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PRECOVERS AND PREENVELOPES UNDER CHANGE OF RINGS

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ABSTRACT. Let $R \to S$ be a ring homomorphism. We analyze the relationship between the precovers (preenvelopes) in the category of right R-modules and the counterparts in the category of right S-modules. Some applications are also given.

1. Introduction. (Pre)covers and (pre)envelopes were introduced in the early 1980s by Enochs [**4**] and, independently, by Auslander and Smalø [**2**]. Let $\mathcal{C}$ be a class of right R-modules and M a right R-module. A homomorphism $\phi : C \to M$ is called a $\mathcal{C}$-*precover* of M if $C \in \mathcal{C}$ and the abelian group homomorphism $\phi_* : \mathrm{Hom}_R(C', C) \to \mathrm{Hom}_R(C', M)$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-precover $\phi : C \to M$ is said to be a $\mathcal{C}$-*cover* if every endomorphism $g : C \to C$ such that $\phi g = \phi$ is an isomorphism. Dually, we have the notions of a $\mathcal{C}$-preenvelope and a $\mathcal{C}$-envelope. In the representation theory of artin algebras, the usual terminology for a precover (respectively, a preenvelope) is a right (respectively, a left) approximation. Accordingly, the usual terminology for a cover (respectively, an envelope) is a minimal right (respectively, a minimal left) approximation. $\mathcal{C}$-covers and $\mathcal{C}$-envelopes may not exist in general, but, if they exist, they are unique up to isomorphism. (Pre)covers and (pre)envelopes provide a common framework for a number of classical notions such as projective covers and injective envelopes and turn out to be extremely fruitful for general module theory as well as for representation theory (see, e.g., [**2, 4, 5, 6, 13**]).

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Let $R \to S$ be a ring homomorphism. Then S becomes a canonical R-bimodule. Thus any right (respectively, left) S-module can be regarded as a right (respectively, left) R-module, and so a homomorphism of right (respectively, left) S-modules can also be regarded as a homomorphism of right (respectively, left) R-modules. Many results on the relationship between the (pre)covers ((pre)envelopes) in the category of R-modules and the counterparts in the category of S-modules have been obtained. For example, Würfel gave a characterization of those rings R such that, for any ring homomorphism $R \to S$, the functor $\mathrm{Hom}_R(S, -)$ preserves injective envelopes [**12**]. Dempsey, Oyonarte and Song furthermore studied those rings R such that for any ring homomorphism $R \to S$, the functor $\mathrm{Hom}_R(S, -)$ preserves injective envelopes or injective covers [**3**]. Zhou investigated relative preenvelopes under almost excellent extensions of rings [**15**]. In the present paper, we will consider more general settings, i.e., we will study the properties of relative (pre)covers and (pre)envelopes under change of rings using the functors $\mathrm{Hom}_R(S, -), \mathrm{Hom}_R(-, S)$ and $- \otimes_R S$.

Throughout this paper, all rings are associative with identity and all modules are unitary. M_R (respectively, ${}_RM$) denotes a right (respectively, left) R-module. $R \to S$ always means a ring homomorphism. For a right R-module M, we write $M^* = \mathrm{Hom}_R(M, S)$, $M^{**} = \mathrm{Hom}_S(\mathrm{Hom}_R(M, S), S)$. There exists a canonical evaluation map $\delta_M : M \to M^{**}$ defined by $\delta_M(x)(f) = f(x)$ for $x \in M$ and $f \in M^*$. M is called S-*reflexive* if δ_M is an isomorphism. All classes of modules are assumed to be closed under isomorphisms. For a class $\mathcal{C}_R$ of right R-modules, we write $\mathrm{Hom}_R(S, \mathcal{C}_R) = \{\mathrm{Hom}_R(S, L) : L \in \mathcal{C}_R\}$, $\mathcal{C}_R \otimes_R S = \{L \otimes_R S : L \in \mathcal{C}_R\}$, $(\mathcal{C}_R)^* = \{M^* : M \in \mathcal{C}_R\}$ and $(\mathcal{C}_R)^{**} = \{M^{**} : M \in \mathcal{C}_R\}$.

Let $R \to S$ be a ring homomorphism, M_R a right R-module and N_S a right S-module.

There are a natural R-homomorphism $\varepsilon_M : \mathrm{Hom}_R(S, M) \to M_R$ defined by $\varepsilon_M(f) = f(1)$ for $f \in \mathrm{Hom}_R(S, M)$ and a natural S-homomorphism $\eta_N : N_S \to \mathrm{Hom}_R(S, N)$ defined by $\eta_N(y)(t) = yt$ for $y \in N$ and $t \in S$. It is not hard to verify that the composition of R-homomorphisms $N_S \overset{\eta_N}{\to} \mathrm{Hom}_R(S, N) \overset{\varepsilon_N}{\to} N_R$ is an identity and the composition of S-homomorphisms $\mathrm{Hom}_R(S, M) \overset{\eta_{\mathrm{Hom}_R(S,M)}}{\to}$ $\mathrm{Hom}_R(S, \mathrm{Hom}_R(S, M)) \overset{(\varepsilon_M)_*}{\to} \mathrm{Hom}_R(S, M)$ is also an identity.

On the other hand, there are a natural S-homomorphism $\mu_N : N \otimes_R S \to N_S$ defined by $\mu_N(x \otimes t) = xt$ for $x \in N$ and $t \in S$ and a natural R-homomorphism $\nu_M : M_R \to M \otimes_R S$ defined by $\nu_M(y) = y \otimes 1$ for $y \in M$. It is easy to check that the composition of R-homomorphisms $N_R \stackrel{\nu_N}{\to} N \otimes_R S \stackrel{\mu_N}{\to} N_S$ is an identity and the composition of S-homomorphisms $M \otimes_R S \stackrel{\nu_M \otimes_R 1}{\to} (M \otimes_R S) \otimes_R S \stackrel{\mu_{M \otimes_R S}}{\to} M \otimes_R S$ is also an identity.

For unexplained concepts and notations, we refer the reader to [**1, 5, 6, 7, 10, 13**].

Let us describe the contents of the article in more detail.

In Section 2, we investigate the (pre)covers and (pre)envelopes under the covariant functor $\mathrm{Hom}_R(S, -)$ and the contravariant functor $\mathrm{Hom}_R(-, S)$. For example, let $R \to S$ be a ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathrm{Hom}_R(S, \mathcal{C}_R))_S \subseteq \mathcal{D}_S$. We prove that:

(i) Assume that $\varphi : M_R \to N_R$ is a right R-homomorphism with $M_R \in \mathcal{C}_R$. Then $\varphi_* : \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, N)$ is a $(\mathrm{Hom}_R(S, \mathcal{C}_R))_S$-precover if and only if $\varphi\varepsilon_M : \mathrm{Hom}_R(S, M) \to N_R$ is a $(\mathrm{Hom}_R(S, \mathcal{C}_R))_R$-precover.

(ii) Assume that $\varphi : M_S \to N_S$ is a right S-homomorphism with $N_S \in \mathcal{D}_S$. Then $\varphi_*\eta_M : M_S \to \mathrm{Hom}_R(S, N)$ is a $(\mathrm{Hom}_R(S, \mathcal{C}_R))_S$-preenvelope if and only if $\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-preenvelope.

As a consequence, we obtain the behavior of cotorsion (injective) precovers and preenvelopes under change of rings.

In the case of the functor $\mathrm{Hom}_R(-, S)$, let $\mathcal{C}_R$ be a class of right R-modules, ${}_S\mathcal{D}$ a class of left S-modules with $(\mathcal{C}_R)^* \subseteq {}_S\mathcal{D}$ and $({}_S\mathcal{D})^* \subseteq \mathcal{C}_R$, $\varphi : M_R \to N_R$ a right R-homomorphism with $N_R \in \mathcal{C}_R$. We obtain that $\varphi^* : N^* \to M^*$ is a $(\mathcal{C}_R)^*$-precover if and only if $\delta_N\varphi : M_R \to N^{**}$ is a $(\mathcal{C}_R)^{**}$-preenvelope.

Section 3 is devoted to the (pre)covers and (pre)envelopes under the covariant functor $- \otimes_R S$. For example, let $\varphi : R \to S$ be a ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathcal{C}_R \otimes_R S)_S \subseteq \mathcal{D}_S$. We prove that:

(i) Suppose that $\varphi : M_R \to N_R$ is a right R-homomorphism with $N_R \in \mathcal{C}_R$. Then $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ is a $(\mathcal{C}_R \otimes_R S)_S$-

preenvelope if and only if $\nu_N\varphi : M_R \to N \otimes_R S$ is a $(\mathcal{C}_R \otimes_R S)_R$-preenvelope.

(ii) Suppose that $\varphi : M_S \to N_S$ is a right S-homomorphism with $M_S \in \mathcal{D}_S$. Then $\mu_N(\varphi \otimes_R 1) : M \otimes_R S \to N_S$ is a $(\mathcal{C}_R \otimes_R S)_S$-precover if and only if $\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-precover.

As a consequence, we get the properties of flat (FP-injective) (pre)covers and (pre)envelopes under localization of rings.

In Section 4, we deal with the (pre)covers and (pre)envelopes under some kinds of special ring homomorphisms such as surjective ring homomorphisms and (almost) excellent extensions of rings. For example, let $R \to S$ be a surjective ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $\varphi : M_S \to N_S$ a right S-homomorphism. We prove that:

(i) If $(\mathrm{Hom}_R(S, \mathcal{C}_R))_S \subseteq \mathcal{D}_S$, then $\varphi : M_S \to N_S$ is a $\mathcal{D}_S$-preenvelope (respectively, $\mathcal{D}_S$-envelope) if and only if $\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-preenvelope (respectively, $\mathcal{C}_R$-envelope).

