

THE TOPOLOGY OF TILE INVARIANTS

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ABSTRACT. In this note, we use techniques in the topology of 2-complexes to recast some tools that have arisen in the study of planar tiling questions. With spherical pictures, we show that the tile counting group associated to a set T of tiles and a set of regions tileable by T is isomorphic to a quotient of the second homology group of a 2-complex built from T . In this topological setting, we derive some well-known tile invariants, one of which we apply to the solution of a tiling question involving modified rectangles.

1. Introduction. Two papers published in 1990, one by Conway and Lagarias [4] and one by Thurston [12], introduced topological and combinatorial group theoretic ideas to the solution of certain tiling questions. About ten years later, a tile counting group was developed in a series of papers by Pak and others (see [6, 7, 8, 9, 11]) in an effort to study the relations among the tiles that must persist in any tiling of a given region. In this note, we show that for a large class of tile sets and tileable regions, the tile counting group is isomorphic to a quotient of the second cellular homology group of a 2-complex associated to the tile set (Theorem 3.1). We then use techniques in low-dimensional topology to revisit the derivation of tile invariants found in [4, 9] and subsequently used to solve tiling problems. Finally, we use tile invariants to solve (in Theorem 5.2) the following tiling question:

For what values of a and b can the modified rectangle $M(a, b)$ (obtained by removing the upper left and lower right squares from an $a \times b$ rectangle in the integer lattice) be tiled by translations of the tiles in the set S pictured in Figure 1?

The general scene in this note is as follows. We have a set of tiles $T = \{t_1, t_2, \dots, t_n\}$ in the integer lattice. We assume each tile t_i consists of a finite union of squares in the lattice and that the resulting region is homeomorphic to a disk. Ribbon tiles, which have been widely studied,

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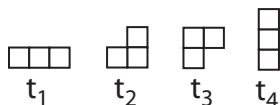
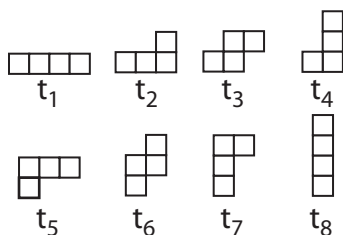
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FIGURE 1. The tile set S .

fit into this category (see [4, 6, 7, 8, 9, 10, 11, 12]). A *ribbon tile* of area n consists of n squares laid out in a path, such that if we start with the bottom left-most square, each step either goes up or to the right. If T_n denotes the set of ribbon tiles having area n , we note that T_n will have 2^{n-1} members. Figure 2 shows the set T_3 of ribbon tile trominoes, and Figure 3 shows the set T_4 of ribbon tile tetrominoes.

FIGURE 2. The set T_3 of ribbon tile trominoes.FIGURE 3. The set T_4 of ribbon tile tetrominoes.

Suppose Γ is a region that can be tiled by a tile set T . By this, we mean that we may cover the region without gaps or overlaps with translations of copies of the tiles in T . We cannot rotate the tiles in T . We restrict our attention to those regions Γ that are homeomorphic to a disk (such regions are also called simply connected polyominoes in tiling circles) because we wish to take the view that tilings represent maps from a disk to a particular 2-complex.

If α represents a particular tiling by T of a region Γ , we let $b_i(\alpha)$ equal the number of copies of tile t_i that appear in the tiling. There

are certain relations among the $b_i(\alpha)$ that hold for all tilings of a given region, and any such relation is called a tile invariant in [9]. One simple relation is an area invariant: If all tiles in T have the same area (number of squares), then for any tiling α of Γ , $\sum_{i=1}^n b_i(\alpha)$ is constant. In [4], Conway and Lagarias prove that, for any tiling α of a given simply connected region Γ by the set T_3 of ribbon tile trominoes given in Figure 2, $b_3(\alpha) - b_2(\alpha)$ is constant. We revisit this result below (Theorem 4.2).

The *tile counting group* $G(T, \mathcal{R})$ associated to a tile set T and a given set of regions \mathcal{R} is an abelian group whose elements correspond to tile invariants. This group is defined in [9] as follows. Begin with all formal integer linear combinations of the tiles in T , and then divide by I the linear span of all relations of the form

$$\sum_{i=1}^n b_i(\alpha)t_i - \sum_{i=1}^n b_i(\beta)t_i,$$

for all tilings α and β of regions Γ in \mathcal{R} . That is,

$$G(T, \mathcal{R}) = \bigoplus_{i=1}^n \mathbb{Z}t_i / I.$$

In a series of papers ([7, 9, 8]), Pak and others determine the tile counting group for the set T_n (for $n \geq 2$) over all simply connected tileable regions. In [6], Korn determines the tile counting group for several other interesting tile sets over various families of tileable regions. We introduce the picture group in the next section to approach the tile counting group topologically.

2. The picture group. Suppose $\mathcal{P} = \langle \mathbf{x} : \mathbf{r} \rangle$ presents the group G . We let F denote the free group generated by the set of letters \mathbf{x} , and we let R denote the normal subgroup of F generated by the words in \mathbf{r} (called relators) so that $G = F/R$. The standard 2-complex $X_{\mathcal{P}}$ associated to \mathcal{P} consists of a single 0-cell c^0 , a 1-cell c_x^1 for each generator $x \in \mathbf{x}$, and a 2-cell c_r^2 for each relator $r \in \mathbf{r}$. The boundary of each 1-cell must be attached to c^0 so that the 1-skeleton of $X_{\mathcal{P}}$, $X_{\mathcal{P}}^1$, is a bouquet of circles. The 2-cell c_r^2 is attached to $X_{\mathcal{P}}^1$ along its boundary according to the word spelled by the relator.

For instance, the standard 2-complex associated to $\mathcal{P} = \langle x, y \mid [x, y] \rangle$ of the group $\mathbb{Z} \times \mathbb{Z}$ is the torus (here $[x, y]$ denotes the word $xyx^{-1}y^{-1}$). The 1-skeleton of the space consists of two circles joined at the 0-cell. The space has a single 2-cell attached to the 1-skeleton according to the word $xyx^{-1}y^{-1}$. We may model this process by identifying opposite edges of a rectangle in the usual way to obtain a torus.

The standard 2-complex $X_{\mathcal{P}}$ associated to a presentation is not so easy to visualize in general, but we may represent elements of $\pi_2(X_{\mathcal{P}})$ with tangible objects called spherical pictures. A good introduction to pictures may be found in [2], but we present the main ideas in this section.

