pi, \mathcal{H}, \mu^1)$ be a* v*-representation of $\mathcal{G}$, $f \in C_{\mathrm{cv}}(\mathcal{G})$ and $\xi, \eta \in \Gamma_{\mathrm{v}}(\mathcal{H})$. Then*

$$\langle \widetilde{\pi}(f)\xi, \eta\rangle = \int f(a)\langle \pi(a)\xi \circ d(a), \eta \circ r(a)\rangle \, d\nu_{\mathrm{v}0}(a)$$

defines a v*-bounded representation of $C_{\mathrm{cv}}(\mathcal{G})$ on $\Gamma_{\mathrm{v}}(\mathcal{H})$, and two equivalent* v*-representations of $\mathcal{G}$ induce equivalent* v*-bounded representations of $C_{\mathrm{cv}}(\mathcal{G})$ as above.*

When $\dim(\mathcal{H}_u)$ *is constant, namely, there is a Hilbert space H with $\mathcal{H}_u \simeq H$, for all $u \in \mathcal{G}^1$,*

$$\widetilde{\pi}(f)\xi(u) = \int f(a)\pi(a)\xi \circ d(a) D_{\mathrm{v}}^{1/2}(a) \, d\lambda_{\mathrm{v}}^u(a),$$

μ^1-almost everywhere, for $f \in C_{\mathrm{cv}}(\mathcal{G})$ and $\xi \in L^2(\mathcal{G}^1, \mu^1, H)$. In general, $\widetilde{\pi}$ is a direct sum of representations on constant fields over all possible dimensions. Similar statements hold for h*-representations $(\pi, \mathcal{H}, \mu^0)$ and the Hilbert space $\Gamma_{\mathrm{h}}(\mathcal{H})$ of square integrable sections with respect to μ^0.*

Consider the measurable field of Hilbert spaces $L^2(\mathcal{G}^2, \lambda_{\mathrm{v}}^u)$ with square integrable sections

$$L^2(\mathcal{G}^2, \nu_{\mathrm{v}}) = \int^{\oplus} L^2(\mathcal{G}^2, \lambda_{\mathrm{v}}^u)\, d\mu^1(u),$$

where μ^1 is a quasi-invariant Borel measure on $\mathcal{G}^1$. Then

$$\pi(a) : L^2(\mathcal{G}^2, \lambda_{\mathrm{v}}^{d(a)}) \longrightarrow L^2(\mathcal{G}^2, \lambda_{\mathrm{v}}^{r(a)}); \quad \pi(a)\xi(b) = \xi(a^{-\mathrm{v}} \cdot_{\mathrm{v}} b)$$

is a v-representation of $\mathcal{G}$, and

$$a \longmapsto \langle \pi(a)\xi \circ d(a), \eta \circ r(a)\rangle = \int \xi(a^{-\mathrm{v}} \cdot_{\mathrm{v}} b)\overline{\eta}(b)\, d\lambda_{\mathrm{v}}^{r(a)}(b)$$

is continuous for $\xi, \eta \in \mathbb{C}_c(\mathcal{G})$ and measurable for $\xi, \eta \in L^2(\mathcal{G}^2, \nu_{\mathrm{v}})$. This is called the v-*left regular representation* of $\mathcal{G}$ with respect to μ^1. Similarly, we could define the h-left regular representation of $\mathcal{G}$ with respect to a quasi-invariant measure μ^0 on $\mathcal{G}^0$.

Lemma 3.7. *The topological algebra $C_{\mathrm{cv}}(\mathcal{G})$ has a left approximate identity in the inductive limit topology. The same holds for $C_{\mathrm{ch}}(\mathcal{G})$.*

Proof. A subset $A \subseteq \mathcal{G}^2$ is d-relatively compact if $A \cap d^{-1}(K)$ is relatively compact for any compact subset K of $\mathcal{G}^1$. An argument similar to [**24,** 2.1.9] shows that $\mathcal{G}^1$ has a fundamental system (U_α) of d-relatively compact neighborhoods. Let $U \supseteq U_\alpha$ for each α and K_α be compact subsets of $\mathcal{G}^1$ such that $K_\alpha \subseteq K_\beta$, for $\alpha \le \beta$, and $\mathcal{G}^1 = \cup_\alpha K_\alpha$. Choose $g_\alpha \in C_{\mathrm{cv}}(\mathcal{G})$ such that $g_\alpha > 0$ on K_α and $\operatorname{supp}(g_\alpha) \subseteq U_\alpha$ and $h_\alpha \in C_c(\mathcal{G}^1)_+$ with

$$h_\alpha(u) = \left(\int g_\alpha d\lambda_{\mathrm{v}}^u\right)^{-1}, \quad \text{for } u \in K_\alpha.$$

Put $f_\alpha = (h_\alpha \circ r)g_\alpha \in C_{\mathrm{cv}}(\mathcal{G})$. Then $\operatorname{supp}(f_\alpha) \subseteq U_\alpha$ and $\lambda^{\mathrm{v}}(f_\alpha) = 1$ on K_α. For $f \in C_{\mathrm{cv}}(\mathcal{G})$ with $K = \operatorname{supp}(f)$, for each α, $\operatorname{supp}(f_\alpha *_{\mathrm{v}} f) \cup \operatorname{supp}(f) \subseteq \overline{U \cdot_{\mathrm{v}} K} =: L$, which is compact. Given $\varepsilon > 0$, there is an α_0 such that, for each α, $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)\alpha_0$, $r(L) \subseteq K_\alpha$ and, for each $(a, b) \in L \times U_\alpha \cap \mathcal{G}^{(2\mathrm{v})}$, $|f(b^{-\mathrm{v}} \cdot_{\mathrm{v}} a) - f(a)| < \varepsilon$. Hence,

$$|f_\alpha *_{\mathrm{v}} f(a) - f(a)| = \left|\int f_\alpha(b)(f(b^{-\mathrm{v}} \cdot_{\mathrm{v}} a) - f(a))\, d\lambda_{\mathrm{v}}^{r(a)}(b)\right| \le \varepsilon,$$

for $a \in L$. The argument for $C_{\mathrm{ch}}(\mathcal{G})$ is similar. □

The above lemma implies that v-left regular representations with respect to all quasi-invariant measures on $\mathcal{G}^1$ induce a faithful family of v-bounded representations of $C_{cv}(\mathcal{G})$. Also, for each quasi-invariant measure μ^1 on $\mathcal{G}^1$, $C_{cv}(\mathcal{G})$ is a generalized Hilbert algebra under the inner product of $L^2(\mathcal{G}^2, \nu_{\rm v}^{-1})$ whose left regular representation is equivalent to the v-left regular representation with respect to μ^1 [**24,** 2.1.10] and, by the Tomita-Takesaki theory, we have a modular function

$$J_{\rm v} : L^2(\mathcal{G}^2, \nu_{\rm v}^{-1}) \longrightarrow L^2(\mathcal{G}^2, \nu_{\rm v}^{-1});$$
$$J_{\rm v}\xi(a) = D_{\rm v}^{1/2}(a)\overline{\xi}(a^{-{\rm v}})$$

and a modular operator

$$\Delta_{\rm v} : L^2(\mathcal{G}^2, \nu_{\rm v}) \cap L^2(\mathcal{G}^2, \nu_{\rm v}^{-1}) \longrightarrow L^2(\mathcal{G}^2, \nu_{\rm v}) \cap L^2(\mathcal{G}^2, \nu_{\rm v}^{-1});$$
$$\Delta_{\rm v}\xi(a) = D_{\rm v}(a)\xi(a).$$

The same observations hold for $C_{ch}(\mathcal{G})$.