(ii) If $(\mathcal{C}_R \otimes_R S)_S \subseteq \mathcal{D}_S$, then $\varphi : M_S \to N_S$ is a $\mathcal{D}_S$-precover (respectively, $\mathcal{D}_S$-cover) if and only if $\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-precover (respectively, $\mathcal{C}_R$-cover).

On the other hand, let S be an excellent extension of a subring R, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $\varphi : M_S \to N_S$ a right S-homomorphism. We show that:

(a) If $(\mathrm{Hom}_R(S, \mathcal{C}_R))_S \subseteq \mathcal{D}_S$ and $N_S \in \mathcal{D}_S$, then $\varphi_* : \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, N)$ is a (respectively, special) $\mathcal{D}_S$-preenvelope if and only if $\varphi : M_R \to N_R$ is a (respectively, special) $\mathcal{C}_R$-preenvelope.

(b) If $(\mathcal{C}_R \otimes_R S)_S \subseteq \mathcal{D}_S$ and $M_S \in \mathcal{D}_S$, then $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ is a (respectively, special) $\mathcal{D}_S$-precover if and only if $\varphi : M_R \to N_R$ is a (respectively, special) $\mathcal{C}_R$-precover.

2. How do the functors $\mathrm{Hom}_R(S, -)$ and $\mathrm{Hom}_R(-, S)$ behave? We start with the following result.

Theorem 2.1. *Let $R \to S$ be a ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and*

$(\mathrm{Hom}_R(S,\mathcal{C}_R))_S \subseteq \mathcal{D}_S$, *$\varphi : M_R \to N_R$ a right R-homomorphism with $M_R \in \mathcal{C}_R$. Consider the following conditions*:

(i) *$\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-precover.*
(ii) *$\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is a $\mathcal{D}_S$-precover.*
(iii) *$\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is a $(\mathrm{Hom}_R(S,\mathcal{C}_R))_S$-precover.*
(iv) *$\varphi\varepsilon_M : \mathrm{Hom}_R(S,M) \to N_R$ is a $(\mathrm{Hom}_R(S,\mathcal{C}_R))_R$-precover.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof.

(i) $\Rightarrow$ (ii). For any $A_S \in \mathcal{D}_S$, we have $A_R \in \mathcal{C}_R$. So we get the exact sequence

$$\mathrm{Hom}_R(A \otimes_S S, M) \longrightarrow \mathrm{Hom}_R(A \otimes_S S, N) \longrightarrow 0.$$

Consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_R(A \otimes_S S, M) & \longrightarrow & \mathrm{Hom}_R(A \otimes_S S, N) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \\ \mathrm{Hom}_S(A, \mathrm{Hom}_R(S,M)) & \longrightarrow & \mathrm{Hom}_S(A, \mathrm{Hom}_R(S,N)). & & \end{array}$$

Since ψ_1 and ψ_2 are standard isomorphisms, we have the exact sequence

$$\mathrm{Hom}_S(A, \mathrm{Hom}_R(S,M)) \longrightarrow \mathrm{Hom}_S(A, \mathrm{Hom}_R(S,N)) \longrightarrow 0.$$

Thus, $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is a $\mathcal{D}_S$-precover since $\mathrm{Hom}_R(S,M) \in \mathcal{D}_S$.

(ii) $\Rightarrow$ (iii) is obvious since $(\mathrm{Hom}_R(S,\mathcal{C}_R))_S \subseteq \mathcal{D}_S$.

(iii) $\Rightarrow$ (iv). Let $G_R \in \mathcal{C}_R$ and $\zeta : \mathrm{Hom}_R(S,G) \to N_R$ be any R-homomorphism. Since $(\mathrm{Hom}_R(S,G))_R \in \mathcal{C}_R$, there exists $\sigma : \mathrm{Hom}_R(S, \mathrm{Hom}_R(S,G)) \to \mathrm{Hom}_R(S,M)$ such that the following diagram is commutative:

$$
\begin{array}{ccccc}
 & & \mathrm{Hom}_R(S,\mathrm{Hom}_R(S,G)) & & \\
 & \overset{\sigma}{\swarrow} & \downarrow \zeta_* & \overset{\varepsilon_{\mathrm{Hom}_R(S,G)}}{\searrow} & \\
\mathrm{Hom}_R(S,M) & \xrightarrow{\varphi_*} & \mathrm{Hom}_R(S,N) & & \mathrm{Hom}_R(S,G). \\
\downarrow \varepsilon_M & & \downarrow \varepsilon_N & \overset{\zeta}{\swarrow} & \\
M_R & \xrightarrow[\varphi]{} & N_R & &
\end{array}
$$

So we have

$$
\begin{aligned}
(\varphi\varepsilon_M)(\sigma\eta_{\mathrm{Hom}_R(S,G)}) &= \varepsilon_N(\varphi_*\sigma)\eta_{\mathrm{Hom}_R(S,G)} = \varepsilon_N\zeta_*\eta_{\mathrm{Hom}_R(S,G)} \\
&= \zeta\varepsilon_{\mathrm{Hom}_R(S,G)}\eta_{\mathrm{Hom}_R(S,G)} = \zeta.
\end{aligned}
$$

Thus, $\varphi\varepsilon_M : \mathrm{Hom}_R(S,M) \to N_R$ is a $(\mathrm{Hom}_R(S,\mathcal{C}_R))_R$-precover.

(iv) $\Rightarrow$ (iii). For any $F_R \in \mathcal{C}_R$, by (iv), we get the exact sequence

$$\mathrm{Hom}_R(\mathrm{Hom}_R(S,F),\mathrm{Hom}_R(S,M)) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(S,F),N) \longrightarrow 0.$$

Observe the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Hom}_R(\mathrm{Hom}_R(S,F),\mathrm{Hom}_R(S,M)) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(S,F),N). \\
\downarrow & \nearrow & \\
\mathrm{Hom}_R(\mathrm{Hom}_R(S,F),M) & &
\end{array}
$$

Then we obtain the exact sequence

$$\mathrm{Hom}_R(\mathrm{Hom}_R(S,F),M) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(S,F),N) \longrightarrow 0.$$

Consider the following commutative diagram:

$$
\begin{array}{ccccc}
\mathrm{Hom}_R(\mathrm{Hom}_R(S,F)\otimes_S S,M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(S,F)\otimes_S S,N) & \longrightarrow & 0 \\
\cong\downarrow & & \cong\downarrow & & \\
\mathrm{Hom}_S(\mathrm{Hom}_R(S,F),\mathrm{Hom}_R(S,M)) & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(S,F),\mathrm{Hom}_R(S,N)). & &
\end{array}
$$

Then we get the exact sequence

$$\mathrm{Hom}_S(\mathrm{Hom}_R(S,F),\mathrm{Hom}_R(S,M)) \to \mathrm{Hom}_S(\mathrm{Hom}_R(S,F),\mathrm{Hom}_R(S,N)) \to 0.$$

So $\mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is a $(\mathrm{Hom}_R(S,\mathcal{C}_R))_S$-precover. $\square$

Corollary 2.2. *Let $R \to S$ be a ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathrm{Hom}_R(S,\mathcal{C}_R))_S \subseteq \mathcal{D}_S$, $\varphi : M_R \to N_R$ a right R-homomorphism with $M_R \in \mathcal{C}_R$ and ε_M an isomorphism. Consider the following conditions*:

(i) $\varphi : M_R \to N_R$ *is a $\mathcal{C}_R$-cover.*
(ii) $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ *is a $\mathcal{D}_S$-cover.*
(iii) $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ *is a $(\mathrm{Hom}_R(S,\mathcal{C}_R))_S$-cover.*
(iv) $\varphi\varepsilon_M : \mathrm{Hom}_R(S,M) \to N_R$ *is a $(\mathrm{Hom}_R(S,\mathcal{C}_R))_R$-cover.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof.

(i) $\Rightarrow$ (ii). If the S-homomorphism $\chi : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,M)$ satisfies $\varphi_*\chi = \varphi_*$, then $\varphi(\varepsilon_M\chi\varepsilon_M^{-1}) = \varepsilon_N\varphi_*\chi\varepsilon_M^{-1} = \varepsilon_N\varphi_*\varepsilon_M^{-1} = \varphi\varepsilon_M\varepsilon_M^{-1} = \varphi$. Thus, $\varepsilon_M\chi\varepsilon_M^{-1}$ is an isomorphism, and so is χ. Therefore, $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is a $\mathcal{D}_S$-cover since φ_* is a $\mathcal{D}_S$-precover by Theorem 2.1.

(ii) $\Rightarrow$ (iii) is evident.

(iii) $\Rightarrow$ (iv). If the R-homomorphism $\omega : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,M)$ satisfies $\varphi\varepsilon_M\omega = \varphi$, then $\varphi_*(\varepsilon_M)_*\omega_* = \varphi_*$. Thus, $(\varepsilon_M)_*\omega_*$ is an isomorphism by (iii). So $\omega = \varepsilon_M\omega_*\varepsilon_M^{-1}$ is an isomorphism. We see that $\varphi\varepsilon_M : M_R \to N_R$ is a $(\mathrm{Hom}_R(S,\mathcal{C}_R))_R$-cover since $\varphi\varepsilon_M$ is a $(\mathrm{Hom}_R(S,\mathcal{C}_R))_R$-precover by Theorem 2.1.