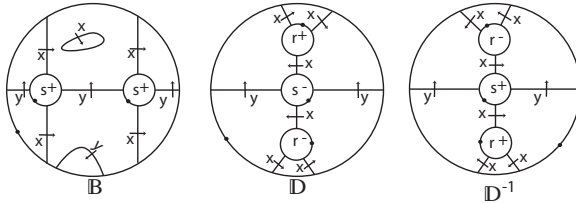


FIGURE 4. Some pictures over $\mathcal{P} = \langle x, y \mid r = x^3, s = [y, x] \rangle$.

A *picture* over a presentation \mathcal{P} has an ambient disk with basepoint, and a finite number $\Delta_1, \Delta_2, \dots, \Delta_k$ of based disks in the interior of the ambient disk. In addition to a basepoint, each Δ_i is labelled with a relator and an exponent of ± 1 . A picture also includes a finite number of arcs. The two ends of an arc are either attached to disk boundaries or joined to each other (creating a loop). No two arcs may intersect, and no arc can intersect the interior of any of the Δ_i disks. Furthermore, each arc gets a label from the generator set and a transverse orientation. The disk and arc labelling must be consistent in this sense: traversing the boundary of the disk Δ_i in the clockwise direction from its basepoint, the word one spells from the arc labels it encounters equals the disk label (using the convention that the exponent of the generator associated to an arc is $+1$ or -1 depending on whether the path around Δ_i is in the direction of the transverse arrow associated to the arc, or against this direction).

A picture in which no arc intersects the boundary of the ambient disk is called a *spherical picture*. A picture \mathbb{B} over \mathcal{P} has a boundary word

$\partial\mathbb{B}$ in F associated with it, read by the generator labels encountered by traversing the boundary of the ambient disk once in the clockwise direction from the ambient basepoint. By a lemma of van Kampen [5] this boundary word is trivial in the group presented by \mathcal{P} .

Lemma 2.1 (van Kampen). *Suppose \mathcal{P} presents the group F/R . A word $w \in F$ is trivial in the group F/R if and only if there exists a picture \mathbb{B} over the presentation \mathcal{P} whose boundary word $\partial\mathbb{B}$ is identically equal to w .*

Figure 4 shows three pictures over $\mathcal{P} = \langle x, y \mid x^3, [y, x] \rangle$ of the group $\mathbb{Z} \times \mathbb{Z}_3$, where we let r denote the relator x^3 , and s denotes the relator $[y, x]$. Note that $\partial\mathbb{B}$ and $\partial\mathbb{D}$ are both equal to $yx^2y^{-1}x^{-2}$ in F .

We define an equivalence relation on pictures over \mathcal{P} by $\mathbb{B}_1 \sim \mathbb{B}_2$ if and only if \mathbb{B}_1 can be transformed to \mathbb{B}_2 (up to planar isotopy) by a finite number of moves of three possible types, called FLOAT, BRIDGE, and FOLD moves. A FLOAT move is the insertion or deletion of a floating arc, an arc that divides the ambient disk into two components, one of which contains the global basepoint and all the disks and remaining arcs of the picture. The picture \mathbb{B} in Figure 4 has two floating arcs, one labeled by y that hits the boundary of the ambient disk, and one loop labeled by x . A BRIDGE move is a local move with respect to two arcs in a picture sharing a generator label as indicated in Figure 5. A FOLD move is the insertion or deletion of a folding pair of disks (see Figure 5). A folding pair of disks consists of two disks, labeled by the same relator but with opposite signs, such that each arc of one disk connects to the other, and such that the disk basepoints are in the same region of the picture.

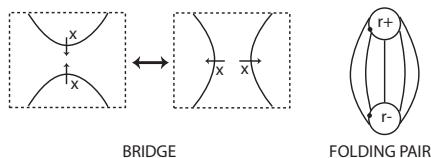


FIGURE 5. BRIDGE and FOLD moves

Note that when two disks of a picture have arcs that meet the boundary of the ambient disk successively as suggested in Figure 6,

and if those arcs are labeled by the same generator but with opposite orientations, we may replace these two arcs by a single arc between the disks by performing a BRIDGE move followed by a FLOAT.

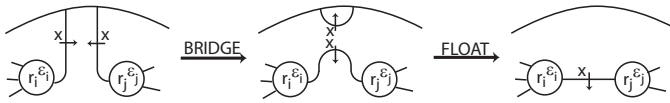


FIGURE 6. Joining two disks by a BRIDGE, FLOAT sequence

We may multiply two pictures by concatenation so that $\partial(\mathbb{B} \cdot \mathbb{D}) = \partial\mathbb{B} \cdot \partial\mathbb{D}$ (see Figure 7). Given $w \in F$, the picture $w \cdot \mathbb{B}$ is obtained by surrounding the ambient basepoint of \mathbb{B} by a series of arcs whose labels spell w (see Figure 7). Note $\partial(w \cdot \mathbb{B}) = w\partial\mathbb{B}w^{-1}$. The inverse \mathbb{D}^{-1} of a picture is obtained by planar reflection of \mathbb{D} and a change of sign of the disks of \mathbb{D} (Figure 4 shows a picture \mathbb{D} and its inverse \mathbb{D}^{-1}).

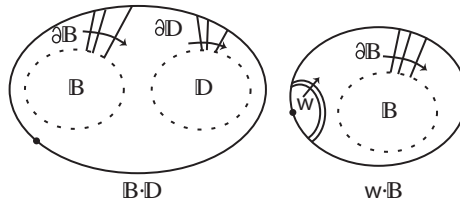


FIGURE 7. The product of two pictures, the F action on Σ .

To get a better feel for the arithmetic of pictures, consider again the pictures from Figure 4. The product of \mathbb{B} (with its floating arcs removed) and \mathbb{D}^{-1} is given in Figure 8. By a series of BRIDGE and FLOAT moves along the lines of Figure 6 this picture is equivalent to the spherical picture at right in Figure 8. That the result is a spherical picture is no accident: if \mathbb{B} and \mathbb{D} are based pictures over \mathcal{P} such that $\partial\mathbb{B}$ and $\partial\mathbb{D}$ determine the same word in the free group F , then $\mathbb{B} \cdot \mathbb{D}^{-1}$ is equivalent to a spherical picture over \mathcal{P} .

Let $[\mathbb{B}]$ denote the equivalence class of $[\mathbb{B}]$, and define $\Sigma = \{[\mathbb{B}] \mid \mathbb{B} \text{ a picture over } \mathcal{P}\}$. It is not difficult to check that Σ forms a group under multiplication given by $[\mathbb{B}] \cdot [\mathbb{D}] = [\mathbb{B} \cdot \mathbb{D}]$. The identity element of Σ is $[\mathbb{O}]$ where \mathbb{O} is the empty picture (consisting of just an ambient

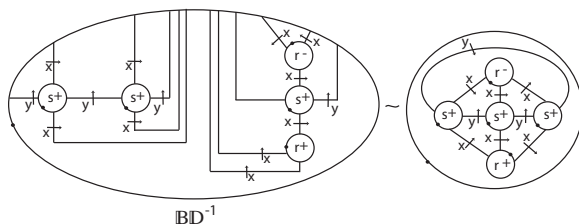


FIGURE 8. The picture $\mathbb{B} \cdot \mathbb{D}^{-1}$ is spherical.

disk). One checks that $[\mathbb{B}]^{-1} = [\mathbb{B}^{-1}]$ and that $w \cdot [\mathbb{B}] = [w \cdot \mathbb{B}]$ defines an F -action on Σ . We call Σ the group of pictures over \mathcal{P} .