Definition 3.8. The full vertical (respectively horizontal) C^*-algebra of $\mathcal{G}$ is the completion of $C_{cv}(\mathcal{G})$ (respectively $C_{ch}(\mathcal{G})$) in $\|\cdot\|^{\rm v}$ (respectively $\|\cdot\|^{\rm v}$).

Lemma 3.9. *Letting $\{L, H\}$ be a representation of $C_{cv}(\mathcal{G})$, there is a unique representation $\{L^1, H^1\}$ of $C_c(\mathcal{G}^1)$ such that*

$$L(hf) = L^1(h)L(f),$$
$$L(fh) = L(f)L^1(h)$$
$$(h \in \mathbb{C}_c(\mathcal{G}^1),\ f \in C_{cv}(\mathcal{G}))$$

where

$$hf(a) = h \circ r(a)f(a), \qquad fh(a) = f(a)h \circ d(a) \quad (a \in \mathcal{G}^2).$$

Moreover, for $f, g \in C_{cv}(\mathcal{G})$, $h \in \mathbb{C}_c(\mathcal{G}^1)$,

$$f{*_{\rm v}}hg = fh{*_{\rm v}}g, \qquad hf{*_{\rm v}}g = h(f{*_{\rm v}}g), \quad (hf)_{\rm v}^* = f_{\rm v}^*h^*,$$

where $h^*(u) = \overline{h}(u)$, for $u \in \mathcal{G}^1$. There is a representation $\{L^0, H^0\}$ of $C_c(\mathcal{G}^0)$ with similar relations to the horizontal convolution.

Proof. We have:

$$\begin{aligned} f{*_{\rm v}}hg(a) &= \int f(a{*_{\rm v}}b)hg(b^{-{\rm v}})d\lambda_{\rm v}^{d(a)}(b) \\ &= \int f(a{*_{\rm v}}b)h(d(a))g(b^{-{\rm v}})d\lambda_{\rm v}^{d(a)}(b) \\ &= \int f(a{*_{\rm v}}b)h(d(a{*_{\rm v}}b))g(b^{-{\rm v}})d\lambda_{\rm v}^{d(a)}(b) \\ &= \int fh(a{*_{\rm v}}b)g(b^{-{\rm v}})d\lambda_{\rm v}^{d(a)}(b) \\ &= fh{*_{\rm v}}g(a), \end{aligned}$$

and the other convolution relations are proved similarly. Since $C_{c{\rm v}}(\mathcal{G})$ has a left approximate identity, the map

$$L^1(h)\bigg(\sum_{i=1}^{n} L(f_i)\xi_i\bigg) = \sum_{i=1}^{n} L(hf_i)\xi_i$$

is well defined and extends to a bounded representation of $C_c(\mathcal{G}^1)$ on the closure of span $\{L(f)\xi : f \in C_{c{\rm v}}(\mathcal{G}), \xi \in H\}$. A similar argument works on $\mathcal{G}^0$ with

$$hf(a) = h \circ r^2(a)f(a), \qquad fh(a) = f(a)h \circ d^2(a),$$

for $h \in C_c(\mathcal{G}^0)$ and $a \in \mathcal{G}^2$. $\square$

Corollary 3.10. *$C^*(\mathcal{G}^1)$ and $C^*(\mathcal{G}^0)$ are subalgebras of the multiplier algebras $M(C^*_{\rm v}(\mathcal{G}))$ and $M(C^*_{\rm v}(\mathcal{G}))$, respectively.*

Similar to [**24,** 2.1.17], every representation of $C_c(\mathcal{G})$ extends to a representation of $B(\mathcal{G})$ of bounded Borel functions on $\mathcal{G}^2$ with vertical or horizontal convolution. For a non singular Borel $\mathcal{G}^1$-set s and $f \in B(\mathcal{G})$ we define $s \cdot_{\rm v} f(a) = \delta_{\rm v}^{1/2}(r(a), s)$ for $a \in r^{-1}(r(s))$, and zero otherwise, and $f \cdot_{\rm v} s(a) = \delta_{\rm v}^{1/2}(d(a), s^{-{\rm v}})$ for $a \in d^{-1}(d(s))$, and zero otherwise. Then

$$\begin{aligned} (s \cdot_{\rm v} (t \cdot_{\rm v} f) &= (st) \cdot_{\rm v} f, \\ (f \cdot_{\rm v} s){*_{\rm v}}g &= f{*_{\rm v}}(s \cdot_{\rm v} g), \\ (s \cdot_{\rm v} f){*_{\rm v}}g &= s \cdot_{\rm v} (f{*_{\rm v}}g) \end{aligned}$$

and

$$(f \cdot_{\mathrm{v}} s)^* = s^{-\mathrm{v}} \cdot_{\mathrm{v}} f^*,$$

for non singular $\mathcal{G}^1$-sets s, t and $f, g \in B(\mathcal{G})$. We denote $B(\mathcal{G})$ with vertical convolution by $B_{\mathrm{v}}(\mathcal{G})$. The same relations hold for $B_{\mathrm{h}}(\mathcal{G})$, that is, $B(\mathcal{G})$ with horizontal convolution. Lemma 6.6 also holds for $B_{\mathrm{v}}(\mathcal{G})$ and $B_{\mathrm{h}}(\mathcal{G})$ and we could find a unique representation V^1 of the Borel ample semigroup of non singular $\mathcal{G}^1$-sets such that

$$L(s \cdot_{\mathrm{v}} f) = V^1(s)L(f),$$
$$L(f \cdot_{\mathrm{v}} s) = L(f)V^1(s),$$

and

$$V^1(s)L^1(h)V^1(s)^* = L^1(h^s),$$

for the non singular $\mathcal{G}^1$-set s, $f \in B_{\mathrm{v}}(\mathcal{G})$ and $h \in C_c(\mathcal{G}^1)$, where $h^s(u) = h(us)$ for $u \in r(s)$, and zero otherwise. The same holds for representations L, L^0 and a representation V^0 of the Borel ample semigroup of non singular $\mathcal{G}^0$-sets [**24,** 2.1.20].