(iv) $\Rightarrow$ (iii). Let the S-homomorphism $\xi : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,M)$ satisfy $\varphi_*\xi = \varphi_*$. Then $\varphi\varepsilon_M\xi = \varepsilon_N\varphi_*\xi = \varepsilon_N\varphi_* = \varphi\varepsilon_M$. Thus, ξ is an isomorphism, and so $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is a $(\mathrm{Hom}_R(S,\mathcal{C}_R))_S$-cover by Theorem 2.1. □

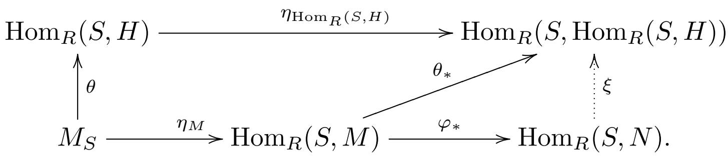
Theorem 2.3. *Let $R \to S$ be a ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathrm{Hom}_R(S,\mathcal{C}_R))_S \subseteq \mathcal{D}_S$, $\varphi : M_S \to N_S$ a right S-homomorphism with $N_S \in \mathcal{D}_S$. Consider the following conditions*:

(i) $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ *is a $\mathcal{D}_S$-preenvelope.*
(ii) $\varphi_*\eta_M : M_S \to \mathrm{Hom}_R(S,N)$ *is a $(\mathrm{Hom}_R(S,\mathcal{C}_R))_S$-preenvelope.*
(iii) $\varphi : M_R \to N_R$ *is a $\mathcal{C}_R$-preenvelope.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

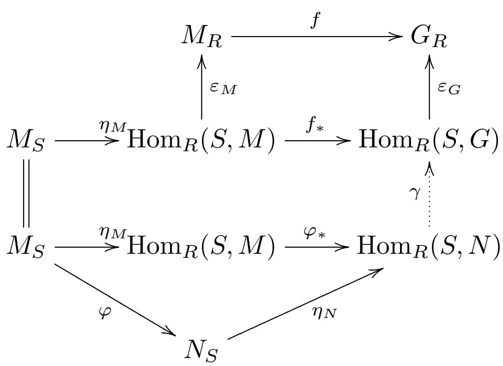
Proof.

(i) $\Rightarrow$ (ii). For any $H_R \in \mathcal{C}_R$, $\mathrm{Hom}_R(S, \mathrm{Hom}_R(S, H)) \in \mathcal{D}_S$ by hypothesis. So, for any S-homomorphism $\theta : M_S \to \mathrm{Hom}_R(S, H)$, there is an S-homomorphism $\xi : \mathrm{Hom}_R(S, N) \to \mathrm{Hom}_R(S, \mathrm{Hom}_R(S, H))$ such that the following diagram commutes.



Hence, we obtain $(\varepsilon_H)_*\xi(\varphi_*\eta_M) = (\varepsilon_H)_*\theta_*\eta_M = (\varepsilon_H)_*\eta_{\mathrm{Hom}_R(S,H)}\theta = \theta$. Thus, $\varphi_*\eta_M : M_S \to \mathrm{Hom}_R(S, N)$ is a $(\mathrm{Hom}_R(S, \mathcal{C}_R))_S$-preenvelope.

(ii) $\Rightarrow$ (iii). For any $G_R \in \mathcal{C}_R$ and any R-homomorphism $f : M_R \to G_R$, by (ii), there exists an S-homomorphism $\gamma : \mathrm{Hom}_R(S, N) \to \mathrm{Hom}_R(S, G)$ such that the following diagram commutes.



So we have

$$(\varepsilon_G\gamma\eta_N)\varphi = \varepsilon_G\gamma\varphi_*\eta_M = \varepsilon_G f_*\eta_M = f\varepsilon_M\eta_M = f.$$

Thus, $\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-preenvelope.

(iii) $\Rightarrow$ (ii). For any $H_R \in \mathcal{C}_R$ and any S-homomorphism $\theta : M_S \to \mathrm{Hom}_R(S, H)$, by (3), there exists an R-homomorphism $\psi : N_R \to \mathrm{Hom}_R(S, H)$ such that $\psi\varphi = \theta$.

Consider the following commutative diagram:

$$
\begin{array}{ccccc}
\mathrm{Hom}_R(S,H) & & \xrightarrow{\quad \eta_{\mathrm{Hom}_R(S,H)} \quad} & & \mathrm{Hom}_R(S,\mathrm{Hom}_R(S,H)) \\
\uparrow{\scriptstyle \theta} & & \nearrow{\scriptstyle \theta_*} & & \uparrow{\scriptstyle \psi_*} \\
M_S & \xrightarrow{\eta_M} & \mathrm{Hom}_R(S,M) & \xrightarrow{\varphi_*} & \mathrm{Hom}_R(S,N).
\end{array}
$$

It follows that $((\varepsilon_H)_*\psi_*)(\varphi_*\eta_M) = (\varepsilon_H)_*\theta_*\eta_M = (\varepsilon_H)_*\eta_{\mathrm{Hom}_R(S,H)}\theta = \theta$. Therefore, $\varphi_*\eta_M : M_S \to \mathrm{Hom}_R(S,N)$ is a $(\mathrm{Hom}_R(S,\mathcal{C}_R))_S$-preenvelope. $\square$

Corollary 2.4. *Let $R \to S$ be a ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathrm{Hom}_R(S,\mathcal{C}_R))_S \subseteq \mathcal{D}_S$, $\varphi : M_S \to N_S$ a right S-homomorphism with $N_S \in \mathcal{D}_S$ and ε_N an isomorphism. Consider the following conditions*:

(i) $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ *is a $\mathcal{D}_S$-envelope.*
(ii) $\varphi_*\eta_M : M_S \to \mathrm{Hom}_R(S,N)$ *is a* $(\mathrm{Hom}_R(S,\mathcal{C}_R))_S$*-envelope.*
(iii) $\varphi : M_R \to N_R$ *is a $\mathcal{C}_R$-envelope.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. Since ε_N is an R-isomorphism and $\varepsilon_N\eta_N = 1$, we have η_N is an S-isomorphism, and so $\varepsilon_N = \eta_N^{-1}$ is also an S-isomorphism. We claim that any R-homomorphism from N to N is also an S-homomorphism. In fact, let $\alpha : N \to N$ be any R-homomorphism and $\alpha_* : \mathrm{Hom}_R(S,N) \to \mathrm{Hom}_R(S,N)$ the induced S-homomorphism. Then $\alpha = \varepsilon_N\alpha_*\varepsilon_N^{-1}$ is an S-homomorphism. Thus, the result can be easily deduced from Theorem 2.3. $\square$

Corollary 2.5. *Let $R \to S$ be a ring homomorphism with ${}_RS$ flat.*

(i) [**5,** page 125, Exercise 5.4.4]. *If $\varphi : M_R \to N_R$ is an injective precover of R-modules, then $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is an injective precover of S-modules.*
(ii) *If $\varphi : M_S \to N_S$ a right S-homomorphism such that N_S is injective and $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is an injective preenvelope of S-modules, then $\varphi : M_R \to N_R$ is an injective preenvelope of R-modules.*

Proof. Since any injective right S-module is an injective right R-module by [**7,** Corollary 3.6A] and $\mathrm{Hom}_R(S,E)$ is an injective right S-module for any injective right R-module E by [**7,** Corollary 3.6B], the result holds by Theorems 2.1 and 2.3. □

Recall that a right R-module M is *cotorsion* [**5,** Definition 5.3.22] if $\mathrm{Ext}_R^1(F,M)=0$ for any flat right R-module F. It is known that every right R-module over any ring R has a cotorsion envelope by [**5,** Theorem 7.4.4] and [**13,** Theorem 3.4.6].

Lemma 2.6. *Let $R\to S$ be a ring homomorphism.*

(i) *If ${}_RS$ is flat, then any cotorsion right S-module is a cotorsion right R-module.*
(ii) *If S_R is flat, then* $\mathrm{Hom}_R(S,A)$ *is a cotorsion right S-module for any cotorsion right R-module A_R.*

Proof.

(i) Let N_S be a cotorsion right S-module and A_R a flat right R-module. Then $A\otimes_R S$ is a flat right S-module. From [**10,** Theorem 11.65], we have $\mathrm{Ext}_R^1(A,N)\cong\mathrm{Ext}_S^1(A\otimes_R S,N)=0$. So N_R is a cotorsion right R-module.

(ii) If B_S is a flat right S-module, then B_R is a flat right R-module. There is an exact sequence $0\to K_S\to P_S\to B_S\to 0$ of right S-modules with P_S projective. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_S(P,\mathrm{Hom}_R(S,A)) & \longrightarrow & \mathrm{Hom}_S(K,\mathrm{Hom}_R(S,A)) & \longrightarrow & \mathrm{Ext}_S^1(B,\mathrm{Hom}_R(S,A)) & \longrightarrow & 0\\
\cong\downarrow & & \cong\downarrow & & & & \\
\mathrm{Hom}_R(P\otimes_S S,A) & \longrightarrow & \mathrm{Hom}_R(K\otimes_S S,A) & \longrightarrow & \mathrm{Ext}_R^1(B\otimes_S S,A)=0. & &
\end{array}$$

So $\mathrm{Ext}_S^1(B,\mathrm{Hom}_R(S,A))=0$. Hence, $\mathrm{Hom}_R(S,A)$ is a cotorsion right S-module. □

Proposition 2.7. *Let $R\to S$ be a ring homomorphism with ${}_RS$ and S_R flat.*

(i) *If* $\varphi : M_R \to N_R$ *is a cotorsion precover of* R*-modules, then* $\varphi_* : \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, N)$ *is a cotorsion precover of* S*-modules.*

(ii) *If* $\varphi : M_S \to N_S$ *is a right* S*-homomorphism such that* N_S *is cotorsion and* $\varphi_* : \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, N)$ *is a cotorsion preenvelope of* S*-modules, then* $\varphi : M_R \to N_R$ *is a cotorsion preenvelope of* R*-modules.*

Proof. It follows from Theorems 2.1, 2.3 and Lemma 2.6. □

We next turn to the special precovers and preenvelopes under the change of rings.