The set $P = \{[\mathbb{P}] \mid \mathbb{P} \text{ a spherical picture over } \mathcal{P}\}$ determines a normal subgroup of Σ which we call the group of spherical pictures over \mathcal{P} .

Pictures over \mathcal{P} give combinatorial representations of continuous maps from the unit disk (or the 2-sphere if the picture is spherical) into $X_{\mathcal{P}}$. One can show that two pictures represent homotopy equivalent maps if and only they determine the same element in the picture group. In fact, the group of pictures is isomorphic to the relative homotopy group $\pi_2(X_{\mathcal{P}}, X_{\mathcal{P}}^1)$, and the group of spherical pictures is isomorphic to $\pi_2(X_{\mathcal{P}})$. We refer the reader to [2] for details. Thus, elements of $\pi_2(X_{\mathcal{P}})$ may be represented by spherical pictures, and elements of $\pi_2(X_{\mathcal{P}}, X_{\mathcal{P}}^1)$ may be represented by (disk) pictures.

The group of spherical pictures $\pi_2(X_{\mathcal{P}})$ has a left $\mathbb{Z}G$ -module structure. One can check that, if $w \in R$ and \mathbb{P} is spherical, then $w \cdot \mathbb{P} \sim \mathbb{P}$. That is, R acts trivially on $\pi_2(X_{\mathcal{P}})$ so we have a well-defined G -action on $\pi_2(X_{\mathcal{P}})$ that extends to give $\pi_2(X_{\mathcal{P}})$ a left $\mathbb{Z}G$ -module structure. When we discuss generators of $\pi_2(X_{\mathcal{P}})$ in what follows we mean module generators.

3. The tile complex. We return now to our tile set $T = \{t_1, t_2, \dots, t_n\}$ in the integer lattice and the task of building a 2-complex associated to the tile set. The 2-complex will be modeled on a group presentation which consists of two generators and n relators, one for each tile in T . In particular, for each t_i chose a basepoint on its boundary, and assign a boundary word in the free group $F = F(x, y)$ from a clockwise path

around its boundary as follows: record the letter y (respectively, y^{-1}) if a vertical edge is traversed to the north (resp. south); and record the letter x (respectively, x^{-1}) if a horizontal edge is traversed to the east (respectively, west). Let r_i ($i = 1, 2, \dots, n$) denote the n boundary words of the n tiles. The group presentation

$$\mathcal{P} = \langle x, y \mid r_1, r_2, \dots, r_n \rangle$$

presents the *tile boundary group* $G_T = F/R$, where R is the normal subgroup of F generated by the r_i . The word r_i depends on the choice of tile basepoint, but words associated to distinct basepoints will be cyclic permutations of each other, and the group determined by the resulting presentation is independent of this choice of basepoint. To any presentation of the tile boundary group we may associate the standard 2-complex, and we are interested in the homotopy and homology groups associated to this complex. Again, these groups are independent of our choice of tile basepoints.

By placing tile basepoints at the southwest corner of each tile, the tile boundary group for the set of ribbon tile trominoes T_3 (Figure 2) has the presentation:

$$\mathcal{P}_3 = \langle x, y \mid [y, x^3], (yx)^2 y^{-2} x^{-2}, y^2 x^2 (y^{-1} x^{-1})^2, [y^3, x] \rangle.$$

Using southwest corners again for the tiles of T_4 (Figure 3), its tile boundary group has the 8 relator presentation

$$\mathcal{P}_4 = \langle x, y \mid [y, x^4], yx^2 yxy^{-2} x^{-3}, yxyx^2 y^{-1} x^{-1} y^{-1} x^{-2}, \dots, [y^4, x] \rangle.$$

Thurston considered the tile boundary group in [12] for dominoes and lozenges, and Conway and Lagarias worked with the group $[F, F]/R$ in [4] (which they called the tile homotopy group), where $[F, F]$ denotes the subgroup of F generated by all commutators $[w, v]$.

Reid studied the Conway/Lagarias tile homotopy group with great success in his paper [10]. In a sense, the tile boundary group F/R we consider in this note (which Reid calls the tile path group of the tile set) is bigger than we need to analyze tiling questions because simply connected regions in the integer lattice have boundary words that live in $[F, F]$. For this reason, Conway, Lagarias and Reid considered $[F, F]/R$. However, it turns out that the tile complex associated to a presentation of F/R is well-suited for interpreting Pak's tile counting group, so we focus on this group in what follows.

Suppose we have some presentation \mathcal{P} of the tile boundary group G_T based on a choice of tile basepoints. Any tiling α by T of a region Γ that is topologically a disk has a natural representation as a based picture over \mathcal{P} and hence determines an element in the picture group $\pi_2(X_{\mathcal{P}}, X_{\mathcal{P}}^1)$. In fact, one may think of the original tiling as a van Kampen diagram over \mathcal{P} , and the corresponding picture as a planar dual. In particular, think of the topological boundary of Γ as the ambient disk, and each tile in the tiling as a disk in the picture. Tile t_i gets replaced with a disk labeled r_i^+ , and two disks share an arc between them if their corresponding tiles share an edge. Any edge of a tile that intersects the boundary of Γ gives rise to an arc from its disk to the ambient disk. The arcs are labeled according to the boundary letter of the associated edge of the tile. If α employs k tiles to tile Γ , the resulting disk picture \mathbb{B}_α has k disks, all with positive exponents.

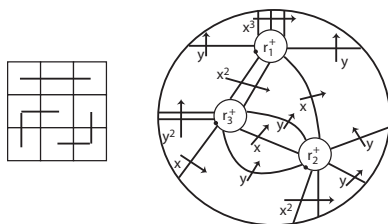


FIGURE 9. Converting a tiling of a region by the tromino set T_3 into a picture over \mathcal{P}_3 .

Two tilings α and β of a given region Γ determine a spherical picture $\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}$ over \mathcal{P} , and the equivalence class $[\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}]$ is an element of the group of spherical pictures, $\pi_2(X_{\mathcal{P}})$.