Theorem 3.11. *If $\mathcal{G}$ is a locally compact second countable* 2*-groupoid with left* 2*-Haar system $\{\lambda_{\mathrm{v}}^u\}$and $\{\lambda_{\mathrm{h}}^x\}$with sufficiently many non singular $\mathcal{G}^1$-sets* (*respectively $\mathcal{G}^0$-sets*)*, then every* v*-bounded* (*respectively* h*-bounded*) *representation of $C_{c\mathrm{v}}(\mathcal{G})$* (*respectively $C_{c\mathrm{h}}(\mathcal{G})$*) *on a separable Hilbert space is the integration of a* v*-representation* (*respectively an* h*-representation*) *of $\mathcal{G}$.*

Proof. The assumption means that, for every measure μ^1 on $\mathcal{G}^1$ (μ^0 on $\mathcal{G}^0$) with induced measure ν_{v} (ν_{h}) every Borel set in $\mathcal{G}$ with positive ν_{v}-measure (ν_{h}-measure) contains a non singular Borel $\mathcal{G}^1$-set ($\mathcal{G}^0$-set) of positive $\mu^1 \circ r$-measure ($\mu^0 \circ r^2$-measure). As is [**24,** 2.1.21], we only need to check the result for factor representations. Let $\{L, H\}$ be a v-bounded factor representation of $C_{c\mathrm{v}}(\mathcal{G})$ with corresponding representations L^1 of $B_{\mathrm{v}}(\mathcal{G})$ and V^1 of the semigroup of non singular G^1-sets. There are a probability measure μ^1 and a Hilbert bundle $(\mathcal{K}, \mathcal{G}^1, \mu^1)$ such that L^1 is unitarily equivalent to the multiplication representation on $\Gamma_{\mathrm{v}}(\mathcal{K})$. We thus may assume that $H = \Gamma_{\mathrm{v}}(\mathcal{K})$ and L^1 is multiplication. An argument as in [**24,** 2.1.21] shows that μ^1 is quasi-invariant and ergodic and

$$V^1(s)\xi(u) = \Delta_{\mathrm{v}}^{1/2}(u \cdot_{\mathrm{v}} s, s)c(u,s)\xi(u \cdot_{\mathrm{v}} s)$$

for $u \in r(s)$ and zero otherwise, where $c(u,s)$ is defined for μ^1–almost everywhere $u \in r(s)$ and is an isometry from $\mathcal{K}_{u\cdot_{\mathrm{v}}s}$ onto $\mathcal{K}_u$ [**13,** page 82, Proposition 1]. Since we have sufficiently many non singular $\mathcal{G}^1$-sets, the set of those $u \in \mathcal{G}^1$ for which $\dim \mathcal{K}_u$ is constant is almost invariant (that is for ν_{v}-almost everywhere a, $r(a)$ is in this set if and only if $d(a)$ is in it). This and ergodicity of μ^1 let us further assume that $H = L^2(\mathcal{G}^1, \mu^1, K)$ for some Hilbert space K. Since the measure induced by the functional $f \mapsto \langle L(f)\xi, \eta\rangle$ is absolutely continuous with respect to $\nu_{\mathrm{v}0}$, there is a Borel function c on $\mathcal{G}^2$ such that

$$\langle L(f)\xi, \eta\rangle = \int fc\, d\nu_{\mathrm{v}0}\,,$$

for $f \in B_{\mathrm{v}}(\mathcal{G})$ and $\xi, \eta \in K$. An argument similar to [**24,** 2.1.21(e)] shows that, when ξ, η are unit vectors, $|c| \le 1$ ν_{v}-almost everywhere; hence,

$$|\langle L(f)\xi, \eta\rangle| \le \int |f|\, d\nu_{\mathrm{v}0}\|\xi\|\|\eta\|,$$

for $f \in C_{\mathrm{cv}}(\mathcal{G})$ and $\xi, \eta \in K$, and the result follows from [**15,** page 106, Theorem 5.4]. The same argument works for h-representations. □

Corollary 3.12. *When $\mathcal{G}$ is second countable with sufficiently many non singular $\mathcal{G}^1$-sets (respectively $\mathcal{G}^0$-sets), every representation of $C_{\mathrm{cv}}(\mathcal{G})$ (respectively $C_{\mathrm{ch}}(\mathcal{G})$) on a separable Hilbert space is* v*-bounded (respectively* h*-bounded) and there is a one-to-one correspondence between $\mathcal{G}^1$-Hilbert bundles (respectively $\mathcal{G}^0$-Hilbert bundles) and separable Hermitian $\mathbb{C}^*_{\mathrm{v}}(\mathcal{G})$-modules (respectively $\mathbb{C}^*_{\mathrm{v}}(\mathcal{G})$-modules) preserving intertwining operators.*

Proof. The first statement is already proved and the second follows from the fact that two v-representations $(\pi_1, \mathcal{H}_1, \mu_1^1)$ and $(\pi_2, \mathcal{H}_2, \mu_2^1)$ giving unitarily equivalent integrated representations are equivalent. □

3.3. Induced representations and reduced C^*-algebras. Let $\mathcal{G} = (\mathcal{G}^2, \mathcal{G}^1, \mathcal{G}^0)$ be a locally compact 2-groupoid with left 2-Haar system $\{\lambda_{\mathrm{v}}^u\}$and $\{\lambda_{\mathrm{h}}^x\}$, and let $\mathcal{H} = (\mathcal{H}^2, \mathcal{H}^1, \mathcal{H}^0)$ be a closed 2-subgroupoid, that is, a 2-subgroupoid such that $\mathcal{H}^i \subseteq \mathcal{G}^i$ is closed for $i = 0, 1, 2$, with left 2-Haar system $\{\sigma_{\mathrm{v}}^u\}$ and $\{\sigma_{\mathrm{h}}^x\}$ such that $\mathcal{G}^1 \subseteq \mathcal{H}^2$ and $\mathcal{G}^0 \subseteq \mathcal{H}^1$. For the equivalence relations $a \sim_{\mathrm{v}} b$ if and only if $d(a) = r(b)$ and $a \cdot_{\mathrm{v}} b \in \mathcal{H}^2$ and $a \sim_{\mathrm{h}} b$ if and only if $d^2(a) = r^2(b)$ and $a \cdot_{\mathrm{h}} b \in \mathcal{H}^2$, for $a, b \in \mathcal{G}^2$, the quotient space $\mathcal{H}\backslash\mathcal{G}$ is Hausdorff and locally compact and the quotient map, $\mathcal{G} \to \mathcal{H}\backslash\mathcal{G}$, is open. Also, there are continuous open surjections from the quotient spaces to $\mathcal{G}^1$ and $\mathcal{G}^0$ induced by d and d^2, respectively (cf., [**24,** 2.2.1]).

Lemma 3.13. *There are Bruhat approximate vertical and horizontal cross-sections for $\mathcal{G}$ over $\mathcal{H}\backslash\mathcal{G}$, that is, non negative continuous functions b_{v}, b_{h} on $\mathcal{G}$ whose supports have compact intersections respectively with $\mathcal{H}^2 \cdot_{\mathrm{v}} K$ and $\mathcal{H}^2 \cdot_{\mathrm{h}} K$ for each compact subset K of $\mathcal{G}^2$ such that*

$$\int b_{\mathrm{v}}(c^{-\mathrm{v}} \cdot_{\mathrm{v}} a)\, d\sigma_{\mathrm{v}}^{r(a)}(c) = 1, \qquad \int b_{\mathrm{h}}(c^{-\mathrm{h}} \cdot_{\mathrm{h}} a)\, d\sigma_{\mathrm{h}}^{r^2(a)}(c) = 1,$$

for each $a \in \mathcal{G}^2$.