Let $\mathcal{C}$ be a class of right R-modules. According to [**5,** Definition 7.1.6] or [**6,** Definition 2.1.12], a $\mathcal{C}$-precover $f : M \to N$ is called *special* if f is an epimorphism and $\ker(f) \in \mathcal{C}^\perp = \{X : \mathrm{Ext}^1_R(C, X) = 0 \text{ for all } C \in \mathcal{C}\}$. Dually, a $\mathcal{C}$-preenvelope $g : M \to N$ is called *special* if g is a monomorphism and $\mathrm{coker}(g) \in {}^\perp\mathcal{C} = \{X : \mathrm{Ext}^1_R(X, C) = 0 \text{ for all } C \in \mathcal{C}\}$.

Proposition 2.8. *Let* $R \to S$ *be a ring homomorphism with* S_R *projective,* $\mathcal{C}_R$ *a class of right* R*-modules,* $\mathcal{D}_S$ *a class of right* S*-modules with* $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ *and* $(\mathrm{Hom}_R(S, \mathcal{C}_R))_S \subseteq \mathcal{D}_S$*,* $\varphi : M_S \to N_S$ *a right* S*-homomorphism.*

(i) *If* $\varphi : M_R \to N_R$ *is a special* $\mathcal{C}_R$*-precover, then* $\varphi_* : \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, N)$ *is a special* $\mathcal{D}_S$*-precover.*

(ii) *If* $\varphi : M_S \to N_S$ *is a special* $\mathcal{D}_S$*-preenvelope, then* $\varphi : M_R \to N_R$ *is a special* $\mathcal{C}_R$*-preenvelope.*

Proof.

(i) By hypothesis, φ is epic. Thus, there is an exact sequence $0 \to K_S \to M_S \to N_S \to 0$ of right S-modules, which induces the exact sequence $0 \to K_R \to M_R \to N_R \to 0$ of right R-modules with $M_R \in \mathcal{C}_R$ and $K_R \in \mathcal{C}_R^\perp$.

Since S_R is projective, we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(S, K) \longrightarrow \mathrm{Hom}_R(S, M) \longrightarrow \mathrm{Hom}_R(S, N) \longrightarrow 0$$

of right S-modules with $\mathrm{Hom}_R(S, M) \in \mathcal{D}_S$.

For any $B_S \in \mathcal{D}_S$, we have $B_R \in \mathcal{C}_R$. Thus, $\mathrm{Ext}^1_S(B, \mathrm{Hom}_R(S,K)) \cong \mathrm{Ext}^1_R(B,K) = 0$ by [**10,** Theorem 11.66]. Hence, $\mathrm{Hom}_R(S,K) \in \mathcal{D}_S^{\perp}$, and so $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is a special $\mathcal{D}_S$-precover.

(ii) There is an exact sequence $0 \to M_S \to N_S \to L_S \to 0$ of right S-modules with $N_S \in \mathcal{D}_S$ and $L_S \in {}^{\perp}\mathcal{D}_S$. So we get the exact sequence $0 \to M_R \to N_R \to L_R \to 0$ of right R-modules with $N_R \in \mathcal{C}_R$.

For any $A_R \in \mathcal{C}_R$, $\mathrm{Hom}_R(S, A_R) \in \mathcal{D}_S$. Utilizing [**10,** Theorem 11.66], we get $\mathrm{Ext}^1_R(L,A) \cong \mathrm{Ext}^1_S(L, \mathrm{Hom}_R(S,A)) = 0$. Thus, $L_R \in {}^{\perp}\mathcal{C}_R$, and so $\varphi : M_R \to N_R$ is a special $\mathcal{C}_R$-preenvelope. □

At the end of this section, we point out that the functor $\mathrm{Hom}_R(-,S)$ converts a (pre)envelope into a (pre)cover under some conditions. The proofs of the next results are essentially dual to those of Theorem 2.1 and Corollary 2.2, and so we leave the proofs to the reader.

Theorem 2.9. *Let $R \to S$ be a ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, ${}_S\mathcal{D}$ a class of left S-modules with $(\mathcal{C}_R)^* \subseteq {}_S\mathcal{D}$ and $({}_S\mathcal{D})^* \subseteq \mathcal{C}_R$, $\varphi : M_R \to N_R$ a right R-homomorphism with $N_R \in \mathcal{C}_R$. Consider the following conditions*:

(i) $\varphi : M_R \to N_R$ *is a $\mathcal{C}_R$-preenvelope.*
(ii) $\varphi^* : N^* \to M^*$ *is a ${}_S\mathcal{D}$-precover.*
(iii) $\varphi^* : N^* \to M^*$ *is a $(\mathcal{C}_R)^*$-precover.*
(iv) $\delta_N\varphi : M_R \to N^{**}$ *is a $(\mathcal{C}_R)^{**}$-preenvelope.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

Corollary 2.10. *Let $R \to S$ be a ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, ${}_S\mathcal{D}$ a class of left S-modules with $(\mathcal{C}_R)^* \subseteq {}_S\mathcal{D}$ and $({}_S\mathcal{D})^* \subseteq \mathcal{C}_R$, $\varphi : M_R \to N_R$ a right R-homomorphism with N_R S-reflexive. Consider the following conditions*:

(i) $\varphi : M_R \to N_R$ *is a $\mathcal{C}_R$-envelope.*
(ii) $\varphi^* : N^* \to M^*$ *is a ${}_S\mathcal{D}$-cover.*
(iii) $\varphi^* : N^* \to M^*$ *is a $(\mathcal{C}_R)^*$-cover.*
(iv) $\delta_N\varphi : M_R \to N^{**}$ *is a $(\mathcal{C}_R)^{**}$-envelope.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

3. How does the functor $-\otimes_R S$ behave? By proofs analogous to the proofs of Theorem 2.1 and Corollary 2.2, one can prove that the functor $-\otimes_R S$ preserves (pre)envelopes under some conditions as follows.

Theorem 3.1. *Let $\varphi : R \to S$ be a ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathcal{C}_R \otimes_R S)_S \subseteq \mathcal{D}_S$, $\varphi : M_R \to N_R$ a right R-homomorphism with $N_R \in \mathcal{C}_R$. Consider the following conditions*:

(i) $\varphi : M_R \to N_R$ *is a $\mathcal{C}_R$-preenvelope.*
(ii) $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ *is a $\mathcal{D}_S$-preenvelope.*
(iii) $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ *is a $(\mathcal{C}_R \otimes_R S)_S$-preenvelope.*
(iv) $\nu_N \varphi : M_R \to N \otimes_R S$ *is a $(\mathcal{C}_R \otimes_R S)_R$-preenvelope.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

Corollary 3.2. *Let $\varphi : R \to S$ be a ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathcal{C}_R \otimes_R S)_S \subseteq \mathcal{D}_S$, $\varphi : M_R \to N_R$ a right R-homomorphism with $N_R \in \mathcal{C}_R$ and ν_N an isomorphism. Consider the following conditions*:

(i) $\varphi : M_R \to N_R$ *is a $\mathcal{C}_R$-envelope.*
(ii) $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ *is a $\mathcal{D}_S$-envelope.*
(iii) $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ *is a $(\mathcal{C}_R \otimes_R S)_S$-envelope.*
(iv) $\varphi : M_R \to N_R$ *is a $(\mathcal{C}_R \otimes_R S)_R$-envelope.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

Let S be a multiplicative subset of a commutative ring R. We can form the ring of fractions $S^{-1}R$. There is a canonical ring homomorphism $R \to S^{-1}R$. For an R-module M, we also can construct the localization of M with respect to S, denoted by $S^{-1}M$, which is an $S^{-1}R$-module, and hence an R-module.

Corollary 3.3. *Let S be a multiplicative subset of a commutative ring R and $\varphi : M_R \to N_R$ an R-homomorphism.*

(i) *If $\varphi : M_R \to N_R$ is a flat preenvelope of R-modules, then $S^{-1}\varphi : S^{-1}M \to S^{-1}N$ is a flat preenvelope of $S^{-1}R$-modules.*

(ii) *If* $S^{-1}\varphi : S^{-1}M \to S^{-1}N$ *is a flat envelope of* R*-modules, then* $S^{-1}\varphi : S^{-1}M \to S^{-1}N$ *is a flat envelope of* $S^{-1}R$*-modules.*

Proof.

(i) Note that any flat $S^{-1}R$-module is a flat R-module and $A \otimes_R S^{-1}R \cong S^{-1}A$ is a flat $S^{-1}R$-module for any flat R-module A. Then Theorem 3.1 applies.