We note that an arbitrary picture over \mathcal{P} , even one with all positive disks, need not correspond to a tiling of a region in the lattice. Also, tilings of other simply connected regions that are not homeomorphic to a disk (e.g., a region that is the one point union of two such disk regions) may be encoded in the picture group. Passing to a more general object to investigate tilings can be helpful in so much as we can bring general tools of topology to bear on the picture group, which is not an easy task, in general.

Let \mathcal{R} consist of all regions homeomorphic to a disk, and suppose $X_{\mathcal{P}}$ is the tile complex associated to some presentation of the tile boundary

group G_T , where $T = \{t_1, \dots, t_n\}$ is the tile set. Let J denote the submodule of $\pi_2(X_{\mathcal{P}})$ generated by all spherical pictures of the form $\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}$ for all tilings α and β for all regions Γ in \mathcal{R} . We let $H_2(X_{\mathcal{P}})$ denote the second cellular homology group of $X_{\mathcal{P}}$.

Theorem 3.1. *The tile counting group $G(T, \mathcal{R})$ is isomorphic to the quotient $H_2(X_{\mathcal{P}})/h(J)$, where $h : \pi_2(X_{\mathcal{P}}) \rightarrow H_2(X_{\mathcal{P}})$ is the Hurewicz homomorphism.*

Proof. The cellular chain groups of $X_{\mathcal{P}}$ are

$$\begin{aligned} C_2(X_{\mathcal{P}}) &= \bigoplus_{i=1}^n \mathbb{Z}c_{r_i}^2, \\ C_1(X_{\mathcal{P}}) &= \mathbb{Z}c_x^1 \oplus \mathbb{Z}c_y^1, \\ C_0(X_{\mathcal{P}}) &= \mathbb{Z}c^0. \end{aligned}$$

Since $X_{\mathcal{P}}$ has a single 0-cell, the first boundary map $\partial_1 : C_1(X_{\mathcal{P}}) \rightarrow C_0(X_{\mathcal{P}})$ is trivial. Since each tile in T determines a closed path in the integer lattice, its boundary word, whatever the basepoint, is an element of $[F, F]$. That is, the exponent sum of each generator in the word r_i is zero. Thus, the second boundary map is also trivial, and it follows that

$$H_2(X_{\mathcal{P}}) = \bigoplus_{i=1}^n \mathbb{Z}c_{r_i}^2$$

and

$$H_1(X_{\mathcal{P}}) = \mathbb{Z} \oplus \mathbb{Z}.$$

We may view the Hurewicz homomorphism through the lens of spherical pictures. Following the notation in [2], given a picture \mathbb{P} over \mathcal{P} , we record the (net) number of occurrences of disks labeled by relator r as follows:

$$\exp_r(\mathbb{P}) = (\#\text{disks in } \mathbb{P} \text{ labeled } r^+) - (\#\text{disks in } \mathbb{P} \text{ labeled } r^-).$$

Note that if $\mathbb{P}_1 \sim \mathbb{P}_2$ then $\exp_r(\mathbb{P}_1) = \exp_r(\mathbb{P}_2)$ since BRIDGE, FLOAT and FOLD moves preserve the (net) number of disks labeled by r in a

picture. Then the Hurewicz homomorphism

$$h : \pi_2(X_{\mathcal{P}}) \rightarrow H_2(X_{\mathcal{P}})$$

is defined by

$$h([\mathbb{P}]) = \sum_{i=1}^n \exp_{r_i}(\mathbb{P}) c_{r_i}^2.$$

Now consider a spherical picture over \mathcal{P} of the form $\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}$, coming from tilings α and β of some region Γ in \mathcal{R} . Since each relator in \mathbb{B}_α has positive exponent, and each relator in \mathbb{B}_β^{-1} has negative exponent, it follows that

$$\exp_{r_i}(\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}) = b_i(\alpha) - b_i(\beta).$$

That is, the homological image of this picture under the Hurewicz map is

$$h([\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}]) = \sum_{i=1}^n (b_i(\alpha) - b_i(\beta)) c_{r_i}^2.$$

Thus, each element in I is naturally identified with an element of $h(J)$, and it follows that

$$G(T, \mathcal{R}) = H_2(X_{\mathcal{P}})/h(J). \quad \square$$

We view Theorem 3.1 primarily as a theoretical result meant to establish a topological interpretation of the tile counting group, not as a result that will be generally useful for calculating the tile counting group. Still, it is useful to explore the connection between these perspectives a bit further by looking at some special cases and revisiting a few well-known examples.

If $\pi_2(X_{\mathcal{P}})$ is generated by spherical pictures that come from the difference of two tilings of a region Γ , that is, if $\pi_2(X_{\mathcal{P}}) = J$, then the tile counting group is isomorphic to $H_2(G_T)$, the second homology group of the tile boundary group. This follows from a theorem of Hopf (see, for instance [3, page 41]) which gives us the following exact sequence of groups

$$\pi_2(X_{\mathcal{P}}) \xrightarrow{h} H_2(X_{\mathcal{P}}) \longrightarrow H_2(G_T) \longrightarrow 0.$$

In fact, all we need to ensure that $G(T, \mathcal{R})$ is isomorphic to $H_2(G_T)$ is that J and $\pi_2(X_{\mathcal{P}})$ have the same image under the Hurewicz map.

An interesting topological question for a tile set is to decide whether this is the case.

In the event that the Hurewicz map $h : \pi_2(X_{\mathcal{P}}) \rightarrow H_2(X_{\mathcal{P}})$ is trivial, in which case the space $X_{\mathcal{P}}$ is called *Cockcroft*, we note that the tile counting group is isomorphic to $H_2(X_{\mathcal{P}})$, a free abelian group of rank equal to the number of tiles in the tile set.

1. *Dominos*. Consider the set $T_2 = \{t_1, t_2\}$ of dominoes on the left in Figure 10. Using southwest corners as basepoints, the domino tile counting group has presentation

$$\mathcal{P}_2 = \langle x, y \mid [y, x^2], [y^2, x] \rangle.$$

Thurston argued in [12] using the universal cover of $X_{\mathcal{P}_2}$ that $\pi_2(X_{\mathcal{P}_2})$ is generated by the spherical picture appearing in Figure 10. This spherical picture results from the difference of two tilings of a 2×2 square, one by two horizontal tiles and one by two vertical tiles, as indicated on the right in Figure 10. Thus, $\pi_2(X_{\mathcal{P}_2}) = J$ in the case of dominoes, and the tile counting group is $H_2(G_{T_2}) = \mathbb{Z} \times \mathbb{Z}_2$.