Proof. This follows from [**4,** page 96, Lemma 1]. □

Consider equivalence relations on $\mathcal{G}^{(2\mathrm{v})}$ and $\mathcal{G}^{(2\mathrm{h})}$, $(a_1, b_1) \sim_{\mathrm{v}} (a_2, b_2)$ if and only if $b_1 = b_2$ and $a_1 \cdot_{\mathrm{v}} a_2^{-\mathrm{v}} \in \mathcal{H}^2$ and $(a_1, b_1) \sim_{\mathrm{h}} (a_2, b_2)$ if and only if $b_1 = b_2$ and $a_1 \cdot_{\mathrm{h}} a_2^{-\mathrm{h}} \in \mathcal{H}^2$, then the quotient spaces $\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{v})}$ and $\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{h})}$ are locally compact 2-groupoids with a set of 1-morphisms $\mathcal{H}\backslash\mathcal{G}^1$ and $\mathcal{H}\backslash\mathcal{G}^0$ with left 2-Haar systems $\{\delta_{\dot{a}} \times \lambda_{\mathrm{v}}^{d(\dot{a})}\}$ and $\{\delta_{\dot{a}} \times \lambda_{\mathrm{h}}^{d^2(\dot{a})}\}$ with a ranging respectively over $\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{v})}$ and $\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{h})}$ (cf., [**24,** 2.2.3]). For $\varphi \in C_c(\mathcal{H})$ and $f \in C_c(\mathcal{G})$,

$$\varphi \cdot_{\mathrm{v}} f(a) = \int \varphi(c) f(c^{-\mathrm{v}} \cdot_{\mathrm{v}} a)\, d\sigma_{\mathrm{v}}^{r(a)}(c),$$
$$f \cdot_{\mathrm{v}} \varphi(a) = \int f(a \cdot_{\mathrm{v}} c)\varphi(c^{-\mathrm{v}})\, d\sigma_{\mathrm{v}}^{d(a)}(c),$$

and

$$\varphi \cdot_{\mathrm{h}} f(a) = \int \varphi(c) f(c^{-\mathrm{h}} \cdot_{\mathrm{h}} a)\, d\sigma_{\mathrm{h}}^{r^2(a)}(c),$$
$$f \cdot_{\mathrm{h}} \varphi(a) = \int f(a \cdot_{\mathrm{h}} c)\varphi(c^{-\mathrm{h}})\, d\sigma_{\mathrm{h}}^{d^2(a)}(c),$$

for $a \in \mathcal{G}^2$. Also, for $\phi \in C_c(\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{v})})$, $\psi \in C_c(\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{v})})$ and $f \in C_c(\mathcal{G})$,

$$\phi \cdot_{\mathrm{v}} f(a) = \int \phi(\dot{a}^{-\mathrm{v}}, a \cdot_{\mathrm{v}} b) f(b^{-\mathrm{v}})\, d\lambda_{\mathrm{v}}^{d(a)}(b),$$
$$f \cdot_{\mathrm{v}} \phi(a) = \int f(b)\phi(\dot{b}, b^{-\mathrm{v}} \cdot_{\mathrm{v}} a)\, d\lambda_{\mathrm{v}}^{r(a)}(b),$$

and

$$\psi \cdot_{\mathrm{h}} f(a) = \int \psi(\dot{a}^{-\mathrm{h}}, a \cdot_{\mathrm{h}} b) f(b^{-\mathrm{h}})\, d\lambda_{\mathrm{h}}^{d^2(a)}(b),$$
$$f \cdot_{\mathrm{h}} \psi(a) = \int f(b)\psi(\dot{b}, b^{-\mathrm{h}} \cdot_{\mathrm{h}} a)\, d\lambda_{\mathrm{h}}^{r^2(a)}(b),$$

for $a \in \mathcal{G}^2$. Then $X_{\mathrm{v}} := C_{c\mathrm{v}}(\mathcal{G})$ is a bimodule over $B_{\mathrm{v}} := C_{c,\mathrm{v}}(\mathcal{H})$ and $E_{\mathrm{v}} := C_{c,\mathrm{v}}(\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{v})})$ with commuting actions on opposite sides, and the action of $C_{c\mathrm{v}}(\mathcal{H})$ as double centralizers on $C_{c\mathrm{v}}(\mathcal{G})$ extends to an action on $C^*_{\mathrm{v}}(\mathcal{G})$, giving a $*$-homomorphism of $C_{c\mathrm{v}}(\mathcal{H})$ into the multiplier algebra $M(C^*_{\mathrm{v}}(\mathcal{G}))$, and the same holds for $C_{c\mathrm{h}}(\mathcal{G})$ (cf., [**24,** 2.2.4]).

Consider X_{v} as a left E_{v}-module and right B_{v}-module with the following vector-valued inner products

$$\langle f, g\rangle_{B_{\mathrm{v}}}(c) = \int \overline{f}(a^{-\mathrm{v}}) g(a^{-\mathrm{v}} \cdot_{\mathrm{v}} c)\, d\lambda_{\mathrm{v}}^{r(c)}(a)$$

and

$$\langle f, g\rangle_{E_{\mathrm{v}}}(\dot{a}, a^{-\mathrm{v}} \cdot_{\mathrm{v}} b) = \int f(a^{-\mathrm{v}} \cdot_{\mathrm{v}} c)\overline{g}(b \cdot_{\mathrm{v}} c) d\sigma_{\mathrm{v}}^{r(a)}(c),$$

for $c \in \mathcal{H}^2$, $a, b \in \mathcal{G}^2$. Then

$$\langle f, gh\rangle_{B_{\mathrm{v}}} = \langle f, g\rangle_{B_{\mathrm{v}}} h, \qquad \langle ef, g\rangle_{B_{\mathrm{v}}} = \langle f, e^* g\rangle_{B_{\mathrm{v}}},$$

and

$$\langle ef, g\rangle_{E_{\mathrm{v}}} = e\langle f, g\rangle_{E_{\mathrm{v}}}, \qquad \langle f, gh\rangle_{E_{\mathrm{v}}} = \langle fh^*, g\rangle_{E_{\mathrm{v}}},$$

for $f, g \in X_{\mathrm{v}}$, $h \in B_{\mathrm{v}}$ and $e \in E_{\mathrm{v}}$, and $f_1\langle g, f_2\rangle_{B_{\mathrm{v}}} = \langle f_1, g\rangle_{E_{\mathrm{v}}} f_2$, for $f_1, f_2, g \in X_{\mathrm{v}}$. The same holds for horizontal spaces and modules. An argument similar to [**24,** 2.2.5] shows the following.