(ii) Since $\nu_{S^{-1}N} : S^{-1}N \to S^{-1}N \otimes_R S^{-1}R$ is an isomorphism by [**10,** Lemma 3.75], the conclusion follows from Corollary 3.2. □

Recall that a right R-module M is *FP-injective* [**11**] if $\mathrm{Ext}^1_R(N, M) = 0$ for all finitely presented right R-modules N.

Lemma 3.4. *Let* $R \to S$ *be a ring homomorphism with* ${}_RS$ *flat. Then any* FP*-injective right* S*-module is an* FP*-injective right* R*-module.*

Proof. Let N_S be an FP-injective right S-module and A_R a finitely presented right R-module. Then $A \otimes_R S$ is a finitely presented right S-module. From [**10,** Theorem 11.65], $\mathrm{Ext}^1_R(A, N) \cong \mathrm{Ext}^1_S(A \otimes_R S, N) = 0$. So N_R is an FP-injective right R-module. □

It is known that every right R-module has an FP-injective preenvelope over any ring R by [**5,** Proposition 6.2.4], but not every right R-module has an FP-injective envelope by [**6,** Corollary 6.3.19].

Recall that R is a *right coherent ring* if every finitely generated right ideal is finitely presented.

Proposition 3.5. *Let* S *be a multiplicative subset of a commutative coherent ring* R *and* $\varphi : M_R \to N_R$ *an* R*-homomorphism.*

(i) *If* $\varphi : M_R \to N_R$ *is an* FP*-injective preenvelope of* R*-modules, then* $S^{-1}\varphi : S^{-1}M \to S^{-1}N$ *is an* FP*-injective preenvelope of* $S^{-1}R$*-modules.*

(ii) *If* $S^{-1}\varphi : S^{-1}M \to S^{-1}N$ *is an* FP*-injective envelope of* R*-modules, then* $S^{-1}\varphi : S^{-1}M \to S^{-1}N$ *is an* FP*-injective envelope of* $S^{-1}R$*-modules.*

Proof. By Lemma 3.4, any FP-injective $S^{-1}R$-module is an FP-injective R-module. By [**9,** Theorem 3.21], $B \otimes S^{-1}R \cong S^{-1}B$ is an FP-injective $S^{-1}R$-module for any FP-injective R-module B since R is a coherent ring. So (i) holds by Theorem 3.1 and (ii) follows from Corollary 3.2. $\square$

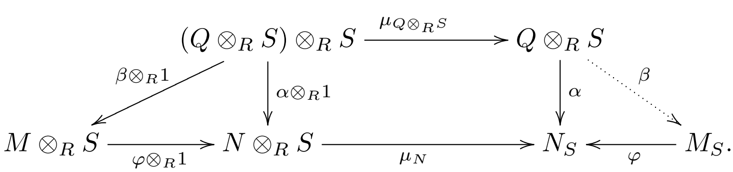
Theorem 3.6. *Let $\varphi : R \to S$ be a ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathcal{C}_R \otimes_R S)_S \subseteq \mathcal{D}_S$, $\varphi : M_S \to N_S$ a right S-homomorphism with $M_S \in \mathcal{D}_S$. Consider the following conditions*:

(i) *$\varphi : M_S \to N_S$ is a $\mathcal{D}_S$-precover.*
(ii) *$\mu_N(\varphi \otimes_R 1) : M \otimes_R S \to N_S$ is a $(\mathcal{C}_R \otimes_R S)_S$-precover.*
(iii) *$\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-precover.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

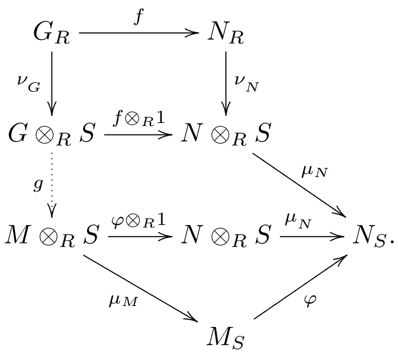
Proof.

(i) $\Rightarrow$ (ii). For any $Q_R \in \mathcal{C}_R$ and any S-homomorphism $\alpha : Q \otimes_R S \to N_S$, by (i), there exists $\beta : Q \otimes_R S \to M_S$ such that the following diagram is commutative.



Thus, we obtain $\mu_N(\varphi \otimes_R 1)(\beta \otimes_R 1)(\nu_Q \otimes_R 1) = \mu_N(\alpha \otimes_R 1)(\nu_Q \otimes_R 1) = \alpha \mu_{Q \otimes_R S}(\nu_Q \otimes_R 1) = \alpha$. So $\mu_N(\varphi \otimes_R 1) : M \otimes_R S \to N_S$ is a $(\mathcal{C}_R \otimes_R S)_S$-precover.

(ii) $\Rightarrow$ (iii). For any $G_R \in \mathcal{C}_R$ and any R-homomorphism $f : G_R \to N_R$, there is an S-homomorphism $g : G \otimes_R S \to M \otimes_R S$ such that the following diagram commutes.



This implies that $\varphi(\mu_M g \nu_G) = \mu_N(\varphi \otimes_R 1) g \nu_G = \mu_N(f \otimes_R 1)\nu_G = \mu_N \nu_N f = f$. So $\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-precover.

The proof of (iii) $\Rightarrow$ (ii) is similar to that of (i) $\Rightarrow$ (ii). $\square$

Corollary 3.7. *Let $\varphi : R \to S$ be a ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathcal{C}_R \otimes_R S)_S \subseteq \mathcal{D}_S$, $\varphi : M_S \to N_S$ a right S-homomorphism with $M_S \in \mathcal{D}_S$ and ν_M an isomorphism. Consider the following conditions*:

(i) *$\varphi : M_S \to N_S$ is a $\mathcal{D}_S$-cover.*
(ii) *$\mu_N(\varphi \otimes_R 1) : M \otimes_R S \to N_S$ is a $(\mathcal{C}_R \otimes_R S)_S$-cover.*
(iii) *$\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-cover.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. Since ν_M is an isomorphism and $\mu_M \nu_M = 1$, we have ν_M is an S-isomorphism. It is easy to see that any R-homomorphism from M to M is also an S-homomorphism, and so the result is a consequence of Theorem 3.6. $\square$

It is known that every right R-module over any ring R has a flat cover by [**5,** Theorem 7.4.4] and every right R-module over a right coherent ring R has an FP-injective cover by [**9,** Theorem 4.9] and the remark following it.

Corollary 3.8. *Let S be a multiplicative subset of a commutative ring R and $\varphi : M_R \to N_R$ an R-homomorphism.*

(i) *$S^{-1}\varphi : S^{-1}M \to S^{-1}N$ is a flat precover (respectively, flat cover) of R-modules if and only if $S^{-1}\varphi : S^{-1}M \to S^{-1}N$ is a flat precover (respectively, flat cover) of $S^{-1}R$-modules.*

(ii) *If R is coherent, then $S^{-1}\varphi : S^{-1}M \to S^{-1}N$ is an FP-injective precover (respectively, FP-injective cover) of R-modules if and only if $S^{-1}\varphi : S^{-1}M \to S^{-1}N$ is an FP-injective precover (respectively, FP-injective cover) of $S^{-1}R$-modules.*

Proof. This is a direct consequence of Theorem 3.6 together with Corollary 3.7. □

Proposition 3.9. *Let $R \to S$ be a ring homomorphism with ${}_RS$ flat, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathcal{C}_R \otimes_R S)_S \subseteq \mathcal{D}_S$, $\varphi : M_S \to N_S$ a right S-homomorphism.*

(i) *If $\varphi : M_S \to N_S$ is a special $\mathcal{D}_S$-precover, then $\varphi : M_R \to N_R$ is a special $\mathcal{C}_R$-precover.*

(ii) *If $\varphi : M_R \to N_R$ is a special $\mathcal{C}_R$-preenvelope, then $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ is a special $\mathcal{D}_S$-preenvelope.*

Proof.

(i) There is an exact sequence $0 \to K_S \to M_S \to N_S \to 0$ of right S-modules with $M_S \in \mathcal{D}_S$ and $K_S \in \mathcal{D}_S^{\perp}$. Hence, we get the exact sequence $0 \to K_R \to M_R \to N_R \to 0$ of right R-modules with $M_R \in \mathcal{C}_R$.

For any $A_R \in \mathcal{C}_R, A \otimes_R S \in \mathcal{D}_S$. By **[10,** Theorem 11.65], we have $\mathrm{Ext}_R^1(A, K) \cong \mathrm{Ext}_S^1(A \otimes_R S, K) = 0$. So $K_R \in \mathcal{C}_R^{\perp}$. Thus $\varphi : M_R \to N_R$ is a special $\mathcal{C}_R$-precover.

(ii) By hypothesis, φ is monic. Thus, there is an exact sequence $0 \to M_S \to N_S \to L_S \to 0$ of right S-modules, which gives rise to the exactness of the sequence $0 \to M_R \to N_R \to L_R \to 0$ of right R-modules with $N_R \in \mathcal{C}_R$ and $L_R \in {}^{\perp}\mathcal{C}_R$.

Since ${}_RS$ is flat, we obtain the exact sequence $0 \to M \otimes_R S \to N \otimes_R S \to L \otimes_R S \to 0$ of right S-modules with $N \otimes_R S \in \mathcal{D}_S$.