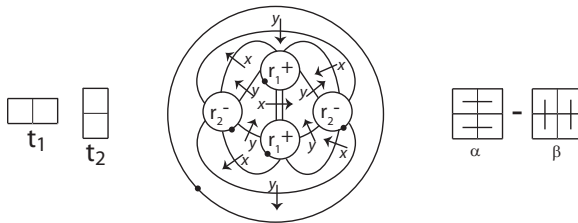


FIGURE 10. $\pi_2(X_{\mathcal{P}_2})$ is generated by a single picture which lives in J .

As Thurston observed, this analysis of $\pi_2(X_{\mathcal{P}_2})$ gives a local replacement property for tilings involving dominoes. Given two tilings α and β of any region Γ homeomorphic to a disk, then $\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}$ determines an element of $\pi_2(X_{\mathcal{P}_2})$ and hence must be expressible in terms of the picture in Figure 10. This translates to the notion that any domino tiling of a region Γ may be obtained from any other tiling of Γ by a finite sequence of “2-flips” that exchange a horizontally tiled 2×2 square portion of Γ with a vertically tiled portion, or vice versa.

2. *Ribbon tiles*. In [11], Sheffield proved the deep result that any two tilings of a simply connected region by T_n can be made to coincide

by a number of “2-flips” chosen from a finite list of local moves. For instance, any tiling of a given region by the set T_3 can be changed into any other tiling of that region by a number of exchanges of two tiles from among the 6 indicated in Figure 11. If a tile set has this local move property with respect to the collection of all tileable regions homeomorphic to a disk, then the local moves generate J . For instance, the 6 spherical pictures determined by the 2-flips in Figure 11 generate J in the case of T_3 . See [6] for other examples of tile sets having a local move property, as well as examples of tile sets that do not have a local move property. We note that Sheffield also proves in [11] the existence of a linear time algorithm for deciding whether a given simply connected region is tileable by T_n .

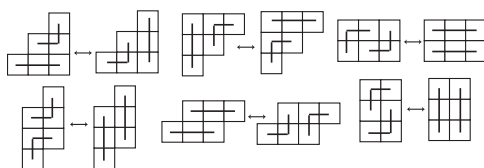


FIGURE 11. These 6 2-flips generate J for the tile set T_3 .

3. *Lozenges.* We may construct tile boundary groups for tile sets in the triangular lattice from the free group on three generators, $F = F(x, y, z)$. For instance, the tile boundary group for the tile set $L = \{l_1, l_2, l_3\}$ consisting of the three lozenges pictured in Figure 12 has the presentation

$$\mathcal{L} = \langle x, y, z \mid r_1 = [x, y], r_2 = [y, z], r_3 = [z, x] \rangle.$$

Theorem 3.1 need not apply to tile sets in the triangular lattice because boundary words of such tiles need not live in $[F, F]$ (consider a triangular tile, for instance). However, the theorem does apply to tile sets whose boundary words do live in $[F, F]$ such as the lozenges.

The tile boundary group for lozenges is isomorphic to \mathbb{Z}^3 . These lozenges were studied in [12] as the dominoes were, and $\pi_2(\widetilde{X}_{\mathcal{L}})$ may be understood by considering the universal cover of $X_{\mathcal{L}}$, $\widetilde{X}_{\mathcal{L}}$, since $\pi_2(X_{\mathcal{L}})$ is isomorphic to $H_2(\widetilde{X}_{\mathcal{L}})$. Here $\widetilde{X}_{\mathcal{L}}$ is the 2-skeleton of a cubical tessellation of \mathbb{R}^3 . By an examination of the 2-dimensional cellular cycles of the universal cover, one checks that the picture in Figure 12

generates $\pi_2(X_{\mathcal{L}})$. This picture is equivalent to one obtained from the difference of two tilings of a hexagonal region consisting of 6 triangles in the lattice, so $\pi_2(X_{\mathcal{L}}) = J$ in this case. Also, since this picture is homologically trivial, $X_{\mathcal{L}}$ is Cockcroft, and the tile counting group for the lozenges is isomorphic to $H_2(X_{\mathcal{L}}) = \mathbb{Z}^3$.

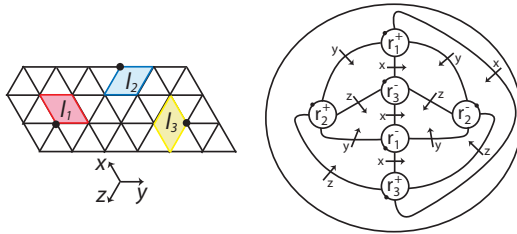


FIGURE 12. Tiling the triangular lattice with lozenges; $\pi_2(X_{\mathcal{L}})$ is generated by a single picture.

More generally, Bogley shows in [1] that the 2-complex associated to

$$\mathcal{P} = \langle x, y, z \mid [x^a, y], [y^b, z], [z^c, x] \rangle$$

is Cockcroft for positive integers a, b, c . This gives an infinite family of tile sets consisting of 3 elongated lozenges in the triangular lattice for which the tile counting group is $H_2(X_{\mathcal{P}}) = \mathbb{Z}^3$. We remark that the topological machinery employed in this example is not necessary, and that one may determine the tile counting group directly.

4. Pictures and tile invariants. Given the tile boundary group presentation

$$\mathcal{P} = \langle x, y \mid r_1, \dots, r_n \rangle,$$

suppose $\{s_1, s_2, \dots, s_k\}$ is a set of words in $F(x, y)$ such that each r_i is in the normal subgroup S of $F(x, y)$ generated by the s_i . That is, each r_i is trivial in the group $H = F/S$. The group H is a quotient of G_T and has presentation

$$\mathcal{Q} = \langle x, y \mid s_1, \dots, s_k \rangle.$$

Let Y be the 2-complex modeled on \mathcal{Q} .

Since each r_i is trivial in H , there is a disk picture \mathbb{B}_{r_i} over \mathcal{Q} having boundary label identically equal to r_i , by van Kampen's lemma. Given

a choice $\{\mathbb{B}_{r_1}, \mathbb{B}_{r_2}, \dots, \mathbb{B}_{r_n}\}$ of such based pictures, we may transform a spherical picture over \mathcal{P} to a spherical picture over \mathcal{Q} . In particular, say \mathbb{P} is a spherical picture over \mathcal{P} with disks $\Delta_1 \cdots \Delta_m$, and disk labels $w_i^{\epsilon_i}$ where $w_i \in \{r_1 \cdots r_n\}$ and $\epsilon_i = \pm 1$. Replace each disk Δ_i in \mathbb{P} with the disk picture $\mathbb{B}_{w_i}^{\epsilon_i}$. The resulting spherical picture, call it $\overline{\mathbb{P}}$, gives an element of $\pi_2(Y)$.

If $\pi_2(Y)$ is understood, this map can tell us information about $\pi_2(X_{\mathcal{P}})$, which in turn may give us tile invariants.