Lemma 3.14. *The linear span of* $\{\langle f,g\rangle_{E_{\mathrm{v}}} : f,g\in X_{\mathrm{v}}\}$ *contains a left approximate identity for* E_{v} *in the inductive limit topology and is dense in* E_{v} *and* $C^*_{\mathrm{v}}(\mathcal{H}\backslash\mathcal{G}^{(2\mathrm{v})}))$. *Similarly, the linear span of* $\{\langle f,g\rangle_{B_{\mathrm{v}}} : f,g\in X_{\mathrm{v}}\}$ *is dense in* B_{v} *and* $C^*_{\mathrm{v}}(\mathcal{H})$. *The same holds for* E_{h} B_{h}.

Corollary 3.15. *The* C^**-algebras* $C^*_{\mathrm{v}}(\mathcal{G}^{(2\mathrm{v})})$ *and* $C^*_{\mathrm{v}}(\mathcal{G}^1)$ *are strongly Morita equivalent. Similarly,* $C^*_{\mathrm{h}}(\mathcal{G}^{(2\mathrm{h})})$ *and* $C^*_{\mathrm{v}}(\mathcal{G}^0)$ *are strongly Morita equivalent.*

Proof. We have already checked that X_{v} has algebraic properties of an E_{v}-B_{v}-imprimitivity bimodule. For $\mathcal{H}=\mathcal{G}^1$, we have

$$\begin{aligned}\langle f,f\rangle_{B_{\mathrm{v}}}(u) &= \int \overline{f}(a^{-\mathrm{v}})f(a^{-\mathrm{v}}\cdot_{\mathrm{v}} r(a))\,d\lambda^u_{\mathrm{v}}(a)\\ &= \int |f(a^{-\mathrm{v}})|^2 d\lambda^u_{\mathrm{v}}(a)\geq 0\end{aligned}$$

and

$$\begin{aligned}\langle f,f\rangle_{E_{\mathrm{v}}}(\dot a, a^{-\mathrm{v}}\cdot_{\mathrm{v}} b) &= \int f(a^{-\mathrm{v}}\cdot_{\mathrm{v}} r(a))\overline{f}(b\cdot_{\mathrm{v}} u)\,d\sigma^{r(a)}_{\mathrm{v}}(u)\\ &= |f(\dot b)|^2\sigma^{r(a)}_{\mathrm{v}}(u) = |f(\dot b)|^2\sigma^{r(a)}_{\mathrm{v}}(K)\geq 0,\end{aligned}$$

for $a,b\in\mathcal{G}^2$, where K is a compact subset of $\mathcal{G}^1$, and the norm conditions

$$\langle fh,fh\rangle_{E_{\mathrm{v}}}\leq \|h\|^2\langle f,f\rangle_{E_{\mathrm{v}}},\qquad \langle ef,ef\rangle_{B_{\mathrm{v}}}\leq\|e\|^2\langle f,f\rangle_{B_{\mathrm{v}}}$$

are satisfied for each $f\in C_{c\mathrm{v}}(\mathcal{G})$, $h\in C_{c\mathrm{v}}(\mathcal{G}^1)$ and $e\in C_{c\mathrm{v}}(\mathcal{G}^{(2\mathrm{v})})$ [**24,** 2.2.7]. Similarly, X_{h} is an E_{h}-B_{h}-imprimitivity bimodule. □

Now by the Rieffel construction, each v-representation of $C^*_{\mathrm{v}}(\mathcal{G}^1)$ induces a v-representation of $C^*_{\mathrm{v}}(\mathcal{G}^{(2\mathrm{v})})$ and then restricts to a v-v-representation of $C^*_{\mathrm{v}}(\mathcal{G})$ which acts on $C^*_{\mathrm{v}}(\mathcal{G}^{(2\mathrm{v})})$ as double centralizers; in other words, the restriction map $P_{\mathrm{v}}: C_{c\mathrm{v}}(\mathcal{G})\to C_{c\mathrm{v}}(\mathcal{G}^1)$ is a generalized conditional expectation in the sense of [**25**]. Similarly, we have a generalized conditional expectation $P_{\mathrm{h}}: C_{c\mathrm{h}}(\mathcal{G})\to C_{c\mathrm{h}}(\mathcal{G}^0)$. More generally, if $\mathcal{G}$ is second countable and $\mathcal{H}$ is a closed 2-subgroupoid such that both $\mathcal{G}$ and $\mathcal{H}$ have sufficiently many non singular Borel sets, it follows from [**24,** Lemma 7.1, 2.2.9-10] that the restriction map from $C_{c\mathrm{v}}(\mathcal{G})$ to $C_{c\mathrm{v}}(\mathcal{H})$ is a generalized conditional expectation, and the same for $C_{c\mathrm{h}}(\mathcal{G})$.

For the representation of $C^*_{\rm v}(\mathcal{G}^1)$ given by multiplication on $L^2(\mathcal{G}^1, \mu^1)$ the induced representation $\operatorname{Ind}\mu^1$ acts on $L^2(\mathcal{G}^1, {\nu_{\rm v}}^{-1})$ by convolution on the left, namely,

$$\langle \operatorname{Ind}\mu^1(f)\xi, \eta\rangle = \int\int\int f(a \cdot_{\rm v} b)\xi(b^{-{\rm v}})\overline{\eta}(a)\, d\lambda^u_{\rm v}(b)\lambda_{{\rm v},u}(a)\, d\mu^1(u),$$

for $f \in C_{\rm cv}(\mathcal{G})$ and $\xi, \eta \in L^2(\mathcal{G}^1, {\nu_{\rm v}}^{-1})$. When μ^1 is quasi-invariant, $\operatorname{Ind}\mu^1$ is just the left regular representation on μ^1. In this case, $\ker(\operatorname{Ind}\mu^1)$ consists of those $f \in C_{\rm cv}(\mathcal{G})$ that $f = 0$ on $\operatorname{supp}({\nu_{\rm v}}^{-1})$ [**24,** 1.1.11]. Since $\mathcal{G}^1$ has a faithful family of quasi-invariant measures [**24,** 1.3.9], $C_{\rm cv}(\mathcal{G})$ has a faithful family of v-bounded representations (consisting of induced representations of such quasi-invariant measures). In particular,

$$\|f\|^{\rm v}_{\rm red} := \sup_{\mu^1} \|\operatorname{Ind}\mu^1(f)\|$$

is a C^*-norm, where μ^1 ranges over all quasi-invariant Borel measures on $\mathcal{G}^1$, and $\|f\|^{\rm v}_{\rm red} \le \|f\|^{\rm v}$, for each $f \in C_{\rm cv}(\mathcal{G})$. Similarly,

$$\|f\|^{\rm h}_{\rm red} := \sup_{\mu^0} \|\operatorname{Ind}\mu^0(f)\| \le \|f\|^{\rm h}$$

is a C^*-norm, where μ^0 ranges over all quasi-invariant Borel measures on $\mathcal{G}^0$. The completions $C^*_{\rm v,red}(\mathcal{G})$ and $C^*_{\rm h,red}(\mathcal{G})$ of $C_{\rm cv}(\mathcal{G})$ and $C_{\rm ch}(\mathcal{G})$ with respect to these C^*-norms are called the vertical and horizontal *reduced C^*-algebras* of $\mathcal{G}$, which are quotients of the vertical and horizontal full C^*-algebras $C^*_{\rm v}(\mathcal{G})$ and $C^*_{\rm h}(\mathcal{G})$ of $\mathcal{G}$.