For any $B_S \in \mathcal{D}_S$, $B_R \in \mathcal{C}_R$. By **[10,** Theorem 11.65], we have $\mathrm{Ext}^1_S(L \otimes_R S, B) \cong \mathrm{Ext}^1_R(L, B) = 0$. Hence, $L \otimes_R S \in {}^{\perp}\mathcal{D}_S$, and so $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ is a special $\mathcal{D}_S$-preenvelope. □

4. Special ring homomorphisms. In this section, we apply the results of the previous sections to study the properties of (pre)covers and (pre)envelopes under some special ring homomorphisms.

We first discuss the (pre)covers and (pre)envelopes under surjective ring homomorphisms. The following lemma is needed.

Lemma 4.1. *Let $R \to S$ be a surjective ring homomorphism and M_S a right S-module. Then $\mathrm{Hom}_R(S, M) \cong M_S \cong M \otimes_R S$.*

Proof. It is routine. □

Theorem 4.2. *Let $R \to S$ be a surjective ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathrm{Hom}_R(S, \mathcal{C}_R))_S \subseteq \mathcal{D}_S$, $\varphi : M_S \to N_S$ a right S-homomorphism.*

(i) *$\varphi : M_S \to N_S$ is a $\mathcal{D}_S$-preenvelope (respectively, $\mathcal{D}_S$-envelope) if and only if $\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-preenvelope (respectively, $\mathcal{C}_R$-envelope).*

(ii) *If $\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-precover (respectively, $\mathcal{C}_R$-cover), then $\varphi : M_S \to N_S$ is a $\mathcal{D}_S$-precover (respectively, $\mathcal{D}_S$-cover).*

Proof.

(i) follows from Theorem 2.3, Corollary 2.4 and Lemma 4.1.

(ii) holds by Theorem 2.1, Corollary 2.2 and Lemma 4.1. □

Immediately, we have

Corollary 4.3. *Let $R \to S$ be a surjective ring homomorphism with ${}_RS$ flat, $\varphi : M_S \to N_S$ a right S-homomorphism.*

(i) *$\varphi : M_S \to N_S$ is an injective preenvelope (respectively, injective envelope) of S-modules if and only if $\varphi : M_R \to N_R$ is an injective preenvelope (respectively, injective envelope) of R-modules.*

(ii) *If* $\varphi : M_R \to N_R$ *is an injective precover* (*respectively, injective cover*) *of* R*-modules, then* $\varphi : M_S \to N_S$ *is an injective precover* (*respectively, injective cover*) *of* S*-modules.*

Corollary 4.4. *Let* $R \to S$ *be a surjective ring homomorphism with* ${}_RS$ *flat and* S_R *projective,* $\varphi : M_S \to N_S$ *a right* S*-homomorphism.*

(i) $\varphi : M_S \to N_S$ *is an* FP*-injective preenvelope* (*respectively,* FP*-injective envelope*) *of* S*-modules if and only if* $\varphi : M_R \to N_R$ *is an* FP*-injective preenvelope* (*respectively,* FP*-injective envelope*) *of* R*-modules.*

(ii) *If* $\varphi : M_R \to N_R$ *is an* FP*-injective precover* (*respectively,* FP*-injective cover*) *of* R*-modules, then* $\varphi : M_S \to N_S$ *is an* FP*-injective precover* (*respectively,* FP*-injective cover*) *of* S*-modules.*

Proof. We first claim that $\mathrm{Hom}_R(S, A)$ is an FP-injective right S-module for any FP-injective right R-module A_R. Indeed, if B_S is a finitely presented right S-module, then B_R is a finitely presented right R-module. Thus, $\mathrm{Ext}^1_S(B, \mathrm{Hom}_R(S, A)) \cong \mathrm{Ext}^1_R(B, A) = 0$ by [**10,** Theorem 11.66]. So $\mathrm{Hom}_R(S, A)$ is an FP-injective right S-module.

In addition, any FP-injective right S-module is an FP-injective right R-module by Lemma 3.4. So the result is a direct consequence of Theorem 4.2. □

Corollary 4.5. *Let* $R \to S$ *be a surjective ring homomorphism with* S_R *flat,* $\varphi : M_S \to N_S$ *a right* S*-homomorphism.*

(i) $\varphi : M_S \to N_S$ *is a cotorsion preenvelope* (*respectively, cotorsion envelope*) *of* S*-modules if and only if* $\varphi : M_R \to N_R$ *is a cotorsion preenvelope* (*respectively, cotorsion envelope*) *of* R*-modules.*

(ii) *If* $\varphi : M_R \to N_R$ *is a cotorsion precover* (*respectively, cotorsion cover*) *of* R*-modules, then* $\varphi : M_S \to N_S$ *is a cotorsion precover* (*respectively, cotorsion cover*) *of* S*-modules.*

Proof. By Lemma 2.6, $\mathrm{Hom}_R(S, A)$ is a cotorsion right S-module for any cotorsion right R-module A_R. In addition, any cotorsion right S-module is a cotorsion right R-module by [**13,** Proposition 3.3.3]. So the result follows from Theorem 4.2. □

Theorem 4.6. *Let $R \to S$ be a surjective ring homomorphism, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathcal{C}_R \otimes_R S)_S \subseteq \mathcal{D}_S$, $\varphi : M_S \to N_S$ a right S-homomorphism.*

(i) *$\varphi : M_S \to N_S$ is a $\mathcal{D}_S$-precover (respectively, $\mathcal{D}_S$-cover) if and only if $\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-precover (respectively, $\mathcal{C}_R$-cover).*
(ii) *If $\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-preenvelope (respectively, $\mathcal{C}_R$-envelope), then $\varphi : M_S \to N_S$ is a $\mathcal{D}_S$-preenvelope (respectively, $\mathcal{D}_S$-envelope).*

Proof.

(i) follows from Theorem 3.6, Corollary 3.7 and Lemma 4.1.

(ii) holds by Theorem 3.1, Corollary 3.2 and Lemma 4.1. □

Following Theorem 4.6, we have

Corollary 4.7. *Let $R \to S$ be a surjective ring homomorphism with S_R flat, $\varphi : M_S \to N_S$ a right S-homomorphism.*

(i) *$\varphi : M_S \to N_S$ is a flat precover (respectively, flat cover) of S-modules if and only if $\varphi : M_R \to N_R$ is a flat precover (respectively, flat cover) of R-modules.*
(ii) *If $\varphi : M_R \to N_R$ is a flat preenvelope (respectively, flat envelope) of R-modules, then $\varphi : M_S \to N_S$ is a flat preenvelope (respectively, flat envelope) of S-modules.*

Corollary 4.8. *Let $R \to S$ be a surjective ring homomorphism with S_R projective, $\varphi : M_S \to N_S$ a right S-homomorphism.*

(i) *$\varphi : M_S \to N_S$ is a projective precover (respectively, projective cover) of S-modules if and only if $\varphi : M_R \to N_R$ is a projective precover (respectively, projective cover) of R-modules.*
(ii) *If $\varphi : M_R \to N_R$ is a projective preenvelope (respectively, projective envelope) of R-modules, then $\varphi : M_S \to N_S$ is a projective preenvelope (respectively, projective envelope) of S-modules.*

Next, we consider the (pre)covers and (pre)envelopes under (almost) excellent extensions of rings. Recall that a ring S is said to be an *almost* *excellent extension* of a subring R [**14**] if the following conditions are satisfied:

(i) S is a *finite normalizing extension* of R, namely, R and S have the same identity and there are elements $s_1, \ldots, s_n \in S$ such that $S = Rs_1 + \cdots + Rs_n$ and $Rs_i = s_iR$ for all $i = 1, \ldots, n$.
(ii) ${}_RS$ is flat and S_R is projective.
(iii) S is *right R-projective*, namely, if M_S is a submodule of N_S and M_R is a direct summand of N_R, then M_S is a direct summand of N_S.

Further, S is called an *excellent extension* of R [**8**] if S is an almost excellent extension of R and S is free with basis $s_1, \ldots, s_n$ as both a right and a left R-module with $s_1 = 1_R$.

Theorem 4.9. *Let S be an almost excellent extension of a subring R, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathrm{Hom}_R(S, \mathcal{C}_R))_S \subseteq \mathcal{D}_S$, $\varphi : M_R \to N_R$ a right R-homomorphism with $M_R \in \mathcal{C}_R$. Then the following are equivalent*:

(i) *$\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-precover.*
(ii) *$\varphi_* : \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, N)$ is a $\mathcal{D}_S$-precover.*
(iii) *$\varphi_* : \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, N)$ is a $(\mathrm{Hom}_R(S, \mathcal{C}_R))_S$-precover.*
(iv) *$\varphi\varepsilon_M : \mathrm{Hom}_R(S, M) \to N_R$ is a $(\mathrm{Hom}_R(S, \mathcal{C}_R))_R$-precover.*

Proof. By Theorem 2.1, it is enough to show that (iv) $\Rightarrow$ (i).

Let $G_R \in \mathcal{C}_R$. By [**15,** Lemma 2.6], there exists a positive integer t such that G_R is isomorphic to a direct summand of $(\mathrm{Hom}_R(S, G))^t$.