This technique of passing to a quotient to arrive at tile invariants was applied by Conway and Lagarias in [4] and by Pak and Muchnik in [8]. Their work was couched in terms of winding numbers of paths in the 1-skeleton of the universal covering of the tile complex. Here we revisit their arguments from the point of view of pictures, and we derive the tile invariants from the fact that the quotient spaces considered are Cockcroft.

Lemma 4.1. *Suppose Y is the 2-complex modeled on $\mathcal{P} = \langle x, y \mid x^a, y^b, (xy)^c \rangle$ where $1/a + 1/b + 1/c \leq 1$. Then Y is Cockcroft.*

This result can be found in the literature (see, for instance, [2, page 174]), but for the reader's convenience, we work through the proof at the end of the section.

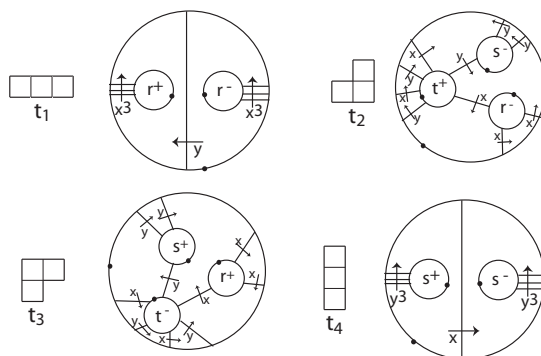


FIGURE 13. The tiles in T_3 determine pictures over $\mathcal{Q} = \langle x, y \mid x^3, y^3, (xy)^3 \rangle$.

Theorem 4.2. (Conway and Lagarias). *Suppose Γ is a region homeomorphic to a disk. For any tiling α of Γ by the set of trominoes T_3 , $b_3(\alpha) - b_2(\alpha) = c(\Gamma)$.*

Proof. Consider the presentation $\mathcal{Q} = \langle x, y \mid r = x^3, s = y^3, t = (xy)^3 \rangle$, and let Y denote the 2-complex modeled on \mathcal{Q} . By Lemma 4.1, Y is Cockcroft. Moreover, each relator in \mathcal{P}_3 determines the trivial element in the group H presented by \mathcal{Q} as demonstrated in Figure 13.

Using these based pictures, any two tilings α and β of a region Γ by T_3 determine an element $[\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}]$ in $\pi_2(X_{\mathcal{P}_3})$ whose image $[\overline{\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}}]$ in $\pi_2(Y)$ must be homologically trivial since Y is Cockcroft. That is, the exponent sum of each relator in $\overline{\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}}$ must be zero. It follows that, for the relator t ,

$$0 = \exp_t(\overline{\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}}) = (b_2(\alpha) - b_3(\alpha)) - (b_2(\beta) - b_3(\beta)).$$

Thus, for any two tilings α and β of Γ , we have $b_2(\alpha) - b_3(\alpha) = b_2(\beta) - b_3(\beta)$. \square

We remark that the relators r_3 and r_4 in the presentation of the tile boundary group for T_3 are consequences of the first two (see Figure 14). That is, r_3 and r_4 are trivial in the group presented by

$$\mathcal{W} = \langle x, y \mid [y, x^3], (yx)^2 y^{-2} x^{-2} \rangle.$$

It would be interesting to know whether the 2-complex $X_{\mathcal{W}}$ associated to this presentation is Cockcroft. If this is the case, then we have a proof of Theorem 4.2 that does not involve the quotient group considered in [4]. Indeed, if α and β are two tilings of a region Γ by T_3 , the spherical picture $\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}$ determines the element $[\overline{\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}}]$ in $\pi_2(X_{\mathcal{W}})$, by using the subpictures of Figure 14. and one checks that

$$\exp_{r_2}([\overline{\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}}]) = (b_2(\alpha) - b_3(\alpha)) - (b_2(\beta) - b_3(\beta)).$$

Theorem 4.3. (Pak and Muchnik). *Suppose Γ is a region homeomorphic to a disk. For any tiling α of Γ by the set of tetrominoes T_4 , $b_2(\alpha) + b_4(\alpha) - b_5(\alpha) - b_7(\alpha) = c(\Gamma)$.*

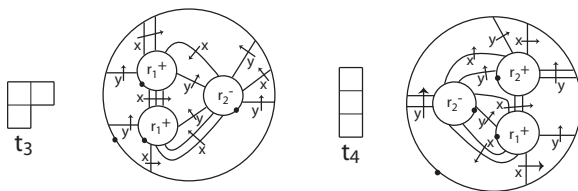


FIGURE 14. Tiles t_3 and t_4 are consequences of t_1 and t_2 .

Proof. Consider the presentation

$$\mathcal{Q} = \langle x, y \mid r = x^4, s = y^4, t = (xy)^2 \rangle,$$

and let Y be the 2-complex modeled on \mathcal{Q} . By Lemma 4.1, Y is Cockcroft. Moreover, each relator in \mathcal{P}_4 determines the trivial element in the group H presented by \mathcal{Q} as demonstrated in Figure 15. The result then follows by the same argument as in the proof of Theorem 4.2, making use of the relator r . In this case, observe that

$$\begin{aligned} \exp_r(\overline{\mathbb{B}_\alpha \cdot \mathbb{B}_\beta^{-1}}) &= (-b_2(\alpha) - b_4(\alpha) + b_5(\alpha) + b_7(\alpha)) \\ &\quad - (-b_2(\beta) - b_4(\beta) + b_5(\beta) + b_7(\beta)). \end{aligned}$$

Since this exponent sum must be 0, it follows that, for any two tilings α and β of Γ , we have $b_2(\alpha) + b_4(\alpha) - b_5(\alpha) - b_7(\alpha) = b_2(\beta) + b_4(\beta) - b_5(\beta) - b_7(\beta)$. \square

We may perhaps use Lemma 4.1 to reverse engineer other interesting tile sets. Consider the group presented by

$$\mathcal{Q} = \langle x, y \mid x^3, y^6, (xy)^2 \rangle.$$

One can check that \mathcal{Q} presents a quotient of the tile boundary group associated with the tiles in Figure 16. Using techniques from this section, it is straightforward to check that, for any tiling α of a region Γ by this tile set, $b_2(\alpha) - b_3(\alpha)$ is constant.

Proof of Lemma 4.1. We first argue that any connected spherical picture over \mathcal{P} must have a local configuration of one of four types in Figure 17. We call such a configuration a dipole.