As we can see from the next example, the C^*-algebra of a 2-groupoid may be independent (up to isomorphism) of the choice of the Haar system (see also [**6**]). In general, one could only expect independence up to Morita equivalence (see the Proposition 3.17).

Example 3.16 (Principal 2-groupoid)**.** Consider the principal 2-groupoid $\mathcal{G} = X^{(5)}$ with the Haar system as in Example 2.11 (ii), coming from a Borel measure α on $X^{(2)}$. When X is uncountable and α is non atomic, then $\mathcal{G}$ has sufficiently many Borel $\mathcal{G}^1$-sets, and any v-representation of $\mathcal{G}$ is a multiple of one-dimensional trivial representation (compare to [**24,** 2.2.12]); hence, for any such choice of α, $C^*_{\rm v}(\mathcal{G})$ is isomorphic to the C^*-algebra of compact operators on a separable Hilbert space.

Proposition 3.17. *If a second countable locally compact groupoid* $\mathcal{G}$ *has two 2-Haar systems* $\{\lambda_{\rm v}^u\}$, $\{\lambda_{\rm h}^x\}$ *and* $\{\sigma_{\rm v}^u\}$, $\{\sigma_{\rm h}^x\}$, *and it has sufficiently many non singular Borel* $\mathcal{G}^1$*-sets (respectively* $\mathcal{G}^0$*-sets) with respect to both systems, then the corresponding* C^**-algebras* $C^*_{\rm v}(\mathcal{G},\lambda)$ *and* $C^*_{\rm v}(\mathcal{G},\sigma)$ *(respectively* $C^*_{\rm h}(\mathcal{G},\lambda)$ *and* $C^*_{\rm h}(\mathcal{G},\sigma)$*) are strongly Morita equivalent.*

Proof. We have $\mathcal{G}\backslash\mathcal{G}^{(2{\rm v})} = \mathcal{G}$ and $X_{\rm v} = C_{c{\rm v}}(\mathcal{G},\lambda)$ is an $E_{\rm v}$-$B_{\rm v}$-imprimitivity bimodule, for $E_{\rm v} = C_{c{\rm v}}(\mathcal{G},\lambda)$ and $B_{\rm v} = C_{c{\rm v}}(\mathcal{G},\sigma)$. The same holds for $X_{\rm v} = C_{c{\rm v}}(\mathcal{G},\sigma)$, $E_{\rm v} = C_{c{\rm v}}(\mathcal{G},\sigma)$ and $B_{\rm v} = C_{c{\rm v}}(\mathcal{G},\lambda)$. The horizontal case is similar. □

3.4. r-Discrete principal 2-groupoids. In this section, we describe the reduced C^*-algebras of r-discrete principal 2-groupoids and find their ideals and masa's.

Lemma 3.18. *Let* $\mathcal{G}$ *be an* r*-discrete 2-groupoid with a 2-Haar system and* $a \in \mathcal{G}^2$. *Let* $L = \operatorname{Ind}\mu^1$ *(respectively* $L = \operatorname{Ind}\mu^0$*) be the representation of* $C_{c{\rm v}}(\mathcal{G})$ *(respectively* $C_{c{\rm h}}(\mathcal{G})$*) induced by the point mass* $\mu^1 = \delta_{d(a)}$ *(respectively* $\mu^0 = \delta_{d^2(a)}$*). Then, for every* $f \in C_{c{\rm v}}(\mathcal{G})$ *(respectively* $f \in C_{c{\rm h}}(\mathcal{G})$*),*

$$f(a) = \langle L(f)\delta_u, \delta_a\rangle = L(f)\delta_u(a),$$

where $u = d(a)$ *(respectively* $u = x := d^2(a)$*) and* δ_u, δ_a *are regarded as unit vectors in* $L^2(\mathcal{G},\lambda_{{\rm v}u})$ *(respectively in* $L^2(\mathcal{G},\lambda_{{\rm h}x})$*). In particular,* $\max\{\|f\|_\infty, \|f\|_2\} \le \|f\|^{\rm v}_{\rm red}$ *(respectively the same for* $\|f\|^{\rm h}_{\rm red}$*), where* $\|.\|_2$ *is the norm in* $L^2(\mathcal{G},\lambda_{{\rm v}u})$ *(respectively in* $L^2(\mathcal{G},\lambda_{{\rm h}x})$*).*

Proof. We have

$$\langle L(f)\delta_u, \delta_a\rangle = \sum_{r(c)=u}\sum_{d(b)=u} f(b\cdot_{\rm v} c)\delta_u(c^{-{\rm v}})\delta_a(b) = f(a),$$

and the rest is proved similarly. □

Now the inclusion map $j_v : C_{c{\rm v}}(\mathcal{G}) \to C_0(\mathcal{G})$ extends to a norm decreasing linear map $j_v : C^*_{{\rm v,red}}(\mathcal{G}) \to C_0(\mathcal{G})$. Let us observe that the latter map is still injective. Consider the surjection $p : C_{c{\rm v}}(\mathcal{G}) \to C_c(\mathcal{G}^1)$, for a quasi-invariant probability measure μ^1 on $\mathcal{G}^1$, the induced representation $\operatorname{Ind}\mu^1$ is the GNS-representation of $\mu^1 \circ p$, namely,

$$\int p(f)\,d\mu^1 = \langle \operatorname{Ind}\mu^1(f)\xi_0, \xi_0\rangle$$

and

$$\operatorname{Ind}\mu^1(f)\xi_0 = f *_{\mathrm{v}} \xi_0 = j_v(f)$$

where $\xi_0 \in L^2(\mathcal{G}, \nu_{\mathrm{v}}{}^{-1})$ is the characteristic function of $\mathcal{G}^1$, and j_v is now considered as the inclusion from $C_{\mathrm{cv}}(\mathcal{G})$ into $L^2(\mathcal{G}, \nu_{\mathrm{v}}{}^{-1})$. Now the above lemma shows that $\operatorname{Ind}\mu^1(g)\xi_0 = j_v(g)$ remains valid for $g \in C^*_{\mathrm{v,red}}(\mathcal{G})$ and if $j_v(g) = 0$, then $\operatorname{Ind}\mu^1(g) = 0$ as ξ_0 is a cyclic vector, and this, being true for all quasi-invariant probability measures μ^1 on $\mathcal{G}^1$, implies that $g = 0$. Also, $\|g\|_\infty \le \|g\|^{\mathrm{v}}_{\mathrm{red}}$, where on the left hand side g is regarded as a continuous function on $\mathcal{G}$. The same observations hold for $C^*_{\mathrm{h,red}}(\mathcal{G})$.