By (iv), we get the exact sequence

$$\mathrm{Hom}_R(\mathrm{Hom}_R(S, G), \mathrm{Hom}_R(S, M)) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(S, G), N) \longrightarrow 0,$$

which induces the exact sequence

$$\mathrm{Hom}_R(\mathrm{Hom}_R(S, G)^t, \mathrm{Hom}_R(S, M)) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(S, G)^t, N) \longrightarrow 0.$$

So we obtain the exact sequence

$$\mathrm{Hom}_R(G, \mathrm{Hom}_R(S, M) \longrightarrow \mathrm{Hom}_R(G, N) \longrightarrow 0,$$

which gives the exactness of the sequence $\mathrm{Hom}_R(G, M) \to \mathrm{Hom}_R(G, N) \to 0$. Thus, $\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-precover. $\square$

Theorem 4.10. *Let S be an excellent extension of a subring R, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathrm{Hom}_R(S, \mathcal{C}_R))_S \subseteq \mathcal{D}_S$, $\varphi : M_S \to N_S$ a right S-homomorphism with $N_S \in \mathcal{D}_S$. Then the following are equivalent*:

(i) $\varphi_* : \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, N)$ *is a $\mathcal{D}_S$-preenvelope.*
(ii) $\varphi_* \eta_M : M_S \to \mathrm{Hom}_R(S, N)$ *is a $(\mathrm{Hom}_R(S, \mathcal{C}_R))_S$-preenvelope.*
(iii) $\varphi : M_R \to N_R$ *is a $\mathcal{C}_R$-preenvelope.*
(iv) $\varphi : M_S \to N_S$ *is a $\mathcal{D}_S$-preenvelope.*

Proof. (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) follow from Theorem 2.3.

(iii) $\Rightarrow$ (i). For any $A_S \in \mathcal{D}_S$, $A_R \in \mathcal{C}_R$. Thus, we get the exact sequence $\mathrm{Hom}_R(N, A) \to \mathrm{Hom}_R(M, A) \to 0$. Since S is an excellent extension of R, we have ${}_R S_R \cong {}_R R^n_R$. Thus $\mathrm{Hom}_R(S, N) \cong N^n_R$ and $\mathrm{Hom}_R(S, M) \cong M^n_R$. So we get the exact sequence

$$\mathrm{Hom}_R(\mathrm{Hom}_R(S, N) \otimes_S S, A) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(S, M) \otimes_S S, A) \longrightarrow 0,$$

which induces the exact sequence

$$\mathrm{Hom}_S(\mathrm{Hom}_R(S, N), \mathrm{Hom}_R(S, A)) \longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_R(S, M), \mathrm{Hom}_R(S, A)) \longrightarrow 0.$$

Since A_S is isomorphic to a direct summand of $\mathrm{Hom}_R(S, A)$ by [**14,** Lemma 1.1], we have the exact sequence

$$\mathrm{Hom}_S(\mathrm{Hom}_R(S, N), A_S) \longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_R(S, M), A_S) \longrightarrow 0.$$

So $\varphi_* : \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, N)$ is a $\mathcal{D}_S$-preenvelope.

(i) $\Rightarrow$ (iv) $\Rightarrow$ (ii) are straightforward by [**14,** Lemma 1.1]. □

Theorem 4.11. *Let S be an excellent extension of a subring R, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathcal{C}_R \otimes_R S)_S \subseteq \mathcal{D}_S$, $\varphi : M_S \to N_S$ a right S-homomorphism with $M_S \in \mathcal{D}_S$. Then the following are equivalent*:

(i) $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ *is a $\mathcal{D}_S$-precover.*
(ii) $\mu_N(\varphi \otimes_R 1) : M \otimes_R S \to N_S$ *is a $(\mathcal{C}_R \otimes_R S)_S$-precover.*
(iii) $\varphi : M_R \to N_R$ *is a $\mathcal{C}_R$-precover.*
(iv) $\varphi : M_S \to N_S$ *is a $\mathcal{D}_S$-precover.*

Proof. (i) $\Rightarrow$ (iv) is easy by **[14,** Lemma 1.1]. (iv) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) hold by Theorem 3.6.

(iii) $\Rightarrow$ (i). For any $A_S \in \mathcal{D}_S$, $A_R \in \mathcal{C}_R$. Therefore, we get the exact sequence $\mathrm{Hom}_R(A, M) \to \mathrm{Hom}_R(A, N) \to 0$. Since ${}_RS_R \cong {}_RR^n_R$, we have $M \otimes_R S \cong M^n_R$ and $N \otimes_R S \cong N^n_R$. So we get the exact sequence

$$\mathrm{Hom}_R(A, \mathrm{Hom}_S(S, M \otimes_R S)) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(S, N \otimes_R S)) \to 0,$$

which induces the exact sequence

$$\mathrm{Hom}_S(A \otimes_R S, M \otimes_R S) \longrightarrow \mathrm{Hom}_S(A \otimes_R S, N \otimes_R S) \longrightarrow 0.$$

Since A_S is isomorphic to a direct summand of $A \otimes_R S$, by **[14,** Lemma 1.1], we have the exact sequence $\mathrm{Hom}_S(A_S, M \otimes_R S) \to \mathrm{Hom}_S(A_S, N \otimes_R S) \to 0$. Thus, $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ is a $\mathcal{D}_S$-precover. $\square$

Theorem 4.12. *Let S be an excellent extension of a subring R, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathcal{C}_R \otimes_R S)_S \subseteq \mathcal{D}_S$, $\varphi : M_R \to N_R$ a right R-homomorphism with $N_R \in \mathcal{C}_R$. Then the following are equivalent*:

(i) *$\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-preenvelope.*
(ii) *$\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ is a $\mathcal{D}_S$-preenvelope.*
(iii) *$\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ is a $(\mathcal{C}_R \otimes_R S)_S$-preenvelope.*
(iv) *$\nu_N\varphi : M_R \to N \otimes_R S$ is a $(\mathcal{C}_R \otimes_R S)_R$-preenvelope.*

Proof. By Theorem 3.1, it is enough to show that (iv) $\Rightarrow$ (i).

Let $G_R \in \mathcal{C}_R$. Since ${}_RS_R \cong {}_RR^n_R$, $G \otimes_R S \cong G^n_R$. By (iv), we get the exact sequence

$$\mathrm{Hom}_R(N \otimes_R S, G \otimes_R S) \longrightarrow \mathrm{Hom}_R(M, G \otimes_R S) \longrightarrow 0,$$

which induces the exact sequence

$$\mathrm{Hom}_R(N \otimes_R S, G)^n \longrightarrow \mathrm{Hom}_R(M, G)^n \longrightarrow 0.$$

Thus, we obtain the exact sequence

$$\mathrm{Hom}_R(N \otimes_R S, G) \longrightarrow \mathrm{Hom}_R(M, G) \to 0,$$

which yields the exactness of the sequence $\mathrm{Hom}_R(N, G) \to \mathrm{Hom}_R(M, G) \to 0$. So $\varphi : M_R \to N_R$ is a $\mathcal{C}_R$-preenvelope. $\square$

Finally, we investigate the special preenvelopes and precovers under (almost) excellent extensions.

Theorem 4.13. *Let S be an almost excellent extension of a subring R, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with* $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ *and* $(\mathrm{Hom}_R(S,\mathcal{C}_R))_S \subseteq \mathcal{D}_S$, $\varphi : M_S \to N_S$ *a right S-homomorphism with $N_S \in \mathcal{D}_S$. Consider the following conditions*:

(i) $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ *is a special $\mathcal{D}_S$-preenvelope.*
(ii) $\varphi : M_S \to N_S$ *is a special $\mathcal{D}_S$-preenvelope.*
(iii) $\varphi : M_R \to N_R$ *is a special $\mathcal{C}_R$-preenvelope.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii)*. Moreover, if S is an excellent extension of R, then* (ii) $\Rightarrow$ (i)*.*

Proof.

(i) $\Rightarrow$ (ii). Since $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is monic, φ is monic by **[14,** Lemma 1.1]. So there is an exact sequence $0 \to M_S \to N_S \to L_S \to 0$ of right S-modules. Applying the functor $\mathrm{Hom}_R(S,-)$ to it, we get the exact sequence

$$0 \to \mathrm{Hom}_R(S,M) \longrightarrow \mathrm{Hom}_R(S,N) \longrightarrow \mathrm{Hom}_R(S,L) \longrightarrow 0$$

of right S-modules with $\mathrm{Hom}_R(S,L) \in {}^{\perp}\mathcal{D}_S$. Since L_S is isomorphic to a direct summand of $\mathrm{Hom}_R(S,L)$. $L_S \in {}^{\perp}\mathcal{D}_S$. So $\varphi : M_S \to N_S$ is a special $\mathcal{D}_S$-preenvelope.

(ii) $\Rightarrow$ (iii) follows from Proposition 2.8 (ii).

(iii) $\Rightarrow$ (ii). By (iii), there is an exact sequence $0 \to M_S \to N_S \to L_S \to 0$ of right S-modules with $L_R \in {}^{\perp}\mathcal{C}_R$. For any $B_S \in \mathcal{D}_S$, $B_R \in \mathcal{C}_R$. Thus, by **[10,** Theorem 11.66], we have $\mathrm{Ext}^1_S(L,\mathrm{Hom}_R(S,B)) \cong \mathrm{Ext}^1_R(L,B) = 0$. But, B_S is isomorphic to a direct summand of $\mathrm{Hom}_R(S,B)$. Hence, $\mathrm{Ext}^1_S(L,B) = 0$, and so $L_S \in {}^{\perp}\mathcal{D}_S$. Thus $\varphi : M_S \to N_S$ is a special $\mathcal{D}_S$-preenvelope.