Suppose we have a connected spherical picture \mathbb{P} over the presentation \mathcal{P} , and we shrink each disk to a vertex. What remains is a labeled

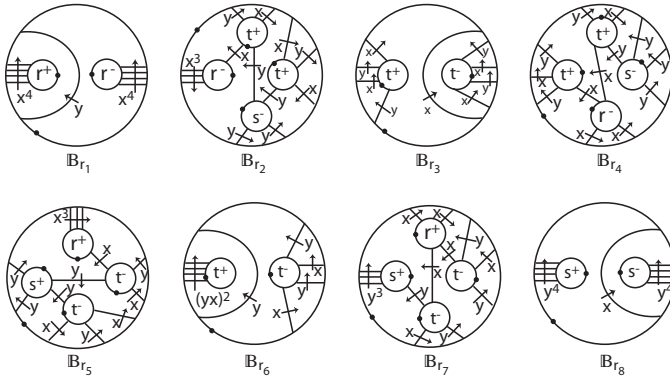


FIGURE 15. Tile $t_i \in T_4$ determines picture B_{r_i} over $\mathcal{Q} = \langle x, y \mid x^4, y^4, (xy)^2 \rangle$.

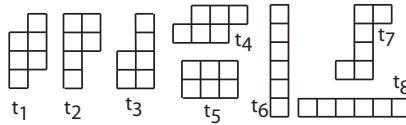


FIGURE 16. A tile set consisting of 6-ominoes.

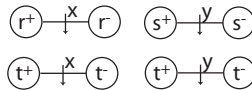


FIGURE 17. Local configurations in a picture over $\mathcal{Q} = \langle x, y \mid r = x^a, s = y^b, t = (xy)^c \rangle$.

graph on the sphere. Each edge gets labeled with x or y ; each vertex by $r = x^a$, $s = y^b$, or $t = (xy)^c$. One can define an angle α at each corner of the graph and subsequently a curvature function γ on the vertices and faces of this graph as follows: $\gamma(v) = 2\pi - \sum \alpha_i$ and $\gamma(f) = 2\pi - \sum (\pi - \alpha_i)$ where, in each case, the sum runs over all angles about the vertex or over all interior angles of the face, respectively. Let V , E and F denote the number of vertices, edges and faces in the resulting graph. Since the graph will have twice as many angles as edges

one can check that

$$\sum_{\text{all } v} \gamma(v) + \sum_{\text{all } f} \gamma(f) = 2\pi(V - E + F) = 2\pi\chi(\mathbb{S}^2) = 4\pi.$$

It follows that some face or vertex of the graph must have positive curvature.

Assign an angle to each corner at each vertex according to this rule: If the corner consists of two x edges, assign the angle $2\pi/a$. If the corner consists of two y edges, assign the angle $2\pi/b$, and if the corner consists of one x edge and one y edge, assign the angle π/c . With this angle assignment, the curvature at each vertex in the graph obtained from \mathbb{P} is 0.

Now, we assume \mathbb{P} has no dipole, and we argue that the curvature of any face will be less than or equal to 0. To see this, note that any vertex labeled r or s can only have t vertices as neighbors, and vice versa. It follows that any face in the graph will be formed by $4k$ vertices for some $k \geq 1$ having vertex label sequence $(rtst)(rtst)\cdots(rtst)$. The curvature of such a face F can be summed by grouping the corners by vertex label. This face will have k vertices labeled by r , k labeled by s and $2k$ labeled by t , so

$$\begin{aligned} \gamma(f) &= 2\pi - k(\pi - 2\pi/a) - k(\pi - 2\pi/b) - 2k(\pi - \pi/c) \\ &= 2\pi(1 - 2k + k(1/a + 1/b + 1/c)) \\ &\leq 2\pi(1 - k) \text{ since } 1/a + 1/b + 1/c \leq 1 \\ &\leq 0 \text{ since } k \geq 1. \end{aligned}$$

But this contradicts the fact that some disk or region must have positive curvature, so \mathbb{P} must have a dipole. By a series of bridge moves the disks in the dipole can be split off from \mathbb{P} . In this way, \mathbb{P} is equivalent to a finite sum of spherical pictures, each one consisting of two disks with the same relator label but oppositely oriented. Thus, the image of $[\mathbb{P}]$ under the Hurewicz map is trivial, and Y is Cockcroft. □

In fact, this argument tells us a bit more, namely, that $\pi_2(Y)$ is generated as a left $\mathbb{Z}\pi_1(Y)$ -module by the spherical pictures in Figure 18.

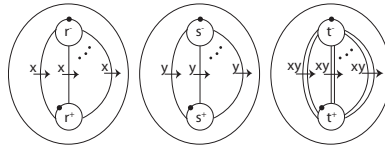


FIGURE 18. Generators of $\pi_2(Y_{\mathcal{Q}})$, where $\mathcal{Q} = \langle x, y \mid r = x^a, s = y^b, t = (xy)^c \rangle$ and $1/a + 1/b + 1/c \leq 1$.

5. A tiling question. In [9, Theorem 7.1], Pak determined which $a \times b$ rectangles could be tiled by the subset $S = \{t_2, t_3, t_5, t_8\}$ of T_4 given in Figure 19. In the spirit of tiling questions involving mutilated checkerboards, we determine which modified rectangles $M(a, b)$ (obtained by removing the upper left and lower right squares of an a by b rectangle) can be tiled by S .

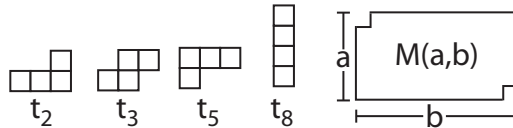


FIGURE 19. The tile set S .

The key to Pak’s argument, and to ours, is that if a region Γ can be tiled by T_4 , then the number of tiles in any tiling of Γ that come from S must be constant, modulo 2. This invariant, called the *height invariant* in [9], can be stated as follows: if Γ can be tiled by T_4 and α is a tiling of Γ by T_4 , then

$$b_2(\alpha) + b_3(\alpha) + b_5(\alpha) + b_8(\alpha) \equiv h(\Gamma) \pmod{2}$$

where $h(\Gamma)$ is a constant depending only on the region Γ . The height invariant can be derived by combining the invariant of Theorem 4.3 with an invariant that comes from a coloring argument. We work through this derivation following the proof of Lemma 5.3. Now we wish to emphasize its usefulness.

In the context of our tiling question and the set S , we say that T_4 *oddly tiles* a region Γ if T_4 tiles Γ in such a way that an odd number of the tiles used come from the complementary set $T_4 \setminus S$. Figure 20

shows three modified rectangles tiled by S , and two regions oddly tiled by T_4 .

The following “odd tiling” lemma gives us a constructive tool for proving the non-tileability of a region Γ by the set S .