A 2-groupoid $\mathcal{G}$ is called essentially v-principal (respectively h-principal) if, for every invariant closed subset F of $\mathcal{G}^1$ (respectively $\mathcal{G}^0$) the set of $u \in F$ (respectively $x \in F$) whose isotropy group $\mathcal{G}^u_u$ (respectively $\mathcal{G}^x_x$) is a singleton, is dense in F. It is called essentially principal if, for every invariant closed subset F of $\mathcal{G}^0$, the set of $x \in F$ whose isotropy groupoid $\mathcal{G}(x)$ is a singleton, is dense in F.

Lemma 3.19. *Let $\mathcal{G}$ be an* r*-discrete essentially* v*-principal* (*respectively* h*-principal*) 2*-groupoids with* 2*-Haar system and $a \in \mathcal{G}^2$. For any quasi-invariant measure μ^1 on $\mathcal{G}^1$* (*respectively μ^0 on $\mathcal{G}^0$*) *with support F, any* v*-representation* (*respectively* h*-representation*) *π on μ^1* (*respectively μ^0*), *and any $f \in C_{\mathrm{cv}}(\mathcal{G})$* (*respectively $f \in C_{\mathrm{ch}}(\mathcal{G})$*) *we have* $\sup_F f \le \|\widetilde{\pi}(f)\|$.

Proof. We only need to deal with the case that, for each $u \in F$ (respectively $x \in F$), $\mathcal{G}^u_u = \{u\}$ (respectively $\mathcal{G}^x_x = \{x\}$). The proof goes by choosing an appropriate sequence of square integrable sections $\{\xi_n\}$ of π such that $\langle \widetilde{\pi}(f)\xi_n, \xi_n\rangle$ tends to $f(u)$ (respectively $f(x)$) as in [**24,** 2.4.4]. □

Let $\mathcal{G}$ be a locally compact groupoid with 2-Haar system. For an invariant open subset U of $\mathcal{G}^1$ (respectively G^0) let $I_{\mathrm{cv}}(U) = \{f \in C_{\mathrm{cv}}(\mathcal{G}) : f(u) = 0\ (u \notin \mathcal{G}_U)\}$ (respectively $I_{\mathrm{ch}}(U) = \{f \in C_{\mathrm{ch}}(\mathcal{G}) :$

$f(x) = 0$ $(x \notin \mathcal{G}_U)\})$ and $I_{\mathrm{v}}(U)$ (respectively I_{h}) be its closure. Let F be the complement of U in $\mathcal{G}^1$ (respectively G^0) then it follows from [**24,** 2.4.5] that $I_{\mathrm{v}}(U)$ (respectively I_{h}) is isomorphic to $C^*_{\mathrm{v,red}}(\mathcal{G}_U)$ (respectively $C^*_{\mathrm{h,red}}(\mathcal{G}_U)$), and it is a closed ideal of $C^*_{\mathrm{v,red}}(\mathcal{G})$ (respectively $C^*_{\mathrm{h,red}}(\mathcal{G})$) whose quotient is isomorphic to $C^*_{\mathrm{v,red}}(\mathcal{G}_F)$ (respectively $C^*_{\mathrm{h,red}}(\mathcal{G}_F)$). If μ^1 (respectively μ^0) is a quasi-invariant measure on $\mathcal{G}^1$ (respectively on $\mathcal{G}^0$) with support F, U is the complement of F, then $I_{\mathrm{v}}(U) = \ker(\operatorname{Ind}\mu^1)$ (respectively $I_{\mathrm{h}} = \ker(\operatorname{Ind}\mu^0)$). This provides a one-to-one correspondence between invariant open subsets of $\mathcal{G}^1$ (respectively G^0) and a family of closed ideals of $C^*_{\mathrm{v,red}}(\mathcal{G})$ (respectively $C^*_{\mathrm{h,red}}(\mathcal{G})$). Both sets are a lattice with respect to inclusion. When $\mathcal{G}$ is r-discrete and essentially v-principal (respectively h-principal), the above correspondence is an order preserving bijection, namely all closed ideals of $C^*_{\mathrm{v,red}}(\mathcal{G})$ (respectively $C^*_{\mathrm{h,red}}(\mathcal{G})$) are of the form $I_{\mathrm{v}}(U)$ (respectively I_{h}) for some invariant open subset U of $\mathcal{G}^1$ (respectively G^0) and the correspondence $U \mapsto I_{\mathrm{v}}(U)$ (respectively I_{h}) preserves inclusion. Indeed, in this case, the surjection p defined after Lemma 3.18 is a conditional expectation and $(\operatorname{Ind}\mu^1$ (respectively $(\operatorname{Ind}\mu^0)$ is the GNS-representation of $\mu^1 \circ p$ (respectively $\mu^0 \circ p$) and so $\|(\operatorname{Ind}\mu^1(f)\| \leq \|\widetilde{\pi}(f)\|$ for $f \in C_{\mathrm{cv}}(\mathcal{G})$ (respectively $\|(\operatorname{Ind}\mu^0(f)\| \leq \|\widetilde{\pi}(f)\|$ for $f \in C_{\mathrm{ch}}(\mathcal{G})$) hence $\ker(\widetilde{\pi})$ is equal to $I_{\mathrm{v}}(U)$ (respectively I_{h}) where U is the complement of the support of μ^1 (respectively μ^0).

Lemma 3.20. *Let $\mathcal{G}$ be an* r*-discrete with* 2*-Haar system. An element g of $C^*_{\mathrm{v,red}}(\mathcal{G})$ (respectively $C^*_{\mathrm{h,red}}(\mathcal{G})$) commutes with each element of $C^*_{\mathrm{v}}(\mathcal{G}^1)$ (respectively $C^*_{\mathrm{h}}(\mathcal{G}^0)$) if and only if it vanishes off the isotropy group bundle $\bigsqcup_{u\in\mathcal{G}^1}\mathcal{G}^u_u$ (respectively $\bigsqcup_{x\in\mathcal{G}^0}\mathcal{G}^x_x$).*

Proof. Since $\mathcal{G}^1$ is open, $C^*_{\mathrm{v}}(\mathcal{G}^1)$ could be considered as a subalgebra of $C^*_{\mathrm{v,red}}(\mathcal{G})$ consisting of those elements vanishing off $\mathcal{G}^1$, and the result follows. The horizontal case is proved similarly. □

Corollary 3.21. *If $\mathcal{G}$ is an* r*-discrete with* 2*-Haar system, $C^*_{\mathrm{v}}(\mathcal{G}^1)$ (respectively $C^*_{\mathrm{h}}(\mathcal{G}^0)$) is a masa in $C^*_{\mathrm{v,red}}(\mathcal{G})$ (respectively $C^*_{\mathrm{h,red}}(\mathcal{G})$) if and only if $\mathcal{G}^1$ (respectively $\mathcal{G}^0$) is the interior of the isotropy group bundle $\bigsqcup_{u\in\mathcal{G}^1}\mathcal{G}^u_u$ (respectively $\bigsqcup_{x\in\mathcal{G}^0}\mathcal{G}^x_x$).*