(ii) $\Rightarrow$ (i). There is an exact sequence $0 \to M_S \to N_S \to L_S \to 0$ of right S-modules with $L_S \in {}^{\perp}\mathcal{D}_S$, which induces the right S-module exact sequence $0 \to \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N) \to \mathrm{Hom}_R(S,L) \to 0$. Since ${}_RS_R \cong {}_RR^n_R$, $\mathrm{Hom}_R(S,L) \cong L^n_R$. Let $A_S \in \mathcal{D}_S$. Then $\mathrm{Hom}_R(S,A) \in \mathcal{D}_S$. So $\mathrm{Ext}^1_S(\mathrm{Hom}_R(S,L),\ \mathrm{Hom}_R(S,A)) \cong \mathrm{Ext}^1_R(L^n,A) \cong \mathrm{Ext}^1_R(L,A)^n \cong \mathrm{Ext}^1_S(L,\mathrm{Hom}_R(S,A))^n = 0$. Since A_S

is isomorphic to a direct summand of $\mathrm{Hom}_R(S,A), \mathrm{Hom}_R(S,L) \in {}^{\perp}\mathcal{D}_S$. It follows that $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is a special $\mathcal{D}_S$-preenvelope since $\mathrm{Hom}_R(S,N) \in \mathcal{D}_S$. □

Theorem 4.14. *Let S be an almost excellent extension of a subring R, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathrm{Hom}_R(S,\mathcal{C}_R))_S \subseteq \mathcal{D}_S$, $\varphi : M_S \to N_S$ a right S-homomorphism with $M_S \in \mathcal{D}_S$. Consider the following conditions*:

(i) $\varphi : M_R \to N_R$ *is a special $\mathcal{C}_R$-precover.*
(ii) $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ *is a special $\mathcal{D}_S$-precover.*
(iii) $\varphi : M_S \to N_S$ *is a special $\mathcal{D}_S$-precover.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)*. Moreover, if S is an excellent extension of R, then* (ii) $\Rightarrow$ (i)*.*

Proof. (i) $\Rightarrow$ (ii) follows from Proposition 2.8 (i).

(ii) $\Rightarrow$ (iii). Since $\varphi_* : \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N)$ is epic, φ is epic by [**14,** Lemma 1.1]. So there is an exact sequence $0 \to K_S \to M_S \to N_S \to 0$ of right S-modules, which induces the exact sequence $0 \to \mathrm{Hom}_R(S,K) \to \mathrm{Hom}_R(S,M) \to \mathrm{Hom}_R(S,N) \to 0$ of right S-modules with $\mathrm{Hom}_R(S,K) \in \mathcal{D}_S^{\perp}$. Since K_S is isomorphic to a direct summand of $\mathrm{Hom}_R(S,K)$, $K_S \in \mathcal{D}_S^{\perp}$. So $\varphi : M_S \to N_S$ is a special $\mathcal{D}_S$-precover.

(ii) $\Rightarrow$ (i). By (ii), there is an exact sequence $0 \to K_R \to M_R \to N_R \to 0$ of right R-modules. For any $A_R \in \mathcal{C}_R, \mathrm{Hom}_R(S,A) \in \mathcal{D}_S$. Also, ${}_RS_R \cong {}_RR^n_R$, $\mathrm{Hom}_R(S,A) \cong A^n_R$. Hence, $\mathrm{Ext}^1_R(A^n,K) \cong \mathrm{Ext}^1_R(\mathrm{Hom}_R(S,A),K) \cong \mathrm{Ext}^1_S(\mathrm{Hom}_R(S,A),\mathrm{Hom}_R(S,K)) = 0$. So $\mathrm{Ext}^1_R(A,K) = 0$. Thus $\varphi : M_R \to N_R$ is a special $\mathcal{C}_R$-precover. □

Theorem 4.15. *Let S be an almost excellent extension of a subring R, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathcal{C}_R \otimes_R S)_S \subseteq \mathcal{D}_S$, $\varphi : M_S \to N_S$ a right S-homomorphism with $M_S \in \mathcal{D}_S$. Consider the following conditions*:

(i) $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ *is a special $\mathcal{D}_S$-precover.*
(ii) $\varphi : M_S \to N_S$ *is a special $\mathcal{D}_S$-precover.*
(iii) $\varphi : M_R \to N_R$ *is a special $\mathcal{C}_R$-precover.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). *Moreover, if S is an excellent extension of R, then* (ii) $\Rightarrow$ (i).

Proof.

(i) $\Rightarrow$ (ii). Since $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ is epic, φ is epic by [**14,** Lemma 1.1]. So there is an exact sequence $0 \to K_S \to M_S \to N_S \to 0$ of right S-modules. Applying the functor $- \otimes_R S$ to it, we get the exact sequence $0 \to K \otimes_R S \to M \otimes_R S \to N \otimes_R S \to 0$ of right S-modules with $K \otimes_R S \in \mathcal{D}_S^{\perp}$. Since K_S is isomorphic to a direct summand of $K \otimes_R S$, $K_S \in \mathcal{D}_S^{\perp}$. So $\varphi : M_S \to N_S$ is a special $\mathcal{D}_S$-precover.

(ii) $\Rightarrow$ (iii) follows from Proposition 3.9 (i).

(iii) $\Rightarrow$ (ii). There is an exact sequence $0 \to K_S \to M_S \to N_S \to 0$ of right S-modules with $K_R \in \mathcal{C}_R^{\perp}$. For any $A_S \in \mathcal{D}_S$, $A_R \in \mathcal{C}_R$. So $\mathrm{Ext}_S^1(A \otimes_R S, K) \cong \mathrm{Ext}_R^1(A, K) = 0$ by [**10,** Theorem 11.65]. But A_S is isomorphic to a direct summand of $A \otimes_R S$, and so $\mathrm{Ext}_S^1(A_S, K_S) = 0$. Thus, $K_S \in \mathcal{D}_S^{\perp}$. Hence, $\varphi : M_S \to N_S$ is a special $\mathcal{D}_S$-precover.

(ii) $\Rightarrow$ (i). There is an exact sequence $0 \to K_S \to M_S \to N_S \to 0$ of right S-modules with $K_S \in \mathcal{D}_S^{\perp}$, which induces the exact sequence $0 \to K \otimes_R S \to M \otimes_R S \to N \otimes_R S \to 0$ of right S-modules. Since S is an excellent extension of R, ${}_R S_R \cong {}_R R_R^n$. Thus, $K \otimes_R S \cong K_R^n$. For any $A_S \in \mathcal{D}_S, A \otimes_R S \in \mathcal{D}_S$, and so we have

$$\begin{aligned} \mathrm{Ext}_S^1(A \otimes_R S, K \otimes_R S) &\cong \mathrm{Ext}_R^1(A, K^n) \cong \mathrm{Ext}_R^1(A, K)^n \\ &\cong \mathrm{Ext}_S^1(A \otimes_R S, K)^n = 0. \end{aligned}$$

But A_S is isomorphic to a direct summand of $A \otimes_R S$. So $K \otimes_R S \in \mathcal{D}_S^{\perp}$, whence $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ is a special $\mathcal{D}_S$-precover. $\square$

Theorem 4.16. *Let S be an almost excellent extension of a subring R, $\mathcal{C}_R$ a class of right R-modules, $\mathcal{D}_S$ a class of right S-modules with $(\mathcal{D}_S)_R \subseteq \mathcal{C}_R$ and $(\mathcal{C}_R \otimes_R S)_S \subseteq \mathcal{D}_S$, $\varphi : M_S \to N_S$ a right S-homomorphism with $N_S \in \mathcal{D}_S$. Consider the following conditions*:

(i) *$\varphi : M_R \to N_R$ is a special $\mathcal{C}_R$-preenvelope.*
(ii) *$\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ is a special $\mathcal{D}_S$-preenvelope.*
(iii) *$\varphi : M_S \to N_S$ is a special $\mathcal{D}_S$-preenvelope.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *Moreover, if S is an excellent extension of R, then* (ii) $\Rightarrow$ (i).

Proof.

(i) $\Rightarrow$ (ii) follows from Proposition 3.9 (ii).

(ii) $\Rightarrow$ (iii). Since $\varphi \otimes_R 1 : M \otimes_R S \to N \otimes_R S$ is monic, φ is monic by **[14,** Lemma 1.1]. So there is an exact sequence $0 \to M_S \to N_S \to L_S \to 0$ of right S-modules, which induces the exact sequence $0 \to M \otimes_R S \to N \otimes_R S \to L \otimes_R S \to 0$ of right S-modules with $L \otimes_R S \in {}^{\perp}\mathcal{D}_S$. Since L_S is isomorphic to a direct summand of $L \otimes_R S$, $L_S \in {}^{\perp}\mathcal{D}_S$. So $\varphi : M_S \to N_S$ is a special $\mathcal{D}_S$-preenvelope.

(ii) $\Rightarrow$ (i). There is an exact sequence $0 \to M_R \to N_R \to L_R \to 0$ of right R-modules, which induces the exact sequence $0 \to M \otimes_R S \to N \otimes_R S \to L \otimes_R S \to 0$ of right S-modules with $L \otimes_R S \in {}^{\perp}\mathcal{D}_S$. For any $A_R \in \mathcal{C}_R$, $A \otimes_R S \in \mathcal{D}_S$. Since ${}_RS_R \cong {}_RR^n_R$, $A \otimes_R S \cong A^n_R$. Hence, $\mathrm{Ext}^1_R(L, A^n) \cong \mathrm{Ext}^1_R(L, A \otimes_R S) \cong \mathrm{Ext}^1_S(L \otimes_R S, A \otimes_R S) = 0$. So $\mathrm{Ext}^1_R(L, A) = 0$. Thus, $\varphi : M_R \to N_R$ is a special $\mathcal{C}_R$-preenvelope. $\square$

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