Lemma 5.1. *If T_4 oddly tiles Γ then S does not tile Γ .*

Proof. In any tiling of a region Γ by T_4 the total number of tiles used is constant. If T_4 oddly tiles Γ , then T_4 tiles Γ with $m + k$ tiles where $m \geq 0$ is the number of tiles from S and $k \geq 1$ is the number of tiles from $T_4 \setminus S$ with k odd. Then $h(\Gamma) \equiv m \pmod{2}$. If S itself tiles Γ , it would have to do so with $m + k$ tiles and we would have $h(\Gamma) \equiv m + k \pmod{2}$ as well. But, since k is odd, this would be a contradiction. \square

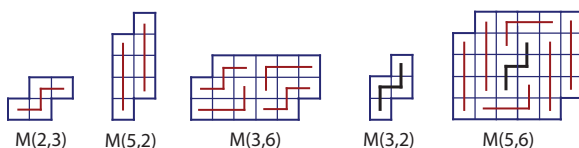


FIGURE 20. Three modified rectangles tiled by S , and two oddly tiled by T_4 .

Theorem 5.2. *Let $a, b > 1$. The set S tiles $M(a, b)$ if and only if*

1. $a \equiv 2 \pmod{4}$ and b is odd; or
2. a is odd and $ab \equiv 2 \pmod{8}$.

Proof. We begin by showing that tilings exist in these two cases. First, suppose $a \equiv 2 \pmod{4}$ and b is odd. Observe that S tiles $M(2, 3)$, as indicated in Figure 20. If S tiles $M(2, b)$, then S tiles $M(2, b + 2)$ by tacking on a copy of the tile t_3 , as suggested in Figure 21. It follows inductively that S tiles $M(2, b)$ for any odd $b \geq 3$. Furthermore, if S tiles $M(a, b)$, then S tiles $M(a + 4, b)$ by Lemma 5.3. Thus, S tiles $M(a, b)$ if $a \equiv 2 \pmod{4}$ and b is odd.

Next, suppose a is odd and $ab \equiv 2 \pmod{8}$. Then either $a \equiv 1 \pmod{4}$ and $b \equiv 2 \pmod{8}$; or $a \equiv 3 \pmod{4}$ and $b \equiv 6 \pmod{8}$. Observe that S tiles $M(5, 2)$ and $M(3, 6)$, as in Figure 20. Using

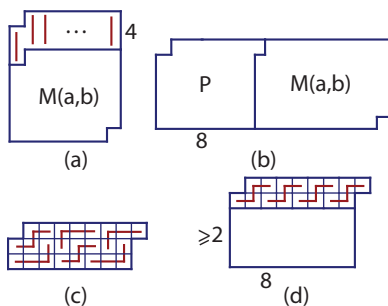


FIGURE 22. (a) Going from $M(a, b)$ to $M(a + 4, b)$; (b) going from $M(a, b)$ to $M(a, b + 8)$; (c) tiling P when $a = 3$; (d) tiling P when $a \geq 4$.

then P is $M(2, 9)$ and may be tiled with 4 copies of the tile t_3 . If $a = 3$, then P may be tiled by S as in Figure 22 (c), and if $a \geq 4$, then P may be tiled by S according to Figure 22 (d), noting that S tiles the rectangle in the figure since its width is 8 and its height is at least 2. (To see this, find a tiling of the 2×8 and 3×8 rectangle by S .) In this way a tiling (respectively, odd tiling) of $M(a, b)$ by S (respectively, T_4) may be extended to a tiling (respectively, odd tiling) of $M(a, b + 8)$ by S (respectively, T_4). \square

We now derive the height invariant by first considering a coloring argument. Following [7, Section 2], “color” the squares of the lattice in horizontal swaths using integers 0, 1, 2, 3 as indicated in Figure 23. Note that, for each tile in T_4 , the sum of its square colors, wherever the tile is placed in the lattice, is constant modulo 4. That is, this coloring produces a well-defined function $\phi : \mathcal{R} \rightarrow \mathbb{Z}_4$, where $\phi(\Gamma)$ equals the sum of the colors in Γ , modulo 4. Moreover, we have $\phi(t_1) = 0$, $\phi(t_2) = 1$, $\phi(t_3) = 2$, $\phi(t_4) = 3$, $\phi(t_5) = 3$, $\phi(t_6) = 0$, $\phi(t_7) = 1$ and $\phi(t_8) = 2$, which yields a tile invariant that Pak calls the shade invariant in [9]. If α denotes a tiling of Γ by T_4 , then

$$b_2(\alpha) + 2b_3(\alpha) + 3b_4(\alpha) + 3b_5(\alpha) + b_7(\alpha) + 2b_8(\alpha) \equiv \phi(\Gamma) \pmod{4}.$$

Combine this shade invariant with the tile invariant in Theorem 4.3 and view the result modulo 2 to obtain the height invariant.

We close this discussion by showing that Theorem 5.2 cannot have been proved using a coloring argument. A *signed tiling* of a region Γ

0	0	0	0	0	0	0
3	3	3	3	3	3	3
2	2	2	2	2	2	2
1	1	1	1	1	1	1
0	0	0	0	0	0	0

FIGURE 23. A coloring of the integer lattice

by a set T of tiles consists of a placement of tiles from T , each given a weight of $+1$ or -1 , in such a way that the sum of the weights of the tiles is $+1$ for each square inside the region Γ , and 0 for each square outside the region. There is a simple connection between coloring arguments and signed tilings, as developed in [4, Theorem 5.2] and [9, Theorem 8.1]. In particular, if a coloring argument can be used to show a region is not tileable by T , then there is no signed tiling of the region by T . This means that (a) if you know a region Γ cannot be tiled by T , and (b) there exists a signed tiling of Γ by T , then the untileability of Γ by T cannot have been proved using a coloring argument.

Lemma 5.4. *If T_4 oddly tiles a simply connected region Γ , then there exists a signed tiling of Γ by S .*

Proof. Figure 24 shows a signed tiling of each tile in $T_4 \setminus S$ by the set S . Then, if T_4 oddly tiles a region Γ , assign weight $+1$ to each tile in the odd tiling that comes from S , and replace each of the tiles from $T_4 \setminus S$ with a signed tiling of that tile by the set S . This produces a signed tiling of Γ by S . □

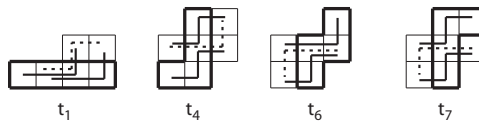


FIGURE 24. Signed tilings of each tile in $T_4 \setminus S$ by the set S .

In the proof of Theorem 5.2 we found that T_4 oddly tiled $M(a, b)$ in various cases. By Lemma 5.4, it follows that there are signed tilings of $M(a, b)$ by S in these cases, and the following result is now immediate.

Theorem 5.5. *Theorem 5.2 cannot be proved with a coloring argument.*

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