A concrete example for which the necessary and sufficient condition of the above corollary fails, is provided by the principal 2-groupoid (Example 2.2 (ii)). In this case, $\mathcal{G}^1 = \{(x,u,x,u,v) : x,u,v \in X\}$ whereas the isotropy group bundle is equal to $\{(x,u,z,u,v) : x,z,u,v \in X\}$, and the above condition holds only when X is a singleton. In this example, $C^*_{\rm v}(\mathcal{G}^1)$ is the algebra of compact operators which fails to be abelian (unless $\mathcal{G}^1$ is a singleton). On the other extreme, for the groupoid bundle $\mathcal{G} = \bigsqcup_{x\in\mathcal{G}^0} \mathcal{G}(x)$ of Example 2.2 (iii), where each $\mathcal{G}(x)$ is a bundle of abelian groups, the above condition always holds. In this example,

$$C^*_{\rm v}(\mathcal{G}^1) = c_0\text{-}\bigoplus_{u\in\mathcal{G}^1} C_0(\widehat{\mathcal{G}}^u_u),$$

where each $\mathcal{G}^u_u$ is an abelian group with Pontryagin dual $\widehat{\mathcal{G}}^u_u$.

In the above corollary, if moreover $\mathcal{G}$ is essentially v-principal (respectively h-principal), the restriction map $p : C^*_{\rm v,red}(\mathcal{G}) \to C^*_{\rm v}(\mathcal{G}^1)$ (respectively $p : C^*_{\rm h,red}(\mathcal{G}) \to C^*_{\rm h}(\mathcal{G}^0)$) is a faithful surjective conditional expectation and there is a one-to-one correspondence between the ample semigroup of compact open $\mathcal{G}^1$-sets (respectively $\mathcal{G}^1$-sets) and the inverse semigroup of partial homeomorphisms of $C^*_{\rm v}(\mathcal{G}^1)$ (respectively $C^*_{\rm h}(\mathcal{G}^0)$) defined by conjugation with respect to the elements in the normalizer of $C^*_{\rm v}(\mathcal{G}^1)$ (respectively $C^*_{\rm h}(\mathcal{G}^0)$) in $C^*_{\rm v,red}(\mathcal{G})$ (respectively $C^*_{\rm h,red}(\mathcal{G})$) (cf., [**24,** 2.4.8]).

REFERENCES

1. M. Amini, *Tannaka-Krein duality for compact groupoids* I, *Representation theory*, Adv. Math. **214** (2007), 78–91.

2. ______, *Tannaka-Krein duality for compact groupoids* II, *Duality*, Oper. Matr. **4** (2010), 573–592.

3. J.C. Baez, *An introduction to n-categories*, in *Category theory and computer science*, Lect. Notes Comp. Sci. **1290**, Springer, Berlin, 1997.

4. N. Bourbaki, *Integration*, Herman, Paris, 1963.

5. R. Brown and K.C.H. Mackenzie, *Determination of a double Lie groupoid by its core diagram*, J. Pure Appl. Alg. **80** (1992), 237–272.

6. M.R. Buneci, *Isomorphic groupoid C^*-algebras associated with different Haar systems*, New York J. Math. **11** (2005), 225–245.

7. A. Buss, R. Meyer and C. Zhu, *Non-Hausdorff symmetries of C^*-algebras*, Math. Ann. **352** (2012), 73–97.

8. ______, *A higher category approach to twisted actions on C^*-algebras*, Proc. Edinb. Math. Soc. **56** (2013), 387–426.

9. A.M. Cegarra and J. Remedios, *The relationship between the diagonal and the bar constructions on a bisimplicial set*, Topol. Appl. **153** (2005), 21–51.

10. A. Connes, *A survey of foliations and operator algebras*, in *Operator algebras and applications*, Part I, American Mathematical Society, Providence, R.I., 1982.

11. J.M.G. Fell, *Weak containment and induced representations of groups*, Canad. J. Math. **14** (1962), 237–268.

12. Philip Green, *The local structure of twisted covariance algebras*, Acta Math. **140** (1978), 191–250.

13. A. Guichardet, *Une caractèrisation des algèbres de von Neumann discrètes*, Bull. Soc. Math. France **89** (1961), 77–101.

14. A. Haefliger, *Orbi-espaces*, in *Sur les Groupes Hyperboliques d'après Mikhael Gromov*, Prog. Math. **83**, Birkhäuser, Boston, 1990.

15. P. Hahn, *Haar measure and convolution algebra on ergodic groupoids*, Ph.D. thesis, Harvard University, Cambridge, MA, 1975.

16. Z.J. Liu, A. Weinstein and P. Xu, *Manin triples for Lie bialgebroids*, J. Diff. Geom. **45** (1997), 547–574.

17. J.-H. Lu and A. Weinstein, *Poisson Lie groups, dressing transformations, and Bruhat decompositions*, J. Diff. Geom. **31** (1990), 501–526.

18. K.C.H. Mackenzie, *Double Lie algebroids and second-order geometry*, I, Adv. Math. **94** (1992), 180–239.

19. R.A. Mehta and X. Tang, *From double Lie groupoids to local Lie* 2*-groupoids*, Bull. Brazilian Math. Soc. **42** (2011), 651–681.

20. P.S. Muhly, *Bundles over groupoids*, in *Groupoids in analysis, geometry, and physics* American Mathematical Society, Providence, RI, 2001.

21. B. Noohi, *Notes on* 2*-groupoids,* 2*-groups and crossed modules*, Homol. Homot. Appl. **9** (2007), 75–106.

22. A.L.T. Paterson, *Groupoids, inverse semigroups, and their operators algebras*, Progr. Math. **170**, Birkhäuser, Boston, 1999.

23. A. Ramsay, *Virtual groups and group actions*, Adv. Math. **6** (1971), 253–322.

24. J. Renault, *A groupoid approach to C^*-algebras*, Lect. Notes Math. **793**, Springer-Verlag, Berlin, 1980.

25. M. Rieffel, *Induced representations of C^*-algebras*, Adv. Math. **13** (1974), 176–257.

26. L. Stefanini, *On the integration of LA-groupoids and duality for Poisson groupoids*, Trav. Math. **17** (2007), 39–59.

27. A. Weinstein, *Coisotropic calculus and Poisson groupoids*, J. Math. Soc. Japan **40** (1988), 705–727.

28. P. Xu, *Momentum maps and Morita equivalence*, J. Diff. Geom. **67** (2004), 289–333.

29. C. Zhu, *Kan replacement of simplicial manifolds*, Lett. Math. Phys. **90** (2009), 383–405.

Department of Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran 14115-134, Iran and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran 19395-5746, Iran
Email address: **mamini@ipm.ir, mamini@modares.ac.ir**